

EULER-POINCARÉ CHARACTERISTIC AND HIGHER ORDER SECTIONAL CURVATURE. I

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ABSTRACT. The following long-standing conjecture of H. Hopf is well known. Let M be a compact orientable Riemannian manifold of even dimension $n \geq 2$. If M has nonnegative sectional curvature, then the Euler-Poincaré characteristic $\chi(M)$ is nonnegative. If M has nonpositive sectional curvature, then $\chi(M)$ is nonnegative or nonpositive according as $n \equiv 0$ or $2 \pmod{4}$. This conjecture for $n = 4$ was proved first by J. W. Milnor and then by S. S. Chern by a different method. The main object of this paper is to prove this conjecture for a general n under an extra condition on higher order sectional curvature, which holds automatically for $n = 4$. Similar results are obtained for Kähler manifolds by using holomorphic sectional curvature, and F. Schur's theorem about the constancy of sectional curvature on a Riemannian manifold is extended.

Introduction. The first and simplest result relating local and global invariants in differential geometry is the Gauss-Bonnet formula. It expresses the Euler-Poincaré characteristic $\chi(M)$ of a compact orientable Riemannian manifold M of even dimension n as an integral of the n th sectional curvature (or the Lipschitz-Killing curvature) times the volume element of M . Of course, for $n = 2$, the n th sectional curvature is the usual sectional curvature.

Over the last three decades, the authors of [5, 6, 9, 10, and 24] have obtained various curvature conditions determining the sign of $\chi(M)$. However, the following long-standing conjecture remains open.

H. HOPF'S CONJECTURE. *If M has nonnegative sectional curvature, then $\chi(M)$ is nonnegative. If M has nonpositive sectional curvature, then $\chi(M)$ is nonnegative when $n \equiv 0 \pmod{4}$, and nonpositive when $n \equiv 2 \pmod{4}$.*

This conjecture cannot be established (see [7 and 13]) just by use of the Gauss-Bonnet formula. For $n = 4$, the conjecture was proven by J. W. Milnor, and then by S. S. Chern in [5], using a different method. In [1], R. L. Bishop and S. I. Goldberg obtained a similar result for Kähler manifolds, using holomorphic sectional curvature instead of sectional curvature. Recently, D. L. Johnson [12] proved the Hopf conjecture for Kähler manifolds of real dimension 6.

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In this paper, we relate the sign of $\chi(M)$ to the sign of higher order sectional curvatures, studying higher order holomorphic sectional curvatures when M is Kähler. The main results are these.

THEOREM A. *Let M be an oriented compact Riemannian manifold of even dimension n , and p be an even integer with $2 \leq p \leq n$. If the p th and $(n - p)$ th curvature operators of M are both nonnegative or both nonpositive, then the Euler-Poincaré characteristic $\chi(M) \geq 0$. If one is nonnegative and the other nonpositive, then $\chi(M) \leq 0$.*

THEOREM B. *Let M be a connected Kähler manifold of real dimension $2n$ with $n \geq 2$, and let p be an even integer with $2 \leq p < 2n$. If M has pointwise constant p th holomorphic sectional curvature, then M has constant p th holomorphic sectional curvature.*

THEOREM C. *Let M be a Kähler manifold of real dimension $2n$ with $n \geq 2$, and let p be an even integer with $2 \leq p < 2n$. If M has constant zero p th holomorphic sectional curvature, then M has constant zero q th holomorphic sectional curvature for all even integers q with $q \geq p$. If M is also compact, then the Euler-Poincaré characteristic $\chi(M)$ is zero.*

In fact, a more general result than Theorem A is proved in §2. However, for $n = 4$, Theorem A can be shown to reduce to the Hopf conjecture. Theorem B is well known when $p = 2$. Restated for a Riemannian manifold M with usual sectional curvature in place of holomorphic sectional curvature, Theorem B is the famous theorem of F. Schur, established for $2 < p < 2n$ by J. R. Thorpe [24]. As originally stated, above, Theorem B was proved by R. M. Naviera [18], using methods different from ours. Theorem C is proved in §5. A similar result for a Riemannian M was obtained by J. R. Thorpe in [24].

1. Curvature operators. Let V be an n -dimensional real inner product space with inner product $\langle \cdot, \cdot \rangle$. For p an integer, $1 \leq p \leq n$, let $\Lambda^p(V)$ or simply Λ^p denote the space of p -vectors of V . We call a p -vector α decomposable if it can be written in the form $u_1 \wedge \cdots \wedge u_p$ with $u_i \in V$. Any basis $E = \{e_1, \dots, e_n\}$ for V induces a basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n\}$ for Λ^p consisting of decomposable p -vectors. An arbitrary p -vector α is of the form

$$\alpha = \sum \alpha_{i_1 \dots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p},$$

where summation extends over all $1 \leq i_1 < \cdots < i_p \leq n$. The coefficients $\alpha_{i_1 \dots i_p}$, skew-symmetric in their indices, are called the Plücker coordinates of α with respect to the basis E . We define an inner product for Λ^p on decomposable p -vectors by

$$\langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p \rangle = \det[\langle u_i, v_j \rangle],$$

where $u_i, v_j \in V$. It is easily seen that the basis for Λ^p induced by an orthonormal basis for V is itself orthonormal.

We identify the Grassmann manifold G of oriented p -dimensional subspaces P of V with the submanifold of Λ^p consisting of decomposable p -vectors of length one

by

$$P \Leftrightarrow u_1 \wedge \cdots \wedge u_p,$$

where $\{u_1, \dots, u_p\}$ is an oriented orthonormal basis for P . Elements of G will be called p -planes.

The following result is well known and will be needed later. For a proof, see Hodge and Pedoe [11, pp. 309ff].

LEMMA 1.1 (THE GRASSMANN QUADRATIC p -RELATIONS). *Choose a basis $\{e_1, \dots, e_n\}$ for V . Then the p -vector α is decomposable if and only if its Plücker coordinates satisfy*

$$(1) \quad \sum_{k=1}^{p+1} (-1)^k \alpha_{i_1 \dots \hat{i}_k \dots i_{p+1}} \alpha_{i_k j_1 \dots j_{p-1}} = 0,$$

for all $1 \leq i_k, j_l \leq n$ where the symbol $\hat{}$ over i_k indicates that the subscript i_k is to be deleted.

Let \mathcal{R}^p , or simply \mathcal{R} for fixed p , denote the vector space of all selfadjoint linear transformations on Λ^p with inner product $\langle T, U \rangle = \text{trace}(T \circ U)$. Elements of \mathcal{R} are called p th (order) curvature operators on V . We associate to each $R \in \mathcal{R}$ its p th (order) sectional curvature function $\sigma_R: G \rightarrow \mathbf{R}$ defined by $\sigma_R(P) = \langle R(P), P \rangle$ for all $P \in G$, where \mathbf{R} is the set of real numbers. Then $R_p \in \mathcal{R}^p$ satisfies the Bianchi identities if

$$(2) \quad \sum_{k=1}^{p+1} \langle (-1)^k R_p(u_1 \wedge \cdots \wedge \hat{u}_k \wedge \cdots \wedge u_{p+1}), u_k \wedge u_{p+2} \wedge \cdots \wedge u_{2p} \rangle = 0,$$

for all $u_1, \dots, u_{2p} \in V$. Let \mathcal{B} denote the space of elements of \mathcal{R} which satisfy the Bianchi identities, and let \mathcal{S} denote the space of elements $R \in \mathcal{R}$ for which $\sigma_R \equiv 0$. Then from Stehney [23, §1] we have the following.

LEMMA 1.2. *With respect to the inner product defined above on \mathcal{R} , $\mathcal{R} = \mathcal{B} \oplus \mathcal{S}$ is an orthogonal decomposition of \mathcal{R} .*

We will need the following consequence of Lemma 1.2.

LEMMA 1.3. *For p an integer with $0 \leq p \leq n$, let $*p: \Lambda^p \rightarrow \Lambda^{n-p}$ be the Hodge star operator. Then the map $\Omega: \mathcal{R}^p \rightarrow \mathcal{R}^{n-p}$ is orthogonal with respect to the above inner product where $\Omega(S) = *pS*(n-p)$ for all $S \in \mathcal{R}^p$.*

PROOF. First, Ω is well defined since the adjoint of $*p$ is

$$(*p)^{-1} = (-1)^{p(n-p)}*(n-p).$$

Thus for $S, T \in \mathcal{R}^p$ we have

$$\begin{aligned} \langle *pS*(n-p), *pT*(n-p) \rangle &= \text{trace}(*pS*(n-p) \circ *pT*(n-p)) \\ &= \text{trace}((-1)^{p(n-p)}*p(S \circ T)*(n-p)) \\ &= \text{trace}(*p(S \circ T)(*p)^{-1}) \\ &= \text{trace}(S \circ T) = \langle S, T \rangle. \end{aligned}$$

Hence the map Ω is orthogonal.

COROLLARY 1.4. *The map $\Omega: \mathcal{R}^p \rightarrow \mathcal{R}^{n-p}$ preserves the orthogonal decomposition given in Lemma 1.2, i.e., Ω preserves the Bianchi identities as well as the operators with identically zero sectional curvature.*

PROOF. In order to show that Ω preserves the orthogonal decomposition, it is sufficient to show that for all $S \in \mathcal{S}^p$, $*pS*(n-p) \in \mathcal{S}^{n-p}$ since we know by Lemma 1.3 that Ω is orthogonal. Now for an $(n-p)$ -plane α in V , $*(n-p)(\alpha)$ is a p -plane in V . Thus for all $S \in \mathcal{S}^p$,

$$\langle *pS*(n-p)(\alpha), \alpha \rangle = (-1)^{p(n-p)} \langle S(*n-p)\alpha, *(n-p)\alpha \rangle = 0,$$

so that $*pS*(n-p) \in \mathcal{S}^{n-p}$.

Although the statements above are true in a general inner product space setting, our main interest is the p th curvature operator arising on a tangent space M_m of a Riemannian manifold M at a point m . Specifically, let R denote the Riemannian curvature tensor at m . Then for even integers $p \geq 2$, the p th curvature operator R_p is defined in Thorpe [26] by

$$\begin{aligned} \langle R_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p \rangle \\ = \frac{1}{2^{p/2} p!} \sum_{\alpha, \beta \in S_p} \text{sgn}(\alpha) \text{sgn}(\beta) R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)}) \\ \cdots R(u_{\alpha(p-1)}, u_{\alpha(p)}, v_{\beta(p-1)}, v_{\beta(p)}), \end{aligned}$$

where $u_i, v_j \in M_m$, and S_p is the permutation group on $(1, \dots, p)$. Also, Thorpe [26] establishes that R_p satisfies the Bianchi identities.

The generalized Gauss-Bonnet theorem expresses the Euler-Poincaré characteristic $\chi(M)$ of a compact oriented Riemannian manifold M of even dimension n as the integral

$$(3) \quad \chi(M) = \frac{2}{c_n} \int_M K dV,$$

where K is the Lipschitz-Killing curvature of M , c_n is the volume of the Euclidean unit n -sphere, and dV is the volume element of M .

The following useful theorem is a composite of the two theorems and Corollary 2 from Thorpe [26]. We write σ_{2k} for $\sigma_{R_{2k}}$.

THEOREM 1.5. *Let M be an oriented Riemannian manifold of even dimension n . Then we have the following.*

(a) *The Lipschitz-Killing curvature K at $m \in M$ satisfies*

$$(4) \quad K(m) = \langle R_n(M_m), M_m \rangle = \frac{p!(n-p)!}{n!} \text{trace}(* (n-p) R_{n-p} * p R_p).$$

(b) *Let $n = 4k$. Then $R_{2k}*(2k) = \pm*(2k)R_{2k}$ if and only if $\sigma_{2k}*(2k)P = \pm\sigma_{2k}P$ for all $2k$ -planes P . Signs above are to be taken consistently, with plus implying that $K \geq 0$, minus that $K \leq 0$.*

(c) *Let M be compact with $n = 4k$, $R_{2k} = \pm*(2k)R_{2k}$, and P_k denoting the k th Pontryagin class of M . Then the plus sign gives*

$$\chi(M) \geq \frac{k!k!}{(2k)!} |P_k| \geq 0,$$

while the minus sign gives $\chi(M) \leq 0$ and $P_k = 0$. Moreover, $\chi(M) = 0$ if and only if M is $2k$ -flat, i.e., $\sigma_{2k} \equiv 0$.

Now we consider the case of a Kähler manifold M with an almost complex structure J and Riemannian curvature operator R .

DEFINITION 1.6. Also denote by J the extension of the structure J from M_m to $\Lambda^p(M_m)$ defined by $J(u_1 \wedge \cdots \wedge u_p) = Ju_1 \wedge \cdots \wedge Ju_p$ for $u_i \in M_m$, extending linearly to $\Lambda^p(M_m)$. Note that this gives $J^2 = (-1)^p$.

LEMMA 1.7. (a) Let R_p be the p th curvature operator on M . Then $R_p = R_p \circ J = J \circ R_p$.

(b) The adjoint of J is $(-1)^p J$.

(c) For even p , J and $*p$ commute.

PROOF. (a) This follows directly from the definition of R_p and the fact that for a Kähler manifold $R = R \circ J = J \circ R$.

(b) This follows directly from the definitions of $\langle \cdot, \cdot \rangle$ and J on $\Lambda^p(M_m)$, and the fact that the adjoint of J on M_m is $-J$.

(c) Let p be even. Now the Hodge star operator $*p: \Lambda^p(M_m) \rightarrow \Lambda^{n-p}(M_m)$ can be defined by

$$(*p\xi) \wedge \eta = \langle \xi, \eta \rangle M_m,$$

for $\xi, \eta \in \Lambda^p(M_m)$, where M_m is the oriented n -plane which is the tangent space of M at m . Let $\xi, \eta \in \Lambda^p(M_m)$. Since $JM_m = M_m$ and the adjoint of J is J , we have

$$\begin{aligned} [(J*pJ)(\xi)] \wedge \eta &= J([*p(J\xi)] \wedge J\eta) \\ &= J(\langle J\xi, J\eta \rangle M_m) \\ &= \langle \xi, \eta \rangle M_m. \end{aligned}$$

Hence $J*pJ = *p$, which implies $*pJ = J*p$.

DEFINITION 1.8. For p even, a p -plane Q is said to be holomorphic if $JQ = Q$. Also, p th holomorphic sectional curvature is p th sectional curvature restricted to holomorphic p -planes.

THEOREM 1.9. Let V be an even-dimensional real inner product space with a complex structure J and Hermitian inner product $\langle \cdot, \cdot \rangle$. Also, let $T: \Lambda^2(V) \rightarrow \Lambda^2(V)$ be a self-adjoint linear operator satisfying the Bianchi identity and $T = T \circ J = J \circ T$. Then we have the following.

(a) T is uniquely determined by its holomorphic sectional curvature, i.e., if T has zero holomorphic sectional curvature, then $T \equiv 0$.

(b) If T has constant holomorphic sectional curvature H , then for all $X, Y, Z, W \in V$ we have

$$\begin{aligned} (5) \quad \langle T(X \wedge Y), Z \wedge W \rangle &= (H/4)[\langle X, Z \rangle, \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle + \langle X, JZ \rangle \langle Y, JW \rangle \\ &\quad - \langle X, JW \rangle \langle Y, JZ \rangle + \langle X, JY \rangle \langle Z, JW \rangle]. \end{aligned}$$

PROOF. This theorem is merely a restatement of Kobayashi-Nomizu [14, vol. II, Chapter IX, Propositions 7.1 and 7.3].

COROLLARY 1.10. *Let V and T satisfy the hypotheses of Theorem 1.9 with $\dim V = 2n$ and T having constant holomorphic sectional curvature H . Let $\{e_1, e_{1^*}, \dots, e_n, e_{n^*}\}$ be an orthonormal basis of V , where $e_{i^*} = Je_i$ for $i = 1, \dots, n$. Then from this basis of V , we have an orthonormal basis B of $\Lambda^2(V)$ given by $B = \{e_{i_1} \wedge e_{i_2} \mid 1 \leq i_1 < i_2 \leq n^*\}$ where $1 < 1^* < \dots < n < n^*\}$. Furthermore, for $Q \in B$ we have*

$$(6) \quad T(Q) = \begin{cases} HQ + \frac{H}{2} \sum_{\substack{Q' \in B, Q' \neq Q \\ Q' \text{ holomorphic}}} Q', & \text{if } Q \text{ is holomorphic,} \\ \frac{1}{4}H(Q + JQ), & \text{if } Q \text{ is not holomorphic.} \end{cases}$$

PROOF. This obviously follows from (5) and the fact that B is an orthonormal basis of $\Lambda^2(V)$.

Finally, we mention the relationship between the k th Chern form and the $(2k)$ th curvature operator on a compact Kähler manifold given in §2 of Gray [10].

THEOREM 1. *Let M be a compact Kähler manifold of real dimension $2n$ with almost complex structure J . For $m \in M$, let $\{e_1, e_{1^*}, \dots, e_n, e_{n^*}\}$ be an orthonormal basis of M_m where $e_{i^*} = Je_i$ for $i = 1, \dots, n$, let γ_k be the k th Chern form (via de Rham's theorem), and let R_{2k} be the $(2k)$ th curvature operator. Then*

$$(7) \quad \begin{aligned} & \frac{(4\pi)^k k!}{(2k)!} \gamma_k(j_1, \dots, j_{2k}) \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq n} \langle R_{2k}(e_{i_1} \wedge e_{i_1^*} \wedge \dots \wedge e_{i_k} \wedge e_{i_k^*}), j_1 \wedge \dots \wedge j_{2k} \rangle, \end{aligned}$$

for all $j_1, \dots, j_{2k} \in M_m$.

PROOF. This follows directly from (2.4), Lemma 2.1, the definition of $(2k)$ th curvature operator in §2 of Gray [10], and the fact that Gray's $(2k)$ th curvature operator is $[(2k)!/2^k]R_{2k}$.

2. Euler-Poincaré characteristic and curvature operators. Let $\Pi_p: \mathcal{R}^p \rightarrow \mathcal{B}^p$ be the orthogonal projection associated with the orthogonal decomposition of Lemma 1.2.

THEOREM 2.1. *Let M be an oriented compact Riemannian manifold of even dimension n , and p be an even integer with $2 \leq p < n$. Let R_p and R_{n-p} be the p th and $(n-p)$ th curvature operators on M . If at each $m \in M$ there exists $\bar{R}_p \in \mathcal{R}^p$ such that $\Pi_p \bar{R}_p = R_p$ with \bar{R}_p and R_{n-p} both positive semidefinite or negative semidefinite operators, then the Euler-Poincaré characteristic $\chi(M) \geq 0$. If semidefinite is replaced by definite at at least one point of M , then $\chi(M) > 0$. If \bar{R}_p and R_{n-p} are of opposite signs as operators, then $\chi(M) \leq 0$ or $\chi(M) < 0$ according as both \bar{R}_p and R_{n-p} are semidefinite or definite (at at least one point of M).*

PROOF. Assume that at each $m \in M$ the above holds with operators \bar{R}_p and R_{n-p} both positive semidefinite. The other cases can be proven similarly. Since $\Pi_p \bar{R}_p = R_p$, there exists $S_p \in \mathcal{S}^p$ with $\bar{R}_p = R_p + S_p$.

Let $\{v_1, \dots, v_k\}$ be an orthonormal basis of $\Lambda^p(M_m)$ consisting of eigenvectors of \bar{R}_p , and let $\{\lambda_1, \dots, \lambda_k\}$ be the corresponding eigenvalues of \bar{R}_p with respect to the given basis. Let $K(m)$ be the Lipschitz-Killing curvature of M at m . Then by Corollary 1.4, Lemma 1.2, and (4) we have

$$\begin{aligned} K(m) &= [p!(n-p)!/n!] \text{trace} \left(\left[*(n-p)R_{n-p}*p \right] \left[\bar{R}_p - S_p \right] \right) \\ &= [p!(n-p)!/n!] \left[\text{trace} \left(*(n-p)R_{n-p}*p\bar{R}_p \right) - \text{trace} \left(*(n-p)R_{n-p}*pS_p \right) \right] \\ &= [p!(n-p)!/n!] \text{trace} \left(*(n-p)R_{n-p}*p\bar{R}_p \right) \\ &= [p!(n-p)!/n!] \sum_{i=1}^k \langle *(n-p)R_{n-p}*p\bar{R}_p(v_i), v_i \rangle \\ &= [p!(n-p)!/n!] \sum_{i=1}^k \lambda_i \langle *(n-p)R_{n-p}*p(v_i), v_i \rangle \\ &= [p!(n-p)!/n!] \sum_{i=1}^k \lambda_i \langle R_{n-p}(*pv_i), *pv_i \rangle \geq 0, \end{aligned}$$

since $\lambda_i \geq 0$ for $i = 1, \dots, k$, and R_{n-p} is positive semidefinite gives $\langle R_{n-p}(\alpha), \alpha \rangle \geq 0$ for all $\alpha \in \Lambda^{n-p}$. Hence by the generalized Gauss-Bonnet formula (3) we have $\chi(M) \geq 0$.

REMARK 2.2. (a) Let the dimension of V be four. Then Thorpe [27] has shown that $R \in \mathcal{O}^2$ has nonnegative (positive) sectional curvature if and only if $R = \Pi_2 \bar{R}$ for some positive semidefinite (definite) $\bar{R} \in \mathcal{O}^2$.

(b) D. Meyer [16] has shown that a compact Riemannian manifold with positive definite 2nd (usual) curvature operator is a homology sphere.

(c) Thorpe [24] characterizes the p th curvature operators R_p of constant p th sectional curvature K as $R_p = K \cdot I$ where I is the identity operator. These are positive or negative definite according as $K > 0$ or $K < 0$.

3. Curvature operators of the form $R_p = cA^p$ on a Kähler manifold. Following Stehney [22] we can define a selfadjoint operator $R_p = cA^p$ on $\Lambda^p(V)$ for $c \in \mathbf{R}$ and a selfadjoint operator A on V by $cA^p(u_1 \wedge \dots \wedge u_p) = c(Au_1 \wedge \dots \wedge Au_p)$ for all $u_i \in V$, extending linearly to $\Lambda^p(V)$. Moreover, $R_p = cA^p$ satisfies the Bianchi identities.

THEOREM 3.1. *Let V be a real inner product space of dimension $2n$, and J be a complex structure on V . For p an even integer with $2 \leq p < 2n$, let $R_p = cA^p$ be an operator on $\Lambda^p(V)$ as defined above with $A^p = J \circ A^p = A^p \circ J$. Also, let $\{v_1, \dots, v_{2n}\}$ be an orthonormal basis of V consisting of eigenvectors of A with corresponding eigenvalues $\lambda_1, \dots, \lambda_{2n}$. Then either $R_p \equiv 0$ or A has exactly p nonzero eigenvalues $\lambda_{i_1}, \dots, \lambda_{i_p}$ ($1 \leq i_1 < \dots < i_p \leq 2n$) with $v_{i_1} \wedge \dots \wedge v_{i_p}$ a holomorphic*

p -plane, so that for all $\gamma \in \Lambda^p(V)$

$$R_p(\gamma) = c\lambda_{i_1} \cdots \lambda_{i_p} \langle \gamma, v_{i_1} \wedge \cdots \wedge v_{i_p} \rangle v_{i_1} \wedge \cdots \wedge v_{i_p}.$$

PROOF. Let $B = \{v_{j_1} \wedge \cdots \wedge v_{j_p} \mid 1 \leq j_1 < \cdots < j_p \leq 2n\}$ for the above orthonormal basis of $\Lambda^p(V)$ consisting of eigenvectors of R_p with corresponding eigenvalues $c\lambda_{j_1} \cdots \lambda_{j_p}$.

Assume that R_p is not identically zero. Then $c \neq 0$ and there exist $1 \leq i_1 < \cdots < i_p \leq 2n$ so that $\lambda_{i_1} \cdots \lambda_{i_p} \neq 0$. Thus A has nonzero eigenvalues $\lambda_{i_1}, \dots, \lambda_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq 2n$. Also, the corresponding eigenvector $v_{i_1} \wedge \cdots \wedge v_{i_p}$ of R_p is a holomorphic p -plane since $R_p = J \circ R_p$.

Now assume there exists $\lambda_j \neq 0$ for some $j \neq i_1, \dots, i_p$. Then $\lambda_{i_1} \cdots \lambda_{i_{p-1}} \lambda_j$ is not zero, and $v_{i_1} \wedge \cdots \wedge v_{i_{p-1}} \wedge v_j$ is an eigenvector of R_p with eigenvalue $\lambda_{i_1} \cdots \lambda_{i_{p-1}} \lambda_j$. Since $R_p = J \circ R_p$, $v_{i_1} \wedge \cdots \wedge v_{i_{p-1}} \wedge v_j$ is also a holomorphic p -plane. Now $v_{i_1} \wedge \cdots \wedge v_{i_p}$ and $v_{i_1} \wedge \cdots \wedge v_{i_{p-1}} \wedge v_j$ are two holomorphic p -planes which intersect in a holomorphic $(p - 1)$ -plane. This is a contradiction since $p - 1$ is odd. Hence $\lambda_j = 0$ for all $j \neq i_1, \dots, i_p$.

Thus all other eigenvalues of R_p are zero other than $c\lambda_{i_1} \cdots \lambda_{i_p}$, and R_p must be of the stated form since B is an orthonormal basis of $\Lambda^p(V)$.

COROLLARY 3.2. *Let M be a Kähler manifold of real dimension $2n$. For p an even integer with $2 \leq p < 2n$, if the p th sectional curvature of M at $m \in M$ is constant at value K , then $K = 0$.*

PROOF. Assume the above is given with $K \neq 0$. Then from Thorpe [24], the p th curvature operator R_p at m is of the form $R_p = K \cdot I^p$ for the identity I . This contradicts Theorem 3.1 since I has $2n$ nonzero eigenvalues. Hence, $K = 0$.

COROLLARY 3.3. *Let M be a compact Kähler manifold of real dimension $2n$. If at each $m \in M$ there exists $c \in \mathbf{R}$, an even integer p with $2 \leq p \leq n$, and a selfadjoint operator A on M_m so that the p th curvature operator R_p of M at $m \in M$ is of the form $R_p = cA^p$, then the Euler-Poincaré characteristic $\chi(M) = 0$, and the n th Chern class γ_n of M is zero.*

PROOF. Let the above at $m \in M$ be given. Then at $m \in M$, the $(2p)$ th curvature operator $R_{2p} = c^2A^{2p}$ by Lemma 2.5 of Stehney [22]. In the following we assume R_p is written in the form of Theorem 3.1.

CASE 1. Let $2 \leq p < n$. Then $2 < 2p < 2n$ and Theorem 3.1 imply that $R_{2p} \equiv 0$ at $m \in M$, since A cannot have both exactly p and $2p$ nonzero eigenvalues. Hence by the trace formula (4) for the Lipschitz-Killing curvature K at m , we have $K(m) = 0$. Also, (4) gives $R_{2n}(m) = 0$.

CASE 2. Let $p = n$. Then $R_{2n} = c^2A^{2n} = (c^2 \det A)I^{2n}$ at $m \in M$ for the identity I . But $\det A = 0$ since A has exactly n nonzero eigenvalues or we must have $c = 0$ by Theorem 3.1. For either situation, we have $R_{2n} \equiv 0$ at m , and formula (4) again gives $K(m) = 0$.

Now since the above occurs at each $m \in M$, $K \equiv 0$ and $R_{2n} \equiv 0$ on M . Hence the generalized Gauss-Bonnet formula (3) gives $\chi(M) = 0$, and formula (7) implies that γ_n is zero also.

COROLLARY 3.4. *Let the hypothesis of Corollary 3.3 be given with p constant on M . Then the k th Chern classes γ_k of M are zero for $k \geq p$, and $\gamma_{p/2}$ is zero if and only if M is p -flat. Also, M is q -flat for all $q \geq 2p$.*

PROOF. From the proof of Corollary 3.3 it follows that the $(2p)$ th curvature operator $R_{2p} \equiv 0$ on M . Thus the q th curvature operator $R_q \equiv 0$ for all $q \geq 2p$ by Thorpe [24, Theorem 6.4] and [26, Lemma of §3]. Hence we have by formula (7) that γ_k is zero for $k \geq p$. Also, $R_q \equiv 0$ on M for all $q \geq 2p$ implies that M is q -flat for all $q \geq 2p$.

Now if M is p -flat, then the above Lemma of Thorpe [26] gives $R_p \equiv 0$ on M , and formula (7) implies that $\gamma_{p/2}$ is zero.

On the other hand, let $\gamma_{p/2}$ be zero and assume M is not p -flat. Suppose at some $m_0 \in M$, R_p is not identically zero and is of the nonzero form stated in Theorem 3.1. In the notation of Theorem 3.1, the holomorphic p -plane $v_{i_1} \wedge \cdots \wedge v_{i_p}$ can be written as $e_1 \wedge e_{1^*} \wedge \cdots \wedge e_r \wedge e_{r^*}$ where $r = p/2$ for an orthonormal basis $\{e_1, e_{1^*}, \dots, e_n, e_{n^*}\}$ of M_{m_0} with $e_{j^*} = J e_j$ for all $j = 1, \dots, n$. Now using formula (7) and Theorem 3.1 we get at m_0 that

$$\begin{aligned} 0 &= (r!)(2\pi)^r \gamma_{p/2}(e_1, e_{1^*}, \dots, e_r, e_{r^*}) \\ &= \frac{(2r)!}{2^r} \sum_{1 \leq j_1 < \cdots < j_r \leq n} \langle R_p(e_{j_1} \wedge e_{j_1^*} \wedge \cdots \wedge e_{j_r} \wedge e_{j_r^*}), \\ &\qquad\qquad\qquad e_1 \wedge e_{1^*} \wedge \cdots \wedge e_r \wedge e_{r^*} \rangle \\ &= \frac{(2r)!}{2^r} c\lambda_{i_1} \cdots \lambda_{i_p}. \end{aligned}$$

But $c\lambda_{i_1} \cdots \lambda_{i_p} \neq 0$ by Theorem 3.1. Thus we have a contradiction, and hence M is p -flat.

COROLLARY 3.5. *Let the hypothesis of Corollary 3.3 be given with p constant on M and $n < p < 2n$. If the $(2n - p)$ th curvature operator R_{2n-p} of M has constant zero holomorphic sectional curvature, then the Euler-Poincaré characteristic $\chi(M) = 0$, and the n th Chern class γ_n is zero.*

PROOF. If $R_p = 0$ at $m \in M$, then by formula (4) the Lipschitz-Killing curvature at m is $K(m) = 0$.

Also, if $R_p \neq 0$ at $m \in M$, then Theorem 3.1, together with its notation and proof, gives that for all $\Lambda^p(M_m)$,

$$R_p(\gamma) = c\lambda_{i_1} \cdots \lambda_{i_p} \langle \gamma, v_{i_1} \wedge \cdots \wedge v_{i_p} \rangle v_{i_1} \wedge \cdots \wedge v_{i_p}.$$

Now using formula (4) and the basis of M_m in Theorem 3.1 we have

$$\begin{aligned} [(2n)! / (p!(2n - p)!)] K(m) &= \text{trace}(* (2n - p) R_{2n-p} * p R_p) \\ &= \sum_{1 \leq j_1 < \cdots < j_p \leq 2n} \langle * (2n - p) R_{2n-p} * p R_p (v_{j_1} \wedge \cdots \wedge v_{j_p}), v_{j_1} \wedge \cdots \wedge v_{j_p} \rangle \\ &= c\lambda_{i_1} \cdots \lambda_{i_p} \langle R_{2n-p} (* p [v_{i_1} \wedge \cdots \wedge v_{i_p}]), * p [v_{i_1} \wedge \cdots \wedge v_{i_p}] \rangle. \end{aligned}$$

Then $K(m) = 0$ since R_{2n-p} has constant zero holomorphic sectional curvature, and $*p(v_{i_1} \wedge \cdots \wedge v_{i_p})$ is a holomorphic $(2n - p)$ -plane by Theorem 3.1 and Lemma 1.7(c).

Thus $K \equiv 0$ on M , and $R_{2n} \equiv 0$ on M by formula (4). Hence the generalized Gauss-Bonnet formula (3) gives $\chi(M) = 0$, and formula (7) implies that γ_n is zero.

4. Constancy of holomorphic sectional curvature. The purpose of this section is to prove the following theorem.

THEOREM 4.1. *Let M be a connected Kähler manifold of real dimension $2n$ with $n \geq 2$, and let p be an even integer with $2 \leq p < 2n$. If M has pointwise constant p th holomorphic sectional curvature, then M has constant p th holomorphic sectional curvature.*

Before we prove Theorem 4.1, we need the following lemma.

LEMMA 4.2. *Let V be a real inner product space of dimension $2n$ for $n \geq 2$ with complex structure J and Hermitian inner product with respect to J . For an even integer p with $2 \leq p < 2n$, let $\{e_1, e_{1^*}, \dots, e_n, e_{n^*}\}$ be an orthonormal basis of V with $e_{i^*} = Je_i$ for $i = 1, \dots, n$, and let $B = \{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n^*\}$. Also, let $T: \Lambda^p(V) \rightarrow \Lambda^p(V)$ be a selfadjoint operator, satisfying the Bianchi identities $T = J \circ T = T \circ J$, with constant holomorphic sectional curvature H . Then for any holomorphic p -plane $Q \in B$,*

$$(8) \quad \langle T(Q), Q' \rangle = \begin{cases} H & \text{if } Q' = Q, \\ \frac{1}{2}H & \text{if } Q' \text{ is holomorphic, and } Q' \cap Q \text{ is} \\ & \text{holomorphic } (p - 2)\text{-plane,} \\ 0 & \text{if } Q' \text{ is not holomorphic, but } Q' \cap Q \\ & \text{contains a holomorphic } (p - 2)\text{-plane.} \end{cases}$$

PROOF. The case when $p = 2$ easily follows from Corollary 1.10, and so in the remainder of the proof we assume that $2 < p < 2n$.

For $2k = p - 2$ and any $1 \leq j_1 < \cdots < j_k \leq n$, let $A = e_{j_1} \wedge e_{j_1^*} \wedge \cdots \wedge e_{j_k} \wedge e_{j_k^*}$ be the holomorphic $(p - 2)$ -plane formed from the given basis of V stated above. Then A^\perp is a holomorphic $(2n - 2k)$ -plane since the inner product is Hermitian. Now define a second-order curvature operator R on A^\perp by

$$\langle R(X \wedge Y), Z \wedge W \rangle = \langle T(A \wedge X \wedge Y), A \wedge Z \wedge W \rangle$$

for all $X, Y, Z, W \in A^\perp$, extending linearly. Then R satisfies the hypothesis of Corollary 1.10, and using formula (6) with the basis of A^\perp taken from the given basis of V we easily get formula (8) for T .

PROOF OF THEOREM 4.1. Let $F(M)$ be the principal $O(2n)$ -bundle of orthonormal frames on M , where $O(2n)$ is the group of orthogonal $2n \times 2n$ matrices. Let $\Pi: F(M) \rightarrow M$ be the projection map.

The curvature form $\Omega = [\Omega_{ij}]$ of the Riemannian connection of M is a smooth 2-form on $F(M)$ with values in the Lie algebra $\mathfrak{o}(2n)$ of real skew-symmetric $2n \times 2n$ matrices. If v, w are tangent vectors at $z = (m; e_1, \dots, e_{2n}) \in F(M)$, with

R the Riemannian curvature operator, we have for $v' = \Pi_*(v)$ and $w' = \Pi_*(w)$ that $\Omega_{i,j}(z)(v, w) = \langle R(e_i \wedge e_j), v' \wedge w' \rangle$.

For $I = \{i_1, \dots, i_p\}$ with i_1, \dots, i_p integers between 1 and $2n$, we define

$$\theta_I = (1/p!) \sum_J \delta_J^I \Omega_{j_1 j_2} \wedge \dots \wedge \Omega_{j_{p-1} j_p},$$

where $J = \{j_1, \dots, j_p\}$ for j_1, \dots, j_p integers between 1 and $2n$, with $\delta_J^I = 1$ (or -1) if the integers i_1, \dots, i_p are distinct and J is an even (or odd) permutation of I , with $\delta_J^I = 0$ otherwise, and with the sum \sum_J taken over all selections J from $\{1, \dots, 2n\}$. Then θ_I is a p -form on $F(M)$ for all I . Also $D\theta_I = 0$ for all I since $D\Omega = 0$ where D is covariant differentiation. Specifically, $D\alpha = d\alpha \circ h$ for an arbitrary form α where d is exterior differentiation and h takes the horizontal part.

The 1-forms ω_i are defined on $F(M)$ for $i = 1, \dots, 2n$ and a tangent vector v at $z = (m; e_1, \dots, e_{2n}) \in F(M)$ by $\omega_i(z)(v) = \langle \Pi_*(v), e_i \rangle$ where $\langle \cdot, \cdot \rangle$ denotes inner product. Note that $D\omega_i = 0$ for all i .

For tangent vectors v_1, \dots, v_p to $F(M)$ at $z \in F(M)$, it follows from the definition of the p th curvature operator that for $I = \{i_1, \dots, i_p\}$ we have

$$\begin{aligned} \theta_I(v_1, \dots, v_p) &= \langle R_p(e_{i_1} \wedge \dots \wedge e_{i_p}), \Pi_*(v_1) \wedge \dots \wedge \Pi_*(v_p) \rangle \\ &= \left\{ \sum_{1 \leq j_1 < \dots < j_p \leq 2n} \langle R_p(e_{i_1} \wedge \dots \wedge e_{i_p}), e_{j_1} \wedge \dots \wedge e_{j_p} \rangle \right. \\ &\quad \left. \times \omega_{j_1} \wedge \dots \wedge \omega_{j_p} \right\} (v_1, \dots, v_p). \end{aligned}$$

Hence, at $z = (m; e_1, \dots, e_{2n}) \in F(M)$ and for $I = \{i_1, \dots, i_p\}$,

$$(9) \quad \theta_I = \sum_{1 \leq j_1 < \dots < j_p \leq 2n} \langle R_p(e_{i_1} \wedge \dots \wedge e_{i_p}), e_{j_1} \wedge \dots \wedge e_{j_p} \rangle \omega_{j_1} \wedge \dots \wedge \omega_{j_p}.$$

Now define the subbundle $U(M)$ of $F(M)$ consisting of the unitary frames on M , i.e., $z = (m; Je_1, \dots, Je_n, e_1, \dots, e_n)$ is an element of $U(M)$ if $\{Je_1, \dots, Je_n, e_1, \dots, e_n\}$ is an orthonormal basis of M_m . Then $U(M)$ has structure group $U(n)$ of unitary matrices on \mathbf{C}^n considered as the subgroup of $O(2n)$ consisting of the matrices commuting with the matrix

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. Note that this representation of $U(n)$ into $O(2n)$, called the real representation of $U(n)$, is given by

$$A + iB \rightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

for $A + iB \in U(n)$, where both A and B are real $n \times n$ matrices.

From §§3 and 4, Chapter IX in [14], it follows that the Riemannian connection of M is a connection in $U(M)$, i.e., the connection form and curvature form Ω of the Riemannian connection of M in $F(M)$, when restricted to $U(M) \subset F(M)$, take values in the Lie algebra $u(n)$ considered as its real representation in $o(2n)$. We may

restrict θ_J , ω_i and all formulas above involving them to $U(M)$ without any alteration. In the following portion of the proof, we only work in $U(M)$ and treat all forms used previously as defined on $U(M)$.

Let $\mathcal{J} = \{J \mid J = \{j_1, \dots, j_p\} \text{ with } 1 \leq j_1 < \dots < j_p \leq 2n\}$. Given $z = (m; e_{1^*}, \dots, e_{n^*}, e_1, \dots, e_n) \in U(M)$, where $e_{i^*} = Je_i$ for $i = 1, \dots, n$, and $J \in \mathcal{J}$, we define $E_J(z) = f_{j_1} \wedge \dots \wedge f_{j_p}$ where

$$f_{j_k} = \begin{cases} e_{j_k^*} & \text{if } j_k \leq n, \\ e_{j_k-n} & \text{if } j_k > n. \end{cases}$$

For all $I, J \in \mathcal{J}$ we define C^∞ functions $R_{I,J}$ on $U(M)$ by $R_{I,J}(z) = \langle R_p(E_I), E_J \rangle$. $J \in \mathcal{J}$ is said to be holomorphic if $E_J(z)$ is a holomorphic p -plane for all $z \in U(M)$. Now let $\omega_J = \omega_{j_1} \wedge \dots \wedge \omega_{j_p}$ for all $J \in \mathcal{J}$. Then on $U(M)$ for $I \in \mathcal{J}$, formula (9) becomes

$$(10) \quad \theta_I = \sum_{J \in \mathcal{J}} R_{I,J} \omega_J.$$

Let H be the C^∞ function on $U(M)$ obtained by taking the C^∞ function H' on M representing the pointwise constant p th holomorphic sectional curvature and composing it with the projection $\Pi: U(M) \rightarrow M$. For $p = 2k$, let $I_0 = \{1, 2, \dots, k, n + 1, n + 2, \dots, n + k\} \in \mathcal{J}$. Then formulas (8) and (10) give for all $z \in U(M)$

$$(11) \quad \begin{aligned} \theta_{I_0}(z) &= H(z) \omega_{I_0}(z) + \frac{1}{2} H(z) \sum_{J \in A} \omega_J(z) \\ &\quad + 0 \cdot \sum_{J \in B} \omega_J(z) + \sum_{J \in C} R_{I_0,J}(z) \omega_J(z), \end{aligned}$$

where

$$A = \{J \in \mathcal{J} \mid J \neq I_0, J \text{ is holomorphic, and } J \cap I_0 \text{ is holomorphic of order } p - 2\},$$

$$B = \{J \in \mathcal{J} \mid J \text{ is not holomorphic, } J \cap I_0 \text{ contains a holomorphic subset of order } p - 2\},$$

$$C = \{J \in \mathcal{J} \mid J \cap I_0 \text{ does not contain a holomorphic subset of order } p - 2\}.$$

Note that $\mathcal{J} = A \cup B \cup C \cup \{I_0\}$. Thus $D\omega_J = 0$ for all $J \in \mathcal{J}$, and (11) gives

$$(12) \quad 0 = D\theta_{I_0} = DH \wedge \left[\omega_{I_0} + \frac{1}{2} \sum_{J \in A} \omega_J \right] + \sum_{J \in C} \left[(DR_{I_0,J}) \wedge \omega_J \right].$$

Since H is constant on the fibers of $U(M)$, dH is a horizontal 1-form and equals DH . Letting $dH = \sum_{i=1}^{2n} H_i \omega_i$ and $DR_{I_0,J} = \sum_{k=1}^{2n} R_{I_0,J,k} \omega_k$ for C^∞ functions $H_i, R_{I_0,J,k}$ on $U(M)$, (12) thus becomes

$$(13) \quad \begin{aligned} 0 &= \sum_{i \notin I_0} H_i \omega_i \wedge \omega_{I_0} + \frac{1}{2} \sum_{J \in A} \left(\sum_{i \notin J} H_i \omega_i \wedge \omega_J \right) \\ &\quad + \sum_{J \in C} \left(\sum_{k \notin J} R_{I_0,J,k} \omega_k \wedge \omega_J \right). \end{aligned}$$

It follows from the definitions of A and C that the intersection of $\{\omega_i \wedge \omega_j | i \notin I_0\}$ and $\{\omega_i \wedge \omega_j | i \notin J, J \in A \cup C\}$ is empty. Hence (13) implies that $H_i \equiv 0$ for each $i \notin I_0$ and $dH = \sum_{i \in I_0} H_i \omega_i$.

Let $I_r = \{1, 2, \dots, \bar{r}, \dots, k, k + 1, n + 1, \dots, n + r, \dots, n + k, n + k + 1\}$ for $r = 1, \dots, k$ where $\bar{}$ over a number indicates that the number is to be deleted. Then repeating the procedure used above for I_0 on each I_r shows that $H_i \equiv 0$ for each $i \notin I_r, r = 0, 1, \dots, k$. Thus $H_i \equiv 0$ for all i and $dH \equiv 0$. Hence from the definition of $H, dH' \equiv 0$, and H' is a constant function on M since M is connected.

5. Constant zero holomorphic sectional curvature. In this section we will examine the consequences of constant zero p th holomorphic sectional curvature on a Kähler manifold M . We need the following lemmas.

LEMMA 5.1. *Let V be a real inner product space of dimension $2n$ with a complex structure J and Hermitian inner product. Let T be a $(2n - 2)nd$ order curvature operator on V satisfying the Bianchi identity, $T = T \circ J = J \circ T$, and having constant holomorphic sectional curvature H . Let $\{e_1, e_{1^*}, \dots, e_n, e_{n^*}\}$ be an orthonormal oriented basis of V , where $e_{i^*} = Je_i$ for all i , and*

$$B = \{e_{i_1} \wedge \dots \wedge e_{i_{2n-2}} | 1 \leq i_1 < \dots < i_{2n-2} \leq n^*, 1 < 1^* < \dots < n < n^*\}.$$

Then for $Q \in B$, formula (6) holds. In particular, if $H = 0$, then T is identically zero.

PROOF. Let $*2, *(2n - 2)$ be the respective Hodge star operators. Then applying Corollary 1.10 to $*(2n - 2)T*2$ and using Corollary 1.4 together with Lemma 1.7 we obtain the above result.

Next we restate the corollary in §2 of Thorpe [26] as follows.

LEMMA 5.2. *Let M be a Riemannian manifold of dimension n , and p, q be positive even integers with $p + q \leq n$. Let $\{e_1, \dots, e_{p+q}\}$ be an orthonormal basis for a $(p + q)$ -plane P at $m \in M$, and $B = \{e_{i_1} \wedge \dots \wedge e_{i_p} | 1 \leq i_1 < \dots < i_p \leq p + q\}$. Then*

$$(14) \quad R_{p+q}(P) = \frac{p!q!}{(p + q)!} \sum_{Q \in B} R_p(Q) \wedge R_q(Q^\perp),$$

where Q^\perp is the oriented orthogonal complement of Q in P , and R_p, R_q, R_{p+q} are the p th, q th, $(p + q)$ th curvature operators of M , respectively.

THEOREM 5.3. *Let M be a Kähler manifold of real dimension $2n$ for $n \geq 2$, and p be an even integer with $2 \leq p < 2n$. Let the p th holomorphic sectional curvature of M at $m_0 \in M$ be identically zero. Then the q th holomorphic sectional curvature of M at m_0 is identically zero for all even $q \geq p$. In particular, at $m_0 \in M$ the Lipschitz-Killing curvature $K(m_0) = 0$, and M is $(2n - 2)$ -flat at m_0 .*

PROOF. Throughout the proof we work only at $m_0 \in M$. Let R_r be the r th curvature operator of M for even integers r with $2 \leq r < 2n$.

In order to prove the theorem, it is sufficient to show that constant zero r th holomorphic sectional curvature of M implies constant zero $(r + 2)nd$ holomorphic sectional curvature of M for an even integer r with $2 \leq r < 2n$. Note that constant

zero $(2n - 2)$ nd holomorphic sectional curvature of M implies that M is $(2n - 2)$ -flat, by Lemma 5.1.

Assume constant zero r th holomorphic sectional curvature of M for an even integer r with $2 \leq r < 2n$. Let P be a holomorphic $(r + 2)$ -plane with an orthonormal basis $\{e_1, e_{1^*}, \dots, e_s, e_{s^*}\}$ where $e_{i^*} = Je_i$ for all i and $2s = r + 2$. Then $P = e_1 \wedge e_{1^*} \wedge \dots \wedge e_s \wedge e_{s^*}$. Extend this basis of P to an orthonormal basis $\{e_1, e_{1^*}, \dots, e_n, e_{n^*}\}$ of M_{m_0} , where $e_{i^*} = Je_i$ for all i . Let $B = \{e_{i_1} \wedge \dots \wedge e_{i_r} \mid 1 \leq i_1 < \dots < i_r \leq s^*\}$, $B' = \{e_{i_1} \wedge \dots \wedge e_{i_r} \mid 1 \leq i_1 < \dots < i_r \leq n^*\}$, and $B'' = \{e_{i_1} \wedge e_{i_2} \mid 1 \leq i_1 < i_2 \leq n^*\}$ where $1 < 1^* < \dots < n < n^*$. Then B, B', B'' are orthonormal bases of $\Lambda^r(P)$, $\Lambda^r(M_{m_0})$, and $\Lambda^2(M_{m_0})$, respectively.

Let $\Pi_p: \Lambda^r(M_{m_0}) \rightarrow \Lambda^r(P)$ be the orthogonal projection map. Note that $J \circ \Pi_p = \Pi_p \circ J$ since P is holomorphic, and $J(\Lambda^r(P)) = \Lambda^r(P)$. We can define an r th-order curvature operator on P by $R'_r = \Pi_p \circ (R_r|_{\Lambda^r(P)})$. Then R'_r satisfies the Bianchi identity, $R'_r = J \circ R'_r = R'_r \circ J$, and R'_r has constant zero holomorphic sectional curvature since R_r does. From Lemma 5.1 it follows that R'_r is identically zero. Hence for all $Q \in B$ we have that $\Pi_p(R_r(Q)) = 0$.

By Lemma 5.2 and formula (14),

$$(15) \quad R_{r+2}(P) = \frac{r!2!}{(r+2)!} \sum_{Q \in B} R_r(Q) \wedge R_2(Q^\perp).$$

For each $Q \in B$ we have

$$R_r(Q) = \sum_{Q' \in B'} \langle R_r(Q), Q' \rangle Q',$$

$$R_2(Q^\perp) = \sum_{Q'' \in B''} \langle R_2(Q^\perp), Q'' \rangle Q'',$$

so

$$R_r(Q) \wedge R_2(Q^\perp) = \sum_{Q' \in B', Q'' \in B''} [\langle R_r(Q), Q' \rangle \langle R_2(Q^\perp), Q'' \rangle (Q' \wedge Q'')].$$

Letting $B''' = B'' \cap \Lambda^2(P)$ and using the above with formula (15), we obtain

$$\langle R_{r+2}(P), P \rangle = \frac{r!2!}{(r+2)!} \sum_{\substack{Q, Q' \in B' \\ Q'' \in B'''}} \langle R_r(Q), Q' \rangle \langle R_2(Q^\perp), Q'' \rangle \langle Q' \wedge Q'', P \rangle.$$

But $\Pi_p(R_r(Q)) = 0$ for all $Q \in B$, i.e., $\langle R_r(Q), Q' \rangle = 0$ for all $Q, Q' \in B$. Hence $\langle R_{r+2}(P), P \rangle = 0$, so that constant zero r th holomorphic sectional curvature of M implies constant zero $(r + 2)$ nd holomorphic sectional curvature of M .

The following Theorem is a corollary of Theorem 5.3.

THEOREM 5.4. *Let M be a compact Kähler manifold of real dimension $2n$ for $n \geq 2$. For an even integer p with $2 \leq p < 2n$, let the p th holomorphic sectional curvature of M be identically zero. Then the Euler-Poincaré characteristic $\chi(M) = 0$ and the $(n - 1)$ st Chern class γ_{n-1} is zero.*

PROOF. By Theorem 5.3 we know that the Lipschitz-Killing curvature K of M is identically zero, and so the generalized Gauss-Bonnet theorem gives $\chi(M) = 0$.

Also by Theorem 5.3 we know that M is $(2n - 2)$ -flat, i.e., R_{2n-2} is identically zero. Hence from Theorem 1.11 and formula (7), it follows that γ_{n-1} is zero.

Our last theorem involves the vanishing of Chern classes for a compact Kähler manifold M .

THEOREM 5.5. *Let M be a compact Kähler manifold of real dimension $2n$ for $n \geq 2$. For an even integer p with $2 \leq p < 2n$, let $R_p(Q) = 0$ for all holomorphic p -planes Q , where R_p is the p th curvature operator of M . Then the Chern classes γ_q of M are all zero for all $q \geq p/2$.*

PROOF. By Theorem 1.11 and formula (7), it is sufficient to show that if $R_r(Q) = 0$ for all holomorphic r -planes Q with r an even integer such that $2 \leq r < 2n$, then $R_{r+2}(Q') = 0$ for all holomorphic $(r + 2)$ -planes Q' .

Let P be a holomorphic $(r + 2)$ -plane with orthonormal basis $\{e_1, e_{1^*}, \dots, e_s, e_{s^*}\}$ where $e_{i^*} = Je_i$ for all i and $2s = r + 2$. Then $P = e_1 \wedge e_{1^*} \wedge \dots \wedge e_s \wedge e_{s^*}$. Let $B = \{e_{i_1} \wedge \dots \wedge e_{i_r} \mid 1 \leq i_1 < \dots < i_r \leq s^* \text{ with } 1 < 1^* < \dots < s < s^*\}$. Then by Lemma 5.2, formula (15) holds.

Assume that $R_r(Q) = 0$ for all holomorphic r -planes Q , and let

$$Q' = X_1 \wedge JX_1 \wedge \dots \wedge X_r \wedge JX_r \wedge Y \wedge Z,$$

where $2t = r - 2$. Then

$$\begin{aligned} 0 &= R_r(X_1 \wedge JX_1 \wedge \dots \wedge X_r \wedge JX_r \wedge (Y - JZ) \wedge (JY + Z)) \\ &= R_r(X_1 \wedge JX_1 \wedge \dots \wedge X_r \wedge JX_r \wedge Y \wedge JY) \\ &\quad + R_r(X_1 \wedge JX_1 \wedge \dots \wedge X_r \wedge JX_r \wedge Z \wedge JZ) \\ &\quad + 2R_r(X_1 \wedge JX_1 \wedge \dots \wedge X_r \wedge JX_r \wedge Y \wedge Z) \\ &= 2R_r(Q'). \end{aligned}$$

Hence $R_r(Q') = 0$ for any r -plane Q' containing a holomorphic $(r - 2)$ -subplane. From this, it is clear that $R_r(Q) = 0$ for all $Q \in B$ since each of these contains a holomorphic $(r - 2)$ -subplane. Thus formula (15) implies that $R_{r+2}(P) = 0$.

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