

CLASSIFYING SPACES FOR FOLIATIONS WITH ISOLATED SINGULARITIES

PETER GREENBERG

ABSTRACT. Let $\Gamma^a \subset \Gamma$ be transitive pseudogroups on \mathbf{R}^n , such that, for any element $g: U \rightarrow V$ of Γ , there is a locally finite subset $S \subset U$, such that $g|_{U-S}$ is an element of Γ^a . We construct $B\Gamma$, up to weak homotopy type, from $B\Gamma^a$ and the classifying spaces of certain groups of germs.

As an application, the classifying space of the pseudogroup of orientation-preserving, piecewise linear homeomorphisms between open subsets of \mathbf{R} is weakly homotopy equivalent to $BR * BR$.

1. Introduction.

1.1 *Main result.* Let $\Gamma^a \subset \Gamma$ be pseudogroups on an n -manifold X , and let $b \in X$. Assume that the Γ -orbit orb of b is the same as the Γ^a -orbit of b , and that for every $g: U \rightarrow V$ in Γ (U, V open in X) there is a locally finite set $S \subset U \cap orb$ such that $g|_{U-S} \in \Gamma^a$. Let G (respectively G^a) be the discrete group of germs of Γ -embeddings (respectively, Γ^a -embeddings) from a neighborhood of b to X , fixing b . In §2 we copy Tsuboi's construction [Ts] of the \mathbf{R}^n -microbundles $G \setminus \mathbf{R}^n$ and $G^a \setminus \mathbf{R}^n$ over BG and BG^a . Removing zero sections, we obtain $\mathbf{R}^n - 0$ microbundles $G \setminus \mathbf{R}^n - 0$ and $G^a \setminus \mathbf{R}^n - 0$. Maps are defined to construct a commutative diagram

$$(1.2) \quad \begin{array}{ccccc} & & G \setminus \mathbf{R}^n & & \\ & \nearrow & \uparrow & \nwarrow & \\ G \setminus \mathbf{R}^n - 0 & \leftarrow & G^a \setminus \mathbf{R}^n - 0 & \rightarrow & G^a \setminus \mathbf{R}^n \\ & \searrow & \downarrow & \swarrow & \\ & & B\Gamma^a & & \end{array}$$

Further, there are maps from each space in (1.2) to $B\Gamma$, commuting with the maps in (1.2). Therefore, there is a map from the homotopy colimit of (1.2) to $B\Gamma$.

1.3 **THEOREM.** *The map from the homotopy colimit of (1.2) to $B\Gamma$ is a weak (homotopy) equivalence.*

The homotopy colimit [BK, T2] of (1.2) is constructed as follows: let Δ^n be the n -simplex. Starting with the disjoint union of the spaces in (1.2), glue in the mapping cylinder of each map. Then attach a copy of $G^a \setminus \mathbf{R}^n - 0 \times \Delta^2$ for each of the four commutative triangles in (1.2). The resulting space is homotopy equivalent to the homotopy pushout of $G \setminus \mathbf{R}^n \leftarrow P \rightarrow B\Gamma^a$, where P is the homotopy pushout of $G \setminus \mathbf{R}^n - 0 \leftarrow G^a \setminus \mathbf{R}^n - 0 \rightarrow G^a \setminus \mathbf{R}^n$.

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1.4 *Discussion.* It has proven difficult to obtain complete information about the topology of classifying spaces of pseudogroups. However, for a pseudogroup Γ^a which is the set of restrictions of elements of a group H of analytic diffeomorphisms of an analytic manifold X , the classifying space is homotopy equivalent to the homotopy quotient $EH \times_H X$ of the action. One might hope that by enlarging the pseudogroup slightly, by adding isolated singularities, one would obtain a pseudogroup Γ , whose classifying space could be understood; Theorem 1.3 tells how to construct $B\Gamma$ by gluing the classifying space BG of the group of germs at the singular points to $B\Gamma^a$.

The main technical difficulty lies in the fact that, a priori, a group of germs does not act on anything larger than a point. Fortunately, a construction of Tsuboi [Ts] builds étale simplicial spaces (that is, simplicial spaces in which all structure maps are étale) which model, up to homotopy, the microbundles associated to groups of germs, and which, moreover, enjoy (when appropriate) étale simplicial maps to the simplicial spaces whose geometrical realizations, after Haefliger [H1] are the classifying spaces for pseudogroups. Segal’s methods [Se3, Appendix] give us sufficient control of the homotopy colimit of (1.2) to prove 1.3.

Formulations of (1.3), for pseudogroups acting on the real line, lead to explicit calculations (1.10).

1.5 COROLLARY. *Let $\Gamma^a \subset \Gamma$ be a simple extension, with singularities in the orbit of $b \in \mathbf{R}$ (2.2), of orientation-preserving pseudogroups on \mathbf{R} . There are spaces L_1, L_2 weakly equivalent to $BG, B\Gamma^a$ respectively, and a commutative diagram*

$$(1.6) \quad \begin{array}{ccc} & L_1 & \\ & \uparrow \Delta & \\ \pi_- & \left(\begin{array}{c} BG^a \\ \downarrow 1 \end{array} \right) & \pi_+ \\ & \downarrow & \\ & L_2 & \end{array}$$

whose homotopy colimit is weakly equivalent to $B\Gamma$.

1.7 REMARK. The maps π_- and π_+ are “restriction to the left” and “restriction to the right” of a singular germ.

In some instances (1.10) the maps $\pi_-, \pi_+ : BG \rightarrow B\Gamma^a$ factor, up to homotopy, through BG^a , via homomorphisms $p_+, p_- : G \rightarrow G^a$. In such cases, let R denote the homotopy pushout of $BG^a \xleftarrow{Bp_-} BG \xrightarrow{Bp_+} BG^a$. Let $j : BG^a \rightarrow R$ be induced by the inclusion $BG^a \rightarrow BG$, and let $i : BG^a \rightarrow B\Gamma$ be the inclusion.

1.8 COROLLARY. *In the situation just described there are models (up to weak equivalence) of $B\Gamma^a, BG^a$ and R and a diagram $B\Gamma^a \xleftarrow{i} BG^a \xrightarrow{j} R$ whose homotopy pushout is weakly equivalent to $B\Gamma$.*

The proofs of 1.7 and 1.8 are postponed to §3.

1.9 BASIC EXAMPLE. The group \mathbf{Z} acts on the line \mathbf{R} as the set of homeomorphisms $g_n(x) = 2^n x, n \in \mathbf{Z}$. Let Γ^a be the pseudogroup of restrictions of the g_n to open subsets of the line. Let Γ be the pseudogroup of homeomorphisms $g : U \rightarrow V, U,$

$V \subseteq \mathbf{R}$ where if $0 \notin U$ then $g \equiv g_n$ for some $n \in \mathbf{Z}$, and if $0 \in U$ then

$$g(x) = \begin{cases} g_n(x), & x \leq 0, \\ g_m(x), & x \geq 0, \end{cases}$$

for some $m, n \in \mathbf{Z}$.

We apply 1.8 to $\Gamma^a \subset \Gamma$.

Since $G^a \simeq \mathbf{Z}$, $G \simeq \mathbf{Z} \times \mathbf{Z}$ we have $R \simeq S^3$, so that $B\Gamma$ is weakly equivalent to the pushout $S^1 \leftarrow S^1 \rightarrow S^3$, that is, to S^3 .

This result can also be obtained from considerations in [H2].

Adding translations to the above Γ , we obtain more interesting pseudogroups which still have small classifying spaces. For example, let A be the ring $a + b\sqrt{2}$, $a, b \in \mathbf{Z}$. The group of units in A is isomorphic to \mathbf{Z} , generated by $u = 1 + \sqrt{2}$. Let H be the affine group of homeomorphisms of \mathbf{R} of the form $g(x) = u^n x + r$, $n \in \mathbf{Z}$, $r \in A$, and let Γ^a be the pseudogroup of restrictions of elements of H to open subsets of \mathbf{R} . Let Γ be the pseudogroup of piecewise-linear homeomorphisms, which on each piece agree with some element of H , and which have isolated bendpoints in A . We apply 1.8 to the pair $\Gamma^a \subset \Gamma$.

Since $G^a \simeq \mathbf{Z}$, $G \simeq \mathbf{Z} \times \mathbf{Z}$ we have $R \simeq S^3$. Thus $B\Gamma$ is weakly equivalent to the homotopy pushout $BH \leftarrow S^1 \rightarrow S^3$, which is homotopy equivalent to $S^3 \vee C$, where C is the homotopy cofiber of $S^1 \rightarrow BH$. It is easy to compute (since H is an extension $\mathbf{Z}^2 \rightarrow H \rightarrow \mathbf{Z}$) that $H_i C \simeq \mathbf{Z}/2$, $i = 1, 2$, and $H_i C = 0$, $i > 2$.

1.10 *Piecewise linear homeomorphisms.* Let Γ^a be the pseudogroup of orientation-preserving affine homeomorphisms between open subsets of the real line, and let Γ be the pseudogroup of orientation-preserving, piecewise linear homeomorphisms, with isolated ‘‘bendpoints’’, between open subsets of the real line. We apply 1.8.

We have $G^a \simeq \mathbf{R}_+$, $G \simeq \mathbf{R}_+ \times \mathbf{R}_+$, where \mathbf{R}_+ is the group of positive real numbers under multiplication. Consequently, R is homotopy equivalent to the join $BR_+ * BR_+$, so that $B\Gamma$ is weakly equivalent to the homotopy pushout of $B\Gamma^a \leftarrow BR_+ \rightarrow BR_+ * BR_+$.

As we prove below (Lemma 1.13), $BR_+ \rightarrow B\Gamma^a$ induces an isomorphism in integral homology. Further, $B\Gamma$ is simply connected (since the topological category whose geometrical realization is $B\Gamma$ has a connected space of morphisms). Hence, $BR_+ * BR_+ \rightarrow B\Gamma$ is a weak homotopy equivalence, and we can calculate immediately:

1.11 COROLLARY. $H_1(B\Gamma; \mathbf{Z}) = H_2(B\Gamma; \mathbf{Z}) = 0$, and for $k \geq 3$,

$$H_k(B\Gamma; \mathbf{Z}) \simeq \bigoplus_{i=1}^{k-2} \bigwedge^i \mathbf{R}_+ \otimes \bigwedge^{k-1-i} \mathbf{R}_+,$$

where $\bigwedge^i \mathbf{R}_+$ is the i th exterior product of \mathbf{R}_+ over \mathbf{Z} , and where the tensor product is over \mathbf{Z} .

Let G be the group of compactly supported piecewise-linear homeomorphisms of \mathbf{R} . Epstein [E, 3.2] showed that G is simple, and hence that $H_1(G; \mathbf{Z}) = 0$. We can compute a little further by applying Mather’s theorem [Ma] that there is a map $BG \rightarrow \Omega B\Gamma$ which induces an isomorphism in homology.

1.12 COROLLARY. $H_2(G; \mathbf{Z}) = \mathbf{R}_+ \otimes \mathbf{R}_+$, and $H_3(G; \mathbf{Z}) = (\wedge^2 \mathbf{R}_+ \otimes \mathbf{R}_+) \oplus (\mathbf{R}_+ \otimes \wedge^2 \mathbf{R}_+)$.

PROOF. [Ma, Corollary 3.5].

Finally, we prove

1.13 LEMMA. $B\mathbf{R}_+ \rightarrow B\Gamma^a$ induces an isomorphism in integral homology.

PROOF. $B\Gamma^a$ is homotopy equivalent to the classifying space of the discrete group Aff of orientation preserving affine homeomorphisms of the real line. We apply the “center kills” technique of Sah [Sa] to prove that the homomorphism $\mathbf{R}_+ \rightarrow \text{Aff}$ induces an isomorphism in homology.

Considering the Leray-Serre spectral sequence of the exact sequence $\mathbf{R} \rightarrow \text{Aff} \rightarrow \mathbf{R}_+$, it suffices to prove that $H_p(\mathbf{R}_+; H_q \mathbf{R}) = 0$, $q > 0$. To begin, if $r \in \mathbf{R}_+$, the map $r: \mathbf{R} \rightarrow \mathbf{R}$ induces $r_q: H_q(\mathbf{R}) \rightarrow H_q(\mathbf{R})$, and hence an isomorphism $(r_q)_*$ of $H_*(\mathbf{R}_+; H_q \mathbf{R})$. But $(r_q)_*$ is the same as the map in homology induced by conjugation by r in the group \mathbf{R}_+ , so that $(r_q)_*$ is the identity.

Recall that $H_q \mathbf{R} = \wedge^q \mathbf{R}$, the q th exterior product of \mathbf{R} over \mathbf{Z} , and that r_q is given on generators by $r_q(s_1 \wedge \cdots \wedge s_q) = rs_1 \wedge \cdots \wedge rs_q$. In particular, 2_q-id is multiplication by $2^q - 1$, which is an isomorphism for $q > 0$ because $H_q \mathbf{R}$ is torsion free. Hence $(2_q - \text{id})_* = (2_q)_* - \text{id}_* = \text{id} - \text{id} = 0$ is an isomorphism of $H_*(\mathbf{R}_+; H_q \mathbf{R})$, $q > 0$, which must therefore be 0.

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1.15 Organization. §2 reformulates and proves the main theorem. §3 proves Corollaries 1.5 and 1.8. §4 gives a geometric interpretation of (1.3).

1.16 NOTATION. We freely use simplicial notation. References are [M, Se1].

Recall [Se1] that a *topological category* is a small category, in which the sets of objects and morphisms have topologies so that the structure maps are continuous. A (topological) category with a (continuous) inverse is called a (topological) groupoid.

The *nerve* of a topological category C is a simplicial space C_* ; a continuous functor $F: C \rightarrow D$ induces a simplicial map $F_*: C_* \rightarrow D_*$. If C is a topological category we denote by $R, D: C_n \rightarrow C_0$ the domain and range maps $D = d_0 \circ \cdots \circ d_n$, $R = d_{n-1} \circ \cdots \circ d_0$.

We employ the two geometric realization functors $|\cdot|$ and $\|\cdot\|$ discussed by Segal [Se2]. If A_* is a simplicial space, $\|A_*\|$ is defined by gluing together the $A_n \times \Delta^n$ by setting $(Fa_n, t_k) \sim (a_n, F^*t_k)$ for every composite of face maps $F: A_n \rightarrow A_k$; $F^*: \Delta^k \rightarrow \Delta^n$ is the induced inclusion. If we further set $(Sa_n, t_k) \sim (a_n, S^*t_k)$ for every composite S of degeneracy maps we obtain $|A_*|$.

The realization functor $\|\cdot\|$ is useful to us because of the following:

1.17 PROPOSITION [Se2, APPENDIX A.1, ii]. *If $f_*: A_n \rightarrow B_n$ is a map of simplicial spaces so that each $f_n: A_n \rightarrow B_n$ is a weak equivalence, then $\|f_*\|: \|A_*\| \rightarrow \|B_*\|$ is a weak equivalence.*

Homotopy colimits are defined as in [BK or T2]. We shall use the following consequence of [T2], (3.5) and (3.11) on multiple homotopy colimits.

1.18 PROPOSITION. *Let D_* be a diagram of simplicial spaces. Denote by D_k the diagram of the spaces of k -simplices of the simplicial spaces in D_* , by $\|D_*\|$ the diagram of the geometric realizations of the simplicial spaces in D_* , and by $\text{hocolim } D_k$ the simplicial space whose space of k -simplices is the homotopy colimit of D_k .*

There is a weak homotopy equivalence from $\|\text{hocolim } D_k\|$ to the homotopy colimit of $\|D_\|$.*

2. A model for $B\Gamma$. We make the definitions necessary for a precise statement of the main theorem.

2.1 THEOREM. *A groupoid Γ étale over its objects X (or a geo Γ on X) is a topological groupoid Γ , whose space of objects is X , such that the domain and range maps $D, R: \Gamma_1 \rightarrow X$ are locally homeomorphisms.*

As described, for example, in [St] one can associate to any pseudogroup Γ on X a geo on X which we also call Γ . The space of morphisms of the geo Γ is the sheaf of germs of elements of the pseudogroup Γ . Further, the geometric realization $\|\Gamma\|$ of the geo is a standard model for the classifying space $B\Gamma$.

A geo is a more general object than a pseudogroup; consider the following example. Let Γ^2 be the geo associated to the pseudogroup of all homeomorphisms between open subsets of \mathbf{R}^2 , and let $\Gamma^{2,1}$ be the full subcategory of Γ^2 whose space of objects is \mathbf{R} , considered as a line in \mathbf{R}^2 . Essentially, $\Gamma^{2,1}$ is the geo associated to the “pseudogroup” whose elements are homeomorphisms between open subset of \mathbf{R} , with germs of extensions to a neighborhood in \mathbf{R}^2 .

2.2 DEFINITION. Let $\Gamma^a \subseteq \Gamma$ be geos on a (topological) n -dimensional manifold X , and let $b \in X$. We say that $\Gamma^a \subseteq \Gamma$ is a simple extension with singularities in the orbit of b provided

(i) $\{Rg(b): g \in \Gamma_1^a, Dg = b\} = \{Rg(b): g \in \Gamma_1, Dg = b\}$,

(ii) if U is an open subset of X , and $g: U \rightarrow \Gamma_1$ is a section of the domain map, then there is a locally finite subset S of $U \cap \{Rg(b); g \in \Gamma_1, Dg = b\}$ such that $g(U - S) \subseteq \Gamma_1^a$.

For the rest of §2, let $\Gamma^a \subset \Gamma$ be a simple extension with singularities in the orbit of b .

2.3 DEFINITION. Let \underline{G} be the discrete groupoid with objects $\underline{G}_0 = \{Rg(b); g \in \Gamma_1, Dg = b\}$ and morphisms $\underline{G}_1 = \{g \in \Gamma_1: Dg \in \underline{G}_0\}$, and structure maps induced from Γ . Let \underline{G}^a be the subcategory of \underline{G} with the same set of objects, and morphisms $\underline{G}_1^a = \underline{G}_1 \cap \Gamma_1^a$.

If Γ^a and Γ act transitively on X , then \underline{G}^a and \underline{G} are Γ^a and Γ , respectively, but with the discrete topology. In any case, if Γ is the geo associated to a pseudogroup then the realizations $\|\underline{G}^a\|$ and $\|\underline{G}\|$ are homotopy equivalent to the classifying space BG^a and BG of the groups of germs at b .

2.4 Models for the microbundles. The spaces $G \setminus \mathbf{R}^n, G^a \setminus \mathbf{R}^n, G \setminus \mathbf{R}^n - 0$ and $G^a \setminus \mathbf{R}^n - 0$ are the geometrical realizations of simplicial spaces which we now define, as in [Ts]. We let d_i and s_j be the face and degeneracy maps both of Γ_* and \underline{G}_* .

2.5 LEMMA. *There exist, for all $k \geq 0$ and $g \in \underline{G}_k$, open subsets $U_g \subseteq \Gamma_k$, homeomorphic to \mathbf{R}^n , such that*

- (i) $g \in U_g$,
- (ii) $U_g \cap (\Gamma_k - \Gamma_k^a) \subseteq \{g\}$,
- (iii) $d_i U_g \subseteq U_{d_i g}$ and $d_i: U_g \rightarrow U_{d_i g}$ is a homeomorphism, onto its image, for all i ,
- (iv) $s_j U_g = U_{s_j g}$ and $s_j: U_g \rightarrow U_{s_j g}$ is a homeomorphism, for all j .

We postpone the proof of 2.5 until the end of §2, and assume the existence of the open sets U_g .

2.6 DEFINITION. Let C_\star be the simplicial space with $C_k = \coprod U_g$, $g \in \underline{G}_k$, and structure maps $d_i: U_g \rightarrow U_{d_i g}$, $s_j: U_g \rightarrow U_{s_j g}$ induced from Γ_\star . Let $I_\star: C_\star \rightarrow \Gamma_\star$ be the étale simplicial map which is inclusion on each U_g .

Let $D_g = \coprod(U_g - \{g\})$, $g \in \underline{G}_k$, and note that the face and degeneracy maps between the C_k restrict to the D_k , making D_\star a subsimplicial space of C_\star . Let $J_\star: D_\star \rightarrow \Gamma_\star$ be the composition of the inclusion $D_\star \rightarrow C_\star$ with I_\star .

Similarly, $C_k^a = \coprod U_g$, $g \in \underline{G}_k^a$, and $D_k^a = \coprod(U_g - \{g\})$, $g \in G_k^a$, are subsimplicial spaces of C_k , and we define $I_\star^a: C_\star^a \rightarrow \Gamma_\star^a$, $J_\star^a: D_\star^a \rightarrow \Gamma_\star^a$ be the étale simplicial maps which are the restriction of I_\star to C_\star^a and D_\star^a .

2.7 REMARK. There is a simplicial map $p_\star: C_\star \rightarrow \underline{G}_\star$, defined by setting $p_k(x) = g$ if $x \in U_g$, and p_\star has a section $\sigma_\star: \underline{G}_\star \rightarrow C_\star$ defined by $\sigma_k(g) = g \in U_g$. By (1.17) $\|p_\star\|: \|C_\star\| \rightarrow \|\underline{G}_\star\|$ is a homotopy equivalence, since each U_g is contractible. Further, $C_k - \sigma_k(\underline{G}_k) = D_k$. With this motivation, we define $G \setminus \mathbf{R}^n = \|C_\star\|$, $G \setminus \mathbf{R}^n - 0 = \|D_\star\|$, $G^a \setminus \mathbf{R}^n = \|C_\star^a\|$, $G^a \setminus \mathbf{R}^n - 0 = \|D_\star^a\|$.

The maps J_k , I_k^a and J_k^a factor through the inclusion $\Gamma_k^a \rightarrow \Gamma_k$, giving a diagram

$$(2.8)_k \quad \begin{array}{ccccc} & & C_k & & \\ & \nearrow & \uparrow & \nwarrow & \\ D_k & \leftarrow & D_k^a & \rightarrow & C_k^a \\ & \searrow & \downarrow & \swarrow & \\ & & \Gamma_k^a & & \end{array}$$

of simplicial spaces and étale simplicial maps (the maps on the upper half of (2.8)_k are inclusions). The I_k , J_k , I_k^a and the inclusions $\Gamma_k^a \rightarrow \Gamma_k$ commute, whenever appropriate, with the maps in (2.8)_k, and therefore glue together to produce a map F from the homotopy colimit of the diagram

$$(2.9) \quad \begin{array}{ccccc} & & \|C_\star\| & & \\ & \nearrow & \uparrow & \nwarrow & \\ \|D_\star\| & \leftarrow & \|D_k^a\| & \rightarrow & \|C_k^a\| \\ & \searrow & \downarrow & \swarrow & \\ & & \|\Gamma^a\| & & \end{array}$$

to $\|\Gamma\|$. We now reformulate (and prove) 1.3.

2.10 THEOREM. *F is a weak equivalence.*

PROOF. Let E_k be the homotopy colimit of (2.8)_k, and let $F_\star: E_\star \rightarrow \Gamma_\star$ be the simplicial map built from the I_\star , J_\star , I_\star^a , J_\star^a and the inclusion $\Gamma_\star^a \rightarrow \Gamma_\star$. By 1.18 it

suffices to prove that each $F_k: E_k \rightarrow \Gamma_k$ is a weak equivalence. By definition of the U_g (2.5)(i) F_k is onto, and E_k is the homotopy colimit of a diagram of spaces étale over Γ_k . Invoking [Se2, A.6] it is enough to show that the fibers of F_k are contractible.

If $g \in \Gamma_k - \Gamma_k^a$, then $g \in U_h$ only if $g = h$, so $g \notin D_k, g \notin C_k^a, g \notin D_k^a$. Hence the fiber $F_k^{-1}(g)$ is a single point, and is therefore contractible. Now suppose that $g \in \Gamma_k^a$.

Consider the top half of diagram (2.8)_k

$$(2.11)_k \quad \begin{array}{ccccc} & & C_k & & \\ & \nearrow & \uparrow & \nwarrow & \\ D_k & \leftarrow & D_k^a & \rightarrow & C_k^a \end{array}$$

Let L_k be the homotopy colimit of (2.11)_k, and let $f_k: L_k \rightarrow \Gamma_k$ be the restriction of F_k to L_k . Since the maps in (2.11)_k are inclusions, the fiber $f_k^{-1}(g), g \in \Gamma_k$, is a disjoint union of components, with one component m_h containing each element h of $I_k^{-1}(g)$.

Further if $g \in \Gamma_k^a$ and $h \in I_k^{-1}(g)$ then $h \in C_k^a \cup D_k$. Therefore, if $g \in \Gamma_k^a, F_k^{-1}(g)$ is the union of pieces which intersect pairwise in the point $g \in \Gamma_k^a$, with one piece M_h for every element h of $I_k^{-1}(g)$. We now prove that each M_h is contractible. There are three cases to consider.

(i) $h = g \in U_g$. Then $h \notin D_k, h \notin D_k^a$, but since $g \in \Gamma_k^a, h \in C_k^a$. Therefore M_h is the homotopy colimit of the diagram $\cdot \leftarrow \cdot \rightarrow \cdot$, and is contractible.

(ii) $h = g \in U_\gamma, g \neq \gamma, \gamma \in \Gamma_k^a$. Since $\gamma \in \Gamma_k^a$ and $h \neq \gamma, h \in D_k, D_k^a$ and so M_k is the homotopy colimit of the diagram

$$\begin{array}{ccccc} & & \cdot & & \\ & \nearrow & \uparrow & \nwarrow & \\ \cdot & \leftarrow & \cdot & \rightarrow & \cdot \\ & \searrow & \downarrow & \swarrow & \\ & & \cdot & & \end{array}$$

which is contractible.

(iii) $h = g \in U_\gamma, g \neq \gamma, \gamma \in \Gamma_k - \Gamma_k^a$. Since $h \in U_\gamma - \{\gamma\}, h \in D_k$. But $\gamma \in \Gamma_k - \Gamma_k^a$ so $h \notin C_k^a$. Thus M_h is the homotopy colimit of the diagram $\cdot \leftarrow \cdot \rightarrow \cdot$ and is contractible.

Since the M_h are all contractible, $F_k^{-1}(g)$ is contractible, so that F_k is a weak equivalence.

PROOF OF LEMMA 2.5. Recall [BK] that the simplicial set \underline{G}_* is filtered by skeleta \underline{G}_*^N , where $\underline{G}_k^N = \underline{G}_k$ if $k \leq N$, and if $k > N, \underline{G}_k^N = \{s(h): h \in \underline{G}_N, s: \underline{G}_N \rightarrow \underline{G}_k \text{ a composition of degeneracy maps}\}$. We define the $U_g, g \in \underline{G}_*^N$, and show that 2.5(i)–(iv) are satisfied, by induction on N .

If $g \in G_0 = \{Rh(b): h \in \Gamma_1, Dh = b\}$ let U_g be an open subset of X which contains g and is homeomorphic to \mathbb{R}^n . Such a U_g exists because X is an n -manifold. If $g \in \underline{G}_k^0$, then $g = s(h)$, with $h \in \underline{G}_0$ and s some composition of degeneracy

maps. Define $U_g = s(U_k)$, where we consider s as the map $s: X \rightarrow \Gamma_k$. Clearly U_g satisfy (i)–(iv).

Suppose that the U_g have been defined so as to satisfy (i)–(iv), for $g \in \underline{U}_*^{N-1}$. For each $g \in \underline{G}_N^N - \underline{G}_N^{N-1}$, let $V_g = \bigcap_{i=1}^N d_i^{-1}(U_{d_i g})$. Because Γ_N is an n -manifold and the d_i are locally homeomorphisms, there is an open subset U_g of V_g , containing g , homeomorphic to \mathbf{R}^n , so that $U_g \cap (\Gamma_N - \Gamma_N^a) \subseteq \{g\}$, and so that $d_i: U_g \rightarrow U_{d_i g}$ is one-to-one. These U_g , for $g \in \underline{G}_N^N$, satisfy (i)–(iv).

If $g \in \underline{G}_k^N - \underline{G}_k^{N-1}$, $k > N$, then there is a unique $h \in \underline{G}_N^N - \underline{G}_N^{N-1}$, and some composition of degeneracy maps $s: \underline{G}_N \rightarrow \underline{G}_k$ so that $g = sh$. Let $U_g = sU_h$. Certainly (i), (ii) hold; we verify (iii), (iv).

For (iv), suppose that $g \in \underline{G}_k^N - \underline{G}_k^{N-1}$. We want to prove that $s_i: U_g \rightarrow \Gamma_{k+1}$ is a homeomorphism onto $U_{s_i g}$. Write $g = sh$, with $h \in \underline{G}_N^N$ and $s: \underline{G}_N \rightarrow \underline{G}_k$ a composition of degeneracy maps. Then $s: U_h \rightarrow U_g$ is a homeomorphism by definition, as is $s_i s: U_h \rightarrow U_{s_i g}$. Therefore, $s_i = (s_i s)s^{-1}: U_g \rightarrow U_{s_i g}$ is a homeomorphism.

To prove (iii), let $g \in \underline{G}_k^N - \underline{G}_k^{N-1}$; we show that $d_i: U_g \rightarrow \Gamma_{k-1}$ is one-to-one, and that $d_i U_g \subseteq U_{d_i g}$. Write $g = sh$ as above. Either there is some composition of degeneracy maps s' , and some j so that $d_i s = s' d_j$, or else $d_i s = s'$. Therefore we can write $d_i s = s' d'$, where d' is either the identity or a face map. In either case, $d': U_h \rightarrow U_{d' h}$ is one-to-one, and $s: U_h \rightarrow U_g$ is by definition a homeomorphism, so $d_i = s' d' s^{-1}: U_g \rightarrow U_{d_i g}$ is one-to-one.

3. Proofs of 1.5 and 1.8. The proofs of 1.5 and 1.8 involve the homotopy colimits of certain diagrams. Recall [BK, T2] that the homotopy colimit of a diagram of topological spaces is a space constructed as follows. To the disjoint union of the spaces in the diagram, attach the mapping cylinders of the maps in the diagram; that is, attach a copy of $A \times \Delta^1$ for every map $A \xrightarrow{f} B$ in the diagram. Then attach, in a similar way, a copy of $A \times \Delta^2$ for every composition $A \xrightarrow{f} B \xrightarrow{g} C$ of maps, and so on.

Note that if the underlying graph of the diagram is finite and has no cycles, the process described above terminates after a finite number of attachments.

3.1 Preliminaries. Consider the homotopy pushout P of $G \setminus \mathbf{R} - 0 \leftarrow G^a \setminus \mathbf{R} - 0 \rightarrow G^a \setminus \mathbf{R}$. Let $\mathbf{R}_+ = \{r \in \mathbf{R} : r > 0\}$, and let $\mathbf{R}_- = \{r \in \mathbf{R} : r < 0\}$. From our construction of $G \setminus \mathbf{R} - 0$ in §2, and because Γ preserves orientation, it follows that $G \setminus \mathbf{R} - 0$ is the disjoint union of spaces $G \setminus \mathbf{R}_+$ and $G \setminus \mathbf{R}_-$, and, likewise, that $G^a \setminus \mathbf{R} - 0 = G^a \setminus \mathbf{R}_+ \amalg G^a \setminus \mathbf{R}_-$. We can therefore regard P as the homotopy colimit of the diagram

$$G \setminus \mathbf{R}_- \leftarrow G^a \setminus \mathbf{R}_- \rightarrow G^a \setminus \mathbf{R} \leftarrow G^a \setminus \mathbf{R}_+ \rightarrow G \setminus \mathbf{R}_+$$

and we reformulate Theorem 1.3 as follows.

3.1. COROLLARY. *Let $\Gamma^a \subseteq \Gamma$ be a simple extension, with singularities in the orbit of $h \in \mathbf{R}$, of orientation-preserving geos on \mathbf{R} . Then there is a weak equivalence from the*

homotopy colimit of the diagram

$$(3.3) \quad \begin{array}{ccccc} & & G \setminus \mathbf{R} & & \\ & \nearrow & \uparrow & \nwarrow & \\ G \setminus \mathbf{R}_- & \leftarrow G^a \setminus \mathbf{R}_- & G^a \setminus \mathbf{R} & \leftarrow G^a \setminus \mathbf{R}_+ & G \setminus \mathbf{R}_+ \\ & \searrow & \downarrow & \swarrow & \\ & & B\Gamma^a & & \end{array}$$

to $B\Gamma$.

PROOF OF 1.5. Let L_1 be the mapping cylinder of the inclusion $G^a \setminus \mathbf{R} \rightarrow G \setminus \mathbf{R}$, and let L_2 be the homotopy colimit of the diagram

$$(3.4) \quad \begin{array}{ccccccc} & & G \setminus \mathbf{R} & & G \setminus \mathbf{R} & & \\ & \nearrow & \uparrow & \nwarrow & \nearrow & \nwarrow & \\ G \setminus \mathbf{R}_- & \leftarrow G^a \setminus \mathbf{R}_- & \rightarrow G^a \setminus \mathbf{R} & \leftarrow G^a \setminus \mathbf{R}_+ & \rightarrow G \setminus \mathbf{R}_+ & & \\ & \searrow & \downarrow & \swarrow & \searrow & \swarrow & \\ & & B\Gamma^a & & & & \end{array}$$

There is a commuting diagram

$$(3.5) \quad \begin{array}{ccc} & L_1 & \\ & \uparrow \Delta & \\ \pi_- & \left(G^a \setminus \mathbf{R} \right) & \pi_+ \\ & \downarrow i & \\ & L_2 & \end{array}$$

where Δ, i, π_- and π_+ are the obvious inclusions. L_1 is homotopy equivalent to $G \setminus \mathbf{R}$ and hence to BG , so 1.5 follows from the following lemmas.

3.6 LEMMA. $B\Gamma^a \rightarrow L_2$ is a weak equivalence.

3.7 LEMMA. The homotopy colimit of (3.5) is weakly equivalent to $B\Gamma$.

PROOF OF 3.6. Let L_3 be the mapping cylinder of $G^a \setminus \mathbf{R} \rightarrow B\Gamma^a$, and let L_4 and L_5 be the homotopy colimits of

$$\begin{array}{ccc} & G \setminus \mathbf{R} & \\ & \uparrow & \\ G \setminus \mathbf{R}_- & \leftarrow G^a \setminus \mathbf{R}_- & \rightarrow G^a \setminus \mathbf{R} \\ & \downarrow & \\ & B\Gamma^a & \end{array} \quad \text{and} \quad \begin{array}{ccc} & G \setminus \mathbf{R} & \\ & \uparrow & \\ G^a \setminus \mathbf{R} & \leftarrow G^a \setminus \mathbf{R}_+ & \rightarrow G \setminus \mathbf{R}_+ \\ & \downarrow & \\ & B\Gamma^a & \end{array}$$

respectively. Then L_2 is the pushout of $L_4 \leftarrow L_3 \rightarrow L_5$. Since $G^a \setminus \mathbf{R}_+ \rightarrow G^a \setminus \mathbf{R}$ and $G \setminus \mathbf{R}_+ \rightarrow G \setminus \mathbf{R}$ are weak equivalences, it is not hard to show that $B\Gamma^a \rightarrow L_i$, $i = 3, 4, 5$, are weak equivalences, whence $B\Gamma^a \rightarrow L_2$ is a weak equivalence.

PROOF OF 3.7. Let K_2 be the homotopy pushout of $G \setminus \mathbf{R} \leftarrow G^a \setminus \mathbf{R} \rightarrow G \setminus \mathbf{R}$. This latter is a subdiagram of (3.4), and π_-, i and π_+ factor through K_2 . Letting K_3 be the homotopy colimit of the diagram

$$\pi_- \left(\begin{array}{c} L_1 \\ \uparrow \Delta \\ G^a \setminus \mathbf{R} \\ \downarrow i \\ K_2 \end{array} \right) \pi_+$$

we see that the homotopy colimit of (3.5) is the pushout of $L_2 \leftarrow K_2 \rightarrow K_3$. But K_3 is homeomorphic to the homotopy colimit of

$$\begin{array}{ccc} G \setminus \mathbf{R} & \xrightarrow{=} & G \setminus \mathbf{R} \\ & \searrow & \downarrow \\ & & G^a \setminus \mathbf{R} \end{array}$$

so the pushout of $L_2 \leftarrow K_2 \rightarrow K_3$ is weakly equivalent to the homotopy colimit of (3.3).

PROOF OF 1.8. The proof of 1.8 requires that the homomorphisms $p_+, p_-: G \rightarrow G^a$ induce maps $p_+ \setminus \mathbf{R}_+: G \setminus \mathbf{R}_+ \rightarrow G^a \setminus \mathbf{R}_+$ and $p_- \setminus \mathbf{R}_-: G \setminus \mathbf{R}_- \rightarrow G^a \setminus \mathbf{R}_-$. We begin, therefore, by slightly altering the open sets U_g used in the construction of the simplicial spaces C_* and C_*^a . Namely, for $g \in \underline{G}_k^a$ define the U_g following the proof of Lemma 2.5. Then, for $g \in \underline{G}_k - \underline{G}_k^a$, let $U_g = \{h \in U_{p_-g}: Dh < Dg\} \cup \{g\} \cup \{h \in U_{p_+g}: Dh > Dg\}$. The U_g still satisfy 2.5(i)–(iv), and the maps $p_+ \setminus \mathbf{R}_+, p_- \setminus \mathbf{R}_-$ are well defined.

Without changing the homotopy type of the homotopy colimit, replace (3.3) with the diagram

$$(3.8) \quad \begin{array}{ccccccc} & & & G \setminus \mathbf{R} & & & \\ & \nearrow & & \uparrow & & \nwarrow & \\ G \setminus \mathbf{R}_- & \leftarrow & G^a \setminus \mathbf{R}_- & \rightarrow & G^a \setminus \mathbf{R} & \leftarrow & G^a \setminus \mathbf{R}_+ \rightarrow G \setminus \mathbf{R}_+ \\ p_- \setminus \mathbf{R}_- \searrow & = & \swarrow & \downarrow & \swarrow & = & \searrow p_+ \setminus \mathbf{R}_+ \\ & & G^a \setminus \mathbf{R}_- & \rightarrow & B\Gamma^a & \leftarrow & G^a \setminus \mathbf{R}_+ \end{array}$$

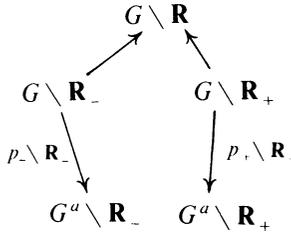
Let L_1 be the homotopy colimit of the subdiagram

$$(3.9) \quad \begin{array}{ccccccc} G \setminus \mathbf{R}_- & \leftarrow & G^a \setminus \mathbf{R}_- & \rightarrow & G^a \setminus \mathbf{R} & \leftarrow & G^a \setminus \mathbf{R}_+ \rightarrow G \setminus \mathbf{R}_+ \\ p_- \setminus \mathbf{R}_- \searrow & \swarrow & & & & \swarrow & \searrow p_+ \setminus \mathbf{R}_+ \\ & & G^a \setminus \mathbf{R}_- & & & & G^a \setminus \mathbf{R}_+ \end{array}$$

and let L_2 be the homotopy colimit of the subdiagram

$$\begin{array}{ccccccc} & & & G \setminus \mathbf{R} & & & \\ & \nearrow & & \uparrow & & \nwarrow & \\ G \setminus \mathbf{R}_- & \leftarrow & G^a \setminus \mathbf{R}_- & \rightarrow & G^a \setminus \mathbf{R} & \leftarrow & G^a \setminus \mathbf{R}_+ \rightarrow G \setminus \mathbf{R}_+ \\ p_- \setminus \mathbf{R}_- \searrow & = & \swarrow & & \swarrow & = & \searrow p_+ \setminus \mathbf{R}_+ \\ & & G^a \setminus \mathbf{R}_- & & & & G^a \setminus \mathbf{R}_+ \end{array}$$

Let $j: L_1 \rightarrow L_2$ be inclusion, and let $i: L_1 \rightarrow B\Gamma^a$ be the map induced by the maps, in (3.8), from the spaces in L_1 to $B\Gamma^a$. There is a diagram $B\Gamma^a \xleftarrow{i} L_1 \xrightarrow{j} L_2$ whose pushout is homotopy equivalent to the homotopy colimit of (3.8), and hence weakly equivalent to $B\Gamma$. Clearly, L_1 is weakly equivalent to BG^a , and L_2 retracts to the homotopy colimit of the diagram



which is weakly equivalent to R .

4 Geometrical interpretation. Let $\Gamma^a \subset \Gamma$ be pseudogroups on \mathbf{R}^n , and suppose that $\Gamma^a \subset \Gamma$ is a simple extension with singularities in the orbit of $b \in \mathbf{R}^n$. Let M be a compact oriented k -manifold with a codimension n Γ -foliation (as described, for example, by charts; see below). We sketch the construction of the classifying map from M to homotopy pushout of $B\Gamma^a \leftarrow P \rightarrow G \setminus \mathbf{R}^n$. Considering $[M] \in H_k B\Gamma$, we describe the image $R_k[M] \in H_{k-1} P$ of the connecting map $R_k: H_k B\Gamma \rightarrow H_{k-1} P$ arising from the Mayer-Vietoris sequence.

There is a (nonunique) subset S of M , compact and contained in finitely many leaves, such that the restriction to $M - S$ of the classifying map $M \rightarrow B\Gamma$ factors, up to homotopy, through $B\Gamma^a$. (One may construct S as follows: suppose that the foliation F is defined via a finite open cover $\{W_i\}$ of M , with submersions $\psi_i: W_i \rightarrow \mathbf{R}^n$, such that the fibers $\psi_i^{-1}(x)$ are leaves of $F|_{W_i}$, and such that, where defined, the $\psi_i \psi_j^{-1}: \psi_j(W_i \cap W_j) \rightarrow \psi_i(W_i \cap W_j)$ are elements of Γ . Let $S_{i,j} \subset \psi_j(W_i \cap W_j)$ be the singular set of $\psi_i \psi_j^{-1}$. We can assume, shrinking the W_i slightly if necessary, that the $S_{i,j}$ are finite sets. Now set $S = \bigcup_{i,j} \psi_j^{-1}(S_{i,j})$.) However S is defined, we can assume that S is the union $S = \bigcup_{j=1}^m K_j \cup \bigcup_{i=1}^l N_i$ where each K_i is a compact leaf and each N_j is a compact $(k - n)$ -submanifold with boundary, of a possibly noncompact leaf L_j .

Let U_i , $1 \leq i \leq l$, and V_j , $1 \leq j \leq m$, be nonintersecting open neighborhoods which retract to the N_i and K_j , respectively. M is covered by the two open sets $\bigcup U_i \cup V_j$ and $M - S$, which intersect in $\bigcup(U_i - N_i) \cup (V_j - K_j)$. We describe the map from M to the homotopy pushout of $B\Gamma^a \leftarrow P \rightarrow G \setminus \mathbf{R}^n$ by giving maps $A: \bigcup U_i \cup V_j \rightarrow G \setminus \mathbf{R}^n$, $B: M - S \rightarrow B\Gamma^a$ and a compatible $C: \bigcup(U_i - N_i) \cup (V_j - K_j) \rightarrow P$.

The retractions of the U_i and the V_j to the N_i and K_j , respectively, and the holonomy microbundles over the N_i and K_j , define the map A . As we observed, S is defined so that $M - S \rightarrow B\Gamma$ factors, up to homotopy, through $B\Gamma^a$. Let $B: M - S \rightarrow B\Gamma^a$ be such a factoring.

We describe $C: \bigcup(U_i - N_i) \cup (V_j - K_j) \rightarrow P$ by describing the restrictions $C_i: U_i - N_i \rightarrow P$ and $C_j: V_j - K_j \rightarrow P$. Recall that P is the homotopy pushout of $G \setminus \mathbf{R}^n - 0 \leftarrow G^a \setminus \mathbf{R}^n - 0 \rightarrow G^a \setminus \mathbf{R}^n$. The map is the composition $C_j: V_j - K_j \rightarrow G \setminus \mathbf{R}^n - 0 \rightarrow P$ which classifies the holonomy microbundle over K_j .

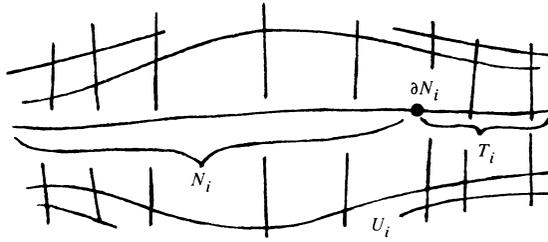


FIGURE 4.1

For each N_i , let $T_i = \partial N_i \times (0, 1)$ be a collar of ∂N_i exterior to N_i in L_i . We may assume (redefining the U_i if necessary) that there is an \mathbf{R}^n -bundle structure $\pi_i: U_i \rightarrow N_i \cup T_i$ with fibers transverse to the leaves of F (see Figure 4.1). Then U_i is the union of

$$Y_i = U_i - (N_i \cup T_i) \text{ and } Z_i = \pi_i^{-1}(T_i), \text{ and } Y_i \cap Z_i = \pi_i^{-1}(T_i) - T_i.$$

Recalling that P is a homotopy pushout, we describe $C_i: U_i - N_i \rightarrow P$ by giving maps $\alpha_i: Y_i \rightarrow G \setminus \mathbf{R}^n - 0$, $\beta_i: Z_i \rightarrow G^a \setminus \mathbf{R}^n$ and a compatible $\gamma_i: \pi_i^{-1}(T_i) - T_i \rightarrow G^a \setminus \mathbf{R}^n - 0$. The map α_i classifies the holonomy microbundle over $N_i \cup T_i$. Over T_i , the holonomy microbundle is nonsingular, since $T_i \subset M - S$, so we get $\beta_i: Z_i \rightarrow G^a \setminus \mathbf{R}^n$, and, by restriction, γ_i .

With proofs of compatibility left to the reader, the map from M to the homotopy pushout $B\Gamma \leftarrow P \rightarrow G \setminus \mathbf{R}$ is defined. To describe the image $R_k[M] \in H_{k-1}P$, keep the notation of the preceding discussion. The open sets $U_i - N_i$, $V_j - K_j$ carry fundamental cycles $[U_i - N_i] \in H_{k-1}(U_i - N_i)$, $[V_j - K_j] \in H_{k-1}(V_j - K_j)$. Then

$$R_k[M] = \sum C_i \cdot [U_i - N_i] + \sum C_j \cdot [V_j - K_j] \in H_{k-1}P.$$

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DEPARTMENT OF MATHEMATICS, NORTH DAKOTA STATE UNIVERSITY, FARGO, NORTH DAKOTA 58105

Current address: Departamento de Matemáticas, Centro de Investigación y Estudios Avanzados del IPN, Apartado Postal 14-740, Mexico 14 DF, CP 07000