A NEW PROOF THAT TEICHMÜLLER SPACE IS A CELL

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ABSTRACT. A new proof is given, using the energy of a harmonic map, that Teichmüller space is a cell.

In [2] the authors developed a new approach to Teichmüller's famous theorem on the dimension of the unramified moduli space for compact Riemann surfaces. Teichmüller's theorem states (roughly) that the space \( \mathcal{T} \) of conformally inequivalent Riemann surfaces of genus \( p, p > 1 \) (with some topological restrictions) is homeomorphic to Euclidean \( \mathbb{R}^{6p-6} \). In proving homeomorphism Teichmüller had put a complete Finsler metric on this space. In [2] we showed that \( \mathcal{T} \) naturally carried the structure of a \( C^\infty \) connected and simply connected differentiable manifold of dimension \( 6p - 6 \). The proof of this was straightforward and used only splitting results for symmetric tensors and a standard existence theorem in elliptic partial differential equations. Using somewhat deeper results from the theory of harmonic functions between Riemannian manifolds and a result of Earle and Eells, we were then able to show that our moduli space \( \mathcal{T} \) was a contractible manifold.

The purpose of this note is to show that there is a straightforward proof that our Teichmüller space is diffeomorphic to \( \mathbb{R}^{6p-6} \). This completes the program of giving the main classical results of Teichmüller strictly in terms of concepts from Riemannian geometry as was formulated in [2, 3, 4].

1. A quick review of the Fischer-Tromba approach to Teichmüller theory. Let \( M \) be a compact oriented surface without boundary. Let \( \mathcal{C} \) denote the space of complex structures compatible with the given orientation, \( \mathcal{D} \) the space of \( C^\infty \) diffeomorphisms, \( \mathcal{D}_0 \) those homotopic (and hence isotopic) to the identity, and \( \mathcal{M}_{-1} \) those Riemannian metrics on \( M \) with Riemann scalar curvature negative one. If \( c = \{ \phi_i, U_i \}, \bigcup U_i = M \), is a complex coordinate atlas for \( M \) and \( f \in \mathcal{D} \), then \( \{ \phi_i \circ f, f^{-1}(U_i) \} \) is a complex coordinate atlas for \( M \) which we designate as \( f^*c \). If \( g \in \mathcal{M}_{-1} \), then for each \( x \in M \), \( g(x) : T_xM \times T_xM \to \mathbb{R} \) is a positive definite symmetric quadratic form on \( M \). By \( f^*g \) we mean the form \( g(f(x))(df(x)\cdot, df(x)\cdot) \).

One can then form the quotient spaces \( \mathcal{M}_{-1}/\mathcal{D}, \mathcal{M}_{-1}/\mathcal{D}_0, \mathcal{C}/\mathcal{D}, \mathcal{C}/\mathcal{D}_0 \). The main result of [2] is

**Theorem 1.1.** The spaces \( \mathcal{T} = \mathcal{M}_{-1}/\mathcal{D}_0 \) and \( \mathcal{C}/\mathcal{D}_0 \) naturally have the structure of a \( C^\infty \) connected and simply connected finite-dimensional manifold of dimension
Moreover there is a naturally defined equivariant diffeomorphism from $\mathcal{M}_1$ to $\mathcal{C}$ which passes to a diffeomorphism of $\mathcal{M}_1/D_0$ with $\mathcal{C}/D_0$. The space $\mathcal{C}/D_0 \cong \mathcal{M}_1/D_0$ is the Teichmüller space of $\mathcal{M}$.

We should remark that the true Riemann space of moduli $R = \mathcal{M}_1/D = \mathcal{C}/\mathcal{D}$ is not a smooth manifold but does have the structure of an algebraic variety.

For purposes of exposition we wish to describe how one puts a differentiable structure on $\mathcal{M}_1/D_0$ and to see what the natural tangent space is to this manifold. To see how the diffeomorphism between $\mathcal{M}_1/D_0$ and $\mathcal{C}/D_0$ is constructed the reader is referred to [3]. Let us think of $\mathcal{M}_1$ as an infinite dimensional submanifold of the space of $C^\infty$ symmetric two tensors $S_2$ on $\mathcal{M}$. For $g \in \mathcal{M}_1$, the tangent space $T_g\mathcal{O}_g$ to the orbit $\mathcal{O}_g$ of $D_0$ at $g$ consists of all symmetric tensors of the form $L_xg$, the Lie derivative of $g$ with respect to some vector field $X$ on $\mathcal{M}$. In this case $X$ will be uniquely determined by $h \in T_g\mathcal{M}_1$. The next splitting result of symmetric tensors is basic to our theory.

**Theorem 1.2.** Every $h \in T_g\mathcal{M}_1$ can be expressed uniquely as a direct sum $h = h^{TT} + L_Xg$ where $h^{TT}$ is a symmetric two tensor on $\mathcal{M}$ which is trace free and divergence free. This implies that in a conformal coordinate system (with respect to the metric $g$), $h^{TT}$ has a local representation as

$$h^{TT} = u dx^2 - u dy^2 - 2v dx dy = \text{Re}\{(u + iv)(dx + idy)^2\}$$

where $u + iv$ is a holomorphic function of the local coordinates $z = x + iy$, and $\text{Re}$ designates the real part.

Thus every $h \in T_g\mathcal{M}_1$ can be expressed uniquely as a direct sum

$$h = \text{Re}(\xi(z)dz^2) + L_Xg$$

where $\xi(z)dz^2$ is a holomorphic quadratic differential on $\mathcal{M}$ with respect to the complex structure induced by $g$. Moreover every such holomorphic quadratic differential occurs in decomposition (1).

Now the $C^\infty$ manifold structure on $\mathcal{M}_1/D_0$ follows as a consequence of fact that $D_0$ acts freely and that as a consequence of the theorem of Riemann-Roch the space of holomorphic quadratic differentials on $\mathcal{M}$ has finite dimension $6p - 6$.

We summarize these facts as

**Theorem 1.3.** The tangent space to the manifold $\mathcal{M}_1/D_0$ at an element $[g] \in \mathcal{M}_1/D_0$ can be naturally identified with those symmetric two tensors which are trace free and divergence free and also (by taking real parts) with the holomorphic quadratic differentials on $\mathcal{M}$, holomorphic with respect to the complex structure induced by $g$.

As we already stated, we are viewing $\mathcal{M}_1$ as a differentiable submanifold of the space of all symmetric tensors $S_2$. There is a natural “weak” $L^2$ Riemannian structure $\langle \langle \ , \ \rangle \rangle$ on $\mathcal{M}_1$, $\langle \langle \ , \ \rangle \rangle_g : T_g\mathcal{M}_1 \times T_g\mathcal{M}_1 \rightarrow R$ defined by

$$\langle \langle h^1, h^2 \rangle \rangle_g = \int_{\mathcal{M}} h^1 \cdot h^2 \, d\mu(g)$$

where, $m$ local coordinates,

$$h^1 \cdot h^2 = g^{ab} g^{cd} h^1_{ac} h^2_{bd}$$
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and where \( g^{ab} \) denotes the local representation of the inverse to the matrix \( \{ g_{ij} \} \) of \( g \), \( d\mu(g) \) is the volume element of \( g \), and where the Einstein summation convention is used. One can also give an intrinsic formulation of (2) avoiding local coordinates, as follows.

Using the metric \( g \) we can transform \( h^1, h^2 \) into 1 : 1 tensors \( H^1, H^2 \) satisfying

\[
g(x)(H^1_x X_x, Y_x) = h^i(x)(X_x, Y_x), \quad i = 1, 2, \]

for all \( X_x, Y_x \in T_x \). Then each \( H^i \) is symmetric with respect to \( g \) and for \( x \in M \) the trace \( \text{tr}(H^1_x H^2_x) \) is a well-defined function (of \( x \)) on \( M \). Then (2) is equivalent to

\[
(2') \quad \langle (h^1, h^2) \rangle_g = \int_M \text{tr}(H^1 H^2) \, d\mu(g).
\]

This \( L^2 \)-Riemannian metric is \( \mathcal{D} \) invariant, a fact which follows immediately from the change of variables formula. Thus \( \mathcal{D} \) acts on \( M_\mathcal{D} \) as a group of isometries.

The important remark is that (1) is an \( L^2 \)-orthogonal decomposition.

2. Dirichlet’s functional on Teichmüller space. Let \( g_0 \in M_\mathcal{D} \) and \([g_0]\) denote its class in \( M_\mathcal{D}/\mathcal{D}_0 \). This fixed \( g_0 \) will act as our base point. Let \( g \in M_\mathcal{D} \) be any other metric and let \( s: M \to M \) be viewed as a map from \((M, g)\) to \((M, g_0)\). Using the metrics \( g \) and \( g_0 \) one defines Dirichlet’s energy functional

\[
E_g(s) = \frac{1}{2} \int_M |ds|^2 \, d\mu(g)
\]

where \( |ds|^2 = \text{trace}_g \, ds \otimes ds \) depends on both metrics \( g \) and \( g_0 \), and again \( d\mu(g) \) is the volume element induced by \( g \).

We may assume that \((M, g_0)\) is isometrically embedded in some Euclidean \( \mathbb{R}^k \), which is possible by the Nash-Moser embedding theorem. Thus we can think of \( s: (M, g) \to (M, g_0) \) as a map into \( \mathbb{R}^k \) with Dirichlet’s integral having the equivalent form

\[
(3') \quad E_g(s) = \frac{1}{2} \sum_{i=1}^k \int_M g(x) \langle \nabla g s^i(x), \nabla g s^i(x) \rangle \, d\mu(g).
\]

For fixed \( g \), the critical points of \( E \) are then said to be harmonic maps. From [1, 5 and 8] we have the following result.

THEOREM 2.1. Given metrics \( g \) and \( g_0 \) there exists a unique harmonic map \( s(g): (M, g) \to (M, g_0) \). Moreover \( s(g) \) depends differentiably on \( g \) in any \( H^r \) topology, \( r > 2 \), and \( s(g) \) is a \( C^\infty \) diffeomorphism.

Consider the function \( g \to E_g(s(g)) \). This function on \( M_\mathcal{D} \) is \( \mathcal{D} \)-invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that \( E_{f^*g}(s(f^*(g))) = E_g(s(g)) \). Let \( c(g) \) be the complex structure associated to \( g \) given by Theorem 1.1. For \( f \in \mathcal{D}_0, f: (M, f^*c(g)) \to (M, c(g)) \) is a holomorphic map, and consequently since the composition of harmonic maps and holomorphic maps is still harmonic, we may conclude, by uniqueness, that \( S(f^*g) = S(g) \circ f \). Since Dirichlet’s functional is invariant under complex holomorphic changes of coordinates, it follows immediately that

\[
E_{f^*g}(s(g) \circ f) = E_g(s(g)).
\]
Consequently for \( [g] \in M_{-1}/D_0 \), define the \( C^\infty \) smooth function \( \tilde{E}: M_{-1}/D_0 \to \mathbb{R} \) by \( \tilde{E}(g) = E_g(s(g)) \). We wish now to prove the main theorem of this note, namely

**Theorem 2.2.** Teichmüller space \( M_{-1}/D_0 \) is \( C^\infty \) diffeomorphic to \( \mathbb{R}^{6p-6} \).

To prove this result it suffices to show that \( \tilde{E} \) has the following properties:

(i) \( \tilde{E} \) is a proper map, i.e. the inverse image of bounded sets in \( \mathbb{R} \) under \( \tilde{E} \) is compact in \( M_{-1}/D_0 \).

(ii) \( [g_0] \) is the only critical point of \( \tilde{E} \).

(iii) \( [g_0] \) is a nondegenerate minimum.

Once (i) through (iii) are established the result follows immediately from the application of the well-known gradient deformations of Morse theory.

The proof of (i) follows from ideas due to Mumford, Schoen and Yau [7], and a result on equicontinuity of harmonic maps (Jost [6, p. 20]). Using a result of Mumford, Schoen and Yau show that \( E: M_{-1}/D_0 \to \mathbb{R} \) is proper; that is, \( E \) is proper on the true space \( \mathcal{R} \) of Riemann moduli.

Now suppose that \( E[g_n] \) is a bounded sequence. It then follows from [7] that \( \{g_n\} \) represents a sequence of a class of metrics in \( M_{-1}/D_0 \) all of whose injectivity radii are strictly bounded below. By a version of Mumford’s theorem due to Tomi-Tromba [10], it follows that there is a subsequence of \( g_n \), call it again \( g_n \) and a sequence of diffeomorphisms \( f_n \in D \) such that \( f_n^*g_n \) converges.

Let \( \gamma_n = f_n^*g_n; r_n = s_n \circ f_n \). Then \( E(\gamma_n, r_n) \) is a bounded sequence of real numbers, the \( \gamma_n \) all have injectivity radii strictly bounded below, and \( r_n: (M, \gamma_n) \to (M, g_0) \) is harmonic. We claim that one can find a subsequence \( f_n \) all of which are in the same homotopy class.

Suppose not. Then there is a subsequence of the \( f_n \) all in distinct homotopy classes. Again call the subsequence \( f_n \). From Jost’s result, the \( r_n = s_n \circ f_n \) are equicontinuous. Since the \( s_n \) are all homotopic to the identity, this gives a contradiction.

Thus we may assume the \( f_n \) are in one homotopy class, \( f_n = h_n \circ f, f \in D \) fixed and \( h_n \in D_0 \). Then necessarily \( h_n^*g_n \) (or more simply \( [g_n] \)) converges. This proves properness on \( M_{-1}/D_0 \).

To show (ii), again let \( s = s(g): (M, g_0) \to (M, g_0) \) be the unique harmonic map determined by \( g \) and \( g_0 \). Let \( \mathcal{N}_g(z) \, dz^2 \) be the quadratic differential defined by

\[
\mathcal{N}_g(z) \, dz^2 = \sum_{i=1}^{k} \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial \bar{z}} \, dz \, d\bar{z},
\]

where \( s^i \) is the \( i \)th component function of \( s: (M, g) \to (M, g_0) \) \( \hookrightarrow \mathbb{R}^k \), and \( z = x + iy \) are local conformal coordinates on \((M, g)\). We next prove

**Theorem 2.3.** \( \mathcal{N}_g(z) \, dz^2 \) is a holomorphic quadratic differential on \((M, c(g))\).

**Proof.** Let \( \Omega \) denote the second fundamental form of \((M, g_0) \subset \mathbb{R}^k \). Thus for each \( p \in M \), \( \Omega(p): T_pM \times T_pM \to T_pM^\perp \). Let \( \Delta \) denote the Laplacian maps from \((M, g)\) to \((M, g_0)\), and \( \Delta \beta \) denote the Laplace-Beltrami operator on functions. Then if \( s \) is harmonic we have

\[
0 = \Delta s = \Delta \beta s + \sum_{j=1}^{2} \Omega(s)(ds(e_j), \, ds(e_j))
\]
with $e_1(p), e_2(p)$ an orthonormal basis for $T_p M$ (w.r.t. the matrix $g$). $N_g$ will be
holomorphic if
\[
\frac{\partial}{\partial z} \left( \sum_{i=1}^k \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z} \right) = 0.
\]
But this equals
\[
2 \sum_{i=1}^k \Delta g s^i \cdot \frac{\partial s^i}{\partial z}
\]
and by (4) we see that this in turn equals
\[
-2 \sum_{i=1}^k \sum_{j=1}^2 \Omega^i(s) \left( ds(e_j), ds(e_j) \right) \cdot \frac{\partial s^i}{\partial z}
\]
\[
= -2 \sum_{i=1}^k \sum_{j=1}^2 \left\{ \sum_{i=1}^k \Omega^i(s) \left( ds(e_j), ds(e_j) \cdot \frac{\partial s^i}{\partial x} \right) + i\Omega(s) \left( ds(e_j), ds(e_j) \cdot \frac{\partial s^i}{\partial y} \right) \right\}.
\]
Since $\Omega(p)$ takes value in $T_p M^\perp$ it follows that both the real and the imaginary
parts of this expression vanish. $G$

From 1.2 we saw that $\xi = \text{Re}(N_g(z) dz^2)$ is a trace free divergence free symmetric
two tensor on $(M, g)$. Let $\rho \in T[g, M_{-1}/D_0$. We know from 1.3 that we may think
of $\rho$ as a trace free divergence free symmetric two tensor. From [10] we have the
following result:

**Theorem 2.4.** $D^2 \tilde{E}([g]) \rho = - \langle \langle \xi, \rho \rangle \rangle_g$. Thus $[g]$ is a critical point of $\tilde{E}$ if
$\rho = 0 \equiv \text{Re}(N_g(z) dz^2)$, or if $N_g(z) dz^2 \equiv 0$.

**Theorem 2.5.** $N_g(z) dz^2 = 0$ implies that $[g] = [g_0]$.

**Proof.** $N_g(z) dz^2 = \left\{ |s_x|^2 - |s_y|^2 + 2i\langle s_x, s_y \rangle \right\} dz^2$. Thus $N_g(z) dz^2$ implies
that $s$ is weakly conformal. Since $s$ is a diffeomorphism it is conformal. Thus
$s: (M, c(g)) \to (M, c([g]))$ is holomorphic and hence $[g] = [g_0]$.

It remains to show (iii). It is clear that since $N_g(z) dz^2 \equiv 0 (s(g_0) = id)$ that $[g_0]$ is a critical point.

Let $\rho, \nu \in T[g_0, M_{-1}/D_0$ be trace free, and divergence free symmetric two tensors. Then a straightforward computation yields

**Theorem 2.6.** The second derivative or Hessian of $\tilde{E}$ at $[g_0]$ is given by the
formula
\[
D^2 \tilde{E}([g_0])(\rho, \nu) = 2 \int_M \rho \cdot \nu \ d\mu(g_0) = 2 \langle \langle \rho, \nu \rangle \rangle_{g_0}.
\]

Thus the Hessian of $\tilde{E}$ at $[g_0]$ is essentially the natural inner product on
$T[g_0, M_{-1}/D_0$ and hence a positive definite quadratic form. This concludes the proof
of our main result 2.2.

**References**

1. J. Eells and L. Lemaire, *Deformations of metrics and associated harmonic maps*, Petodi Memo-
   rial Volume, and preprint IHES.
2. A. E. Fischer and A. J. Tromba, *On a purely Riemannian proof of the structure and dimension

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