

A NEW PROOF THAT TEICHMÜLLER SPACE IS A CELL

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ABSTRACT. A new proof is given, using the energy of a harmonic map, that Teichmüller space is a cell.

In [2] the authors developed a new approach to Teichmüller's famous theorem on the dimension of the unramified moduli space for compact Riemann surfaces. Teichmüller's theorem states (roughly) that the space \mathcal{T} of conformally inequivalent Riemann surfaces of genus p , $p > 1$ (with some topological restrictions) is homeomorphic to Euclidean \mathbf{R}^{6p-6} . In proving homeomorphism Teichmüller had put a complete Finsler metric on this space. In [2] we showed that \mathcal{T} naturally carried the structure of a C^∞ connected and simply connected differentiable manifold of dimension $6p - 6$. The proof of this was straightforward and used only splitting results for symmetric tensors and a standard existence theorem in elliptic partial differential equations. Using somewhat deeper results from the theory of harmonic functions between Riemannian manifolds and a result of Earle and Eells, we were then able to show that our moduli space \mathcal{T} was a contractible manifold.

The purpose of this note is to show that there is a straightforward proof that our Teichmüller space is diffeomorphic to \mathbf{R}^{6p-6} . This completes the program of giving the main classical results of Teichmüller strictly in terms of concepts from Riemannian geometry as was formulated in [2, 3, 4].

1. A quick review of the Fischer-Tromba approach to Teichmüller theory. Let M be a compact oriented surface without boundary. Let \mathcal{C} denote the space of complex structures compatible with the given orientation, \mathcal{D} the space of C^∞ diffeomorphisms, \mathcal{D}_0 those homotopic (and hence isotopic) to the identity, and \mathcal{M}_{-1} those Riemannian metrics on M with Riemann scalar curvature negative one. If $c = \{\varphi_i, U_i\}$, $\bigcup U_i = M$, is a complex coordinate atlas for M and $f \in \mathcal{D}$, then $\{\varphi_i \circ f, f^{-1}(U_i)\}$ is a complex coordinate atlas for M which we designate as f^*c . If $g \in \mathcal{M}_{-1}$, then for each $x \in M$, $g(x): T_x M \times T_x M \rightarrow \mathbf{R}$ is a positive definite symmetric quadratic form on M . By f^*g we mean the form $g(f(x))(df(x)\cdot, df(x)\cdot)$.

One can then form the quotient spaces $\mathcal{M}_{-1}/\mathcal{D}$, $\mathcal{M}_{-1}/\mathcal{D}_0$, \mathcal{C}/\mathcal{D} , $\mathcal{C}/\mathcal{D}_0$. The main result of [2] is

THEOREM 1.1. *The spaces $\mathcal{T} = \mathcal{M}_{-1}/\mathcal{D}_0$ and $\mathcal{C}/\mathcal{D}_0$ naturally have the structure of a C^∞ connected and simply connected finite-dimensional manifold of dimension*

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6p - 6. Moreover there is a naturally defined equivariant diffeomorphism from M_{-1} to C which passes to a diffeomorphism of M_{-1}/D_0 with C/D_0 . The space $C/D_0 \cong M_{-1}/D_0$ is the Teichmüller space of M .

We should remark that the true Riemann space of moduli $R = M_{-1}/D = C/D$ is not a smooth manifold but does have the structure of an algebraic variety.

For purposes of exposition we wish to describe how one puts a differentiable structure on M_{-1}/D_0 and to see what the natural tangent space is to this manifold. To see how the diffeomorphism between M_{-1}/D_0 and C/D_0 is constructed the reader is referred to [3]. Let us think of M_{-1} as an infinite dimensional submanifold of the space of C^∞ symmetric two tensors S_2 on M . For $g \in M_{-1}$, the tangent space $T_g O_g$ to the orbit O_g of D_0 at g consists of all symmetric tensors of the form $L_X g$, the Lie derivative of g with respect to some vector field X on M . In this case X will be uniquely determined by $h \in T_g M_{-1}$. The next splitting result of symmetric tensors is basic to our theory.

THEOREM 1.2. *Every $h \in T_g M_{-1}$ can be expressed uniquely as a direct sum $h = h^{TT} + L_X g$ where h^{TT} is a symmetric two tensor on M which is trace free and divergence free. This implies that in a conformal coordinate system (with respect to the metric g), h^{TT} has a local representation as*

$$h^{TT} = u dx^2 - u dy^2 - 2v dx dy = \text{Re}\{(u + iv)(dx + i dy)^2\}$$

where $u + iv$ is a holomorphic function of the local coordinates $z = x + iy$, and Re designates the real part.

Thus every $h \in T_g M_{-1}$ can be expressed uniquely as a direct sum

$$(1) \quad h = \text{Re}(\xi(z) dz^2) + L_X g$$

where $\xi(z) dz^2$ is a holomorphic quadratic differential on M with respect to the complex structure induced by g . Moreover every such holomorphic quadratic differential occurs in decomposition (1).

Now the C^∞ manifold structure on M_{-1}/D_0 follows as a consequence of fact that D_0 acts freely and that as a consequence of the theorem of Riemann-Roch the space of holomorphic quadratic differentials on M has finite dimension $6p - 6$.

We summarize these facts as

THEOREM 1.3. *The tangent space to the manifold M_{-1}/D_0 at an element $[g] \in M_{-1}/D_0$ can be naturally identified with those symmetric two tensors which are trace free and divergence free and also (by taking real parts) with the holomorphic quadratic differentials on M , holomorphic with respect to the complex structure induced by g .*

As we already stated, we are viewing M_{-1} as a differentiable submanifold of the space of all symmetric tensors S_2 . There is a natural "weak" L_2 Riemannian structure $\langle\langle \cdot, \cdot \rangle\rangle$ on M_{-1} , $\langle\langle \cdot, \cdot \rangle\rangle_g: T_g M_{-1} \times T_g M_{-1} \rightarrow R$ defined by

$$(2) \quad \langle\langle h^1, h^2 \rangle\rangle_g = \int_M h^1 \cdot h^2 d\mu(g)$$

where, m local coordinates,

$$h^1 \cdot h^2 = g^{ab} g^{cd} h_{ac}^1 h_{bd}^2$$

and where $\{g^{ab}\}$ denotes the local representation of the inverse to the matrix $\{g_{ij}\}$ of g , $d\mu(g)$ is the volume element of g , and where the Einstein summation convention is used. One can also give an intrinsic formulation of (2) avoiding local coordinates, as follows.

Using the metric g we can transform h^1, h^2 into 1 : 1 tensors H^1, H^2 satisfying

$$g(x)(H_x^i X_x, Y_x) = h^i(x)(X_x, Y_x), \quad i = 1, 2,$$

for all $X_x, Y_x \in T_x M$. Then each H^i is symmetric with respect to g and for $x \in M$ the trace $\text{tr}(H_x^1 H_x^2)$ is a well-defined function (of x) on M . Then (2) is equivalent to

$$(2') \quad \langle\langle h^1, h^2 \rangle\rangle_g = \int_M \text{tr}(H^1 H^2) d\mu(g).$$

This L_2 -Riemannian metric is \mathcal{D} invariant, a fact which follows immediately from the change of variables formula. Thus \mathcal{D} acts on \mathcal{M}_{-1} as a group of isometries.

The important remark is that (1) is an L_2 -orthogonal decomposition.

2. Dirichlet's functional on Teichmüller space. Let $g_0 \in \mathcal{M}_{-1}$ and $[g_0]$ denote its class in $\mathcal{M}_{-1}/\mathcal{D}_0$. This fixed g_0 will act as our base point. Let $g \in \mathcal{M}_{-1}$ be any other metric and let $s: M \rightarrow M$ be viewed as a map from (M, g) to (M, g_0) . Using the metrics g and g_0 one defines Dirichlet's energy functional

$$(3) \quad E_g(s) = \frac{1}{2} \int_M |ds|^2 d\mu(g)$$

where $|ds|^2 = \text{trace}_g ds \otimes ds$ depends on both metrics g and g_0 , and again $d\mu(g)$ is the volume element induced by g .

We may assume that (M, g_0) is isometrically embedded in some Euclidean \mathbf{R}^k , which is possible by the Nash-Moser embedding theorem. Thus we can think of $s: (M, g) \rightarrow (M, g_0)$ as a map into \mathbf{R}^k with Dirichlet's integral having the equivalent form

$$(3') \quad E_g(s) = \frac{1}{2} \sum_{i=1}^k \int_M g(x) \langle \nabla_g s^i(x), \nabla_g s^i(x) \rangle d\mu(g).$$

For fixed g , the critical points of E are then said to be *harmonic maps*. From [1, 5 and 8] we have the following result.

THEOREM 2.1. *Given metrics g and g_0 there exists a unique harmonic map $s(g): (M, g) \rightarrow (M, g_0)$. Moreover $s(g)$ depends differentiably on g in any H^r topology, $r > 2$, and $s(g)$ is a C^∞ diffeomorphism.*

Consider the function $g \rightarrow E_g(s(g))$. This function on \mathcal{M}_{-1} is \mathcal{D} -invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that $E_{f^*g}(s(f^*(g))) = E_g(s(g))$. Let $c(g)$ be the complex structure associated to g given by Theorem 1.1. For $f \in \mathcal{D}_0$, $f: (M, f^*c(g)) \rightarrow (M, c(g))$ is a holomorphic map, and consequently since the composition of harmonic maps and holomorphic maps is still harmonic, we may conclude, by uniqueness, that $S(f^*g) = S(g) \circ f$. Since Dirichlet's functional is invariant under complex holomorphic changes of coordinates, it follows immediately that

$$E_{f^*(g)}(s(g) \circ f) = E_g(s(g)).$$

Consequently for $[g] \in \mathcal{M}_{-1}/\mathcal{D}_0$, define the C^∞ smooth function $\tilde{E}: \mathcal{M}_{-1}/\mathcal{D}_0 \rightarrow \mathbf{R}$ by $\tilde{E}([g]) = E_g(s(g))$. We wish now to prove the main theorem of this note, namely

THEOREM 2.2. *Teichmüller space $\mathcal{M}_{-1}/\mathcal{D}_0$ is C^∞ diffeomorphic to \mathbf{R}^{6p-6} .*

To prove this result it suffices to show that \tilde{E} has the following properties:

(i) \tilde{E} is a proper map, i.e. the inverse image of bounded sets in \mathbf{R} under \tilde{E} is compact in $\mathcal{M}_{-1}/\mathcal{D}_0$.

(ii) $[g_0]$ is the only critical point of \tilde{E} .

(iii) $[g_0]$ is a nondegenerate minimum.

Once (i) through (iii) are established the result follows immediately from the application of the well-known gradient deformations of Morse theory.

The proof of (i) follows from ideas due to Mumford, Schoen and Yau [7], and a result on equicontinuity of harmonic maps (Jost [6, p. 20]). Using a result of Mumford, Schoen and Yau show that $E: \mathcal{M}_{-1}/\mathcal{D}_0 \rightarrow \mathbf{R}$ is proper; that is, E is proper on the true space \mathcal{R} of Riemann moduli.

Now suppose that $\tilde{E}[g_n]$ is a bounded sequence. It then follows from [7] that $\{g_n\}$ represents a sequence of a class of metrics in $\mathcal{M}_{-1}/\mathcal{D}_0$ all of whose injectivity radii are strictly bounded below. By a version of Mumford's theorem due to Tomi-Tromba [10], it follows that there is a subsequence of g_n , call it again g_n and a sequence of diffeomorphisms $f_n \in \mathcal{D}$ such that $f_n^*g_n$ converges.

Let $\gamma_n = f_n^*g_n; r_n = s_n \circ f_n$. Then $E(\gamma_n, r_n)$ is a bounded sequence of real numbers, the γ_n all have injectivity radii strictly bounded below, and $r_n: (M, \gamma_n) \rightarrow (M, g_0)$ is harmonic. We claim that one can find a subsequence f_n all of which are in the same homotopy class.

Suppose not. Then there is a subsequence of the f_n all in distinct homotopy classes. Again call the subsequence f_n . From Jost's result, the $r_n = s_n \circ f_n$ are equicontinuous. Since the s_n are all homotopic to the identity, this gives a contradiction.

Thus we may assume the f_n are in one homotopy class, $f_n = h_n \circ f$, $f \in \mathcal{D}$ fixed and $h_n \in \mathcal{D}_0$. Then necessarily $h_n^*g_n$ (or more simply $[g_n]$) converges. This proves properness on $\mathcal{M}_{-1}/\mathcal{D}_0$.

To show (ii), again let $s = s(g): (M, g_0) \rightarrow (M, g_0)$ be the unique harmonic map determined by g and g_0 . Let $\mathcal{N}_g(z) dz^2$ be the quadratic differential defined by

$$\mathcal{N}_g(z) dz^2 = \sum_{i=1}^k \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z} dz^2$$

where s^i is the i th component function of $s: (M, g) \rightarrow (M, g_0) \hookrightarrow \mathbf{R}^k$, and $z = x+iy$ are local conformal coordinates on (M, g) . We next prove

THEOREM 2.3. $\mathcal{N}_g(z) dz^2$ is a holomorphic quadratic differential on $(M, c(g))$.

PROOF. Let Ω denote the second fundamental form of $(M, g_0) \subset \mathbf{R}^k$. Thus for each $p \in M$, $\Omega(p): T_p M \times T_p M \rightarrow T_p M^\perp$. Let Δ denote the Laplacian maps from (M, g) to (M, g_0) , and Δ_β denote the Laplace-Beltrami operator on functions. Then if s is harmonic we have

$$(4) \quad 0 = \Delta s = \Delta_\beta s + \sum_{j=1}^2 \Omega(s)(ds(e_j), ds(e_j))$$

with $e_1(p), e_2(p)$ an orthonormal basis for T_pM (w.r.t. the matrix g). \mathcal{N}_g will be holomorphic if

$$\frac{\partial}{\partial \bar{z}} \left(\sum_{i=1}^k \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z} \right) = 0.$$

But this equals

$$2 \sum_{i=1}^k \Delta_\beta s^i \cdot \frac{\partial s^i}{\partial z}$$

and by (4) we see that this in turn equals

$$\begin{aligned} & -2 \sum_{i=1}^k \sum_{j=1}^2 \Omega^i(s) (ds(e_j), ds(e_j)) \cdot \frac{\partial s^i}{\partial z} \\ & = -2 \sum_{i=1}^k \sum_{j=1}^2 \left\{ \sum \Omega^i(s) \left(ds(e_j), ds(e_j) \frac{\partial s^i}{\partial x} \right) + i \Omega(s) \left(ds(e_j), ds(e_j) \frac{\partial s^i}{\partial y} \right) \right\}. \end{aligned}$$

Since $\Omega(p)$ takes value in T_pM^\perp it follows that both the real and the imaginary parts of this expression vanish. \square

From 1.2 we saw that $\xi = \text{Re}(\mathcal{N}_g(z) dz^2)$ is a trace free divergence free symmetric two tensor on (M, g) . Let $\rho \in T_{[g]}M_{-1}/\mathcal{D}_0$. We know from 1.3 that we may think of ρ as a trace free divergence free symmetric two tensor. From [10] we have the following result:

THEOREM 2.4. $D\tilde{E}([g])\rho = -\langle \xi, \rho \rangle_g$. Thus $[g]$ is a critical point of \tilde{E} if $\rho = 0 \equiv \text{Re}(\mathcal{N}_g(z) dz^2)$, or if $\mathcal{N}_g(z) dz^2 \equiv 0$.

THEOREM 2.5. $\mathcal{N}_g(z) dz^2 = 0$ implies that $[g] = [g_0]$.

PROOF. $\mathcal{N}_g(z) dz^2 = \{|s_x|^2 - |s_y|^2 + 2i\langle s_x, s_y \rangle\} dz^2$. Thus $\mathcal{N}_g(z) dz^2$ implies that s is weakly conformal. Since s is a diffeomorphism it is conformal. Thus $s: (M, c(g)) \rightarrow (M, c(g_0))$ is holomorphic and hence $[g] = [g_0]$.

It remains to show (iii). It is clear that since $\mathcal{N}_g(z) dz^2 \equiv 0$ ($s(g_0) = \text{id}$) that $[g_0]$ is a critical point.

Let $\rho, \nu \in T_{[g_0]}M_{-1}/\mathcal{D}_0$ be trace free, and divergence free symmetric two tensors. Then a straightforward computation yields

THEOREM 2.6. The second derivative or Hessian of \tilde{E} at $[g_0]$ is given by the formula

$$D^2\tilde{E}([g_0])(\rho, \nu) = 2 \int_M \rho \cdot \nu d\mu(g_0) = 2\langle \rho, \nu \rangle_{g_0}.$$

Thus the Hessian of \tilde{E} at $[g_0]$ is essentially the natural inner product on $T_{[g_0]}M_{-1}/\mathcal{D}_0$ and hence a positive definite quadratic form. This concludes the proof of our main result 2.2.

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