THE MORAVA $K$-THEORIES OF SOME CLASSIFYING SPACES

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Abstract. Let $P$ be a finite abelian $p$-group with classifying space $BP$. We compute, in representation theoretic terms, the Morava $K$-theories of the stable wedge summands of $BP$. In particular, we obtain a simple, and purely group theoretic, description of the rank of $K(s)^*(BG)$ for any finite group $G$ with an abelian $p$-Sylow subgroup. A minimal amount of topology quickly reduces the problem to an algebraic one of analyzing truncated polynomial algebras as modular representations of the semigroup $M_n(\mathbb{Z}/p)$.

1. Main results. For a fixed prime $p$, there exists a sequence of cohomology theories $K(1)^*$, $K(2)^*$, ..., with $K(s)^*$ periodic with period $2(p^s - 1)$. These are the Morava $K$-theories [13], and generalize ordinary complex $K$-theory in the sense that $K(1)^*$ is one of the $(p - 1)$ isomorphic summands of $K^*(\mathbb{Z}; \mathbb{Z}/p)$. Recently it has been becoming clear that they play a central role in homotopy theory—see, for example, the work of M. Hopkins and J. Smith [6]. $K(s)^*$ has further computational virtues: $K(s)^*(X)$ is always a free module over the coefficient ring $K(s)^*$, and there is a Kunneth isomorphism.

A major outstanding problem is to find good models for the spaces representing these theories. Related to this is the question of finding, for finite groups $G$, a group theoretic description for the rings $K(s)^*(BG)$, analogous to Atiyah's isomorphism [2]:

$$\hat{R}(G) \cong K(BG).$$

The author, together with J. Harris, recently analyzed stable wedge decompositions of classifying spaces of abelian $p$-groups [5]. In this paper we compute, in representation theoretic terms, the Morava $K$-theories of the resulting wedge summands. As a consequence, we obtain a very simple, and purely group theoretic, description of the rank of $K(s)^*(BG)$ for any finite group $G$ with an abelian $p$-Sylow subgroup.

To state our results, we establish some notation. It is convenient to let $K(0)^*$ be rational cohomology. We let $K(s)^*(X)$ always denote the reduced $s$th Morava $K$-theory of $X$. If $X$ is a space or spectrum, we let $k_s(X)$ equal the rank of $K(s)^*(X)$ as a $K(s)^*$-module, and, by abuse of notation, we let $k_s(G) = k_s(BG_+)$ for a finite group $G$. ($BG_+$ is the union of $BG$ with a disjoint basepoint.)
If $P$ is a finite $p$-group, let $\mathbb{Z}/p[\text{End}(P)]$ be the ring with basis the semigroup $\text{End}(P)$. $\text{End}(P)$ acts on $BP$ and thus on $BP_+$. It is easy to show that an idempotent $e$ in $\mathbb{Z}/p[\text{End}(P)]$ yields a stable wedge summand of $BP_+$, $eBP_+$, such that, e.g., $H_q(eBP_+; \mathbb{Z}/p) = e_*H_q(BP_+; \mathbb{Z}/p)$. In [5], we show that, if $P$ is abelian, every stable wedge summand of $BP_+$ is homotopic to one of this form.

Note that $\text{End}(P)$ acts diagonally on the set $P^s$ so that, letting $\mathbb{Z}/p[P^s]$ denote the $\mathbb{Z}/p$-vector space with basis $P^s$, $\mathbb{Z}/p[P^s]$ is a $\mathbb{Z}/p[\text{End}(P)]$-module.

**Theorem 1.1.** Let $P$ be a finite abelian $p$-group and let $e$ be an idempotent in $\mathbb{Z}/p[\text{End}(P)]$. Then

$$k_s(eBP_+) = \dim e\mathbb{Z}/p[P^s].$$

Now let $G$ be a finite group with $p$-Sylow subgroup $P$. By transfer arguments, $BG_+$ is a stable wedge summand of $BP_+$, localized at $p$. In our situation we can be more explicit. Let $W = N_G(P)/C_G(P)$ and let $e_w = |W|^{-1}\sum_{w \in W} w$ in $\mathbb{Z}/p[\text{End}(P)]$. Then, if $P$ is abelian, $BG_+ = e_wBP_+$ [5]. Furthermore, $e_w\mathbb{Z}/p[P^s] = \mathbb{Z}/p[P^s]^W$, a vector space with a basis corresponding to $W$-orbits in $P^s$. Theorem 1.1 thus implies

**Theorem 1.2.** If $G$ is a finite group with an abelian $p$-Sylow subgroup $P$, and $W = N_G(P)/C_G(P)$, then $k_s(G) = |P^s/W|$.

**Remarks 1.3.** (i) Doug Ravenel has noted that $k_s(G)$ is finite for all finite groups $G$ [12].

(ii) Using Atiyah’s theorem and some representation theory, one can show [7] that for any finite group $G$,

$$k_1(G) = \text{number of conjugacy classes of } p\text{-elements in } G.$$  

It is an amusing exercise using the Sylow theorems to check that this result is compatible with Theorem 1.2 above.

When $P = (\mathbb{Z}/p)^n$ one can identify $\text{End}(P)$ with the matrix ring $M_n(\mathbb{Z}/p)$, $P^s$ with the set $M_{n,s}(\mathbb{Z}/p)$ of $n \times s$ matrices over $\mathbb{Z}/p$, and the action of $\text{End}(P)$ on $P^s$ with matrix multiplication. In particular, letting $s = n$ in Theorem 1.1 implies the following.

**Corollary 1.4.** Let $M$ be an irreducible right $\mathbb{Z}/p[M_n(\mathbb{Z}/p)]$-module, let $P_M$ be the associated principal indecomposable module (i.e., its projective cover), and let $X_M$ be the associated wedge summand of $B(\mathbb{Z}/p)^n$. Then

$$k_n(X_M) = \dim P_M.$$  

Thus, no summand of $B(\mathbb{Z}/p)^n$ is $K(n)$-acyclic.

At the other extreme, we show

**Theorem 1.5.** Of the $p^n - 1$ distinct indecomposable spectra that appear as wedge summands of $B(\mathbb{Z}/p)^n$, exactly $(p - 1)n$ are not acyclic in $K$-theory. For each such summand $X$, $k_1(X) = 1$.

Further analysis of $\mathbb{Z}/p[M_{n,s}(\mathbb{Z}/p)]$ as a $\mathbb{Z}/p[M_n(\mathbb{Z}/p)]$-module yields the next results.
Theorem 1.6. (1) For each pair \((n, s)\), there exists a linear function \(\alpha_{n,s} : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}\) such that, if \(X\) is any wedge summand of \(B(\mathbb{Z}/p)^s_+\),

\[ k_s(X) = \alpha_{n,s}(k_0(X), \ldots, k_n(X)). \]

Thus \(k_s(X)\) is determined for all \(s\) by \(k_0(X), \ldots, k_n(X)\).

(2) If \(P\) is a finite abelian \(p\)-group and \(X\) is a wedge summand of \(BP\), then \(\sum_{s=0}^{\infty} k_s(X) t^s\) is a rational function with poles contained in the set \(\{ p^{-s} | s = 0, 1, 2, \ldots \} \).

(3) If \(X\) is as in (2), the sequence \(k_1(X), k_2(X), k_3(X), \ldots\) converges, in the \(p\)-adic topology, to an element of \(\mathbb{Z}(p)\).

We note that the linear functions \(\alpha_{n,s}\) will be made more explicit in the course of the proof.

Finally we note that J. F. Adams, J. Gunawardena and H. Miller have shown [1]:

\[
\mathbb{Z}/p\left[ M_{n,s}(\mathbb{Z}/p) \right] = \text{Hom}_A \left( H^*(B(\mathbb{Z}/p)^s_+), H^*(B(\mathbb{Z}/p)^n_+) \right)
\]

where \(A\) is the mod \(p\) Steenrod algebra. Combined with our observations, this yields the amusing corollary:

**Corollary 1.7.** If \(X\) is a stable wedge summand of \(B(\mathbb{Z}/p)^n\), for some \(n\), then

\[ k_s(X) = \dim \text{Hom}_A \left( H^*(B(\mathbb{Z}/p)^s_+), H^*(X) \right). \]

The organization of the paper is as follows. Theorem 1.1 is proved in §3 after we discuss \(K(s)^*(BP_+)\) in §2. Theorem 1.5 is proved in §4, Theorem 1.6 in §5. §6 contains some explicit calculations, e.g. \(k_s(GL_2(\mathbb{F}_q))\) where \(q = p^d\), and \(k_s(L(n))\) where \(L(n) = \sum_{s=0}^{\infty} \text{SP}^s(S)/\text{SP}^{p^{-1}s}(S)\).

We illustrate the ideas of the proof of Theorem 1.1 by sketching the argument in the case \(P = (\mathbb{Z}/p)^n\).

\[
K(s)^*(B\mathbb{Z}/p^s_+) = K(s)^*[x]/(x^p),
\]

so that \(K(s)^*(B(\mathbb{Z}/p)^n_+) = K(s)^*[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)\). The right action of \(M_n(\mathbb{Z}/p)\) on \(K(s)^*(B(\mathbb{Z}/p)^n_+)\) is determined by the formal group law for \(K^*(s)\)—modulo some high-degree error terms, it is the standard action on a truncated polynomial algebra. We are left needing to show the purely algebraic result:

**Theorem 1.8.** \((\mathbb{Z}/p[M_{n,s}(\mathbb{Z}/p)])^*\) and \((\mathbb{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p))^*\) have the same irreducible composition factors as right \(\mathbb{Z}/p[M_n(\mathbb{Z}/p)]\)-modules.

The relevance of this result to Theorem 1.1 comes from the following elementary but handy observation, which will be used numerous times in our arguments.

**Lemma 1.9.** Let \(R\) be a finite-dimensional algebra over a field \(F\), and let \(M\) and \(N\) be finitely generated right \(R\)-modules. The following conditions are equivalent:

(i) \(\dim_F M e = \dim_F N e\) for all idempotents \(e \in R\).

(ii) \(M\) and \(N\) have the same irreducible composition factors.

Either condition is implied by

(iii) There exist \(R\)-module filtrations of \(M\) and \(N\) such that the associated graded objects are isomorphic as \(R\)-modules.
This project had its genesis during a recent visit to the University of Washington, when Doug Ravenel wrote down the numbers 1, 3, 3, 5, 2, 2 in my presence. For that, and for subsequent conversations, I give him hearty thanks. Thanks are also due to Dave Carlisle and Reg Wood for aid with the proof of Theorem 1.5.

2. $K(s)\ast (BP_+).$ In this section we describe, up to filtration, the End($P$)-module $K(s)\ast (BP_+)$, where $P$ is any finite abelian $p$-group. (See also the proof of Theorem 4.9 in [10].)

$M_n(Z/p)$ acts on the right of the polynomial algebra $S_n = Z/p[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are dual to the standard basis of $(Z/p)^n$. For a fixed $s$, the ideal generated by the $p^s$ powers is a submodule. We let $S_{n,s}$ denote the quotient $M_n(Z/p)$ algebra $Z/p[x_1, \ldots, x_n]/(x_1^{p^s}, \ldots, x_n^{p^s})$. More generally, note that $M(n_1, \ldots, n_r)$ acts on $\otimes_{i=1}^r S_{n_i,i,s}$ where $n = n_1 + \cdots + n_r$ and $M(n_1, \ldots, n_r)$ is the subsemigroup of $M_n(Z/p)$ consisting of matrices preserving the flag

$$(Z/p)^n \subseteq (Z/p)^{n-1} \subseteq \cdots \subseteq (Z/p)^1.$$ (Here $(Z/p)^m \subseteq (Z/p)^n$ is the inclusion of the last $m$ coordinates.)

Now suppose that $P = \prod_{i=1}^r (Z/p^i)^{n_i}$. We define a “standard” action of End($P$) on $\otimes_{i=1}^r S_{n_i,i,s}$ as follows: If Tor($P, Z/p$) is identified with $(Z/p)^n$ in the obvious way, then it is easily checked that Tor($\alpha, Z/p$) $\in M(n_1, \ldots, n_r)$ for $\alpha \in$ End($P$). This defines a map of semigroups End($P$) $\to M(n_1, \ldots, n_r)$, and thus a right action of End($P$) on $\otimes_{i=1}^r S_{n_i,i,s}$.

**Proposition 2.1.** If $P = \prod_{i=1}^r (Z/p^i)^{n_i}$, then, as algebras,

$$K(s)\ast (BP_+) = K(s)\ast \bigotimes_{i=1}^r S_{n_i,i,s},$$

and the End($P$) action is the standard one, modulo terms of higher degree.

**Corollary 2.2.** With $P$ as above, and $e \in Z/p[End(P)]$ an idempotent,

$$k_s(eBP_+) = \text{dim} \left[ \bigotimes_{i=1}^r S_{n_i,i,s} \right] e.$$

We collect the results about Morava $K$-theories which imply Proposition 2.1. A good reference is [13, §4].

We start with some generalities. If $E$ is an $MU$-oriented ring spectrum, then $E*((CP^\infty)^n) = E*[[x_1, \ldots, x_n]]$, where each $x_i$ has degree 2. The multiplication $CP^\infty \times CP^\infty \to CP^\infty$ defines a formal group law $F$ over $E*$.

The semigroup $M_n(Z)$ acts on $Z^n$ and thus on $(CP^\infty)^n = K(Z^n, 2)$. The formal group law $F$ determines the induced action on $E*$ cohomology: If $A = (a_{ij}) \in M_n(Z)$ then

$$A*(x_i) = \sum_{j=1}^n [a_{ij}]_F x_i.$$
Since \( F(x, y) = x + y \) modulo higher-order polynomials in \( x \) and \( y \), we conclude

**Lemma 2.3.** Let \( E^*((CP_\infty)^{n}) = E^*[x_1, \ldots, x_n] \) be filtered by degree, i.e. let \( F_k E^*[x_1, \ldots, x_n] = \{ f(x_1, \ldots, x_n) | f(x, \ldots, x) \text{ is divisible by } x^k \} \). Then this is a decreasing filtration by sub-\( M_n(\mathbb{Z}) \)-algebras and the associated graded \( M_n(\mathbb{Z}) \)-algebra is isomorphic to \( E^* \otimes \mathbb{Z}[x_1, \ldots, x_n] \) with the standard action.

Now suppose that \( P \) is a finite abelian group of rank \( n \). Then \( P \) fits into a short exact sequence

\[
0 \to \mathbb{Z}^n \to \mathbb{Z}^n \to P \to 0,
\]

and any endomorphism \( \alpha: \mathbb{Z}^n \to P \) can be extended to a diagram:

\[
0 \to \mathbb{Z}^n \to \mathbb{Z}^n \to P \to 0
\]

where \( A_0, A_1 \in M_n(\mathbb{Z}) \). This, in turn, induces a map of fibrations:

\[
(S^1)^n \to BP \xrightarrow{\delta} (CP_\infty)^n
\]

\[
\downarrow A_0 \downarrow \alpha \downarrow A_1
\]

\[
(S^1)^n \to BP \xrightarrow{\delta} (CP_\infty)^n,
\]

and Gysin sequence techniques allow for a computation of \( E^*(BP) \) as an \( \text{End}(P) \)-module.

Specializing to the case of interest, we have \( K(s)_* = \mathbb{Z}/p[v_s, v_s^{-1}] \), where \( v_s \) has degree \( 2(p^s - 1) \). From \([13, \S 4]\) we have that \( \delta^*: K(s)^*(CP_\infty)_+ \to K(s)^*(B(\mathbb{Z}/p')_+) \) can be identified with the projection map \( K(s)^*[x] \to K(s)^*[x]/(x^{p^n}) \). Combining this with the Kunneth isomorphism,

\[
K(s)^*(X) \otimes_{K(s)^*} K(s)^*(Y) = K(s)^*(X \land Y),
\]

yields the algebra structure of \( K(s)^*(BP_+) \) and that \( \delta^* \) is epic in the diagram:

\[
K(s)^*((CP_\infty)_+) \xrightarrow{\delta^*} K(s)^*((CP_\infty)_+)
\]

\[
\downarrow \delta^* \downarrow \delta^*
\]

\[
K(s)^*(BP_+) \xrightarrow{\alpha^*} K(s)^*(BP_+)
\]

Proposition 2.1 now follows from Lemma 2.3 together with the observation that \( \text{Tor}(\alpha, \mathbb{Z}/p) = A_1 \otimes \mathbb{Z}/p: (\mathbb{Z}/p)^n \to (\mathbb{Z}/p)^n \).

**3. Proof of Theorem 1.1.** Our proof consists first of a number of reductions.

**Reduction to the case \( P = (\mathbb{Z}/p')^n \).** We need the following theorem from \([5]\).

**Theorem 3.1.** Let \( P = \prod_{i=1}^r (\mathbb{Z}/p')^{n_i} \). Any idempotent \( e \in \mathbb{Z}/p[\text{End}(P)] \) is conjugate to one of the form \( e_i \otimes \cdots \otimes e_r \), where \( e_i \) is an idempotent in \( \mathbb{Z}/p[M_{n_i}(\mathbb{Z}/p')] \).

With notation as in this last theorem, we have

\[
e^* \mathbb{Z}/p[PS] = \bigotimes e_i \mathbb{Z}/p[(\mathbb{Z}/p')^{n_i}]
\]
and, by the Kunneth isomorphism for $K(s)^*$,

$$(3.2b) \quad k_s(eBP_+) = \prod_{i=1}^r k_s\left(e_iB(Z/p^i)^{n_i}\right).$$

It follows that Theorem 1.1 for all $P$ follows from the theorem for $P$ of the form $(Z/p^n)^n$.

Reduction to the case $P = (Z/p)^n$. The filtration $(Z/p^n)^n \supset (Z/p^{n-1})^n \supset \cdots \supset (Z/p)^n$ is preserved by $M_n(Z/p^n)$. This induces a filtration of $Z/p[(Z/p^n)^{ns}]$, and the associated graded module is isomorphic to $Z/p[(Z/p)^{ns}]$ where $M_n(Z/p^n)$ acts on $(Z/p)^n$ via mod $p$ reduction.

Let $e \in Z/p[M_n(Z/p^n)]$ be an idempotent and let $\bar{e} \in Z/p[M_n(Z/p^n)]$ be its mod $p$ reduction. The above comments imply

$$(3.3a) \quad eZ/p[(Z/p^n)^{ns}] = \bar{e}Z/p[(Z/p)^{ns}].$$

An inspection of Corollary 2.2 yields

$$(3.3b) \quad k_s(eB(Z/p)^n) = k_s(\bar{e}B(Z/p)^n).$$

Remark 3.4. In [5] it is shown that $H^*(eB(Z/p)^n) = H^*(\bar{e}B(Z/p)^n)$ as graded vector spaces, and that the multiplicity of $eB(Z/p^n)$ in $B(Z/p^n)^n$ equals that of $\bar{e}B(Z/p^n)$ in $B(Z/p^n)^n$.

Reduction to the case $s = 1$. Let $V = (Z/p)^n$ and let $e \in Z/p[End(V)]$ be an idempotent. Since $Z/p[V^s] = Z/p[V]^s$ naturally, we have

$$(3.5a) \quad eZ/p[V^s] = eZ/p[V]^s.$$

With notation as in §2, we claim that

$$(3.5b) \quad S_{n,s}e \simeq \left(S_{n,1}^e\right)e.$$  

(3.5b) will follow from the next lemma.

Lemma 3.6. $S_{n,s}$ can be filtered by right $M_n(Z/p)$-submodules so that the associated graded module is isomorphic to $S_{n,s}^e$.

Proof. We use the notation: $S_{n,s}(V^*) = S_{n,s}$ and $\xi: S_{n,s}(V^*) \to S_{n,s}(V^*)$ is the $p$th power map. Then $S_{n,s}(V^*)$ is filtered by the subalgebras $A_i \equiv S_{n,s-1}(\xi_i(V^*))$. Then, for $i = 1, \ldots, s$,

$$A_{i-1} / A_i \simeq S_{n,1}(V^*)$$

naturally, so that

$$E_0(S_{n,s}(V^*)) \simeq S_{n,1}(V^*)^s$$

as $End(V)$-modules.

Finally we are left needing to prove Theorem 1.1 in the special case when $P = (Z/p)^n$ and $s = 1$. More precisely, we need to show that

$$\dim eZ/p[(Z/p)^n] = \dim S_{n,1}e$$

for all idempotents $e \in Z/p[M_n(Z/p^n)]$. 

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View $V = (\mathbb{Z}/p)^n$ as a restricted Lie algebra with trivial bracket and $p$th power map. Let $U(V)$ denote the universal enveloping algebra.

**Proposition 3.7.** Let $E_0(\mathbb{Z}/p[V])$ be the graded algebra associated to the filtration of $\mathbb{Z}/p[V]$ by the augmentation ideal. Then there is a natural isomorphism of algebras

$$U(V) = E_0(\mathbb{Z}/p[V]).$$

This is a special case of a more general theorem provided by D. Quillen in [11]. (For completeness, we give an explicit proof below.)

Assuming this proposition, we are nearly done. Note that $S_{n,1} = U(V^*) = U(V)^*$ as right $\text{End}(V)$-modules. Thus Proposition 3.7 implies that

$$(e\mathbb{Z}/p[V])^* = S_{n,1} e$$

for all idempotents $e \in \mathbb{Z}/p[\text{End}(V)]$. The proof of Theorem 1.1 is complete.

**Proof of Proposition 3.7.** $U(V)$ is isomorphic to $\mathbb{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ with the standard left $M_n(\mathbb{Z}/p)$ action. Let $e_i \in V$ be the $i$th standard basis vector, and let $\Theta(x_i) = e_i - 0 = 0 \in \mathbb{Z}/p[V]$.

**Lemma 3.8.** $\Theta$ extends to an isomorphism of algebras

$$\Theta: \mathbb{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p) \to \mathbb{Z}/p[V].$$

**Proof.** To show that $\Theta$ extends to an algebra map, it suffices to check that $\Theta(x_i)^p = 0$. This is okay: $(e_i - 0)^p = e_i^p - 0^p = 0 - 0 = 0$ in $\mathbb{Z}/p[V]$. By dimension counting, to show that $\Theta$ is an isomorphism, it suffices to show that $\Theta$ is onto. This is easy: $\{e_i, \ldots, e_n\}$ is a set of algebra generators for $\mathbb{Z}/p[V]$ and $e_i = \Theta(1 + x_i)$.

Now note that if $\mathbb{Z}/p[V]$ is filtered by the augmentation ideal, and $\mathbb{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ is filtered by degree, then $\Theta$ is a filtration-preserving map (i.e., $\Theta(x_i)$ is in the augmentation ideal). The proof of Proposition 3.7 is completed with

**Lemma 3.9.** $\Theta$ is an $M_n(\mathbb{Z}/p)$-module map, up to filtration. More precisely, $\Theta(Ax) = A\Theta(x)$ modulo terms of higher filtration, for all $A \in M_n(\mathbb{Z}/p)$, $x \in \mathbb{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$.

**Proof.** Since $M_n(\mathbb{Z}/p)$ acts on both $\mathbb{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ and $\mathbb{Z}/p[V]$ via algebra maps, it suffices to check the result when $x = x_i$. Furthermore, it suffices to assume that $A$ is an “elementary” matrix, i.e. we can assume that $A$ is diagonal, a permutation, or the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 0 \\ 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 \\ I_{n-2} \\ \end{pmatrix}.$$ 

If $A$ is diagonal or a permutation, then $\Theta(Ax_i) = A\Theta(x_i)$. To check the last possibility it suffices to assume that $n = 2$ (so we are considering $\mathbb{Z}/p[x, y]/(x^p, y^p)$), $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \end{pmatrix}$ and $x_i = x$. In this case, straightforward calculation shows that $A\Theta(x) = \Theta(Ax) = \Theta(xy)$, which is in higher filtration.
4. K-theory. In this section we use Theorem 1.1 to prove Theorem 1.5, i.e. we compute \( k_1(X) \) where \( X \) is an indecomposable wedge summand of \( B(\mathbb{Z}/p)^n \). By [5], such spectra are in one-to-one correspondence with the conjugacy classes of primitive idempotents \( e \in \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \). These, in turn, correspond to the \( p^n \) distinct (absolutely) irreducible \( \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \)-modules, so that, if \( e \) corresponds to \( S \), then

\[
\dim M_e = \text{multiplicity of } S \text{ in } M
\]

for any \( \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \)-module \( M \).

Recall that \( S_{n,1} = \mathbb{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p) \). By the results in §3, \( S_{n,1} \) and \((\mathbb{Z}/p)((\mathbb{Z}/p)^n)^*\) have the same irreducible composition factors as right \( \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \)-modules. We need to analyze these factors.

Let \( S_{n,1}(d) \) be the homogeneous elements in \( S_{n,1} \) of degree \( d \). Theorem 1.5 follows from

**Proposition 4.1.** The \( \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \)-modules \( S_{n,1}(d), d = 0, \ldots, (p - 1)n, \) are all distinct and irreducible.

We first show that the \( S_{n,1}(d) \) are all distinct. For that, we work with \( \mathbb{Z}/p((\mathbb{Z}/p)^n) \).

**Lemma 4.2.** There exists an orthogonal idempotent decomposition \( 1 = \sum e_i \in \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \) such that \( \dim e_i \mathbb{Z}/p((\mathbb{Z}/p)^n) = 1 \) for all \( i \).

**Proof.** Let \( D_n \subset M_n(\mathbb{Z}/p) \) be the subsemigroup of diagonal matrices. The idempotent decomposition takes place in \( \mathbb{Z}/p[D_n] \). For \( n = 1 \), the proposition (and thus the lemma) are easy to verify: the \( \mathbb{Z}/p[D_1(\mathbb{Z}/p)] \)-modules \( S_{1,1}(d) \) are irreducible (they are one-dimensional) and distinct (they are powers of the determinant representation). For the general case, note that \( \mathbb{Z}/p[D_n] = \mathbb{Z}/p[D_1]^\otimes_n \), so tensoring the \( n = 1 \) decomposition yields the lemma.

**Remark 4.3.** Geometrically, our argument here corresponds to the following. \( B\mathbb{Z}/p \), stably decomposes: \( B\mathbb{Z}/p = X_0 \vee \cdots \vee X_{p-1} \) with \( k_1(X_i) = 1 \) (\( X_0 = S^0 \) and \( X_{p-1} = B\Sigma_p \)). Thus \( B(\mathbb{Z}/p)^n \) decomposes into \( p^n \) summands,

\[
B(\mathbb{Z}/p)^n = \bigvee_{0 \leq i_1, \ldots, i_{n-1} \leq p-1} (X_{i_1} \wedge \cdots \wedge X_{i_{n-1}}),
\]

each of which has \( k_1 = 1 \).

Given \( I = (i_1, \ldots, i_n) \) with \( 0 \leq i_j \leq p - 1 \), let \( e_I \in \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \) be the idempotent corresponding to \( X_{i_1} \wedge \cdots \wedge X_{i_n} \). By Lemma 4.2, in a primitive idempotent decomposition of \( e_I \) there will be a unique idempotent \( e_I \) such that \( e_I \mathbb{Z}/p((\mathbb{Z}/p)^n)^* \) \( \neq 0 \). The number of irreducibles appearing in \( \mathbb{Z}/p((\mathbb{Z}/p)^n)^* \) will correspond to the number of conjugacy classes of \( e_I \).

The group of permutation matrices conjugates the \( e_I \)’s to others, and thus does the same for the \( e_I \)'s. In particular, if \( p = 2 \), this yields an upper bound of \( n + 1 \) on the number of conjugacy classes of the \( e_I \). But a lower bound is given by the number of \( S_{n,1}(d) \) again \( n + 1 \). Proposition 4.1 has thus been proved in this case.

**Example 4.4.** \( B\mathbb{Z}/3 \simeq X \vee B\Sigma_3 \), stably and localized at 3. Then \( B\Sigma_3 \) is a wedge summand in \( X \wedge X \) (see [5, §7]).
This example illustrates, geometrically, why our argument for \( p = 2 \) fails for larger primes. (The example shows that \( e_{(0,2)} \) is conjugate to \( e_{(1,1)} \), even though \( e_{(0,2)} \) is not conjugate to \( e_{(1,1)} \).) Instead, we prove directly that the modules \( S_{n,1}(d) \) are irreducible. The author learned of this proof from Dave Carlisle and Reg Wood.

For \( I = (i_1, \ldots, i_n) \) with \( 0 \leq i_j \leq p - 1 \), let \( |I| = i_1 + \cdots + i_n \), and let \( x' = x_1^{i_1} \cdots x_n^{i_n} \in S_{n,1} \). The irreducibility of \( S_{n,1}(d) \) follows from a characteristic \( p \) version of Lemma 2.4 of [3].

**Lemma 4.5.** Given \( I, J \) with \( |I| = |J| = d \), there exists \( \Theta_{I,J} \in \mathbb{Z}/p[\mathcal{M}_n(\mathbb{Z}/p)] \) such that

\[
\Theta_{I,J}(x^K) = \begin{cases} x^I & \text{if } K = I, \\ 0 & \text{if } K \neq I, \ |K| = d. \end{cases}
\]

**Proof.** The existence of \( \Theta_{I,J} \) follows exactly as in [3]. (In fact, \( \Theta_{I,I} \in \mathbb{Z}/p[D_n] \).) Armed with the \( \Theta_{I,J} \), it suffices to show that there exists \( \Psi_{I,J} \) such that

\[
\Psi_{I,J}(x') = x' + \text{other terms},
\]

since we can then let \( \Theta_{I,J} = \Theta_{J,J} \circ \Psi_{I,J} \circ \Theta_{I,I} \). To show the existence of \( \Psi_{I,J} \) it suffices to assume that \( n = 2 \), \( I = (i + 1, j) \), and \( J = (i, j + 1) \). This is then easy to verify:

\[
(x + ay)^{i+1-j} y^j = a(i + 1)x'y^{j+1} + \text{other terms},
\]

so letting \( a = (i + 1)^{-1} \in (\mathbb{Z}/p)^* \) yields a linear substitution \( \Psi_{I,J} \).

### 5. The \( k \)-sequence.

In this section, we study the sequence of numbers \( k_0(X), k_1(X), k_2(X), \ldots \) where \( X \) is a wedge summand of \( BP_+ \), and prove Theorem 1.6. By the reductions (3.2b) and (3.3b) of §3, it suffices to prove Theorem 1.6 in the case when \( P = (\mathbb{Z}/P)^n \), except that statement (2) of the theorem needs to be strengthened to

\[(2') \text{ If } A \text{ is a summand of } B(\mathbb{Z}/p)^n \text{ then, for any } r = 1, 2, \ldots , \]

\[
\sum_{s=0}^{\infty} k_{rs}(X) t^s \text{ is a rational function with poles contained in the set } \{ p^{-s} | s = 0, 1, 2, \ldots \}.
\]

Using Theorem 1.1, we need to study the sequence

\[
\dim e(\mathbb{Z}/p[\mathcal{M}_n(\mathbb{Z}/p)]) | s = 0, 1, 2, \ldots \}
\]

where \( e \) is an idempotent in \( \mathbb{Z}/p[\mathcal{M}_n(\mathbb{Z}/p)] \). We examine the structure of \( \mathcal{M}_n(\mathbb{Z}/p) \) as a left \( \mathcal{M}_n(\mathbb{Z}/p) \)-set.

If \( V \) is a subspace of \( (\mathbb{Z}/p)^4 \), let \( M_V \subset \mathcal{M}_n(\mathbb{Z}/p) \) be the set of all matrices with rows which are vectors in \( V \).

**Lemma 5.1.** The \( M_V \) satisfy the following properties:

1. \( M_V \) is a left \( \mathcal{M}_n(\mathbb{Z}/p) \)-set.
2. \( M_V \) is filtered by rank as a left \( \mathcal{M}_n(\mathbb{Z}/p) \)-set.
3. If \( V \) and \( V' \) are isomorphic subspaces of \( (\mathbb{Z}/p)^4 \), then \( M_V = M_{V'} \) as filtered left \( \mathcal{M}_n(\mathbb{Z}/p) \)-sets.
4. \( M_{V+V'} = M_{V\cap V'} \).

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DEFINITIONS 5.2. (1) Let \( a_{s,k} \) = number of \( k \)-planes in \((\mathbb{Z}/p)^n\)

\[
\frac{(p^s - 1) \cdots (p^{s-k+1} - 1)}{(p^k - 1) \cdots (p - 1)}.
\]

(2) For \( 0 \leq k \leq n \), let \( M'_n,k \subset M_{n,k}(\mathbb{Z}/p) \) be the set of matrices of rank < \( k \), and let \( N_{n,k} \) be the \( \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \)-module \( \mathbb{Z}/p[M_{n,k}(\mathbb{Z}/p)]/\mathbb{Z}/p[M'_n,k] \).

With these definitions, Lemma 5.1 has the following corollary.

**Corollary 5.3.** As left \( \mathbb{Z}/p[M_n(\mathbb{Z}/p)] \)-modules,

\[
\bigoplus_{k=0}^{n} a_{s,k} N_{n,k}
\]

have the same irreducible composition factors.

Theorem 1.6 will now follow from an examination of the coefficients \( a_{s,k} \). Let \( A_n \) be the \((n + 1) \times (n + 1)\) matrix with \((s,k)\) entry \( a_{s,k} \). Note that \( A_n \) is lower triangular, and thus invertible. The linear function \( \alpha_{n,s} : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z} \) of Theorem 1.6(1) will be the composite

\[
(5.4) \quad \mathbb{Z}^{n+1} \xrightarrow{A_n^{-1}} \mathbb{Z}^{n+1} \xrightarrow{(a_{1,0}, \ldots, a_{1,n})} \mathbb{Z}.
\]

To prove statements (2') and (3) we use the following observation:

**Lemma 5.5.** Let \( \alpha_0, \alpha_1, \ldots \) be a sequence of elements in \( \text{Hom}(\mathbb{Z}^m, \mathbb{Z}) \). Let \( v_1, \ldots, v_m \) form a basis for \( \mathbb{Z}^m \).

(1) Suppose the sequence \( \alpha_0(v), \alpha_1(v), \ldots \) converges, in the \( p \)-adic topology, to an element in \( \mathbb{Z}_{(p)} \) for \( v = v_1, \ldots, v_m \). Then the same is true for all \( v \in \mathbb{Z}^m \).

(2) Suppose \( f_1(t) = \sum_{s=0}^\infty \alpha_s(v)t^s \) is a rational function with poles contained in the set \( \{ p^{-s} \mid s = 0, 1, 2, \ldots \} \) for \( v = v_1, \ldots, v_m \). Then the same is true for all \( v \in \mathbb{Z}^m \).

Applying this lemma to the basis \( \{ A_n e_0, \ldots, A_n e_n \} \) where \( \{ e_0, \ldots, e_n \} \) is the standard basis of \( \mathbb{Z}^{n+1} \), we see that statements (2') and (3) of Theorem 1.6 follow from the next two lemmas.

**Lemma 5.6.** For all \( r, k \), the function \( \sum_{s=0}^\infty a_{rs,k} t^s \) is a rational function with poles contained in the set \( \{ p^{-s} \mid s = 0, 1, 2, \ldots \} \).

**Lemma 5.7.** For all \( k \), the sequence \( a_{0,k}, a_{1,k}, a_{2,k}, \ldots \) converges, in the \( p \)-adic topology, to an element of \( \mathbb{Z}_{(p)} \).

**Proof of Lemma 5.6.**

\[
\sum_{s=0}^\infty a_{r,s} t^s = \frac{1}{(p^k - 1) \cdots (p - 1)} \sum_{s=0}^\infty (p^{rs} - 1) \cdots (p^{rs-k+1} - 1)t^s
\]

which is a linear combination of functions of the form

\[
\sum_{s=0}^\infty p^{ast} t^s = \frac{1}{(p^s - 1) \cdots (p - 1)}.
\]
Proof of Lemma 5.7. In the $p$-adic topology,
\[
\lim_{s \to \infty} a_{s,k} = \lim_{s \to \infty} \frac{(p^s - 1) \cdots (p^{s-k} - 1)}{(p^k - 1) \cdots (p - 1)} = \frac{1}{(1 - p^k) \cdots (1 - p)}.
\]

Remark 5.8. There is a recursion relation, easily verified,
\[
a_{s+1,k} - a_{s,k} = p^{s-k+1} a_{s,k-1}.
\]
This implies, for example, that $p^{s-n+1}$ divides $(k_{s+1}(X) - k_s(X))$ where $X$ is a wedge summand of $B(\mathbb{Z}/p)^n$. This can be improved slightly: noting that $N_{n,n} \simeq \mathbb{Z}/p[\text{GL}_n(\mathbb{Z}/p)]$, where the projection $\mathbb{Z}/p[\text{M}_n(\mathbb{Z}/p)] \to \mathbb{Z}/p[\text{GL}_n(\mathbb{Z}/p)]$ defines the module structure, we see that $p^{n(s)} (= \text{order of the } p\text{-Sylow subgroup of } \text{GL}_n(\mathbb{Z}/p))$ divides $\dim e N_{n,n}$ for all idempotents $e \in \mathbb{Z}[\text{M}_n(\mathbb{Z}/p)]$. Thus we have that if $n \geq 2$, $p^{s-n+2}$ divides $(k_{s+1}(X) - k_s(X))$, with $X$ as above.

6. Further remarks and examples. As a practical matter, Theorem 1.2 is easiest to use when $W$ acts freely on $\mathcal{X} - \{0\}$. In this case,
\[
k_s(G) = \left[ |\mathcal{X}|^s - 1 \right] / |W| + 1.
\]

Examples 6.1. (1) $G = \text{GL}_2(\mathbb{F}_q)$, $q = p^n$. $P$ is the unipotent subgroup of upper triangular matrices with 1’s on the diagonal, $C_G(P)$ includes the constant diagonal matrices, and $N_C(P)$ is the set of all upper triangular matrices. It is easy to check that $W \simeq \mathbb{F}_q^* \text{ acting in the usual way on } \mathcal{X} = \mathbb{F}_q$. Thus
\[
k_s(\text{GL}_2(\mathbb{F}_q)) = \frac{(q^s - 1)}{(q - 1)} + 1.
\]
(2) $G = \text{SL}_2(\mathbb{F}_q)$ or $\text{PSL}_2(\mathbb{F}_q)$, $q > 2$. This is as in (1), except that now $W \simeq \text{ squares in } \mathbb{F}_q^*$. Thus
\[
k_s(\text{SL}_2(\mathbb{F}_q)) = k_s(\text{PSL}_2(\mathbb{F}_q)) = \frac{2(q^s - 1)}{(q - 1)} + 1.
\]

Example 6.2. With $p = 3$, let $W$ be a 2-Sylow subgroup of $\text{GL}_2(\mathbb{Z}/3)$ (of order 16). Let $G = (\mathbb{Z}/3)^2 \rtimes W$. Then $P = (\mathbb{Z}/3)^2$, and $W$ has four distinct subgroups of order 2 each of which is the isotropy subgroup of two distinct elements of $(\mathbb{Z}/3)^2$. Counting $W$-orbits in $P^s$ having various isotropy subgroups leads to
\[
k_s(G) = \left[ (3^s + 2)^2 + 7 \right] / 16.
\]

The next examples use the methods of §5.

Example 6.3. If $X$ is one of the $(p-1)$ indecomposable summands of $B\mathbb{Z}/p$, then $k_s(X) = (p^s - 1) / (p - 1)$.

Example 6.4. As in [5, §7], we use the notation $X_{i,j}$, $0 \leq i, j \leq p-1$, to denote the $p^2$ distinct irreducible summands of $B(\mathbb{Z}/p)^2$, and $S_{i,j}$ to denote the corresponding irreducible $\mathbb{Z}/p[\text{M}_2(\mathbb{Z}/p)]$-module. Here $X_{i,0} = X_i$ of Remark 4.3 (i.e. a summand of $B\mathbb{Z}/p$), and $S_{i,j} = S_{i,0} \otimes (\text{det})^j$. $X_{0,0} = S^0$. In [4], D. J. Glover computed the dimensions of the projective covers of the $S_{i,j}$. This amounts to computing $k_s(X_{i,j})$ for all $i$ and $j$, and, by Remark 5.8, $k_s(X_{i,j})$ can also be immediately computed. We read off the following table.
Table 1

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>$k_1(X_{i,j})$</th>
<th>$k_2(X_{i,j})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i, 0)</td>
<td>$0 &lt; i \leq p - 1$</td>
<td>$1$</td>
</tr>
<tr>
<td>(0, p - 1)</td>
<td>$1$</td>
<td>$2p + 1$</td>
</tr>
<tr>
<td>(i, j)</td>
<td>$i + j = p - 1$ and $i, j &gt; 0$</td>
<td>$1$</td>
</tr>
<tr>
<td>(0, j)</td>
<td>$0 &lt; j &lt; p - 1$</td>
<td>$0$</td>
</tr>
<tr>
<td>(p - 1, j)</td>
<td>$0 &lt; j \leq p - 1$</td>
<td>$p$</td>
</tr>
<tr>
<td>(i, j)</td>
<td>all other $(i, j) \neq (0, 0)$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

By (5.4), $k_s(X_{i,j}) = a_{s,2}k_2(X_{i,j}) + a_{s,1}[1 - (p^{s-1} - 1)/(p - 1)]k_1(X_{i,j})$ for $(i, j) \neq (0, 0)$.

**Example 6.5.** Let $L(n) = \Sigma^{-n}\text{SP}^p(S)/\text{SP}^p(S)$ [9]. $L(n)$ is an indecomposable summand of $B(\mathbb{Z}/p)^n$ and corresponds to the Steinberg representation of $\mathbb{Z}/p[\text{GL}_n(\mathbb{Z}/p)]$ pulled back to $\mathbb{Z}/p[\text{M}_n(\mathbb{Z}/p)]$. We first note that $k_s(L(n)) = 0$ for $s < n$. Welcher showed this topologically in [14] (see also [10, §4]). For an algebraic proof, it suffices to show that $e_nN_{n,k} = 0$ for $k < n - 1$, where $e_n \in \mathbb{Z}/p[\text{GL}_n(\mathbb{Z}/p)]$ is a Steinberg idempotent, since $e_nB(\mathbb{Z}/p)^n \simeq L(n) \vee L(n - 1)$. The relevant computation is easy and appears in [8]. With this information, it follows that

$$k_n(L(n)) = \dim e_nN_{n,n} = p^{(\frac{n}{2})}.$$ 

We conclude

$$k_s(L(n)) = a_{s,n}p^{(\frac{n}{2})}.$$

**Remark 6.6.** With this formula, one can check that, for all $s$,

$$\sum_{n=0}^s (-1)^r k_s(L(n)) = 0.$$ 

The author has recently discovered [15] that formulae like this occur whenever $K(s)^*(\quad)$ is applied to a “spacelike” resolution of a spectrum (e.g. the $L(n)$ sequence of [9]).

**Remark 6.7.** Suppose that $X$ is an indecomposable summand of $B(\mathbb{Z}/p)^n$ such that $k_s(X) = 0$ for $s < n$ (e.g. $X = L(n)$). Call such a summand regular. As in the last example, it follows that if $X$ is regular, then $k_s(X) = a_{s,n}k_n(X)$, and $k_n(X)$ will be the dimension of an indecomposable projective $\mathbb{Z}/p[\text{GL}_n(\mathbb{Z}/p)]$-module. We conjecture that almost all the indecomposable summands of $B(\mathbb{Z}/p)^n$ are regular. More precisely, we conjecture that, with $r(n, p) = \text{number of regular summands of } B(\mathbb{Z}/p)^n$, $\lim_{p \to \infty} r(n, p)/p^n = 1$. This is true for $n \leq 2$.

**References**


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