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BRANCHED COVERINGS OF 2-COMPLEXES AND DIAGRAMMATIC REDUCIBILITY

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ABSTRACT. The condition that all spherical diagrams in a 2-complex be reducible is shown to be equivalent to the condition that all finite branched covers be aspherical. This result is related to the study of equations over groups. Furthermore large classes of 2-complexes are shown to be diagrammatically reducible in the above sense; in particular, every 2-complex has a subdivision which admits a finite branched cover which is diagrammatically reducible.

A consequence of the classical Riemann-Hurwitz formula for Riemann surfaces [**6**, p. 301] is the fact that a finite branched cover of an aspherical Riemann surface is also aspherical. However the property of asphericity of a 2-complex implying asphericity of finite branched covers fails for general 2-complexes. In fact we shall prove (Theorem 4.5 below) that this property holds precisely for those 2-complexes which are *diagrammatically reducible*.

A 2-complex X is said to be diagrammatically reducible (abbreviated $\mathrm{DR}(X)$) if for every combinatorial map $f\colon S^2 \to X$ of the 2-sphere into X (a "spherical diagram" in X) there is a pair of faces in the domain with an edge in common which are mapped mirrorwise by f across that edge. A closely related notion was introduced by Lyndon and Schupp [**13**] but this notion appears in its present form in a paper by Sieradski [**14**]. In that same paper Sieradski asked a question which would have implied that all classical knot complements in the 3-sphere are homotopy equivalent to diagrammatically reducible 2-complexes. We shall prove this latter assertion (Theorem 6.5) making use of our amalgamation theorem for diagrammatic reducibility (Theorem 5.4).

The characterization of diagrammatic reducibility in terms of branched covers follows from a lifting theorem which may be of general interest. If $f\colon X \to Y$ is a combinatorial map of combinatorial 2-complexes, then f induces a map of the link complexes of vertices $L_f\colon L_X \to L_Y$ (the "star graphs" or "coinitial graph" in group theoretical language). The map f is said to be *reduced* if L_f is an immersion. We prove (Theorem 4.2) that if $f\colon X \to Y$ is a reduced map of 2-complexes with X finite, then there is a commutative diagram (see below) where $\pi\colon Z \to Y$ is a finite branched cover and where g is injective off the zero skeleton of X. This result is

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itself a consequence of a theorem of Marshall Hall's on free groups. To our knowledge this is the first application of M. Hall's theorem to 2-complexes.

$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ X \arrow[ur, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

The organization of the paper is as follows. In §1 we review what is known about diagrammatic reducibility, its relation to solving equations over groups, and the weight test and its relation to the conjugacy problem for hyperbolic 2-complexes. In §2 we introduce the basic notions related to branched covers and give examples related to group theory. The examples (Fibonacci groups, Higman presentations) have a long history in the literature, but our way of viewing them in terms of branched covers is new. In §3 we prove that every finite 2-complex has a subdivision which admits hyperbolic branched covers and in §4 we establish the lifting theorem quoted above and prove our characterization of diagrammatic reducibility in terms of branched covers. In §5 we prove amalgamation theorems for constructing new diagrammatically reducible 2-complexes out of old ones. The applications are contained in §6. Among these are a strengthening of the characteristic property of the Higman-Neumann-Neuman imbedding of a free group of countable rank in one of rank 2 and the application to classical knots mentioned earlier.

An appendix has been included to relate our notion of diagrammatic reducibility to the notion of diagrammatic asphericity [**21**]. It is appropriate to say here that showing a 2-complex is diagrammatically reducible is very much stronger than proving it is diagrammatically aspherical and has implications about equations over groups that the latter does not.

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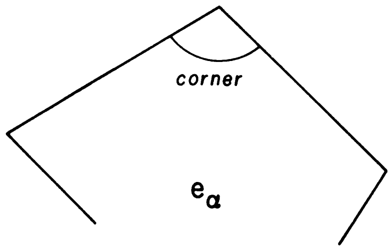
Notation. If A is a set, let $\bar{A}$ be a set disjoint from A, in 1-1 correspondence with A by the map $a \mapsto \bar{a}$. A presentation $\mathscr{P}$ is a symbol $\langle A \mid S \rangle$ where S is a subset of the free monoid on the alphabet $A \cup \bar{A}$. The symbol $K(\mathscr{P})$ denotes the 2-complex canonically associated to $\mathscr{P}$ and $G(\mathscr{P})$ denotes $\pi_1(K(\mathscr{P}))$, the group of the presentation.

If G is a group and $x \in G$, $\bar{x}$ (or x^{-1} if space permits) denotes the inverse of x. If S is a subset of G, then $\langle S \rangle$ denotes the subgroup generated by S and $\langle\langle S \rangle\rangle_G$ (or $\langle\langle S \rangle\rangle$, if G is clear) denotes the normal closure of S in G. If $x \in F$, x^G denotes the conjugacy class of x in G.

1. Diagrammatic reducibility.

1.1 We work in the category of combinatorial 2-complexes and combinatorial maps. A cellular map f: $X \to Y$ of two CW complexes is *combinatorial* if the restriction of f to each open cell is a homeomorphism onto its image. A 2-complex X is called combinatorial if the attaching map $S^1 \xrightarrow{f_\alpha} X^{(1)}$ of each 2-cell e_α of X is combinatorial for suitable subdivision S^1_α of S^1 (depending on e_α). The number of vertices in this subdivision S^1_α is denoted $d(e_\alpha)$ and is called the degree of e_α. The face e_α has $d(e_\alpha)$ (unoriented) corners [**5**, §2], where each corner is incident with one vertex of $f_\alpha^{-1}(X^{(1)})$; thus each corner of a 2-complex X is incident with one face and with one vertex of X.



Each unoriented corner represents two oriented corners with opposite orientations. Corners are invariant under combinatorial maps and give rise to a functor $X \mapsto F_X$ from combinatorial 2-complexes and combinatorial maps to free groups, where F_X is the free group with free basis consisting of a choice of one oriented corner from each pair of oppositely oriented corners.

From now on we shall understand 2-*complex to mean combinatorial* 2-*complex.*

1.2 With each 2-complex X is associated a graph L_X, the link of the zero skeleton $X^{(0)}$ in X. Thus L_X can be defined as the boundary of a regular neighborhood of $X^{(0)}$ in X. Since we have need of a specific cell structure on L_X, we recall the following construction of it due to J. H. C. Whitehead [**17**]. It has come to be known as the "star graph" or even "coinitial graph". Let E denote the set of *oriented* edges of X. Thus E is equipped with a fixed point free involution $e \mapsto \bar{e}$ associating to the edge e the edge $\bar{e}$ with same carrier but opposite orientation. We take E as the *vertex* set for the graph L_X. Each attaching map f_α of a 2-cell e_α of X determines a cyclic word w_α (not necessarily reduced) in the free semigroup in the alphabet E. If the syllable ab occurs in the spelling of w_α, we join the vertex a to the vertex $\bar{b}$ of L_X by an edge corresponding to this syllable. Hence the unoriented edges connecting a to $\bar{b}$ in L_X correspond 1-1 to occurrences of syllables ab or $\bar{b}\bar{a}$ in the cyclic words w_α determined by attaching maps of 2-cells of X. In particular, unoriented edges of L_X correspond 1-1 to unoriented corners of X.

From the definition, it is clear that the assignment $X \mapsto L_X$ is a functor from 2-complexes and combinatorial maps to graphs and nondegenerate morphisms of graphs.

1.3 A combinatorial map $f\colon X \to Y$ of 2-complexes is *reduced* if the induced map $L_f\colon L_X \to L_Y$ is an *immersion* [**16**]. Recall that a nondegenerate map $g\colon \Gamma \to \Gamma'$ of graphs is an immersion if for each vertex v of Γ the induced map $g\colon \mathrm{Star}_\Gamma(v) \to \mathrm{Star}_{\Gamma'}(g(v))$ is injective, where $\mathrm{Star}_\Gamma(v)$ is the set of oriented edges of Γ with initial vertex v.

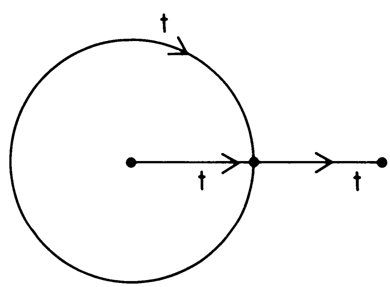
The only way a nondegenerate morphism of graphs $g\colon \Gamma \to \Gamma'$ can fail to be an immersion is for two distinct edges of Γ to be folded [**16**]. That is there exist oriented edges $e \neq e'$ of Γ with the same initial vertex and such that $ge = ge'$.

A combinatorial map $f\colon C \to X$ is called a *spherical diagram* if C is some cell structure on S^2. We have the following criterion.

1.4 LEMMA. *The spherical diagram $f\colon C \to X$ is not reduced iff there is a pair of distinct open 2-cells α and β of C with an edge e in the closure of each and an involutory homeomorphism g of $\alpha \cup \beta \cup e$ such that g interchanges α and β, $g \mid e$ is the identity of e, and $f \mid \beta = f \cdot g \mid \alpha$.*

Of necessity g maps α in an orientation reversing manner onto β. We abbreviate the criterion 1.4 by saying there is a pair of faces mapped mirrorwise by f across a common face. In this case we can cut out α, β, and e and paste along the boundary to obtain a finite collection of 2-spheres glued together at vertices and a combinatorial map to X.

1.5 The 2-complex X is called diagrammatically reducible (abbreviated DR(X)) if no spherical diagram $f\colon C \to X$ is reduced. In this case the cut-and-paste operation indicated above produces an explicit null homotopy of the map f. Since a transversality argument shows that $\pi_2(X)$ is generated as a $\pi_1(X)$ module by classes of spherical diagrams, it follows that diagrammatic reducibility implies asphericity for 2-complexes. The converse however is false. The simplest counterexample is a reduced spherical diagram in the dunce hat $K(\mathscr{P})$ where $\mathscr{P} = \langle t \mid tt^{-1} \rangle$, sketched below:



(The second cell is at infinity, the unbounded region in the plane. The author has become accustomed to call this diagram "my favorite diagram" [**4**].)

1.6 There is an intimate relation between diagrammatic reducibility and solving equations over groups [**3**, **5**]. Rather than recall the procedure in general how a 2-complex determines systems of equations over groups [**5**, 2.7] we give a specific example which should make the process clear. The 2-complex $X = K(\mathscr{P})$ can be seen to be diagrammatically reducible, where $\mathscr{P} = \langle x, y, z, w \mid xy\bar{x}\bar{y}zw\bar{z}\bar{w} \rangle$. Consider the "equation"

$$E = a_1xa_2ya_3\bar{x}a_4\bar{y}a_5za_6wa_7\bar{z}a_8\bar{w} = 1 \tag{1.6.1}$$

where a_i are elements of some group A $(1 \leqslant i \leqslant 8)$ and where x, y, z and w are variables. Then it follows from DR(X) that equation (1.6.1) can be solved in an overgroup of A. In fact much more is true. The natural homomorphism $\phi\colon A \to A\langle x, y, z, w\rangle/\langle\langle E\rangle\rangle$ (where $A\langle x, y, z, w\rangle$ is the free product of A with the free group freely generated by x, y, z and w) satisfies "Property G" [**5**, 2.8.3]. Here a homomorphism $\phi\colon A \to B$ of groups satisfies "Property G" if, for any number n and elements $a_1, a_2, \ldots, a_n$ in A satisfying $1 \in \prod_{i=1}^n \phi(a_i)^B$, one has $1 \in \prod_{i=1}^n a_i^A$.

Briefly one says that DR(X) implies that all systems of equations over an arbitrary coefficient group A modeled on X satisfy "Property G", and hence are solvable in an overgroup of A. The reverse implication is invalid: The real projective plane is not aspherical, but the associated equation satisfies "Property G" [5, 5.4]. We shall need the connection of DR(X) with "Property G" only in §6 in our discussion of the *HNN* imbedding. However it was this connection which motivated us at first to study diagrammatic reducibility.

There is a simple "weight test" for diagrammatic reducibility which we recall since we need it to describe hyperbolic 2-complexes. A *weight* on a 2-complex S is a real valued function on the corners of X. We write $\gamma < A$ to indicate the corner γ of X is incident with the 2-cell A of X and $\gamma < v$ to indicate that γ is incident at the vertex v of X.

1.7 THEOREM [5, 4.7]. *The 2-complex X is diagrammatically reducible if there is a weight w on X satisfying*

1.7.1 *Curvature Condition. For each 2-cell A of X one has $\sum_{\gamma < A} w(\gamma) \leqslant d(A) - 2$, and*

1.7.2 *Link Condition. For each nontrivial cyclically reduced circuit $\gamma_1\gamma_2 \cdots \gamma_n$ in the link complex L_X one has $\sum_{i=1}^{n} w(\gamma_i) \geqslant 2$. Observe here that corners of X correspond 1-1 to edges of L_X. The condition that the circuit be cyclically reduced means $\gamma_{i+1} \neq \bar{\gamma}_i$ for $1 \leqslant i \leqslant n-1$, and $\gamma_n \neq \bar{\gamma}_1$.*

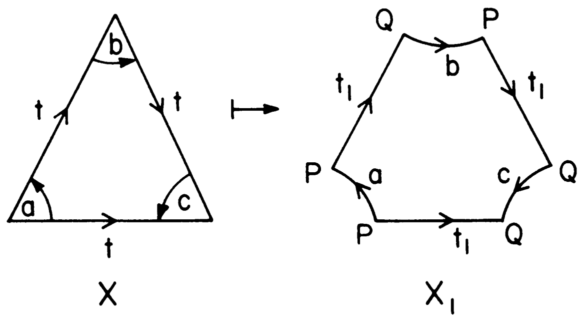
1.8 On the basis of this weight test we define a hyperbolic 2-complex to be a pair (X, w), where X is a 2-complex and w is a nonnegative weight on X satisfying the link condition 1.7.2 and satisfying the curvature condition 1.7.1 with strict inequality for each 2-cell A. In [5, Appendix], we proved an isoperimetric inequality for finite hyperbolic 2-complexes, leading to a solution of the conjugacy problem for their fundamental groups. To relate these notions to the theory of Riemann surfaces, we observe that surfaces of nonpositive Euler characteristic admit weights satisfying 1.7.1 and 1.7.2, whereas surfaces of negative Euler characteristic admit weights satisfying 1.7.2 and satisfying 1.7.1 with strict inequalities.

1.9 Finally we recall a result of Howie, whose proof is sketched in [5, 6.10], that a 2-complex X is diagrammatically reducible iff every subdivision of X is diagrammatically reducible. Thus the notion of diagrammatic reducibility is a combinatorial invariant for 2-complexes.

2. Branched covers.

2.1 DEFINITION. A combinatorial map f: $Y \to X$ of 2-complexes is a branched covering if f_0: $Y_0 \to X_0$ is a covering map. Here $X_0 = X - X^{(0)}$, $Y_0 = Y - Y^{(0)}$ and $f_0 = f \mid Y_0$. We shall only be concerned with finite branched coverings, where f_0: $Y_0 \to X_0$ is a finite sheeted covering map.

To construct branched covers we need to reformulate the definition slightly. Let N be a regular neighborhood of $X^{(0)}$ in X, so the boundary of N can be identified with the link complex L_X. Let $X_1 = X - \mathring{N}$, so the inclusion $X_1 \subset X_0$ is a homotopy equivalence. Thus coverings of X_1 are the same as those of X_0. Given a finite covering $U \xrightarrow{\pi} X_1$ there are in general several ways to extend it to a branched cover of X. One way is to attach a cone to each connected component L' of $\pi^{-1}(L_X)$ (recall $L_X \subset X_1$) and extend π conically. This works since L' is a finite cover of $\pi(L')$, a connected component of L_X. We call the branched cover of X constructed in this way the "conical extension of π: $U \to X_1$". If on the other hand X has only one vertex, so $X = K(\mathscr{P})$ for some presentation $\mathscr{P}$, then it is more natural to extend π: $U \to X_1$ to a branched cover Y of X in a different way, by identifying all cone points of the conical extension to a single vertex. In this case $Y = K(\mathscr{Q})$ for some presentation $\mathscr{Q}$. In particular, if $\pi^{-1}(L_X)$ is connected, then the extension of π to a branched cover of X is unique up to isomorphism and is of the form $K(\mathscr{Q})$. Since X_1 has an explicit cell structure whose edges correspond 1-1 to the union of the edges of X with those of L_X, this often makes the construction of explicit branched covers very easy.



2.2 Example. Let $X = K(\mathscr{P})$, with $\mathscr{P} = \langle t \mid tt\bar{t}\rangle$, the dunce hat. We label the oriented corners a, b, c as shown above. Then X_1 has two vertices P and Q, four edges a, b, c, t_1 and one 2-cell with attaching map $at_1bt_1c\bar{t}_1$. Since the edge "t_1" is a maximal tree of $X_1^{(1)}$, to construct an abelian cover of X_1 it suffices to choose elements $\phi(a)$, $\phi(b)$, and $\phi(c)$ in an abelian group A such that $\phi(a) + \phi(b) + \phi(c) = 0$. If the elements $\phi(a), \phi(b), \phi(c)$ generate A, the cover $U \xrightarrow{\pi} X_1$ of X_1 will be connected and $\pi^{-1}(L_X)$ will also be connected here.

As a specific example, if $A = \mathbf{Z}_n$ and $\phi(b) = \phi(c) = 1$ while $\phi(a) = n - 2$, then the branched cover Y of X is uniquely determined and $Y = K(\mathscr{Q})$, with

$$\mathscr{Q} = \langle t_1, t_2, \dots, t_n \mid t_it_{i+1}\bar{t}_{i+2};\ i \ (\operatorname{mod} n)\rangle.$$

The group $G(\mathscr{Q})$ of this presentation is the Fibonacci group $F(2, n)$ due originally to Conway [**1**] when $n = 5$.

2.3 The general Fibonacci groups $F(r, n)$ [**12**] occur as $G(\mathscr{Q})$ where

$$\mathscr{Q} = \langle t_1, t_2, \dots, t_n \mid t_it_{i+1} \cdots t_{i+r}\bar{t}_{i+r+1};\ i \ (\operatorname{mod} n)\rangle.$$

The complex $K(\mathscr{Q})$ is easily seen to be a branched cyclic cover of $K(\mathscr{P})$, where $\mathscr{P} = \langle t \mid t^r\bar{t}\rangle$.

2.4 There is no need however to restrict oneself to the covers of X_1 in 2.2 giving rise to the Fibonacci groups. For example, the complex $Y = K(\mathscr{Q})$ occurs as a branched cyclic cover of the dunce hat, where

$$\mathscr{Q} = \langle t_1, t_2, \dots, t_n \mid t_{i+2}t_i\bar{t}_{i+1};\ i \ (\operatorname{mod} n)\rangle.$$

If $n = 5$, this is a presentation of the binary icosahedral group, as John Stallings pointed out to me. This group does not appear in the list of finite Fibonacci groups $F(2, n)$.

2.5 We discussed the "Higman presentations" $\mathscr{H}_n$, $n \geqslant 2$, in [5, 4.20], where

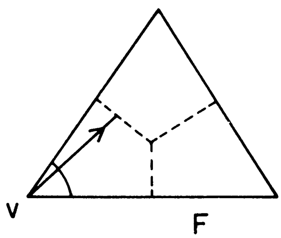
$$\mathscr{H}_n = \langle x_1, x_2, \ldots, x_n \mid x_{i+1}x_i\bar{x}_{i+1}\bar{x}_i\bar{x}_i;\ i \ (\mathrm{mod}\, n)\rangle.$$

For $n = 2$ or 3, $G(\mathscr{H}_n)$ is trivial, while for $n \geqslant 4$, $G(\mathscr{H}_n)$ is an infinite perfect group having no proper subgroups of finite index [7]. It can be checked that $K(\mathscr{H}_n)$ is a branched n-fold cyclic covering of $K(\mathscr{P})$, where $\mathscr{P} = \langle t \mid tt\bar{t}\bar{t}\bar{t}\rangle$. The complex $K(\mathscr{H}_n)$ was shown to be diagramatically reducible for $n \geqslant 4$ [5].

We shall next analyze the geometry of X, X_0, and X_1 more carefully for X an arbitrary 2-complex.

2.6 The dual 1-skeleton of a 2-complex X is the largest subgraph D_X of the first barycentric subdivision X' of X whose vertices are barycenters of 1-cells and 2-cells of X. Thus each face A of X contributes $d(A)$ edges to D_X. The dual 1-skeleton is a functor from 2-complexes and combinatorial maps to graphs and nondegenerate morphisms of graphs.

There is a canonical deformation retraction η: $X_1 \to D_X$ obtained by joining a vertex v of X linearly in a face F containing v to a point of D_X.



In particular η induces by restriction a morphism of graphs η_X: $L'_X \to D_X$ where L'_X is the first barycentric subdivision of L_X, the link complex. The following three properties of η_X are easily checked.

2.7 The map η_X is natural in combinatorial maps $X \to Y$, so one has a commutative diagram of graphs:

$$\begin{tikzcd} L'_X \arrow[r, "\eta_X"] \arrow[d, "f"] & D_X \arrow[d, "f"] \\ L'_Y \arrow[r, "\eta_Y"] & D_Y \end{tikzcd}$$

2.8 If X has no monogons (so $d(A) > 1$ for each face A of X), then η_X is an immersion of graphs.

2.9 X_1 is isomorphic to the mapping cylinder of the map η_X: $L'_X \to D_X$.

In particular, 2.9 has an important consequence for the sequel.

2.10 PROPOSITION. *Covering spaces of X_1 correspond* 1-1 *to those of D_X.*

Thus one has a construction for all finite branched covers of X. First construct the finite covers of D_X, a graph. This is a problem about subgroups of finite index in a free group. Next use the deformation retraction η: $X_1 \to D_X$ to construct all finite covers of X_1. Take the conical extensions of such covers (2.1) and finally make identifications ad libidum in the fibers over $X^{(0)}$.

2.11 PROPOSITION. *Let f: $Y \to X$ be a branched covering.*

(a) *If X is diagrammatically reducible, so is Y.*

(b) *If X satisfies the curvature and link conditions for weight w, then so does Y for the pull-back weight $f^*(w)$.*

(c) *If (X, w) is hyperbolic, then so is (Y, f^*w).*

PROOF. All of these assertions are consequences of the fact that f induces an immersion $L_Y \to L_X$ of link complexes, which in turn preserves reduced paths [**16**]. Thus, for example in 2.11(a), if g: $S^2 \to Y$ is a reduced spherical diagram, then $f \circ g$: $S^2 \to X$ will also be reduced. The remaining statements are proved similarly.

2.12 It is worth giving an example of a finite branched cover Y of an aspherical 2-complex X which is not itself aspherical. One may take X to be the dunce hat and $Y = K(\mathscr{Q})$ where $\mathscr{Q} = \langle x, y \mid xx\bar{y}, yy\bar{x} \rangle$. It is easily checked that Y is a 2-fold branched cover of X. But $\pi_1(Y) = \mathbf{Z}_3$ so (by P. A. Smith's theorem) Y cannot be aspherical. This phenomenon will be explained in §4, when reduced diagrams will be related to branched covers.

Here is a curious consequence of 2.9.

2.13 PROPOSITION. *Let X be a 2-complex and let Y be a triangulable subset of $X_0 = X - X^{(0)}$. Then Y is aspherical.*

PROOF. If Y is compact, then Y may be assumed to be a subset of X_1. Now X_1 collapses onto D_X, a graph, whence X_1 is diagrammatically reducible. Since diagrammatic reducibility is invariant under subdivision and is inherited by subcomplexes, it follows that Y is diagrammatically reducible and hence aspherical.

In the general case, Y may be triangulated; so every finite subcomplex of Y is aspherical, by the preceding paragraph. It follows that Y is aspherical.

3. Existence of hyperbolic covers

3.1 THEOREM. *Suppose that X is a finite 2-complex such that $d(F) \geqslant 3$ for each face F of X. Then X has a finite branched cover which admits a hyperbolic structure.*

Observe that any cell structure C on S^2 with only two vertices and no monogons provides a counterexample to extending 3.1, for any branched cover of C is itself the 2-sphere. Observe also that for any 2-complex X the first barycentric subdivision X' satisfies $d(F) = 3$ for all faces F of X'.

3.2 COROLLARY. *If X is a finite 2-complex then X', the first barycentric subdivision of X, has a finite branched cover which admits a hyperbolic structure.*

PROOF OF 3.1. We may assume X is connected. We can find a positive number ε such that for any face F of X, $\varepsilon < (d(F) - 2)/d(F)$. This is true because X is finite and $d(F) \geqslant 3$ for all faces F. We set $w(\gamma) = \varepsilon$ for all corners γ of X. Pick a number N such that $N\varepsilon \geqslant 2$ and list all the finite number of nontrivial cyclically reduced circuits of the link complex L_X of length $< N$, say $C_1, C_2, \ldots, C_k$.

Since η_X is an immersion by 2.8 it follows that each of $\eta_X(C_1), \eta_X(C_2), \ldots, \eta_X(C_k)$ is a nontrivial cyclically reduced circuit in D_X. Choose a base point v for D_X. By connecting some vertex in the circuit $\eta_X(C_i)$ to v in D_X by an arc α_i, we construct elements $x_1, x_2, \ldots, x_k$ of $\pi_1(D_X, v) = G$. These elements depend on the choice of α_i, but their conjugacy classes in G are independent of choices. By M. Hall's theorem [**16**, 6.3], we can find a subgroup M of finite index in G with $x_i \notin M$, $1 \leqslant i \leqslant k$. Furthermore, by taking the intersection of finitely many conjugate subgroups, we may assume in addition that $M \triangleleft G$. Then no conjugate of x_i lies in M.

Let $Y_1 \xrightarrow{\pi} X_1$ be the finite regular cover of X_1 (see 2.1 for notation) corresponding to $M \triangleleft G$ and extend π to a finite branched cover π: $Y \to X$, by the conical extension, for definiteness. We claim that (Y, π^*w) is a hyperbolic 2-complex. The curvature condition is automatic, since (X, w) satisfies the curvature condition. It remains to check the link condition.

Suppose then that Z is a nontrivial cylically reduced circuit in L_Y. Then $\pi(Z)$ is a nontrivial cyclically reduced circuit in X. On the other hand $\pi(Z) \neq C_i$ for $1 \leqslant i \leqslant k$, for none of the circuits C_i lift to closed paths in Y_1; otherwise some conjugate of some x_i would lie in M, contrary to choice of M. Thus the length of $\pi(Z)$ is greater than or equal to N. If $Z = \gamma_1\gamma_2 \cdots \gamma_n$ with γ_i corners, $n \geqslant N$, it follows that $\sum_{i=1}^{n} w(\gamma_i) \geqslant N\varepsilon \geqslant 2$. Thus the link condition is satisfied and (Y, π^*w) is a hyperbolic structure. This completes the proof of Theorem 3.1.

4. Lifting theorem.

4.1 THEOREM. *Let f: $\Gamma \to \Delta$ be an immersion of graphs, where Γ is finite. Then there is a finite covering $\Delta_1 \xrightarrow{\pi} \Delta$ and a commutative diagram*

$$\begin{tikzcd} & \Delta_1 \arrow[d, "\pi"] \\ \Gamma \arrow[ur, "f_1"] \arrow[r, "f"'] & \Delta \end{tikzcd}$$

where f_1 is injective.

This is a graphical form of Marshall Hall's theorem on free groups. The proof for the special case when Δ is a 1-vertex graph can be found in [**16**, 6.1]. The reduction of the general case to the 1-vertex case is given in [**19**, Appendix].

4.2. THEOREM. *Let $f\colon X \to Y$ be a reduced combinatorial map of the 2-complexes X and Y with X finite. Then there is a finite branched cover $\pi\colon Z \to Y$ and a commutative diagram*

$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ X \arrow[ur, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

where $g|(X - X^{(0)})$ is injective.

PROOF. Since f is reduced it follows that f induces immersions of graphs $L_X \xrightarrow{f} L_Y$ and $D_X \xrightarrow{f} D_Y$. Since D_X is finite, it follows from 4.1 that there is a commutative diagram

(4.2.1)
$$\begin{tikzcd} & \Gamma \arrow[d, "\pi"] \\ D_X \arrow[ur, "g"] \arrow[r, "f"'] & D_Y \end{tikzcd}$$

where π is a finite covering and where g is injective. Since coverings of D_Y are the same as coverings of Y_1 by 2.9 it follows we can thicken up the diagram (4.2.1) to obtain a commutative diagram

(4.2.2)
$$\begin{tikzcd} & Z_1 \arrow[d, "\pi"] \\ X_1 \arrow[ur, "g"] \arrow[r, "f"'] & Y_1 \end{tikzcd}$$

where $\pi\colon Z_1 \to Y_1$ is a finite covering. Next extend π by conical extension (2.1) to a finite branched cover $\pi\colon Z \to Y$ and extend the map $g\colon X_1 \to Z_1$ also conically to a map $g\colon X \to Z$ so that the diagram

(4.2.3)
$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ X \arrow[ur, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

is commutative. It remains to check that $g|(X - X^{(0)})$ is injective. But this will follow if we can show that $g\colon X_1 \to Z_1$ is injective. Since X_1 and Z_1 are mapping cylinders over D_X and Γ respectively, it suffices to show that $g\colon X_1 \to Z_1$ is injective on some neighborhood of D_X in X_1. However $f\colon X_1 \to Y_1$ is locally injective, so it follows from (4.2.2) that $g\colon X_1 \to Z_1$ is also locally injective. Since $g\colon D_X \to \Gamma$ is injective, it follows that $g\colon X_1 \to Z_1$ is injective on some neighborhood of D_X in X_1, as required.

4.3 LEMMA. *Let $f\colon C \to X$ be a combinatorial map of 2-complexes where C is a cell structure on S^2. If f is injective on $C - C^{(0)}$, then the Hurewicz map $\pi_2(X) \to H_2(X)$ is nonzero.*

PROOF. One sees that f induces a nonzero map $H_2(C) \to H_2(X)$, so the class of f in $\pi_2(X)$ is detected in $H_2(X)$.

4.4 DEFINITION. We recall that a connected 2-complex X is called Cockcroft if the Hurewicz map $\pi_2(X) \xrightarrow{h_X} H_2(X)$ is zero.

4.5 THEOREM. *The following assertions are equivalent for a connected 2-complex X:*

4.5.1 *X is diagrammatically reducible.*

4.5.2 *Every finite branched cover of X is aspherical.*

4.5.3 *Every finite branched cover of X is Cockcroft.*

PROOF. 4.5.1 $\Rightarrow$ 4.5.2 follows from 2.11(a).

4.5.2 $\Rightarrow$ 4.5.3 follows since aspherical 2-complexes are Cockcroft.

4.5.3 $\Rightarrow$ 4.5.1: Suppose that every finite branched cover of X is Cockcroft and let $f\colon C \to X$ be a spherical diagram. If f were reduced, it would follow from 4.2 that there is a commutative diagram

$$\begin{tikzcd} & Y \arrow[d, "\pi"] \\ C \arrow[ur, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

where $\pi\colon Y \to X$ is a finite branched cover and where $g\colon C \to Y$ is injective when restricted to $C - C^{(0)}$. It follows from Lemma 4.3 that Y is not Cockcroft, contrary to hypothesis.

Thus every spherical diagram in X is reduced, and X is diagramatically reducible.

5. Amalgamation theorems. Theorem 4.5 is a powerful tool which can be exploited to prove certain 2-complexes are diagrammatically aspherical. The arguments also use the next result which is a generalized version of a theorem of J. H. C. Whitehead **[18]**.

5.1 THEOREM. *Let Z be a connected CW complex which is the union of two subcomplexes X and Y. Assume that each connected component of X and of Y is aspherical. Assume also that each connected component C_0 of $X \cap Y$ is aspherical and the inclusions $C_0 \subseteq X_0$ and $C_0 \subseteq Y_0$ induce injections in π_1; here X_0 and Y_0 are connected components of X and Y respectively. Then Z is aspherical.*

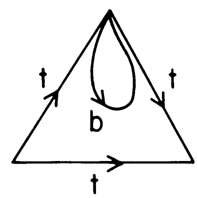
The proof makes use of the Mayer-Vietoris homology sequence for the universal cover $\tilde{Z}$ of Z together with the fact that the universal covers of X_0, Y_0, and C_0 all are imbedded as subcomplexes of $\tilde{Z}$. One deduces that $\tilde{H}_*(\tilde{Z}) = 0$, so by Whitehead's theorem $\tilde{Z}$ is contractible.

5.2 EXAMPLE. A naive attempt to generalize 5.1, replacing "aspherical" everywhere by "diagrammatically reducible" fails as the next example shows. Let X be the dunce hat:

$$X = K(\mathscr{P}), \quad \mathscr{P} = \langle t \mid tt\bar{t}\rangle.$$

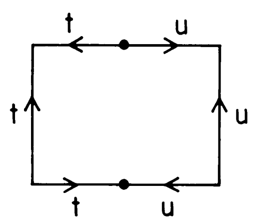
Bore out the interior of a disc D from the 2-cell of X as shown below to get $A = X - \mathring{D}$.

(5.2.1)

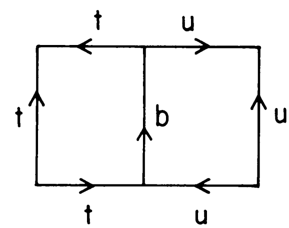


Thus $A = K(\langle t, b \mid tbt\bar{t}\rangle)$. Let Z be the double of A along the closure of the edge b; $Z = A \amalg_B A'$, where $A \cong A'$ and where $B = K(\langle b \mid \rangle)$, a circle. One sees that A (and hence A') collapses onto a graph; so A is diagrammatically reducible, as in B. In addition the inclusions $B \subset A$ and $B \subset A'$ are injective on π_1. However Z is not diagrammatically reducible.

To see this, observe that Z is a subdivision of $W = K(\langle t, u \mid t\bar{t}t = u\bar{u}u\rangle)$, as the next two diagrams indicate:

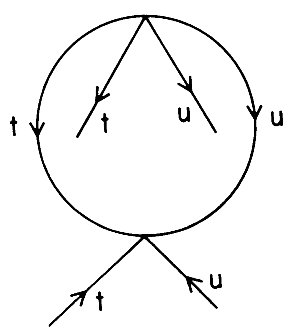


W



Z

However here is a reduced spherical diagram in W:



This example was also discovered independently by Sieradski (private communication).

Thus some additional conditions are needed to obtain an amalgamation theorem for diagrammatic reducibility.

5.3 DEFINITION. The subgraph A of the 2-complex X is said to be 2-sided in X if A has a neighborhood U homeomorphic to $A \times I$, $I = [0,1]$. If A_0 is a connected component of A, identified with $A \times \{1/2\}$ under the homeomorphism, and if X_0 is a connected component of $X - A$ containing $A_0 \times \{\delta\}$, $\delta = 0$ or 1, then the composite map $A_0 \overset{\cong}{\to} A_0 \times \{\delta\} \subset X_0$ is called "pushing A_0 to one side in X".

5.4 THEOREM. *Let A be a subgraph of $X^{(1)}$ where X is a connected 2-complex. Assume that all components of $X - A$ are diagrammatically reducible and A is two-sided in X. Assume further that all maps obtained by pushing connected components of A to one side in X induce injections on π_1. Then X is diagrammatically reducible.*

PROOF. Let ρ: $Y \to X$ be a finite connected branched cover of X. By 4.5 it suffices to prove that Y is aspherical. We identify a neighborhood of A in X with $A \times I$ and identify A with $A \times \{1/2\}$. By subdividing we may assume $A \times I$ is a subcomplex of X and by passing to a smaller product neighborhood of A, we may assume that π is unbranched over $A_\delta = A \times \{\delta\}$; $\delta = 0,1$. Let $B_\delta = \pi^{-1}(A_\delta)$, a 2-sided subgraph of Y, and let $B = B_0 \cup B_1$. Observe that all connected components of $Y - B$ are aspherical, by 4.5 (the connected components over $A \times I$ are aspherical since $A \times I$ is diagrammatically aspherical, A being a graph). Let C be a connected component of B and let Y_0 be a connected component of Y containing C_0, the push of C to one side in Y. Let $X_0 = \rho(H_0)$ and $D_0 = \rho(C_0)$ and consider the following commutative diagram of fundamental groups (for suitable choice of base points):

(5.4.1)
$$\begin{tikzcd} \pi_1(C_0) \arrow[r] \arrow[d, "\rho_*"', "1\text{-}1"] & \pi_1(Y_0) \arrow[d, "\rho_*"] \\ \pi_1(D_0) \arrow[r, "1\text{-}1"] & \pi_1(X_0) \end{tikzcd}$$

The left vertical arrow is injective, since C_0 can be assumed to avoid all branch points, whence $\rho | C_0$: $C_0 \to D_0$ is a covering. The bottom arrow is injective by the hypothesis of the theorem if X_0 is homeomorphic to one of the components of $X - A$, and it is trivially so if X_0 is a component of $A \times I$. It follows that the top arrow in 5.4.1 is injective. Thus Y is aspherical by Theorem 5.1.

5.5 EXAMPLE. Let X be a 2-complex with no monogons. Let $X_1 = X - \mathring{N}$, with N a regular neighborhood of $X^{(0)}$ in X, so X_1 contains L_X, the link complex. Let Z be the double of X_1 along L_X obtained by gluing two isomorphic copies of X_1 along L_X. Then Z is diagrammatically reducible. To see this observe that X_1 collapses onto D_X, a graph. Obviously L_X is 2-sided in Z. Also since X has no monogons, the canonical map η_X: $L_X \to D_X$ of 2.8 is an immersion, so induces on each connected component an injection on π_1. It follows from 5.4 that Z is diagrammatically reducible.

This result is certainly false in general if X possesses monogons. For one can take X to be a cell structure on S^2 with one vertex, one edge, and two monogons. The complex Z constructed above is homeomorphic to S^2 in this example.

Another hypothesis on X which will guarantee that $Z = X_1 \coprod_{L_X} X_1$ is diagrammatically reducible is to assume that $H_2(X) = 0$. In this case, the map of a connected component of L_X into X_1 induces an injection on π_1 by the Kervaire Conjecture, which is valid for free groups. We omit the details.

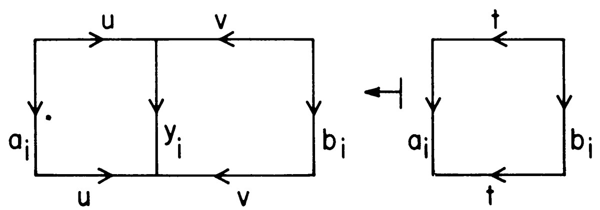
Theorem 5.4 is very powerful, as the next result shows.

5.6 THEOREM. *Let F be a free group with free basis S. Let ϕ: $G_1 \xrightarrow{\cong} G_2$ be an isomorphism of two subgroups G_1 and G_2 of F. Let $\{a_i, i \in I\}$ be a free basis for G_1, where a_i is a word, not necessarily reduced, in the alphabet $S \cup \bar{S}$, and let b_i be a representative word of $\phi(a_i)$, not necessarily reduced. Then the 2-complex $K(\mathscr{P})$ is diagrammatically reducible, where $\mathscr{P} = \langle t, s;\ s \in S \mid ta_it^{-1} = b_i;\ i \in I\rangle$.*

PROOF. Let $X = K(\langle S \mid \rangle)$ and $Y = K(\langle y_i, i \in I \mid \rangle)$, both bouquets of circles. Form the identification space Z of $X \coprod (Y \times I)$ by attaching $Y \times \{0\}$ to X by subdividing the circle y_i and attaching it by the word a_i, and by attaching $Y \times \{1\}$ to X by subdividing the circle y_i and attaching it by the word b_i. The space Z is diagrammatically reducible by 5.4. If we identify all vertices of Z to a single vertex, the resulting space W is still diagrammatically reducible. Furthermore $W = K(\mathscr{Q})$, where

$$\mathscr{Q} = \langle S, u, v, y_i, i \in I \mid uy_iu^{-1} = a_i, vy_iv^{-1} = b_i, i \in I\rangle$$

(5.6.1)



Clearly $K(\mathscr{Q})$ is obtained from a subdivision of $K(\mathscr{P})$, by pinching vertices as one sees by inspecting (5.6.1). It follows that $K(\mathscr{P})$ is diagrammatically reducible.

5.7 EXAMPLE. It follows from 5.6 that the 2-complex $K(\mathscr{P})$ is diagrammatically reducible, where $\mathscr{P} = \langle x, y, t \mid tx\bar{x}x\bar{t} = y\bar{y}y\rangle$. Compare this with Example 5.2. From this result and from 1.6 it follows that the equation over the group G,

$$a_1ta_2xa_3\bar{x}a_4xa_5\bar{t}a_6\bar{y}a_7ya_8\bar{y} = 1$$

has a solution in an overgroup of G. Here a_i, $1 \leqslant i \leqslant 8$, are elements of G and t, x, and y are indeterminates.

5.8 REMARK. Theorem 5.6 says that certain HNN presentations are diagrammatically aspherical. There is an analogous result for free-product-with-amalgamation presentations (indeed, for graphs of free groups) which we leave to the reader to formulate with a caution: the amalgamation relations must have the form of a conjugation by new stable letter.

5.9 We want to discuss another amalgamation result, the connected sum of two 2-complexes, since this appeared in Sieradski's paper [**14**].

Let X be a connected 2-complex and let F be a face (= closed 2-cell) of X. Imbed a closed disc D in F so that $D \cap X^{(1)}$ is precisely one point on the boundary ∂D of D. See (5.2.1) for a picture of this. Let us call the pair (X, D), "suitable". If (Y, D) is another suitable pair we may amalgamate by boring out $\mathring{D}$ and identifying along ∂D:

$$Z = (X - \mathring{D}) \amalg_{\partial D} (Y - \mathring{D})$$

Let us denote Z by $X \# Y$, an abuse of notation, since Z depends on the pairs (X, D) and (Y, D).

5.10 PROPOSITION. *If X and Y are diagrammatically reducible, then so is $X \# Y$.*

PROOF. Let $X_1 = X - \mathring{D}$ and $Y_1 = Y - \mathring{D}$ and let $A = \partial D$. Since diagrammatic reducibility is preserved under subdivision and under taking subcomplexes, it follows that X_1 and Y_1 are diagrammatically reducible, as is the graph A. Let $\{v\} = \partial D \cap X^{(1)} = \partial D \cap Y^{(1)}$, the unique vertex of A. Let $p\colon \tilde{Z} \to Z$ be a finite branched cover of $Z = X \# Y$. We may assume that $p^{-1}(v)$ consists of precisely one point, $\tilde{v}$, if necessary by identifying together all points of the fiber $p^{-1}(v)$. Then $p^{-1}(A) = B$ is a bouquet of n circles all attached together at $\tilde{v}$, where n is the generic degree of the cover. Theorem 5.1 will imply that $\tilde{Z}$ is aspherical provided we can show that the inclusions $B \subset \tilde{X}_1 = \operatorname{def} p^{-1}(X_1)$ and $B \subset \tilde{Y}_1 = \operatorname{def} p^{-1}(Y_1)$ are injective on π_1. Thus by symmetry we can forget about Y and consider only the inclusion $B \subset \tilde{X}_1$.

Observe that $p\colon \tilde{X}_1 \to X_1$ can be extended to a branched cover $p\colon \tilde{X} \to X$. To do this, just make each of the n-circles of B the boundary of a disc and map that disc homeomorphically onto D.

If $\pi_1(B) \to \pi_1(\tilde{X}_1)$ were not injective there would be a reduced planar connected simply connected diagram K in $\tilde{X}_1$ with boundary label a cyclically reduced nontrivial word in $\pi_1(B)$. Choosing K to have the minimal number of faces among all such diagrams it follows we may assume K is a reduced disc diagram $f\colon K \to \tilde{X}_1$ with boundary label $f \mid \partial K$ a cyclically reduced nontrivial word in $\pi_1(B)$. Now each letter of the boundary label $f \mid \partial K$ bounds a unique disc in $p^{-1}(D)$. Thus $f\colon K \to \tilde{X}_1$ can be extended to a reduced spherical diagram in $\tilde{X}$. But $\tilde{X}$ is a branched cover of X and X is diagrammatically reducible. By 2.11, so is $\tilde{X}$ diagrammatically reducible. This contradiction shows that $\pi_1(B) \to \pi_1(\tilde{X}_1)$ is injective, and the proof is complete.

6. Applications.

6.1 We recall some terminology. A group homomorphism $\phi\colon G \to H$ is said to be *normally convex* [**15**] if for any subset S of G one has

$$\phi^{-1}(\langle\langle \phi(S) \rangle\rangle_H) = \langle\langle S \rangle\rangle_G.$$

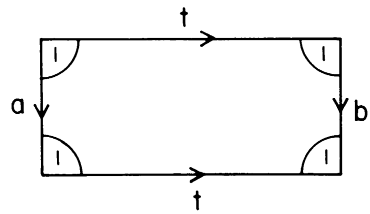
In particular this implies that ϕ is injective and for any $N \triangleleft G$, the induced homomorphism $G/N \to H/\langle\langle \phi(N) \rangle\rangle_H$ is injective. If $\phi\colon G \to H$ satisfies "Property G" (see 1.6) then it is normally convex.

In [**9**], Higman, Neumann and Neumann produce a normally convex imbedding of a free group F_∞ of countable rank into a free group F_2 of rank 2. This imbedding plays an important role in other group theoretic imbedding theorems, e.g. [**8**]. We shall establish below a strengthening of the property of normal convexity and give a geometric interpretation for the HNN imbedding.

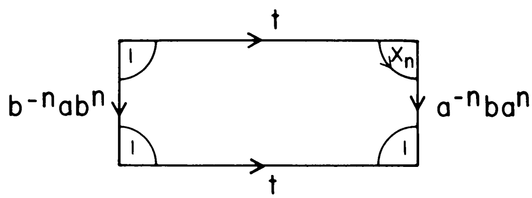
6.2 THEOREM. *There is an imbedding of F_∞ in F_2 which satisfies "Property* G."

PROOF. Let $F = F(a, b)$ be the free group with free basis $\{a, b\}$ and consider the subgroups $G_1 = \langle b^{-n}ab^n, n \geqslant 0\rangle$ and $G_2 = \langle a^{-n}ba^n, n \geqslant 0\rangle$ of F. Both are free on the indicated set of generators, so let $\phi\colon G_1 \xrightarrow{\cong} G_2$ be given by $\phi(b^{-n}ab^n) = a^{-n}ba^n$; $n \geqslant 0$. By Theorem 5.6 the 2-complex $K(\mathscr{P})$ is diagrammatically reducible, where $\mathscr{P} = \langle a, b, t \mid t^{-1}b^{-n}ab^nt = a^{-n}ba^n;\ n \geqslant 0\rangle$. This implies that the corresponding system of equations of groups satisfies "Property G". We shall give an explicit specialization of the corner variables in these equations as shown in the following diagrams of 2-cells of $K(\mathscr{P})$:

n=0:



n>0:



It follows that the natural homomorphism of groups

$$\begin{aligned} F_\infty &= \operatorname{def}\langle x_n, n \geqslant 1 \mid \rangle \\ &\to \langle a, b, t, x_n;\ n \geqslant 1 \mid t^{-1}at = b,\ t^{-1}b^{-n}ab^nt = x_na^{-n}ba^n;\ n \geqslant 1\rangle \\ &= \operatorname{def} H \end{aligned}$$

satisfies "Property G". But using Tietze transformations we see that $H \cong \langle a, t \mid \rangle = F(a, t)$. The indicated homomorphism $F_\infty \to F(a, t)$ is exactly the HNN imbedding.

6.3 REMARK. I know of essentially only one example of normally convex homomorphism $\phi\colon G \to H$ that does not satisfy Property G. If $F(a, b)$ is free on $\{a, b\}$, the natural homomorphism $F(a, b) \to F(a, b, t)/\langle\langle at^2bt^{-1}\rangle\rangle$ is normally convex, by a theorem of Howie's [**11**] but does not satisfy "Property G" by a result of this author's [**4**]. Why this should be so is a great mystery.

6.4 We now come to the application to knot complements in the 3-sphere. From the work of Haken [**22**] it follows that if M is a bounded Haken manifold there is a sequence of (possibly disconnected) 3-manifolds $M = M_0, M_1, \ldots, M_n$ such that M_j

is obtained from M_{j-1} by splitting M_{j-1} along a bounded properly imbedded two-sided incompressible surface F_j and such that each connected component of M_n is a ball. Using our Theorem 5.4, the fact the bounded surfaces have graphs as spines, and an obvious induction we deduce

6.5 THEOREM. *Every bounded Haken manifold has the homotopy type of a diagrammatically reducible* 2*-complex.*

6.6 REMARK. Since tame knot complements are bounded Haken manifolds, Theorem 6.5 applies to show they are homotopy equivalent to DR 2-complexes. This can be viewed as a solution to a question raised by Sieradski [**14**]. Sieradski asked whether it was possible to find a presentation $\mathscr{P}$ of the fundamental group of a tame knot which satisfied his "coloring test;" that is, so that $K(\mathscr{P})$ possessed a weight (1.7) satisfying the link and curvature conditions and so that w takes values in $\{0, 1\}$. This conjecture implies all tame knot complements have the homotopy type of DR 2-complexes; since it is DR that Sieradski is interested in, it is in this sense that 6.5 answers his question.

6.7 REMARK. Howie has brought to my attention that the argument for 6.5 is the same argument used in [**21**] to prove that bounded Haken manifolds are diagrammatically aspherical. This latter is a very much weaker result and has no implications about equations over groups.

Appendix. I had at one time considered including an appendix relating the notion of diagrammatic reducibility (DR) to the notion of diagrammatic asphericity (DA) discussed in [**21**]. However I have recently shown conclusively that there is no relationship between DA and the solution of equations over groups, my main interest here (compare 1.6 above). Thus I believe it is more important to inform the reader of new developments in equations over groups rather than to include here a scholarly survey of another attempt to formulate a notion of combinatorial homotopy which is demonstrably irrelevant to the problem at hand.

To state my result let me recall the notion of a Kervaire complex due to S. Brick [**20**]. A 2-complex X is called "Kervaire" if all equations over all coefficient groups, whose words in the variable letters are the attaching maps of the 2-cells of X, are solvable in an overgroup of the coefficient group. Thus, for example, the dunce hat $K\langle x \mid xx\bar{x}\rangle$ is Kervaire since Howie has shown that the equation $axbxc\bar{x} = 1$ is solvable in an overgroup of G, where $a, b, c \in G$, the coefficient group [**11**]. Recall also that a 2-complex X is called DA [**21**] if, given a spherical diagram in X, some sequence of diamond moves exists which splits off a 2-sphere component with precisely two faces. My result is

THEOREM. *There is a* 2*-complex* Z *which is both Cockcroft and DA, but not Kervaire.*

The example, Z, I construct is given as $Z = X \coprod_\Gamma X$, where $X = K\langle x, y, z \mid x^2, y^2, t = xy\rangle$ and $\Gamma = K\langle t \mid \ \rangle$. The details will appear in my forthcoming article, *Amalgamations and the Kervaire problem.*

To complete the justification of my assertion that DA is unrelated to solving equations over groups, it suffices to exhibit a Kervaire complex which is not DA. For this one can take the example $X = K(\mathscr{P})$, where $\mathscr{P} = \langle x, y \mid y^{-2}xyx, x \rangle$. Chiswell has shown that X is not DA [**21**]. However X is Kervaire, as one sees without difficulty by reducing to the fact that the dunce hat is Kervaire.

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