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SEMISTABILITY AT ∞, ∞-ENDED GROUPS AND GROUP COHOMOLOGY

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ABSTRACT. A finitely presented group G, is *semistable at* ∞ if for some (equivalently any) finite complex X, with $\pi_1(X) = G$, any two proper maps $r, s\colon [0,\infty) \to \tilde{X}$ ($\equiv$ the universal cover of X) that determine the same end of $\tilde{X}$ are properly homotopic in $\tilde{X}$.

If G is semistable at ∞, then $H^2(G; ZG)$ is free abelian. 0- and 2-ended groups are all semistable at ∞.

THEOREM. *If $G = A *_C B$ where C is finite and A and B are finitely presented, semistable at ∞ groups, then G is semistable at ∞.*

THEOREM. *If $\alpha\colon C \to D$ is an isomorphism between finite subgroups of the finitely presented semistable at ∞ group H, then the resulting HNN extension is semistable at ∞.*

Combining these results with the accessibility theorem of M. Dunwoody gives

THEOREM. *If all finitely presented 1-ended groups are semistable at ∞, then all finitely presented groups are semistable at ∞.*

1. Introduction. If K is a locally finite connected CW complex, then proper maps $r, s\colon [0,\infty) \to K$ *converge to the same end of* K if for any compact set $C \subset K$ there exists an integer $N(C) > 0$ such that $r([N,\infty))$ and $s([N,\infty))$ lie in the same component of $K - C$. If $E(K)$ is the set of all proper maps $[0,\infty) \to K$, then $\mathcal{E}(K)$, *the set of ends of* K, is E modulo the equivalence relation: $r \sim s$ if r and s converge to the same end of K. The *number of ends* of K is the cardinality of $\mathcal{E}$.

Let G be a finitely presented group, and X a finite complex with $\pi_1(X) = G$. Each of the following definitions is independent of the finite complex X, so long as $\pi_1(X) = G$. The *number of ends* of G is the number of ends of $\tilde{X}$, the universal cover of X. G is *semistable at* ∞ if for any two proper maps $r, s\colon [0,\infty) \to \tilde{X}$, that converge to the same end of $\tilde{X}$, r and s are properly homotopic. A finitely presented group G has $0, 1, 2$ or ∞ ends [**F**]. 0- and 2-ended groups are semistable at ∞. [$\mathbf{M}_1$]–[$\mathbf{M}_5$] provide a collection of 1-ended groups that are semistable at ∞. The semistability at ∞ of ∞-ended groups has been set aside until now. M. Dunwoody's accessibility theorem [**D**] provides an algebraic result which we combine with our geometric results to obtain

THEOREM. *If all finitely presented* 1*-ended groups are semistable at* ∞, *then all finitely presented groups are semistable at* ∞.

THEOREM. *If all finitely presented* 1*-ended groups K have free abelian $H^2(K; ZK)$, then all finitely presented groups G have free abelian $H^2(G; ZG)$.*

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In [**GM**] we showed that if G is semistable at ∞, then $H^2(G; ZG)$ is free abelian. It is unknown if all finitely presented groups are semistable at ∞, or if all finitely presented groups G are such that $H^2(G; ZG)$ is free abelian. In [**S**], J. Stallings showed that a finitely presented group G has more than one end if and only if G is an amalgamated free product $A *_C B$ and C is finite of index ≥ 2 in both A and B, or G is an HNN extension, $\langle A, t \colon t^{-1}C_1t = C_2\rangle$, where the C_i are isomorphic finite subgroups of A. If $G = A *_C B$ where C is finite, or $G = \langle A, t \colon t^{-1}C_1t = C_2\rangle$, where the C_i are finite, we say *G factors over a finite group.* Call A and B *the factors of G* when $G = A \ *_C B$ and A *the factor of G* when $G = \langle A, t \colon t^{-1}C_1t = C_2\rangle$. Assume G has more than one end. By Stallings' theorem, G factors over a finite group. If a factor of G has more than one end, then it too factors over a finite group. The group G is called *accessible* if the process of successively decomposing factors with more than one end terminates after a fintie number of steps. In [**D**], Dunwoody proves that all finitely presented groups are accessible.

2. Preliminaries. Let $G = A *_C B$ be an amalgamated free product. In [**S**], Stallings discusses the pregroup structure of G. In particular we find the following definition: $g_1g_2\cdots g_n$ is a *reduced word* in G if

(i) Each g_i is an element of $A - C$ or $B - C$, and

(ii) $g_i \in A - C$ if and only if $g_{i+1} \in B - C$ for any $i \in \{1, 2, \ldots, n-1\}$.

By Theorem 32.A.4.5 and Example 3.A.5.3 of [**S**] we have

PROPOSITION 1. *Every element of $G - C$ is represented by a reduced word in G. If $u_1u_2\cdots u_k$ and $v_1v_2\cdots v_m$ are reduced words in G representing the same element of G, then $k = m$ and for each i, $v_i^{-1}\cdots v_1^{-1}u_1\cdots u_i$ is an element of C.* $\square$

Proposition 1 provides much of the geometry in what is to come.

If $\langle g_1, \ldots, g_n \colon r_1, \ldots, r_m\rangle$ is a presentation for the group G, let X be the standard finite 2-complex obtained from this presentation. Each edge of $\tilde{X}$, the universal cover of X, corresponds to one of the letters $g_1^{\pm}, \ldots, g_n^{\pm}$. A proper map $r \colon [0, \infty) \to \tilde{X}$ is called an *edge path to ∞ at $*$* if $r(0) = *$ and r restricted to $[n, n+1]$, for each $n \in \{0, 1, \ldots\}$, is an edge of $\tilde{X}$. r can be represented as $(a_1, a_2, \ldots)$ at $*$, where each $a_i \in \{g_1^{\pm}, \ldots, g_n^{\pm 1}\}$. Note that the initial point $*$ of r must be specified to distinguish r from its translates under covering transformations of $\tilde{X}$. To see that G is semistable at ∞, it suffices to show: Any two edge paths to ∞ in $\tilde{X}$ that converge to the same end are properly homotopic.

3. The main theorems.

THEOREM 1. *If $G = A *_C B$, where C is finite, and A and B are finitely presented, semistable at ∞ groups, then G is semistable at ∞.*

PROOF. The following excessive presentation is a geometric convenience. Let $P_1 = \langle a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k \colon r_1, \ldots, r_l, t_1, \ldots, t_q, s_1, \ldots, s_p\rangle$ be a presentation for G where $c_1, \ldots, c_k$ generates C, $P_2 = \langle a_1, \ldots, a_n, c_1, \ldots, c_k \colon r_1, \ldots, r_l\rangle$ is a presentation for A, and $P_3 = \langle b_1, \ldots, b_m, c_1, \ldots, c_k \colon t_1, \ldots, t_q\rangle$ is a presentation for B. Furthermore, assume no a_i or b_j is in C. Let X be the standard finite 2-complex obtained from P_1 such that $\pi_1(X) = P_1$. Let $\tilde{X}$ be the universal cover of X. Let Y and Z be the finite complexes obtained from P_2 and P_3 respectively. $\tilde{X}$

contains a disjoint collection of copies of $\tilde{Y}$ (respectively $\tilde{Z}$), one for each element of G/A (respectively G/B).

LEMMA 1. *Assume $u = (u_1, \ldots, u_f)$ is an edge path in $\tilde{X}$. Let x and y be the initial and end point of u respectively. If Λ is a copy of $\tilde{Y}$ (or $\tilde{Z}$) in $\tilde{X}$, and u intersected with Λ is $\{x, y\}$, then there is an edge path d in the letters $c_1^{\pm 1}, \ldots, c_k^{\pm 1}$ from x to y.*

PROOF. Since Λ is a copy of $\tilde{Y}$, and $u \cap \Lambda = \{x, y\}$, we must have u_1 and u_f in $\{b_1^{\pm 1}, \ldots, b_m^{\pm 1}\}$. Let β_1 be the maximal subpath of u, with initial edge u_1, and each subsequent edge in $\{b_1^{\pm 1}, \ldots, b_m^{\pm 1}, c_1^{\pm 1}, \ldots, c_k^{\pm 1}\}$. If u does not equal β_1, then the edge of u following the last edge of β_1 (call it e) must be in $\{a_1^{\pm 1}, \ldots, a_n^{\pm 1}\}$. Let α_1 be the maximal subpath of u, with initial edge e, such that each edge of α_1 is in $\{a_1^{\pm 1}, \ldots, a_n^{\pm}, c_1^{\pm 1}, \ldots, c_k^{\pm 1}\}$. Continuing, we have rewritten u as $(\beta_1, \alpha_1, \ldots, \beta_{t-1}, \alpha_{t-1}, \beta_t)$. (Since the last edge of u is in $\{b_1^{\pm}, \ldots, b_m^{\pm}\}$, we must end with a β.) If v is an initial or end point of α_i for any $i \in \{1, \ldots, t-1\}$, call v a *piercing point* of u. Since x and y are in Λ, there is an edge path from x to y in the letters $\{a_1^{\pm}, \ldots, a_n^{\pm}, c_1^{\pm}, \ldots, c_k^{\pm}\}$. If $t \neq 1$, then β_1 and β_t are not in C since $u \cap \Lambda = \{x, y\}$. Also (if $t \neq 1$) $\beta_1 \alpha_1 \beta_2 \alpha_2 \cdots \beta_t$ is not reduced. Hence some $\alpha_i \in \{\alpha_1, \ldots, \alpha_{t-1}\}$ or $\beta_i \in \{\beta_2, \ldots, \beta_{t-1}\}$ must be in C. If we replace this subpath of u by an edge path τ in the letters $\{c_1^{\pm 1}, \ldots, c_k^{\pm}\}$, we obtain a new edge path from x to y, call it u'.

Note that $\tau \cap \Lambda$ is the empty set. u' can be written in terms of maximal subpaths, using two fewer subpaths than the number needed for u. The piercing points of u' are a subset of the piercing points of u. (This is used in Remark 1, below). Continuing, we will have an edge path from x to y in the letters $\{b_1^{\pm 1}, \ldots, b_m^{\pm 1}, c_1^{\pm 1}, \ldots, c_k^{\pm 1}\}$. But x and y are in Λ, a copy of $\tilde{Y}$. Hence there is an edge path from x to y in the letters $\{c_1, \ldots, c_k\}$. □

REMARK 1. The above process of replacing a subpath of first u, then u', etc. by edge paths in the letters $\{c_1^{\pm 1}, \ldots, c_k^{\pm 1}\}$ is always done so that the replacement edge paths connect two piercing points of u. If the order of the group C is α, our subpath replacement process can be interpreted homologically *for loops* in $\tilde{X}$ as follows: if u is a loop in $\tilde{X}$ then u is homologous, in the α-fold star of the image of u, to $d_1 + d_2 + \cdots + d_z$ where each d_i is a loop entirely in the edges $\{a_1^{\pm 1}, \ldots, a_n^{\pm 1}, c_1^{\pm 1}, \ldots, c_k^{\pm 1}\}$ or the edges $\{b_1^{\pm 1}, \ldots, b_m^{\pm 1}, c_1^{\pm 1}, \ldots, c_k^{\pm 1}\}$.

LEMMA 2. *Assume u, d, x, y and Λ are as in Lemma 1. If $\Lambda, \Lambda_1, \ldots, \Lambda_p$ are the copies of $\tilde{Y}$ and $\tilde{Z}$ in $\tilde{X}$ that meet u, then u is homotopic to d* rel.$\{0,1\}$ *by a homotopy H, with image in $\bigcup_{i=1}^{p} \Lambda_i$. (Note that we do not include Λ in this union!)*

PROOF. Again assume Λ is a copy of $\tilde{Y}$. Recall, u was written as $(\beta_1, \alpha_1, \ldots, \beta_{t-1}, a_{t-1}, \beta_t)$. One of the paths δ of $\{\alpha_1, \ldots, \alpha_{t-1}\} \cup \{\beta_2, \ldots, \beta_{t-1}\}$, when considered as an element of G, fell in C. Hence we have an edge path, τ, in the letters $\{c_1^{\pm 1}, \ldots, c_k^{\pm 1}\}$ with the same initial point and end point as δ.

Each of $\alpha_1, \ldots, \alpha_{t-1}, \beta_2, \ldots, \beta_{t-1}$ fall in some Λ_i, $i \in \{1, \ldots, p\}$. Hence δ is in some Λ_i, and the loop $\delta\tau^{-1}$ must be in this Λ_i. Thus δ is homotopic rel.$\{0,1\}$ to τ in this Λ_i. Continuing, we have u is homotopic rel.$\{0,1\}$ to β, an edge path in the letters $\{b_1^{\pm 1}, \ldots, b_m^{\pm 1}, c_1^{\pm}, \ldots, c_k^{\pm 1}\}$, where the initial and terminal edges of β are u_1 and u_f respectively. (Recall $u = (u_1, \ldots, u_f)$, and u_1 and u_f are in $\{b_1^{\pm}, \ldots, b_m^{\pm 1}\}$.)

Hence β is homotopic rel.$\{0,1\}$ to d in the copy of $\tilde{Z}$ containing u_1 and u_f. This copy of $\tilde{Z}$ is one of the Λ_i, $i \in \{1, \dots, p\}$. □

Assume $u, d, x, y, \Lambda, \Lambda_1, \dots, \Lambda_p$, and H are as in Lemma 2. Assume u' is an edge path in $\tilde{X}$ with initial point x' and end point y'. Assume $u' \cap \Lambda = \{x', y'\}$, and that $\Lambda, \Lambda'_1, \dots, \Lambda'_q$ are the copies of $\tilde{Y}$ and $\tilde{Z}$ that meet u'. By Lemma 2, u' is homotopic rel.$\{0,1\}$ to d', an edge path from x' to y' in the letters $c_1^{\pm 1}, \dots, c_k^{\pm 1}$, by a homotopy H', with the image in $\bigcup_{i=1}^{q} \Lambda'_i$.

LEMMA 3. *If there is no edge path from x to x' in the letters $c_1^{\pm 1}, \dots, c_k^{\pm 1}$, then no Λ_i meets any Λ'_j. In particular, there are finite complexes F and F' containing the images of H and H' respectively, such that $F \cap F'$ is the empty set.*

PROOF. If $\Lambda_i \cap \Lambda'_j$ is nonempty, choose a vertex v in $\Lambda_i \cap \Lambda'_j$. Let w (respectively w') be a subpath of u (u') from x (x') to a vertex of Λ_i (Λ'_j). Let e (e') be an edge path in Λ_i (Λ'_j) from the end point of w (w') to v. Then $w'e'e^{-1}w^{-1}$ is an edge path from x' to x whose intersection with Λ is $\{x', x\}$. Lemma 1 now contradicts our hypothesis on x and x'. □

LEMMA 4. *If r is an edge path to ∞ in $\tilde{X}$, Λ is a copy of $\tilde{Y}$ or $\tilde{Z}$ in $\tilde{X}$, and r meets Λ in an infinite set of vertices V, then r is properly homotopic to an edge path to ∞ with image in Λ, which also passes through each vertex in V.*

PROOF. Represent r as $(\beta, \alpha_1, u_1, \alpha_2, u_2, \dots)$ at $*$, where α_i is either a trivial path or a maximal subpath of r contained in Λ and u_i is a subpath of r such that $u_i \cap \Lambda = \{x_i, y_i\}$, where x_i is the initial point of u_i and y_i is the end point of u_i. Let H_i be the homotopy rel.$\{0,1\}$ of u_i to d_i described in Lemma 2, where d_i is an edge path in the letters $c_1^{\pm 1}, \dots, c_k^{\pm 1}$. Let α be the order of C. By Lemma 3, if $x_i \notin St^{\alpha}(x_j)$, then the image of H_i misses the image of H_j. Hence by combining the H_i we have a proper homotopy of r to $(\beta, \alpha_1, d_1, \alpha_2, d_2, \dots)$, and $(\beta, \alpha_1, d_1, \alpha_2, d_2, \dots)$ is properly homotopic to $(\alpha_1, d_1, \alpha_2, d_2, \dots)$. □

To finish Theorem 1 we show

LEMMA 5. *If r and s are edge paths to ∞ in $\tilde{X}$ which converge to the same end of $\tilde{X}$, then r and s are properly homotopic.*

PROOF. We consider two cases.

Case 1. Assume r meets Λ, a copy of $\tilde{Y}$ or $\tilde{Z}$, in the infinite set of vertices $\{v_1, v_2, \dots\}$. By Lemma 4, r is properly homotopic to t, an edge path to ∞ with image in Λ such that t passes through each v_i. Since r and s converge to the same end, s is properly homotopic to an edge path to ∞ of the form $(\delta_1, \alpha_1, \alpha_1^{-1}, \delta_2, \alpha_2, \alpha_2^{-1}, \dots)$ where each δ_i and α_i is an edge path of $\tilde{X}$, $s = (\delta_1, \delta_2, \dots)$, and the union of the end ponts of all α_i contains an an infinite subset W, of $\{v_1, v_2, \dots\}$. By Lemma 4, s is properly homotopic to u, an edge path to ∞ in Λ that passes through each vertex of W. Now u and t converge to the same end in Λ since they both pass through each vertex of W, and are thus properly homotopic in Λ. Hence r and s are properly homotopic in $\tilde{X}$.

Cases 2. Assume neither r nor s is properly homotopic to an edge path to ∞ that meets a copy of $\tilde{Y}$ or $\tilde{Z}$ in an infinite number of vertices. Since r and s converge to the same end in $\tilde{X}$ we may assume that $r = (\alpha_1, \alpha_2, \dots)$ and $s = (\beta_1, \beta_2, \dots)$, where the initial point of α_i equals the initial point of β_i, for all I. The edge loop $\alpha_i\beta_i^{-1}$ is homotopically trivial by a homotopy H_i, whose image F_i lies in the union of all copies of $\tilde{Y}$ and $\tilde{Z}$ that meet the loop $\alpha_i\beta_i^{-1}$. Since neither r nor s meets a copy of $\tilde{Y}$ or $\tilde{Z}$ in an infinite number of vertices, a given compact set can meet only finitely many F_i. Thus by patching together the homotopies H_i we have a proper homotopy of r to s. □

Theorem 2 is the analogue of Theorem 1, with amalgamated free product replaced by HNN extension. The proof of Theorem 2 is in direct analogy with that of Theorem 1. We provide a proof of Lemma 6, which is our analogue for Lemma 1. The remaining details can be filled paralleling Lemmas 2–5.

THEOREM 2. *If C_1 and C_2 are finite subgroups of the finitely presented group A, $m\colon C_1 \to C_2$ is an isomorphism, and A is semistable at ∞, then the HNN extension G of A with respect to m is semistable at ∞.*

PROOF. Let $\langle a_1, \ldots, a_n, d_1, \ldots, d_p, e_1, \ldots, e_p, t\colon r_1, \ldots, r_l, t^{-1}d_it = e_i\rangle$ be a presentation for G, where $\langle a_1, \ldots, a_n, d_1, \ldots, d_p, e_1, \ldots, e_p\colon r_1, \ldots, r_l\rangle$ is a presentation for A, $d_1, \ldots, d_p$ and $e_i, \ldots, e_p$ are generators for C_1 and C_2 respectively, $a_i \in A - (C_1 \cup C_2)$ and $m(d_i) = e_i$. Let X be the standard 2-complex obtained from the above presentation of G, and Y the standard 2-complex obtained from the above presentation of A.

LEMMA 6. *Assume $u = (u_1, \ldots, u_k)$ is an edge path in $\tilde{X}$ with initial point x and end point y. If Λ is a copy of $\tilde{Y}$ in $\tilde{X}$ and $u \cap \Lambda = \{x, y\}$, then there is an edge path from x to y in the letters $d_1^{\pm 1}, \ldots, d_p^{\pm 1}$ or $e_1^{\pm 1}, \ldots, e_p^{\pm 1}$.*

PROOF. u_1 and u_k are in the set $\{t, t^{-1}\}$. Rewrite u as $(\beta_1, \alpha_1, \ldots, \beta_{q-1}, \alpha_{q-1}, \beta_q)$ where $\beta_i = t^{k(i)}$ and α_i is an edge path in the letters $\{a_1^{\pm 1}, \ldots, a_n^{\pm 1}, d_1^{\pm 1}, \ldots, d_p^{\pm 1}, e_1^{\pm 1}, \ldots, e_p^{\pm 1}\}$. Call z a *piercing point* of u if z is not in $\{x, y\}$ and z is a vertex of an edge of u which is labeled $t^{\pm 1}$. There is an edge path from x to y in the letters $\{a_1^{\pm 1}, \ldots, a_n^{\pm 1}, d_1^{\pm 1}, \ldots, d_p^{\pm 1}, e_1^{\pm 1}, \ldots, e_p^{\pm 1}\}$. By the pregroup structure of an HNN extension (see [S]), one of the α_i represents an element of C_1 or C_2, and if $\alpha_i \in C_1$, then it is preceded and followed by t^{-1} and t respectively; if $\alpha_i \in C_2$, then it is preceded and followed by t and t^{-1} respectively. If $\alpha_i \in C_1$, replace $t^{-1}\alpha_it$ by a word in the letters $\{e_1^{\pm 1}, \ldots, e_p^{\pm 1}\}$. If $\alpha_i \in C_2$, then replace $t\alpha_it^{-1}$ by a word in the letters $\{d_1^{\pm 1}, \ldots, d_p^{\pm 1}\}$. Call the resulting edge path u^1. The piercing points of u^1 are a subset of the piercing points of u. Hence (unless $u = t\alpha_1t^{-1}$ or $u = t^{-1}\alpha_1t$), we have $u^1 \cap \Lambda = \{x, y\}$. The edge path u^1 contains two fewer copies of $t^{\pm 1}$ than u. Define u^i inductively. By induction on the number of times $t^{\pm 1}$ appears in u^i, we have, for some i, $u^i = tvt^{-1}$ where $v \in C_2$ or $u^i = t^{-1}vt$ where $v \in C_1$. Hence there is an edge path from x to y in the letters $\{d_1^{\pm 1}, \ldots, d_p^{\pm 1}\}$ or $\{e_1^{\pm 1}, \ldots, e_p^{\pm 1}\}$. □

REMARK 2. Let δ be the order of the group C_1. In Lemma 6, α_i being in C_1 means there is an edge path τ_i, in the letters $\{d_1^{\pm 1}, \ldots, d_p^{\pm 1}\}$ such that $\alpha_i\tau_i^{-1}$ is a loop. Furthermore $t^{-1}\alpha_it = t^{-1}\tau_it \in C_2$. Hence there is an edge path γ_i in the letters $\{e_1^{\pm 1}, \ldots, e_p^{\pm 1}\}$ such that $t^{-1}\tau_it\gamma_i^{-1}$ is a loop. Both τ_i and γ_i connect piercing points of u. Using 2-cells of $\tilde{X}$ corresponding to the relations $t^{-1}d_it = e_i$ of G, we see the loop $t^{-1}\tau_it\gamma_i^{-1}$ is homotopically trivial in the δ-fold star of a piercing point of u. This, along with the subpath replacement process of Lemma 6, can be interpreted homologically *for loops* in $\tilde{X}$ as follows: If u is a loop in $\tilde{X}$, then u is homologous, in the δ-fold star of the image of u, to $d_1 + d_2 + \cdots + d_z$, where each d_i is a loop in the edges $\{a_1^{\pm 1}, \ldots, a_n^{\pm 1}, d_1^{\pm 1}, \ldots, d_1^{\pm 1}, \ldots, d_1^{\pm 1}, e_1^{\pm 1}, \ldots, e_p^{\pm 1}\}$.

THEOREM 3. *If all finitely presented* 1*-ended groups are semistable at* ∞*, then all finitely presented groups are semistable at* ∞.

PROOF. Apply induction to Dunwoody's Theorem along with Theorems 1 and 2, and the fact that all finite groups and all 2-ended groups are semistable at ∞. □

4. The cohomology of ∞-ended groups. An inverse sequence $\{G_i, h_i\}$ of groups and homomorphisms is *semistable* (sometimes called Mittag-Leffler) if for each integer $n > 0$, there is an integer $M(n)$ such that the images of all groups G_k, $k > M(n)$, in G_n are equal. Let G be a finitely presented group, and X be a finite CW complex with $\pi_1(X) = G$. Let $\{C_i\}$ be an exhausting sequence of compact sets in $\tilde{X}$, the universal cover of X. In [**GM**] we showed

PROPOSITION A. *$H^2(G; ZG)$ is free abelian if and only if the inverse sequence of groups $\{H_1(\tilde{X} - C_i)\}$ is semistable.*

In [$\mathbf{M}_1$, Theorem 2.1] we showed

PROPOSITION B. *G is semistable at ∞ if and only if for each end $[r] \in \mathcal{E}(\tilde{X})$* (*see* §1), *the inverse sequence of groups $\{\pi_1(\tilde{X} - c_i), r\}$ is semistable.*

$\{H_1(\tilde{X} - C_i)\}$ semistable means that for each compact set $C \subset \tilde{X}$ there is a compact set $D(C) \subset \tilde{X}$ such that for any third compact set E and loop α in $\tilde{X} - D$, α is homologous in $\tilde{X} - C$ to a sum of loops, each with image in $\tilde{X} - E$.

THEOREM 4. *If $G = A *_C B$, where C is finite, and A and B are finitely presented with $H^2(A; ZA)$ and $H^2(B; ZB)$ free abelian, then $H^2(G; ZG)$ is free abelian.*

PROOF. Let P_1 be the presentation for G described in Theorem 1. Also, let X, Y, and Z be as in Theorem 1. Let F be compact in $\tilde{X}$. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_l$ be the copies of $\tilde{Y}$ and $\tilde{Z}$ in $\tilde{X}$ that intersect F.

Since $H^2(A, ZA)$ and $H^2(B; ZB)$ are free abelian, for each $i \in \{1, \ldots, l\}$ there is a compact set $D_i \subset \Lambda_i$ such that if E is compact in $\tilde{X}$ and α_i is a loop in $\Lambda_i - D_i$ then α_i is homologous in $\Lambda_i - F$ to a sum of loops in $\Lambda_i - E$. If α is the order of the group C, we show $D(F)$ is the α-fold star of the union of the sets $D_1, \ldots, D_l$. Let E be compact in $\tilde{X}$ and β a loop in $\tilde{X} - D$. By Remark 1, β is homologous in $\tilde{X} - \bigcup_{i=1}^{l} D_i$ to a sum of loops $d_1 + \cdots + d_z$, where each d_i is in a copy of $\tilde{Y}$ or $\tilde{Z}$. If d_i lies in $\Lambda_j - D_j$ then d_i is homologous in $\Lambda_j - F$ to a sum of loops in $\Lambda_j - E$. If d_i lies in a copy Λ of $\tilde{Y}$ or $\tilde{Z}$ not in the set $\{\Lambda_1, \ldots, \Lambda_l\}$, then Λ misses F and d_i is homotopically trivial in Λ. Thus β is homologous in $\tilde{X} - F$ to a sum of loops each with image in $\tilde{X} - E$. □

Similarly, Theorem 5 below follows from Remark 2.

THEOREM 5. *If C_1 and C_2 are finite subgroups of the finitely presented group A, $m \colon C_1 \to C_2$ is an isomorphism, and $H^2(A, ZA)$ is free abelian, then the HNN extension G of A with respect to m is such that $H^2(G; ZG)$ is free abelian.* □

THEOREM 6. *If all* 1*-ended finitely presented groups* K *have free abelian* $H^2(K; ZK)$, *then all finitely presented groups* G *have free abelian* $H^2(G; ZG)$.

PROOF. Apply induction to Dunwoody's Theorem, along with Theorems 4 and 5, and the fact that all finite groups and all 2-ended groups K have free abelian $H^2(K; ZK)$. □

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