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CONJUGACY CLASSES IN ALGEBRAIC MONOIDS

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ABSTRACT. Let M be a connected linear algebraic monoid with zero and a reductive group of units G. The following theorem is established.

THEOREM. *There exist affine subsets* $M_1, \ldots, M_k$ *of* M, *reductive groups* $G_1, \ldots, G_k$ *with antiautomorphisms* *, *surjective morphisms* $\theta_i \colon M_i \to G_i$, *such that:* (1) *Every element of* M *is conjugate to an element of some* M_i, *and* (2) *Two elements* a, b *in* M_i *are conjugate in* M *if and only if there exists* $x \in G_i$ *such that* $x\theta_i(a)x^* = \theta_i(b)$. *As a consequence, it is shown that* M *is a union of its inverse submonoids.*

Introduction. The objects of study in this paper are connected linear algebraic monoids M with zero. This means by definition that the underlying set of M is an irreducible affine variety and that the product map is a morphism (i.e. a polynomial map). We will further assume that the group of units G of M is reductive. This means [**1**, **3**] that the unipotent radical of G is trivial. Then by [**6**, **10**], M is unit regular, i.e. $M = E(M)G$ where $E = E(M) = \{e \in M | e^2 = e\}$. In this paper we study the conjugacy classes of M. An initial study was made by the author [**8**], where the general problem was reduced to nilpotent elements. The approach here is quite different, yielding a more complete answer. To be precise, we show that there exist affine subsets $M_1, \ldots, M_k$ of M, reductive groups $G_1, \ldots, G_k$ with antiautomorphisms *, surjective morphisms $\theta_i \colon M_i \to G_i$, $i = 1, \ldots, k$, such that: (1) Every element of M is conjugate to an element of some M_i, and (2) If $a, b \in M_i$, then a is conjugate to b in M if and only if there exists $x \in G_i$ such that $x\theta_i(a)x^* = \theta_i(b)$. As an application of this result, we show that M is a union of its inverse submonoids. An inverse semigroup is a semigroup S with the property that for each $a \in S$, there exists a unique $\bar{a} \in S$ such that $a\bar{a}a = a$ and $\bar{a}a\bar{a} = \bar{a}$. See [**2**]. Finally in §3, we use our main results to briefly analyze the conjugacy classes of nilpotent elements.

1. Preliminaries. Throughout this paper Z^+ will denote the set of all positive integers and K an algebraically closed field. Let G be a connected linear algebraic group defined over K. The *radical* $R(G)$ is the maximal closed connected normal solvable subgroup of G and the *unipotent radical* $R_u(G)$ is the group of unipotent elements of $R(G)$. We will assume throughout that G is a *reductive* group, i.e., $R_u(G) = 1$. Then $R(G) \subseteq C(G)$, the center of G. Moreover $G = R(G)G_0$ where $G_0 = (G, G)$ is a semisimple group, i.e. $R(G_0) = 1$. Also [**3**, Theorem 27.5] G_0 is a product of the simple closed normal subgroups of G. In particular we have the following.

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FACT 1.1. If H is a closed normal subgroup G, then $G = HC_G(H)$. If H_1, H_2, H_1', H_2' are closed normal subgroups of G with $G = H_1H_2 = H_1'H_2'$ then

$$G = (H_1 \cap H_1')(H_1 \cap H_2')(H_2 \cap H_1')(H_2 \cap H_2')R(G).$$

A connected diagonalizable subgroup of G is called a *torus.* Let T be a maximal torus of G. Then

$$R(G) \subseteq C(G) \subseteq C_G(T) = T.$$

$W = N_G(T)/T$ is called the *Weyl group* of G and is finite. A maximal closed connected solvable subgroup of G is called a *Borel subgroup.* Let B_1, B_2 be Borel subgroups of G with $T \subseteq B_1 \cap B_2$. Then [**3**, Theorem 28.3] G is expressible as the following disjoint union:

$$G = \bigcup_{\sigma \in W} B_1 \sigma B_2.$$

This is called the *Bruhat decomposition* of G. A subgroup of G containing a Borel subgroup is called *parabolic.* Let P be a parabolic subgroup of G with $T \subseteq P$. Then there exists a parabolic subgroup P^- of G such that $T \subseteq P^-$ and $L = P \cap P^-$ is a reductive group. P^- is called the *opposite parabolic subgroup* of P relative to T and L is called a *Levi factor* of P. If $U = R_u(P)$, then [**1**, **3**], $P = LU$ is a semidirect product. This is called the *Levi decomposition* of P. By Fact 1.1, we have

FACT 1.2. Let $G = G_1G_2$ where G_1, G_2 are closed connected normal subgroups of G. Let P be a parabolic subgroup of G. Then $P_i = P \cap G_i$ is a parabolic subgroup of G_i ($i = 1, 2$) and $P = P_1P_2$. If $P = LU$ is a Levi decomposition of P, then $P_i = L_iU_i$, $L = L_1L_2$, $U = U_1U_2$, where $L_i = L \cap G_i$, $U_i = U \cap G_i$, $i = 1, 2$.

The following result follows from [**1**, Theorem 28.7].

FACT 1.3. Let P, Q be parabolic subgroups of G with Levi decompositions, $P = L_1U_1$, $Q = L_2U_2$ such that $T \subseteq L_1 \cap L_2$. Then

$$P \cap Q = (U_1 \cap U_2)(U_1 \cap L_2)(L_1 \cap U_2)(L_1 \cap L_2).$$

By a (linear) *algebraic monoid*, we mean a monoid M such that the underlying set is an affine variety and the product map is a morphism. The identity component of M is denoted by M^c. We will assume that M is *connected* (i.e. $M = M^c$) and that M has a zero. We will further asssume that the group of units G of M is reductive. Then by [**6**, **10**], M is unit regular, i.e. $M = E(M)G$. Here $E(M)$ is the idempotent set of M. We fix a maximal torus T of G. If $\Gamma \subseteq E(\overline{T})$, then we let

$$C_G^r(\Gamma) = \{a \in G | ae = eae \text{ for all } e \in \Gamma\},$$
$$C_G^l(\Gamma) = \{a \in G | ea = eae \text{ for all } e \in \Gamma\}.$$

Then $C_G(\Gamma) = C_G^r(\Gamma) \cap C_G^l(\Gamma)$ is a reductive group. If $e \in E(\overline{T})$, then by [**5**, **7**], $C_G^r(e), C_G^l(e)$ are opposite parabolic subgroups of G. We let

$$G_e^r = \{a \in G | ae = e\}^c, \quad G_e^l = \{a \in G | ea = e\}^c,$$
$$G_e = \{a \in G | ae = ea = e\}^c = G_e^r \cap C_G(e),$$
$$\hat{G}_e = \{a \in G | ae = ea = e\}.$$

Since $G_e \triangleleft C_G(e)$, we have by Fact 1.1,

$$C_G(e) = G_eC_G(G_e).$$

In particular, $\hat{G}_e = G_e\hat{T}_e$. Now $eC_G(e)$ is the group of units of eMe by [**4**]. In particular, $eC_G^l(e) = eC_G(e)$. Hence we have the surjective homomorphism: $a \to ea$ from $C_G^l(e)$ onto the reductive group $eC_G(e)$. Thus

$$R_u(C_G^l(e)) \subseteq G_e^l \lhd C_G^l(e).$$

Similarly

$$R_u(C_G^r(e)) \subseteq G_e^r \lhd C_G^r(e).$$

Since $C_G^r(e) = R_u(C_G^r(e))C_G(e)$, we get

$$G_e^r = R_u(C_G^r(e))G_e.$$

LEMMA 1.4. *Let $e, f \in E(\overline{T})$. Then*

$$C_G(e,f) = (G_e \cap G_f)(G_f \cap C_G(T_e))(G_e \cap C_G(T_f))C_G(T_e,T_f).$$

PROOF. Now $C_G^r(f) \cap C_G(e)$ is a parabolic subgroup of $C_G(e)$ with Levi factor $C_G(e,f)$. Since $C_G(e) = G_eC_G(G_e)$, we have by Fact 1.2,

$$C_G(e,f) = [C_G(f) \cap G_e][C_G(f) \cap C_G(G_e)].$$

Similarly

$$C_G(e,f) = [C_G(e) \cap G_f][C_G(e) \cap C_G(G_f)].$$

Since $C_G(G_e) \subseteq C_G(T_e)$, $C_G(G_f) \subseteq C_G(T_f)$, we are done by Fact 1.1.

LEMMA 1.5. *Let $e, f \in E(\overline{T})$. Then*
(i) $G_e^r \cap C_G^r(f) = [G_e^r \cap C_G(T_f)][G_e^r \cap G_f^r]$,
(ii) $G_e^r \cap C_G^l(f) = [G_e^r \cap C_G(T_f)][G_e^r \cap G_f^l]$.

PROOF. We prove (i), as the proof of (ii) is similar. By Fact 1.3,

$$C_G^r(e) \cap C_G^r(f) = [R_u(C_G^r(e)) \cap C_G^r(f)][C_G(e) \cap C_G^r(f)].$$

Since $R_u(C_G^r(e)) \subseteq G_e^r$, we obtain

$$G_e^r \cap C_G^r(f) = [R_u(C_G^r(e)) \cap C_G^r(f)][G_e \cap C_G^r(f)].$$

By Facts 1.2, 1.3,

$$\begin{aligned} R_u(C_G^r(e)) \cap C_G^r(f) &= [R_u(C_G^r(e)) \cap R_u(C_G^r(f))][R_u(C_G^r(e)) \cap C_G(f)] \\ &\subseteq [G_e^r \cap G_f^r][R_u(C_G^r(e)) \cap G_f][R_u(C_G^r(e)) \cap C_G(G_f)] \\ &\subseteq [G_e^r \cap G_f^r][G_e^r \cap C_G(T_f)]. \end{aligned}$$

Now $C_G(e) \cap C_G^r(f)$ is a parabolic subgroup of $C_G(e)$ with Levi decomposition

$$[C_G(e,f)][R_u(C_G^r(f)) \cap C_G(e)].$$

So by Fact 1.2,

$$\begin{aligned} G_e \cap C_G^r(f) &= [G_e \cap C_G(f)][G_e \cap R_u(C_G^r(f))] \\ &\subseteq [G_e \cap C_G(f)][G_e^r \cap G_f^r]. \end{aligned}$$

By Lemma 1.4,

$$C_G(e,f) = (C_G(e) \cap G_f)(C_G(e) \cap C_G(T_f)).$$

Since $G_e \cap C_G(f) \lhd C_G(e,f)$ and since the radical of $G_e \cap C_G(f)$ is contained in $T_e \subseteq G_e \cap C_G(T_f)$, we obtain

$$G_e \cap C_G(f) = (G_e \cap G_f)(G_e \cap C_G(T_f)).$$

Since $G_e^r \cap G_f^r \lhd G_e^r \cap C_G^r(f)$, the result follows.

LEMMA 1.6. *Let $e \in E(\overline{T})$. Then $C_G(T_e) = T_e C_G(G_e)$.*

PROOF. Since $G_e \lhd C_G(e)$,

$$C_G(G_e) \subseteq C_G(T_e) \subseteq C_G(e) = G_e C_G(G_e).$$

So

$$C_G(T_e) = C_G(G_e)[G_e \cap C_G(T_e)] = C_G(G_e)T_e.$$

LEMMA 1.7. *Let $e, f \in E(\overline{T})$. Then*

$$C_G^r(e) \cap C_G^l(f) = [G_e^r \cap C_G(T_f)][C_G(T_e, T_f)][G_f^l \cap C_G(T_e)][G_e^r \cap G_f^l].$$

PROOF. By Fact 1.3,

$$C_G^r(e) \cap C_G^l(f) = [G_e^r \cap G_f^l][G_e^r \cap C_G(f)][C_G(e) \cap G_f^l]C_G(e, f).$$

Now $G_e^r \cap G_f^l \lhd C_G^r(e) \cap C_G^l(f)$. Also if $a \in G_f^l \cap C_G(T_e)$, $b \in G_e^r \cap C_G(T_f)$, then $a^{-1}b^{-1}ab \in G_e^r \cap G_f^l$. Moreover $C_G(T_e, T_f)$ normalizes $G_e^r \cap C_G(T_f)$ and $G_f^l \cap C_G(T_e)$. So we are done by Lemmas 1.4, 1.5.

LEMMA 1.8. *Let $e, f \in E(\overline{T})$, $a \in G_e^r$, $b \in C_G(T_e)$. If $ab \in C_G^r(f)$, then $a, b \in C_G^r(f)$. If $ab \in C_G^l(f)$, then $a, b \in C_G^l(f)$.*

PROOF. Suppose $ab \in C_G^r(f)$. Now $a = a_1 a_2$ for some $a_1 \in R_u(C_G^r(e))$, $a_2 \in G_e$. Then $a_2 b \in C_G(e)$. So by Fact 1.3, $a_1, a_2 b \in C_G^r(f)$. Then $a_2 b \in C_G(e) \cap C_G^r(f)$. So by Fact 1.2, $a_2 b = uv$ for some $u \in G_e \cap C_G^r(f)$, $v \in C_G(G_e) \cap C_G^r(f)$. So $u^{-1}a_2 = vb^{-1} \in G_e \cap C_G(T_e) = T_e \subseteq T \subseteq C_G^r(f)$. So $b \in C_G^r(f)$. Hence $a \in C_G^r(f)$.

PROPOSITION 1.9. *Let $\Gamma \subseteq E(\overline{T})$, $e_1, \ldots, e_{k+1} = f \in \Gamma$. Let $V = C_G(\Gamma)$, $Y_0 = G_f^l$, $Y_1 = G_{e_1}^r$, $Y_i = C_G(e_1, \ldots, e_{i-1}) \cap G_{e_i}^r$ for $i = 2, \ldots, k+1$. Then*

$$Y_0 \cdots Y_{k+1} \cap V = \prod_{i=1}^{k+1} V_{e_i}.$$

PROOF. We prove by induction on k. So first let $k = 0$, $a \in G_f^l$, $b \in G_f^r$ such that $ab \in V \subseteq C_G(f)$. Then $af = abf = fabf = f$. So $a \in G_f$. Similarly $b \in G_f$. So

$$ab \in G_f \cap V = G_f \cap \hat{V}_f = V_f(G_f \cap T) = V_f.$$

So let $k > 0$, $a \in Y_0 \cdots Y_{k+1} \cap V$. Then $a = y_0 \cdots y_{k+1}$, $y_i \in Y_i$. Now $y_1, \ldots, y_{k+1}$, $a \in C_G^r(e_1)$. Thus $y_0 \in C_G^r(e_1) \cap G_f^l$. By Lemma 1.5, there exist $\bar{y}_0 \in G_f^l \cap C_G(T_{e_1})$, $u \in G_{e_1}^r$ such that $y_0 = \bar{y}_0 u$. So $\bar{y}_1 = uy_1 \in G_{e_1}^r$ and $a = \bar{y}_0 \bar{y}_1 y_2 \cdots y_{k+1}$. Thus without loss of generality, we may assume that $y_0 \in C_G(T_{e_1}) \cap G_f^l$. For $i = 2, \ldots, k+1$, we can factor by Lemma 1.5,

$$y_i = c_i y_i', \quad c_i \in G_{e_1}, \quad y_i' \in C_G(e_1, \ldots, e_{i-1}) \cap C_G(T_{e_1}) \cap G_{e_i}^r.$$

Let

$$d_i = y_2 \cdots y_{i-1} c_i (y_2 \cdots y_{i-1})^{-1}, \qquad i = 3, \ldots, k+1.$$

Then

$$y_1' = y_1 d_{k+1} \cdots d_3 c_2 \in G_{e_1}^r, \quad y_1'' = y_0 y_1' y_0^{-1} \in G_{e_1}^r.$$

Clearly

$$a = y_0 y_1' y_2' \cdots y_{k+1}' = y_1'' y_0 y_2' \cdots y_{k+1}'.$$

Moreover $y_0y_2'\cdots y_{k+1}' \in C_G(T_{e_1})$. By Lemma 1.8, $y_1'', y_0y_2'\cdots y_{k+1}' \in V$. So $y_1'' \in V_{e_1}$. By the induction hypothesis $y_0y_2'\cdots y_{k+1}' \in V_{e_2}\cdots V_{e_{k+1}}$. This completes the proof.

2. Main section. We fix a connected linear algebraic monoid M with zero 0 and a reductive group of units G. As usual two elements $a, b \in M$ are *conjugate* ($a \sim b$) if $x^{-1}ax = b$ for some $x \in G$. Note that for $a \in M$, $g \in G$, $ag \sim ga$. We fix a maximal torus T of G. Let $W = N_G(T)/T$ denote the Weyl group of G. We let $\mathcal{R}, \mathcal{L}, \mathcal{H}$ denote the usual Green's relations on M [**2**]. If $a, b \in M$, then $a \mathcal{R} b$ means $aM = bM$, $a \mathcal{L} b$ means $Ma = Mb$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. Let $e \in E(\overline{T})$, $\sigma = nT \in W$. Then we let

$$e^\sigma = \sigma^{-1}e\sigma = n^{-1}en \in E(\overline{T}).$$

We let

$$M_{e,\sigma} = eC_G(e^\theta | \theta \in \langle\sigma\rangle)\sigma.$$

Our first result is that every element of M is conjugate to an element of some $M_{e,\sigma}$. In preparation, we prove

LEMMA 2.1. *Let $e \in E(\overline{T})$, $\sigma = nT \in W$, $k \in Z^+$, $x, y \in C_G(e^{\sigma^j} | 0 \le j \le k-1)$, $x \in G^l_{e^{\sigma^k}}$. Then $exyn \sim eyn$.*

PROOF. We prove by induction on k. First let $k = 1$. Then

$$\begin{aligned} exyn &= xyen \sim yenx = yne^\sigma x \\ &= yne^\sigma = yen = eyn. \end{aligned}$$

In general let $k > 1$. Then

$$exyn = xeyn \sim eynx = eynxn^{-1}y^{-1}yn = ex'yn$$

where $x' = ynxn^{-1}y^{-1} \in C_G(e^{\sigma^j} | 0 \le j \le k-2) \cap G^l_{e^{\sigma^{k-1}}}$. So by the induction hypothesis $ex'yn \sim eyn$.

THEOREM 2.2. *Every element of M is conjugate to an element of some $M_{e,\sigma}$.*

PROOF. Let $a \in M$. By [**8**, Corollary 2.3], there exists a maximal torus T_1 of G, $e, f \in E(\overline{T}_1)$ such that $e \mathcal{R} a \mathcal{L} f$. Since all maximal tori of G are conjugate, we can assume that $T = T_1$. There exists $\theta = mT \in W$ such that $e^\theta = f$. Thus $e \mathcal{R} em \mathcal{L} f$. So $em \mathcal{H} a$. Since $eC_G(e)$ is the $\mathcal{H}$-class of e, we see that $a \in eC_G(e)m = eC_G(e)\theta$. Suppose inductively that $a \in eC_G(e^{\theta^j} | j = 0, \ldots, k)\theta$. Let $H = C_G(e^{\theta^j} | j = 0, \ldots, k)$. So there exists $x \in H$ such that $a = exm$. By [**5**], $C^l_H(e^{\theta^{k+1}})$, $C^r_H(\theta e \theta^{-1})$ are parabolic subgroups of H containing T. By the Bruhat decomposition there exists $\pi = n_1T \in W(H)$, $x_1 \in C^l_H(e^{\theta^{k+1}})$, $x_2 \in C^r_H(\theta e\theta^{-1})$ such that $x = x_1n_1x_2$. So

$$exm = ex_1n_1x_2m \sim (m^{-1}x_2m)ex_1n_1m.$$

Now $m^{-1}x_2m \in C_G(e^{\theta^j} | j = 1, \ldots, k+1) \cap C^r_G(e)$. So

$$m^{-1}x_2me = ze \quad \text{for some } z \in C_G(e^{\theta^j} | j = 0, \ldots, k+1).$$

Thus

$$a \sim ezx_1n_1m, \qquad zx_1 \in C^l_H\left(e^{\theta^{k+1}}\right).$$

Let $\lambda = \pi\theta = n_1 m T \in W$. We claim that $e^{\lambda^j} = e^{\theta^j}$ for $j = 0, \ldots, k+1$. For $j = 0$, this is obvious. So assume $e^{\theta^j} = e^{\lambda^j}$, $j \leq k$. Then $\pi \in C_W(e^{\theta^j})$. So

$$e^{\lambda^{j+1}} = \left(e^{\theta^j}\right)^{\lambda} = \left(e^{\theta^j}\right)^{\pi\theta} = \left(e^{\theta^j}\right)^{\theta} = e^{\theta^{j+1}}.$$

Thus $y = zx_1 \in C^l_H(e^{\lambda^{k+1}})$. Hence $y = y_1 y_2$ for some $y_1 \in H^l_{e^{\lambda^{k+1}}}$, $y_2 \in C_H(e^{\lambda^{k+1}})$. By Lemma 2.1,

$$a \sim e y_1 y_2 n_1 m \sim e y_2 n_1 m, \qquad y_2 \in C_G\left(e^{\lambda^j} | j = 0, \ldots, k+1\right).$$

Continuing this process, we see that there exist $\sigma = nT \in W$ and $u \in C_G(e^{\sigma^j} | 0 \leq j \leq |W|) = C_G(e^{\gamma} | \gamma \in \langle\sigma\rangle)$ such that $a \sim eun$. Then clearly $eun \in M_{e,\sigma}$. This completes the proof of the theorem.

Schein [**13**] has shown that the full transformation semigroup on any set is a union of its inverse subsemigroups. The corresponding result, for the full matrix semigroup over a field, follows from the Fitting decomposition.

THEOREM 2.3. (i) *If F is a commutative, idempotent submonoid of M, then $FN_G(F)$ is the maximal unit regular inverse submonoid of M with idempotent set F.*

(ii) *If F is a subsemilattice of $E(\overline{T})$ with $1 \in F$, then*

$$FN_G(F) = FC_G(F)N_W(F).$$

(iii) *If $e \in E(\overline{T})$, $\sigma \in W$, $F = \langle 1, e^{\theta} | \theta \in \langle\sigma\rangle\rangle$, then $M_{e,\sigma} \subseteq FN_G(F)$.*

(iv) *M is a union of its unit regular inverse submonoids.*

PROOF. (i) That $FN_G(F)$ is a submonoid of M is obvious. Let $a \in FN_G(F)$, $a^2 = a$. So $a = fu$ for some $f \in F$, $u \in N_G(F)$. Then $fuf = f$. Since M is a matrix semigroup and f, ufu^{-1} commute, we see that $f = ufu^{-1}$. Thus $a = fu = fuf = f \in F$. So F is the idempotent set of $FN_G(F)$. It follows that $FN_G(F)$ is the maximal unit regular submonoid of M with idempotent set F. Since F is commutative, it follows [**2**] that $FN_G(F)$ is an inverse semigroup.

(ii) Let $a \in N_G(F)$. Clearly $T \subseteq C_G(F)$. So $aTa^{-1} \subseteq C_G(aFa^{-1}) = C_G(F)$. So T, aTa^{-1} are maximal tori of $C_G(F)$. Hence $b^{-1}aTa^{-1}b = T$ for some $b \in C_G(F)$. Hence $b^{-1}a \in N_G(T) \cap N_G(F)$. So $a = b(b^{-1}a) \in C_G(F)N_W(F)$.

(iii), (iv) follow from (ii) and Theorem 2.2.

Now fix $e \in E(\overline{T})$, $\sigma = nT \in W$. Let $f = e^{\sigma}$, $\alpha + 1$ the order of σ. Let

$$V = C_G(e^{\theta} | \theta \in \langle\sigma\rangle).$$

So V is a reductive group, $T \subseteq V$, $V^{\sigma} = V$, $M_{e,\sigma} = eV\sigma$. Now $\hat{V}_e = \{a \in V | ae = ea = e\} = \hat{T}_e V_e$ is a closed normal subgroup of V. Let

$$\Omega = \prod_{\theta \in \langle\sigma\rangle} \hat{V}_e^{\theta} = \prod_{\theta \in \langle\sigma\rangle} \hat{V}_{e^{\theta}}.$$

Then Ω is a closed normal subgroup of V. If $x \in V$, let $x^* = nx^{-1}n^{-1} \in V$. Then $\Omega^* = \Omega$. So * induces an antiautomorphism * on the reductive group $G_{e,\sigma} = V/\Omega$. Define $\xi \colon M_{e,\sigma} \twoheadrightarrow G_{e,\sigma}$ as follows: If $a = evn \in M_{e,\sigma}$, $v \in V$, then, $\xi(a) = v\Omega \in G_{e,\sigma}$. Since $\hat{V}_e \subseteq \Omega$, ξ is well defined. Note further that if $G_{e,\sigma}$ is replaced by $eV/e\Omega$ (which is isomorphic to $G_{e,\sigma}$ as an abstract group), then ξ would also be a morphism of varieties.

THEOREM 2.4. *Let $a, b \in M_{e,\sigma}$. Then a is conjugate to b in M if and only if there exists $x \in G_{e,\sigma}$ such that $x\xi(a)x^* = \xi(b)$.*

PROOF. For $a, b \in M_{e,\sigma}$, define $a \equiv b$ if $x\xi(a)x^* = \xi(b)$ for some $x \in G_{e,\sigma}$. We are to show that $\equiv \; = \; \sim$. Let

$$A = \{a \in V | eun \sim eaun \text{ for all } u \in V\}.$$

Clearly $\hat{V}_e \subseteq A$. Let $a, b \in A$. Then for $u \in V$, $eabun \sim ebun \sim eun$. So $A^2 \subseteq A$. Now let $a \in A$, $u \in V$. Then

$$\begin{aligned} e(nan^{-1})un &\sim an^{-1}unen = ea(n^{-1}un)n \\ &\sim e(n^{-1}un)n = n^{-1}unen \sim eun. \end{aligned}$$

Thus $nAn^{-1} \subseteq A$. It follows that $\Omega \subseteq A$. Now let $m_1, m_2 \in M_{e,\sigma}$ such that $m_1 \equiv m_2$. Let $m_1 = eun$, $m_2 = evn$ where $u, v \in V$. Then there exists $x \in V$ such that $v \in \Omega xunx^{-1}n^{-1}$. Since $\Omega \subseteq A$,

$$\begin{aligned} m_1 = evn &\sim exunx^{-1}n^{-1}n = exunx^{-1} \\ &= xeunx^{-1} \sim eun = m_2. \end{aligned}$$

This shows that $\equiv \; \subseteq \; \sim$.

Conversely let $m_1, m_2 \in M_{e,\sigma}$ such that $m_1 \sim m_2$. Then there exists $X_1 \in G$ such that

$$X_1 m_1 = m_2 X_1. \tag{1}$$

Let $m_1 = eun$, $m_2 = evn$ where $u, v \in V$. Then by (1),

$$X_1 e \,\mathcal{R}\, X_1 eun = evnX_1 \,\mathcal{R}\, e$$

So $X_1 e = eX_1 e$ and $X_1 \in C_G^r(e)$. Also by (1),

$$fX_1 = n^{-1}enX_1 \,\mathcal{L}\, m_2 X_1 = X_1 m_1 \,\mathcal{L}\, m_1 \,\mathcal{L}\, f.$$

Thus $X_1 \in C_G^r(e) \cap C_G^l(f)$. By Lemma 1.7, $X_1 \in X[G_e^r \cap G_f^l]$ for some

$$X \in [C_G(T_e) \cap G_f^l][C_G(T_e, T_f)][C_G(T_f) \cap G_e^r].$$

Since $m_1 = em_1$, $m_2 = m_2 f$, we see by (1) that

$$Xeun = evnX. \tag{2}$$

Now $X = axb$ for some

$$a \in C_G(T_e) \cap G_f^l, \quad x \in C_G(T_e, T_f), \quad b \in C_G(T_f) \cap G_e^r. \tag{3}$$

So by (2), $eaxun = evnxb$. Then $eaxu = evnxbn^{-1}$ and $axu, vnxbn^{-1} \in C_G(e)$. So

$$axu = vnxbn^{-1}z \quad \text{for some } z \in \hat{G}_e. \tag{4}$$

Now $nxbn^{-1} \in C_G(T_e)$. So by Lemma 1.6, $nxbn^{-1} = t\eta$ for some $t \in T$, $\eta \in C_G(G_e)$. So $vt \in V \subseteq C_G(e)$. So $vt = v'v''$ for some $v' \in C_V(V_e) \subseteq C_G(T_e)$, $v'' \in V_e \subseteq G_e$. Also $u = u'u''$ for some $u' \in C_V(V_e) \subseteq C_G(T_e)$, $u'' \in V_e \subseteq G_e$. Then

$$axu = vnxbn^{-1}z = vt\eta z = v'v''\eta z = v'\eta v''z.$$

So

$$axu' = v'\eta(v''z(u'')^{-1}). \tag{5}$$

Let $z' = v''z(u'')^{-1}$. Then $z' \in \hat{G}_e$. So $z'h = hz' = h$ for all $h \in E(M)$ with $h \leq e$. Now $axu', v', \eta \in C_G(T_e)$. So by (5), $z' \in C_G(T_e)$. Thus $z'h = hz'$ for all $h \in E(\overline{T})$ with $h \geq e$. So for any maximal chain Γ of $E(\overline{T})$ with $e \in \Gamma$, $z' \in C_G(\Gamma) = T \subseteq V$. Let $u_1 = u'(z')^{-1}v'' \in V$. Then by (5),

$$axu_1 = v'\eta v'' = v'v''\eta = vt\eta = vnxbn^{-1}.$$

Also $z = (v'')^{-1}z'u'' \in V \cap \hat{G}_e = \hat{V}_e$. So

$$(6) \qquad axu_1 = vnxbn^{-1}, \qquad u_1, v \in V, \ z \in \hat{V}_e.$$

Now $xb \in C^r_G(e)$. So $nxbn^{-1} \in C^r_G(\sigma e\sigma^{-1})$. Thus $ax \in C^r_G(\sigma e\sigma^{-1})$. By (3), Lemma 1.8, $a, x \in C^r_G(\sigma e\sigma^{-1})$. So $x \in C^r_G(\sigma e\sigma^{-1}) \cap C_G(T_e, T_f)$. Hence we can factor

$$(7) \qquad x = y_1x_1 \quad \text{for some } y_1 \in G^r_{\sigma e\sigma^{-1}} \cap C_G(T_e, T_f), \ x_1 \in C_G(T_{\sigma e\sigma^{-1}}, T_e, T_f).$$

Also $a \in C^r_G(\sigma e\sigma^{-1}) \cap C_G(T_e) \cap G^l_f$. So working within $C_G(T_e)$ and applying Lemma 1.5, we can factor

$$(8) \quad a = c_1a_1 \quad \text{for some } c_1 \in G^r_{\sigma e\sigma^{-1}} \cap G^l_f \cap C_G(T_e), \ a_1 \in C_G(T_e, T_{\sigma e\sigma^{-1}}) \cap G^l_f.$$

Now by (6),

$$c_1a_1y_1x_1u_1 = vny_1x_1bn^{-1}.$$

So

$$(9) \qquad wa_1x_1u_1 = vny_1x_1n^{-1}$$

where

$$w = c_1a_1y_1a_1^{-1}[(a_1x_1u_1)(nb^{-1}n^{-1})(a_1x_1u_1)^{-1}] \in G^r_{\sigma e\sigma^{-1}}.$$

Suppose now inductively that

$$(10) \qquad x = y_1 \cdots y_kx_k,$$

where

$$(11) \qquad \begin{aligned} &y_i \in C_G(T_f, T_{\sigma^j e\sigma^{-j}} \mid j = 0, \ldots, i-1) \cap G^r_{\sigma^i e\sigma^{-i}}, \qquad i = 1, \ldots, k, \\ &x_k \in C_G(T_f, T_{\sigma^j e\sigma^{-j}} \mid j = 0, \ldots, k). \end{aligned}$$

Further assume that there exist

$$\begin{aligned} &w_i \in C_G(T_{\sigma^j e\sigma^{-j}} \mid i+1 \leq j \leq k) \cap G^r_{\sigma^i e\sigma^{-i}}, \qquad i = 1, \ldots, k, \\ &a_k \in C_G(T_{\sigma^i e\sigma^{-i}} \mid i = 0, \ldots, k) \cap G^l_f \end{aligned}$$

such that

$$(12) \qquad w_k \cdots w_1a_kx_ku_1 = vny_kx_kn^{-1}.$$

Note that (7)–(9) show (10)–(12) to be valid for $k = 1$. Now

$$ny_kx_kn^{-1} \in C^r_G(\sigma^{k+1}e\sigma^{-k-1}).$$

So by (12), $w_k \cdots w_1a_1x_k \in C^r_G(\sigma^{k+1}e\sigma^{-k-1})$. Repeated use of Lemma 1.8 shows that $w_1, \ldots, w_k, a_k, x_k \in C^r_G(\sigma^{k+1}e\sigma^{-k-1})$. So by Lemma 1.5, we can factor for

$i = 1, \ldots, k$,

$$\begin{aligned}
w_i &= q_i w_i', & w_i' &\in C_G(T_{\sigma^j e \sigma^{-j}} | i+1 \le j \le k+1) \cap G^r_{\sigma^i e \sigma^{-i}}, \\
& & q_i &\in G^r_{\sigma^{k+1} e \sigma^{-k-1}} \cap C_G(T_{\sigma^j e \sigma^{-j}} | i+1 \le j \le k), \\
a_k &= c_{k+1} a_{k+1}, & a_{k+1} &\in C_G(T_{\sigma^j e \sigma^{-j}} | 0 \le j \le k+1) \cap G^l_f, \\
& & c_{k+1} &\in C_G(T_{\sigma^j e \sigma^{-j}} | 0 \le j \le k) \cap G^r_{\sigma^{k+1} e \sigma^{-k-1}}, \\
x_k &= y_{k+1} x_{k+1}, & x_{k+1} &\in C_G(T_f, T_{\sigma^j e \sigma^{-j}} | 0 \le j \le k+1), \\
& & y_{k+1} &\in C_G(T_f, T_{\sigma^j e \sigma^{-j}} | 0 \le j \le k) \cap G^r_{\sigma^{k+1} e \sigma^{-k-1}}.
\end{aligned}$$

Let

$$\begin{aligned}
q_i' &= w_k' \cdots w_{i+1}' q_i (w_k' \cdots w_{i+1}')^{-1} \in G^r_{\sigma^{k+1} e \sigma^{-k-1}}, \qquad i = 1, \ldots, k-1, \\
c_{k+1}' &= w_k' \cdots w_1' c_{k+1} (w_k' \cdots w_1')^{-1} \in G^r_{\sigma^{k+1} e \sigma^{-k-1}}, \\
y_{k+1}' &= w_k' \cdots w_1' a_{k+1} y_{k+1} (w_k' \cdots w_1' a_{k+1})^{-1} \in G^r_{\sigma^{k+1} e \sigma^{-k-1}}, \\
p &= q_k q_{k-1}' \cdots q_1' c_{k+1}' y_{k+1}' \in G^r_{\sigma^{k+1} e \sigma^{-k-1}}.
\end{aligned}$$

Then

$$w_k \cdots w_1 a_k x_k = p w_k' \cdots w_1' a_{k+1} x_{k+1}.$$

So by (12),

$$w_{k+1}' \cdots w_1' a_{k+1} x_{k+1} u_1 = v n y_{k+1} x_{k+1} n^{-1}$$

where

$$w_{k+1}' = v n y_k^{-1} n^{-1} v^{-1} p \in G^r_{\sigma^{k+1} e \sigma^{-k-1}}.$$

This completes the induction step in (10)–(12). So (10) is valid for all $k \in Z^+$. In particular it is valid for $k = \alpha$, where $\sigma^{\alpha+1} = 1$. Then

$$(13) \qquad x = y_1 \cdots y_\alpha x_\alpha, \qquad x_\alpha \in C_G(e^\theta | \theta \in \langle \sigma \rangle) = V.$$

Now by (4), (6)

$$(14) \qquad axu = vnxbn^{-1} z, \qquad z \in \hat{V}_e.$$

Let $Y_0 = G^l_f$, $Y_1 = G^r_{\sigma e \sigma^{-1}}$,

$$Y_i = C_G(\sigma^j e \sigma^{-j} | j = 1, \ldots, i-1) \cap G^r_{\sigma^i e \sigma^{-i}}, \qquad i \ge 2.$$

Then Y_j normalizes Y_i for $j \ge i \ge 1$. Also V normalizes Y_i for all i. By (3), (11) we see that

$$(15) \qquad a \in Y_0, \qquad nbn^{-1} \in Y_1, \qquad y_i \in Y_i, \quad 1 = 1, \ldots, \alpha.$$

Also, since $\sigma^{\alpha+1} = 1$, we see by (11) that

$$(16) \qquad n y_i n^{-1} \in Y_{i+1}, \quad i = 1, \ldots, \alpha - 1, \qquad n y_\alpha n^{-1} \in V_e.$$

Since $x_\alpha, u, v \in V$ and $V^\sigma = V$, we see by (13)–(16),

$$\begin{aligned}
v(x_\alpha u x_\alpha^*)^{-1} &= v n x_\alpha n^{-1} u^{-1} x_\alpha^{-1} \\
&= axuz^{-1} nb^{-1} x^{-1} x_\alpha n^{-1} u^{-1} x_\alpha^{-1} \\
&= [a y_1 \cdots y_\alpha x_\alpha u z^{-1} u^{-1} x_\alpha^{-1}](x_\alpha u) \\
&\quad \times [nb^{-1} x_\alpha^{-1} (y_\alpha^{-1} \cdots y_1^{-1}) x_\alpha n^{-1}](x_\alpha u)^{-1} \\
&\in \hat{V}_e Y_0 Y_1 \cdots Y_\alpha.
\end{aligned}$$

Since $\sigma^\alpha e \sigma^{-\alpha} = f$, we see by Proposition 1.9 that $v(x_\alpha u x_\alpha^*)^{-1} \in \Omega$. Thus $m_1 = eun \equiv evn = m_2$. This completes the proof of the theorem.

The proof of the above theorem shows

COROLLARY 2.5. *Let $a, b \in M_{e,\sigma}$. Then $a \sim b$ if and only if there exists $x \in V = C_G(e^\theta | \theta \in \langle\sigma\rangle)$ such that $x^{-1}ax = b$.*

COROLLARY 2.6. *Let $D = eC_G(e)$ denote the group of units of eMe, $h \in E(\overline{eT})$, $\theta = mT \in C_W(e)$. Then $M_{h,\theta} = (eMe)_{h,e\theta}$ and $G_{h,\theta} \cong D_{h,e\theta}$. If $a, b \in M_{h,\theta}$, then a is conjugate to b in M if and only if a is conjugate to b in eMe.*

PROOF. Let

$$V = C_G(h^\gamma | \gamma \in \langle\theta\rangle), \quad Y = C_D(h^\gamma | \gamma \in \langle\theta\rangle).$$

Let $a \in Y$. Then $a = ex$ for some $x \in C_G(e)$. For $\gamma \in \langle\theta\rangle$,

$$xh^\gamma = xeh^\gamma = ah^\gamma = h^\gamma a = h^\gamma ex = h^\gamma x.$$

So $x \in C_V(e)$ and $Y = eC_V(e)$. Now $V = V_h C_V(V_h) = V_h C_V(e)$. Hence

$$M_{h,\theta} = hV\theta = hC_V(e)\theta = heC_V(e)\theta = hY\theta = (eMe)_{h,e\theta}.$$

Let $\Omega = \prod_{\gamma \in \langle\theta\rangle} \hat{V}_{h^\gamma}$. Since $V = V_h C_V(e)$, $h \le e$,

$$G_{h,\theta} = V/\Omega \cong C_V(e)/C_\Omega(e) \cong eV/eC_\Omega(e).$$

By Proposition 1.9,

$$C_\Omega(e) = \prod_{\gamma \in \langle\theta\rangle} [\hat{V}_{h^\gamma} \cap C_G(e)].$$

It follows that $eV/eC_\Omega(e) = D_{h,e\theta}$. We are now done by Theorem 2.4.

CONJECTURE 2.7. Let $a, b \in eMe$. Then a is conjugate to b in M if and only if a is conjugate to b in eMe.

CONJECTURE 2.8. Let $\mathcal{Y} = \{M_{e,\sigma} | e \in E(\overline{T}), \sigma \in W\}$, $\mathcal{Y}_0$ the set of maximal elements (with respect to inclusion) of $\mathcal{Y}$. Then if $Y_1, Y_2 \in \mathcal{Y}_0$, $a \in Y_1$, $b \in Y_2$, $a \sim b$, then $Y_1^\theta = Y_2$ for some $\theta \in W$.

Let $g \in G$. Then the map: $x \to gx^{-1}g^{-1}$ is an antiautomorphism of G. We will call such an antiautomorphism an *inner antiautomorphism.*

EXAMPLE 2.9. Let $n \in Z^+$, $M = \mathcal{M}_n(K)$. Let $h = \prod_{\theta \in \langle\sigma\rangle} e^\theta$, r the rank of h. Then $G_{e,\sigma} \cong \mathrm{GL}(r, K)$ and * is an inner antiautomorphism.

EXAMPLE 2.10. Let $M = \{A \otimes B | A, B \in \mathcal{M}_2(K)\}$. Then the possibilities for $G_{e,\sigma}$ are $G, \mathrm{SL}(2,K), \mathrm{PGL}(2,K), G_m, \{1\}$. In all cases, * is inner.

CONJECTURE 2.11. If the simple components of G are all of type A_l, then * is necessarily inner.

By [**3**, Theorem 27.4], an antiautomorphism of a semisimple group is the composition of an inner antiautomorphism and an automorphism determined by an automorphism of the Dynkin diagram of the group.

CONJECTURE 2.12. For all $t \in R(G_{e,\sigma})$, $t^* = t^{-1}$ and hence * is completely determined by its action on the semisimple group $G'_{e,\sigma} = (G_{e,\sigma}, G_{e,\sigma})$.

3. Nilpotent elements. We continue from [**8**] the analysis of conjugacy classes of nilpotent elements of M. It was shown in [**8**] that the conjugacy classes of *minimal* nilpotent elements (in the $\mathcal{J}$-class ordering) is always finite. Renner [**12**] has introduced the finite fundamental inverse monoid $\mathrm{Ren}(M) = N_G(\overline{T})/T$ and used it to generalize the Bruhat decomposition to M. We easily have

PROPOSITION 3.1. *Let $e \in E(\overline{T})$, $\sigma = nT \in W$, $k \in Z^+$. Then the following conditions are equivalent:*
(i) $a^k = 0$ *for some* $a \in M_{e,\sigma}$,
(ii) $M_{e,\sigma}^k = 0$,
(iii) $(e\sigma)^k = 0$ *in* $\mathrm{Ren}(M)$,
(iv) $e^\sigma \cdots e^{\sigma^k} = 0$.

Since $V = C_G(e^\theta | \theta \in \langle\sigma\rangle)$ is a reductive group, we see that any closed normal subgroup of V containing T, must equal V. Thus

PROPOSITION 3.2. *Let $e \in E(\overline{T})$, $\sigma \in W$. Then $G_{e,\sigma}$ is trivial if and only if $T = \prod_{\theta\in\langle\sigma\rangle} T_{e^\theta}$.*

In particular, we see that $G_{e,\sigma}$ trivial implies that $e\sigma$ is nilpotent. If the groups $G_{e,\sigma}$ are trivial for all nilpotent $e\sigma$, then by Theorems 2.2, 2.4, the number of conjugacy classes of nilpotent elements in M is finite.

CONJECTURE 3.3. The number of conjugacy classes of nilpotent elements of M is finite if and only if the groups $G_{e,\sigma}$ are trivial for all nilpotent $e\sigma$.

EXAMPLE 3.4. If $M = \mathcal{M}_n(K)$, then we see by Example 2.9 that the groups $G_{e,\sigma}$ are trivial for nilpotent $e\sigma$.

EXAMPLE 3.5. Let $G_0 = \{A \otimes (A^{-1})^t | A \in \mathrm{SL}(3, K)\}$, $G = K^* G_0$, $M = \overline{KG_0}$. Let $S = M \setminus G$. Then

$$E(S) = \{e \otimes f | e^2 = e, f^2 = f \in \mathcal{M}_3(K), ef^t = f^t e = 0\}.$$

In particular

$$e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in E(M).$$

Also if

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in W(G),$$

then $e^\sigma = f$ and $(e\sigma)^2 = 0$. The group $G_{e,\sigma}$ can be seen to be the one dimensional torus with * being given by $x \to x^{-1}$. Thus by Theorem 2.4, the number of conjugacy classes of nilpotent elements of M is infinite. However if C denotes the center of G, then the number of conjugacy classes of nilpotent elements in M/C is finite.

EXAMPLE 3.6. Suppose $\mathrm{char}\, K \neq 2$, $n \in Z^+$, $n \geq 2$. For $r \in Z^+$, let J_r denote the $r \times r$ matrix

$$\begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}.$$

Let G_0 consist of all $A \in \mathrm{SL}(2n+1, K)$ such that

$$A^t \begin{bmatrix} 1 & 0 \\ 0 & J_{2n} \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & J_{2n} \end{bmatrix}.$$

Thus [**3**, §7.2], G_0 is the special orthogonal group of type B_n. Let $G = K^*G_0$, $M = \overline{KG_0}$. Then

$$e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix} \in E(M).$$

If

$$\sigma = \begin{bmatrix} \pm 1 & 0 \\ 0 & J_{2n} \end{bmatrix} \in W(G),$$

then $e^\sigma = f$ and $(e\sigma)^2 = 0$. It can be seen that $G_{e,\sigma} \cong \mathrm{PGL}(n, K)$ with the antiautomorphism * on $G_{e,\sigma}$ given by $A \to J_n A^t J_n$. Thus by Theorem 2.4, the number of conjugacy classes of nilpotent elements of M is infinite. This gives a counterexample to [**8**, Conjectures 4.5, 4.6]. Note also that for $n \geq 3$, * is not inner.

The above examples suggest

CONJECTURE 3.7. Suppose that the center of G is one dimensional. Then the number of conjugacy classes of nilpotent elements of M is finite if and only if $\mathrm{Ren}(M)$ is isomorphic to the symmetric inverse semigroup of some finite set.

REFERENCES

1. R. W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, Wiley, 1985.
2. J. M. Howie, *An introduction to semigroup theory*, Academic Press, 1976.
3. J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, 1981.
4. M. S. Putcha, *Green's relations on a connected algebraic monoid*, Linear and Multilinear Algebra **12** (1982), 205–214.
5. ——, *A semigroup approach to linear algebraic groups*, J. Algebra **80** (1983), 164–185.
6. ——, *Reductive groups and regular semigroups*, Semigroup Forum **30** (1984), 253–261.
7. ——, *Determinant functions on algebraic monoids*, Comm. Algebra **11** (1983), 695–710.
8. ——, *Regular linear algebraic monoids*, Trans. Amer. Math. Soc. **290** (1985), 615–626.
9. ——, *Linear algebraic monoids*, monograph (to appear).
10. L. E. Renner, *Reductive monoids are von Neumann regular*, J. Algebra **93** (1985), 237–245.
11. ——, *Classification of semisimple algebraic monoids*, Trans. Amer. Math. Soc. **292** (1985), 193–224.
12. ——, *Analogue of the Bruhat decomposition for algebraic monoids*, J. Algebra **101** (1986), 303–338.
13. B. M. Schein, *A symmetric semigroup of transformations is covered by its inverse subsemigroups*, Acta Math. Acad. Sci. Hungar. **22** (1971), 163–171.

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