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REPRESENTATIONS OF HYPERHARMONIC CONES

SIRKKA-LIISA ERIKSSON

ABSTRACT. Hyperharmonic cones are ordered convex cones possessing order properties similar to those of hyperharmonic functions on harmonic spaces. The dual of a hyperharmonic cone is defined to be the set of extended real-valued additive and left order-continuous mappings ($\not\equiv \infty$). The second dual gives a representation of certain hyperharmonic cones in which suprema of upward directed families are pointwise suprema, although infima of pairs of functions are not generally pointwise infima. We obtain necessary and sufficient conditions for the existence of a representation of a hyperharmonic cone in which suprema of upward directed families are pointwise suprema and infima of pairs of functions are pointwise infima.

0. Introduction. Hyperharmonic cones (see [**8**]) are ordered convex cones possessing order properties similar to those of positive hyperharmonic functions on harmonic spaces. Their theory continues the algebraic axiomatization of potential theory developed notably by G. Mokobodzki [**10**, **11**]; D. Sibony [**13**]; N. Boboc, Gh. Bucur, A. Cornea [**3**]; and M. G. Arsove, H. Leutwiler [**1**]. Since the cancellation law fails in nontrivial hyperharmonic cones ($\neq \{0\}$), it is not possible to extend them to vector lattices. However, many results for superharmonic structures have generalizations to hyperharmonic cones.

Any hyperharmonic cone can be represented as a cone of extended real-valued continuous functions on a locally compact Hausdorff space [**8**, Theorem 7.4]. However, in this representation, the supremum of an upward directed family of functions, or the infimum of two functions, is not necessarily a pointwise supremum, or infimum, respectively. Since, for positive hyperharmonic functions on harmonic spaces, the suprema of upward directed families and the infima of pairs of functions are always pointwise suprema and infima, respectively, it is natural to ask whether there exists a representation for hyperharmonic cones with these properties. In dealing with this question the additive, left order continuous mappings (called hyperharmonic morphisms) play an essential role. The family of extended real-valued hyperharmonic morphisms ($\not\equiv \infty$) on a hyperharmonic cone W is called the dual of W. Such duals are not always hyperharmonic cones. We show that under a suitable distributivity assumption on W the dual is a hyperharmonic cone having the same distributivity property. Then, imposing an additional separation property, we find that the dual gives a representation in which suprema of upward directed

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families are pointwise suprema, although infima of pairs of functions are not generally pointwise infima. In our main result (Theorem 4.4), we obtain necessary and sufficient conditions for the existence of a representation of W in which suprema of upward directed families are pointwise suprema and infima of pairs of functions are pointwise infima.

Some results in this work are generalizations of corresponding results of H-cones (see [**3**]), since H-cones can be extended to hyperharmonic cones. It should also be noted that H-integrals (i.e., additive, left order-continuous functions, finite on a dense set) on an H-cone are cancellable hyperharmonic morphisms on some hyperharmonic cone [**8**, Theorem 6.24] and [**9**, Proposition 2.2].

1. Hyperharmonic cones. Let $(W, +, \leq)$ be a partially ordered abelian semigroup with a neutral element 0 and having the properties

$$u \geq 0, \tag{1.1}$$

$$u \leq v \Rightarrow u + w \leq v + w \tag{1.2}$$

for all $u, v, w \in W$. Along with the given initial order ($\leq$) another partial order $\preceq$, called *specific order*, is defined by $u \preceq v$ if and only if $v = u + u'$ for some $u' \in W$. Lattice operations with respect to initial order are denoted by $\wedge, \vee$ and, with respect to specific order, by $\curlyvee, \curlywedge$.

A structure $(W, + \leq)$ satisfying (1.1) and (1.2) is called an *ordered convex cone* if it admits an operation of multiplication by strictly positive real numbers such that for all $\alpha, \beta \in \mathbf{R}_+ \backslash \{0\}$ and $x, y, \in W$,

$$\alpha(x + y) = \alpha x + \alpha y, \quad (\alpha + \beta)x = \alpha x + \beta y,$$
$$(\alpha\beta)x = \alpha(\beta x), \quad 1 \cdot x = x, \quad x \leq y \Rightarrow \alpha x \leq \alpha y.$$

DEFINITION 1.1. An ordered convex cone $(W, +, \leq)$ is called a *hyperharmonic cone* if the following axioms hold:

(H1) any nonempty upward directed family $F \subset W$ has a least upper bound $\bigvee F$ satisfying $\bigvee(x + F) = x + \bigvee F$ for all $x \in W$,

(H2) any nonempty family $F \subset W$ has a greatest lower bound $\bigwedge F$ satisfying $\bigwedge(x + F) = x + \bigwedge F$ for all $x \in F$,

(H3) for any $u, v_1, v_2 \in W$ such that $u \leq v_1 + v_2$ there exist $u_1, u_2 \in W$ satisfying $u = u_1 + u_2$, $u_1 \leq v_1, u_2 \leq v_2$.

As noted in Lemma 2.2 of [**8**], it follows from (H1), (H2) that any nonempty set has a least upper bound (which, however, need not be translation invariant).

Axioms (H2) and (H3) ensure that for all $u, v \in W$ the set $\{w \in W | u \leq v + w\}$, has a least element m such that $m \preceq u$ [**8**, Theorem 2.3]. Partially ordered abelian semigroups satisfying this property and (1.1), (1.2) are called hyperharmonic structures by M. G. Arsove and H. Leutwiler [**1**]. Many essential properties of the operator

$$S_v u = \min\{w \in W | u \leq v + w\}$$

are proved by them in [**1**, pp. 95–114]. For the properties of $S_v u$ in hyperharmonic cones we refer to [**8**, p. 34]. We mention only one:

$$v \preceq u \Rightarrow u = S_v u + v \tag{1.3}$$

[**8**, Lemma 3.2(k)].

An element $u \in W$ is called *cancellable* if $x + u \leq y + u$ implies that $x \leq y$ for all $x, y \in W$. Cancellable elements in hyperharmonic cones are the same as elements cancellable with respect to specific order [**8**, Theorem 3.9]. In terms of the definition

$$\underline{u} = \bigwedge_{n \in \mathbf{N}} \frac{u}{n}, \qquad u \in W,$$

there is a useful characterization of cancellability in hyperharmonic cones, namely:

$$u \text{ is cancellable } \Leftrightarrow \underline{u} = 0 \tag{1.4}$$

[**8**, Theorem 3.9]. The element u ($u \in W$) has the further interesting properties:

$$\underline{u} + u = u, \tag{1.5}$$

$$v \leq \underline{u} \Leftrightarrow v + u = u, \tag{1.6}$$

$$v \leq u \Rightarrow \underline{v} \leq \underline{u} \Rightarrow \underline{v} \preceq u \Rightarrow \underline{v} + u = u \tag{1.7}$$

[**8**, Lemma 3.8].

An important tool for handling uncancellable elements is

PROPOSITION 1.2. *If $(W, +, \leq)$ is a hyperharmonic cone and u any element of W, then $(\underline{u} + W, +, \leq)$ is also a hyperharmonic cone. Moreover, u is cancellable in $\underline{u} + W$* [**8**, *Proposition* 4.1].

The cancellability of u in $\underline{u} + W$ means that

$$u + x \leq u + y \Leftrightarrow \underline{u} + x \leq \underline{u} + y. \tag{1.8}$$

A main result concerning hyperharmonic cones and specific order is

THEOREM 1.3. *Let $(W, +, \leq)$ be a hyperharmonic cone. Then $(W, +, \preceq)$ is also a hyperharmonic cone, and the least specific upper bound of any family in W is translation invariant* [**8**, *Theorem* 4.16].

2. The dual of a hyperharmonic cone. Let $(V, +, \leq)$ and $(W, +, \leq)$ be hyperharmonic cones. A mapping $\varphi\colon W \to V$ is called *left order-continuous* if $\varphi(\bigvee F) = \bigvee_{f \in F} \varphi(f)$ for any nonempty upward directed family $F \subset W$.

Note that any left order-continuous mapping is also increasing. If a mapping $\varphi\colon W \to V$ is both additive and left order-continuous then it is positively homogeneous, i.e., $\varphi(\alpha u) = \alpha\varphi(u)$ for all $\alpha \in \mathbf{R}_+ \backslash \{0\}$ [**8**, p. 38].

An additive and left order-continuous mapping $\varphi\colon W \to V$ is called a *hyperharmonic morphism* [**8**, Definition 6.1]. In the sequel we consider only the case $V = \overline{\mathbf{R}}_+$. The set of hyperharmonic morphisms $\varphi\colon W \to \overline{\mathbf{R}}_+$ with the property $\varphi(0) = 0$ (equivalently $\varphi \not\equiv \infty$) is called the dual of W and denoted by W^*. Although the dual is not necessarily a hyperharmonic cone [**8**, Remark 6.17(b)], in the most important cases it is.

THEOREM 2.1. *Let $(W, +, \leq)$ be a hyperharmonic cone satisfying the following axiom:*

$$\bigvee_{f \in F} (s \wedge f) = s \wedge \left(\bigvee F\right) \tag{H4}$$

for any nonempty upward directed set $F \subset W$ and all $s \in W$. Then, under the usual pointwise operations and partial order, W^* is a hyperharmonic cone [**8**, *Corollary* 6.16].

We show, in fact, that axiom (H4) holds in W^*. This will enable us to develop further results in the theory of duals of hyperharmonic cones. We first present some preliminary results.

PROPOSITION 2.2. *Let W be a hyperharmonic cone and $\mu_1, \mu_2 \colon W \to \overline{\mathbf{R}}_+$ hyperharmonic morphisms. The mapping $\lambda \colon W \to \overline{\mathbf{R}}_+$ given by*

$$\lambda(s) = \inf\{\mu_1(s_1) + \mu_2(s_2) | s_1 + s_2 = s\}, \qquad s \in W, \tag{2.1}$$

is additive and increasing.

PROOF. Using the decomposition (H3), one easily sees that λ is increasing. Subadditivity of λ (i.e., $\lambda(s+t) \leq \lambda(s) + \lambda(t)$) is obvious. The proof of superadditivity (i.e., $\lambda(s+t) \geq \lambda(s) + \lambda(t)$) is lengthy, and the method is the same as in [**8**, Theorem 6.10].

Applying ideas similar to those used in the theory of H-cones, [**7**], we can even show that the mapping λ is left order-continuous. The proof is based on the following three results.

PROPOSITION 2.3. *Suppose that W is a hyperharmonic cone. Let $\mu_1, \mu_2 \colon W \to \overline{\mathbf{R}}_+$ be hyperharmonic morphisms, and let $\lambda \colon W \to \overline{\mathbf{R}}_+$ be given by* (2.1). *If $\lambda(u) < \infty$ for some $u \in W$, then there exist $s_1, s_2 \in W$ with $s_1 + s_2 = u$ satisfying the conditions $\lambda(s_1) = \mu_1(s_1)$, $\lambda(s_2) = \mu_2(s_2)$, and hence $\lambda(u) = \mu_1(s_1) + \mu_2(s_2)$.*

PROOF. We merely sketch the proof since it is basically the same as [**7**, Proposition 1.2]. Assume that $u \in W$ and $\lambda(u) < \infty$, and choose a sequence $(\varepsilon_n)_{n \in \mathbf{N}}$ in $\mathbf{R}_+ \backslash \{0\}$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. From (2.1) it follows that there exist $s_{1n}, s_{2n} \in W$ with $s_{1n} + s_{2n} = u$ such that $\mu_1(s_{1n}) + \mu_2(s_{2n}) \leq \lambda(u) + \varepsilon_n$.

The desired s_1, s_2 can now be taken as

$$s_1 = \bigwedge_{n \geq 1} \left(\bigvee_{k \geq n} s_{1k} \right), \quad s_2 = \bigvee_{n \geq 1} \left(\bigwedge_{k \geq n} s_{2k} \right).$$

Note that these always exist in a hyperharmonic cone W and that $u = s_1 + s_2$ by virtue of Theorem 1.3.

LEMMA 2.4. *Suppose that W is a hyperharmonic cone. Let μ be a hyperharmonic morphism from W into $\overline{\mathbf{R}}_+$ and φ an additive and increasing mapping from W into $\overline{\mathbf{R}}_+$ such that $\varphi \leq \mu$. Then the set*

$$C = \{u \in W | \varphi(u) = \mu(u) < \infty\}$$

has the following properties:

(a) *C is a specifically solid convex subcone of W (i.e., a convex subcone of W such that if $u \in C$ and $t \preceq u$ then $t \in C$);*

(b) *if F is an upward directed subfamily of C, then $\varphi(\bigvee F) = \mu(\bigvee F)$;*

(c) *if $s, t \in C$, then $s \vee t \in C$.*

PROOF. Property (c) follows from $s \vee t \preceq s + t$ (see Corollary 4.9 in [**8**]), and the other properties are obtained by easy calculations.

LEMMA 2.5. *Let W be a hyperharmonic cone and C any specifically solid convex subcone of W. The mapping $B\colon W \to W$ defined by*

$$Bu = \bigvee\{s \in C | s \leq u\}, \qquad u \in W,$$

is additive, increasing, contractive (i.e., $Bu \leq u$) and idempotent (i.e., $B(Bu) = Bu$).

PROOF. Let u and v be arbitrary elements of W. If $s \in C$ and $s \leq u + v$ then, by (H3), there exist $s_1, s_2 \in W$ satisfying the conditions $s = s_1 + s_2, s_1 \leq u, s_2 \leq v$. Since C is specifically solid, the elements s_1 and s_2 belong to C. Hence we have $s = s_1 + s_2 \leq Bu + Bv$, so that $B(u + v) \leq Bu + Bv$. Since the reverse inequality is obvious, the mapping B is additive. The other assertions are trivial.

We can now establish the left order-continuity of λ.

THEOREM 2.6. *Let W be a hyperharmonic cone and $\mu_1, \mu_2\colon W \to \overline{\mathbf{R}}_+$ hyperharmonic morphisms. Then the mapping $\lambda\colon W \to \overline{\mathbf{R}}_+$ given by (2.1) is left order-continuous, so that $\lambda = \mu_1 \wedge \mu_2$.*

PROOF. We shall use ideas similar to those used in the theory of H-cones (see Theorem 1.4 in [7]). Let us put

$$C_i = \{u \in W | \lambda(u) = \mu_i(u) < \infty\} \qquad (i = 1, 2).$$

Lemma 2.4 ensures that C_i is a specifically solid convex subcone of W. Hence, by Lemma 2.5, the mapping $B_i\colon W \to W$ $(i = 1, 2)$ defined by

$$B_i u = \bigvee\{s \in C_i | s \leq u\}, \qquad u \in W,$$

is additive, increasing, contractive and idempotent.

Suppose now that u is an arbitrary element of W and $F \subset W$ is an upward directed family with $u = \bigvee F$. Assume that $\sup_{t \in F} \lambda(t)$ is finite. Otherwise, the left order-continuity of λ at u is trivial. For any $t \in F$, we put $v_t = \underline{t} + S_t(B_1 t + B_2 t)$, and in view of Proposition 2.3 there exist elements $t_i \in C_i$ $(i = 1, 2)$ such that $t = t_1 + t_2$. We show that the family $(v_t)_{t \in F}$ is directed upwards. Since $B_i t_i = t_i$ and B_i $(i = 1, 2)$ is additive, we have

$$B_1 t + B_2 t = t_1 + t_2 + B_1 t_2 + B_2 t_1 = t + B_1 t_2 + B_2 t_1,$$

whence $t \preceq B_1 t + B_2 t$. By (1.3) this implies

$$t + S_t(B_1 t + B_2 t) = B_1 t + B_2 t = t + B_1 t_2 + B_2 t_1,$$

and an application of (1.8) yields

$$v_t = \underline{t} + S_t(B_1 t + B_2 t) = \underline{t} + B_1 t_2 + B_2 t_1. \tag{2.2}$$

Taking $r, t \in F$ such that $r \leq t$ $(= t_1 + t_2)$, we have elements $r', r'' \in W$ such that $r = r' + r''$ and $r' \leq t_1, r'' \leq t_2$. Since the mapping B_i $(i = 1, 2)$ is contractive and increasing, there results

$$B_1 r' + B_2 r' \leq r' + B_2 t_1, \qquad B_1 r'' + B_2 r'' \leq B_1 t_2 + r''.$$

Adding these inequalities yields

$$\begin{aligned} r + v_r = B_1 r + B_2 r &\leq r + B_1 t_2 + B_2 t_1 \\ &\leq r + \underline{t} + B_1 t_2 + B_2 t_1 = r + v_t. \end{aligned}$$

Since according to (1.7) $\underline{r} + t = t$, we have $\underline{r} + \underline{t} = \underline{t}$ by (1.8). Hence we get

$$v_r = \underline{r} + v_r \leq \underline{r} + v_t = v_t,$$

which shows that the family $(v_t)_{t\in F}$ is directed upwards.

Next, let $u_i = \bigvee_{t\in F} B_i t$ $(i = 1, 2)$ and $v = \bigvee_{t\in F} v_t$. Lemma 2.4, (b) and (c), leads to $\mu_i(B_i t) = \lambda(B_i t)$ and $\mu_i(u_i) = \lambda(u_i)$ for $i = 1, 2$. From the equality (2.2) we infer that

$$t + v_t = B_1 t + B_2 t \tag{2.3}$$

for all $t \in F$. Hence we have

$$\lambda(t) + \lambda(v_t) = \lambda(B_1 t) + \lambda(B_2 t) = \mu_1(B_1 t) + \mu_2(B_2 t) \tag{2.4}$$

for all $t \in F$. Taking suprema over $t \in F$ in (2.3) and (2.4), we obtain $u+v = u_1+u_2$ and

$$\begin{aligned}\sup_{t\in F} \lambda(t) + \sup_{t\in F} \lambda(v_t) &= \mu_1(u_1) + \mu_2(u_2) = \lambda(u_1) + \lambda(u_2)\\ &= \lambda(u) + \lambda(v) \geq \sup_{t\in F} \lambda(t) + \lambda(v).\end{aligned}$$

Since $\sup_{t\in F} \lambda(t)$ was assumed finite, there results $\sup_{t\in F} \lambda(v_t) \geq \lambda(v)$, and the theorem follows.

A key result concerning the dual is the following

THEOREM 2.7. *Let W be a hyperharmonic cone. Then axioms* (H1) *and* (H4) *hold in W^*, and every nonempty subset of W^* has a greatest lower bound. Hence, if W satisfies* (H4), *then W^* is a hyperharmonic cone satisfying* (H4).

PROOF. The final assertion follows from the preceding one simply by observing that the assumption of (H4) in W forces W^* to be a hyperharmonic cone. This fact was established in Theorem 6.15 of [**8**], and the methods used in proving Theorem 6.15 show that for any hyperharmonic cone W axiom (H1) holds in W^* and that every nonempty subset of W^* has a greatest lower bound. There remains only to show that (H4) holds in W^*. Let $(\mu_\alpha)_{\alpha\in I}$ be an upward directed family in W^* and φ an arbitrary element of W^*. Put $\mu = \bigvee_{\alpha\in I} \mu_\alpha$ and $\lambda_\alpha = \varphi \wedge \mu_\alpha$ for $\alpha \in I$. Obviously, we have $\varphi \wedge \mu \geq \bigvee_{\alpha\in I} \lambda_\alpha$. To obtain the reverse inequality, we only have to consider the case $\sup_{\alpha\in I} \lambda_\alpha(s) < \infty$, for a given $s \in W$. Then Proposition 2.3 ensures that for any $\alpha \in I$ there exist elements s_α and t_α such that $s = s_\alpha + t_\alpha$ and $\lambda_\alpha(s_\alpha) = \varphi(s_\alpha)$, $\lambda_\alpha(t_\alpha) = \mu_\alpha(t_\alpha)$. Let $\alpha \in I$ be fixed. From $\lambda_\alpha \geq \lambda_\beta$ for any $\beta \leq \alpha$, we obtain

$$\varphi(s_\beta) = \lambda_\beta(s_\beta) \leq \lambda_\alpha(s_\beta) \leq \varphi(s_\beta),$$

whence $\varphi(s_\beta) = \lambda_\alpha(s_\beta)$ for any $\beta \leq \alpha$. By virtue of Lemma 2.4(a), this yields

$$\varphi\left(\mathop{\curlyvee}_{\beta\in J} s_\beta\right) = \lambda_\alpha\left(\mathop{\curlyvee}_{\beta\in J} s_\beta\right)$$

for any finite subset J of $\{\beta \in I | \beta \leq \alpha\}$. Setting $u_\alpha = \curlyvee_{\beta\leq\alpha} s_\beta$ and $v_\alpha = \curlywedge_{\beta\leq\alpha} t_\beta$, we have $\varphi(u_\alpha) = \lambda_\alpha(u_\alpha)$ and $\mu_\alpha(v_\alpha) = \lambda_\alpha(v_\alpha)$ in view of Lemma 2.4. Moreover, $s = u_\alpha + v_\alpha$ by Theorem 1.3.

Now let $u = \curlyvee_{\alpha\in I} u_\alpha$ and $v = \curlywedge_{\alpha\in I} v_\alpha$. Then we have

$$\varphi(u) \geq \sup_{\alpha\in I} \lambda_\alpha(u) \geq \sup_{\alpha\in I} \lambda_\alpha(u_\alpha) = \sup_{\alpha\in I} \varphi(u_\alpha) = \varphi(u).$$

Since $v \preceq v_\alpha$, Lemma 2.4(a) results in $\mu_\alpha(v) = \lambda_\alpha(v)$ for all $\alpha \in I$ and thus $\mu(v) = \sup_{\alpha\in I} \lambda_\alpha(v)$. Finally, using the equality $s = u + v$, we conclude that

$$\begin{aligned}(\varphi \wedge \mu)(s) &\geq \sup_{\alpha\in I} \lambda_\alpha(s) = \sup_{\alpha\in I} \lambda_\alpha(u) + \sup_{\alpha\in I} \lambda_\alpha(v)\\ &= \varphi(u) + \mu(v) \geq (\varphi \wedge \mu)(s).\end{aligned}$$

COROLLARY 2.8. *If W is a hyperharmonic cone satisfying* (H4), *then W^{**} is also a hyperharmonic cone satisfying* (H4).

REMARK. Any H-cone can be extended to a hyperharmonic cone satisfying (H4) [**9**, Proposition 2.2]. Hence the results in this section are generalizations of the corresponding results in the theory of H-cones.

3. Imbedding a hyperharmonic cone in its second dual. Any hyperharmonic cone can be represented as a cone of extended real-valued continuous functions on a locally compact Hausdorff space [**8**, Theorem 7.4]. In this representation a least upper bound for an upward directed family is not necessarily a pointwise supremum. We show, however, that under certain assumptions on the cone, the second dual gives a representation in which these two suprema are the same.

Throughout this section we assume that W is a hyperharmonic cone satisfying (H4).

We need the following result [**8**, Theorem 6.20]:

THEOREM 3.1. *Let W be a hyperharmonic cone satisfying* (H4) *and D a solid convex subcone of W. If $\varphi\colon D \to \overline{\mathbf{R}}_+$ is additive and left order-continuous, then it can be extended to a hyperharmonic morphism $\varphi^*\colon W \to \overline{\mathbf{R}}_+$ by setting*

$$\varphi^*(u) = \sup\{\varphi(t) | t \in D,\ t \leq u\}, \qquad u \in W.$$

We shall examine the evaluation map $s \to \tilde{s}$ from W into W^{**} defined by $\tilde{s}(\mu) = \mu(s)$, $\mu \in W^*$, $s \in W$. Note that $\tilde{s}\colon W^* \to \overline{\mathbf{R}}_+$ is indeed a hyperharmonic morphism, since for any nonempty upward directed family $F \subset W^*$

$$\tilde{s}\left(\bigvee F\right) = \left(\bigvee F\right)(s) = \sup_{t\in F} \mu(s) = \sup_{t\in F} \tilde{s}(\mu)$$

in view of Theorem 6.15 and Corollary 6.16 in [**8**].

Obviously, the mapping $s \to \tilde{s}$ is additive and increasing. Moreover, for any $\mu \in W^*$ and any nonempty upward directed family $F \subset W$ we have

$$\widetilde{\bigvee F}(\mu) = \mu\left(\bigvee F\right) = \sup_{t\in F} \mu(f) = \sup_{t\in F} \tilde{f}(\mu) = \left(\bigvee_{f\in F} \tilde{f}\right)(\mu).$$

Hence, least upper bounds for upward directed families F in W remain the same when we identify s with $\tilde{s}$ for all $s \in W$. Furthermore, $\bigvee \tilde{F}$ is a pointwise supremum for any upward directed family $F \subset W$.

In order to show that $\tilde{s} \wedge \tilde{t} = \widetilde{s \wedge t}$ for all $s, t \in W$, we need the following Proposition 3.2, which extends Proposition 2.3.4 of N. Boboc, Gh. Bucur, A. Cornea in [**3**] and requires new methods to deal with the case of infinite valued μ

and uncancellable elements in W. It will be convenient here, and in what follows, to use the notations W_μ and $\overline{W}_\mu$ for the solid convex subcones of W defined by (3.1)

$$W_\mu = \{s \in W | \mu(s) < \infty\}, \quad \overline{W}_\mu = \left\{ s \in W | s = \bigvee_{r \in W_\mu} (s \wedge r) \right\} \qquad (\mu \in W^*).$$

PROPOSITION 3.2. *Let $\mu\colon W \to \overline{\mathbf{R}}_+$ be a hyperharmonic morphism and u, v elements of W. If $\mu(u \wedge v) < \infty$, then there exist hyperharmonic morphisms $\mu_1, \mu_2\colon W \to \overline{\mathbf{R}}_+$ such that $\mu = \mu_1 + \mu_2$ and $\mu(u \wedge v) = \mu_1(u) + \mu_2(v)$.*

PROOF. Let $\mu\colon W \to \overline{\mathbf{R}}_+$ be a hyperharmonic morphism and u, v elements of W with $\mu(u \wedge v) < \infty$. Define the mapping $\mu_1 : W \to \overline{\mathbf{R}}_+$ by

$$\mu_1(s) = \sup_{n \in \mathbf{N}} (\mu((s + n(u \wedge v)) \wedge nv) - n\mu(u \wedge v)) \qquad (s \in W).$$

We show that μ_1 is a hyperharmonic morphism. Let s and t be arbitrary elements of W. The inequality

$$(s + n(u \wedge v)) \wedge nv + (t + m(u \wedge v)) \wedge mv \leq (s + t + (n + m)(u \wedge v)) \wedge (n + m)v$$

for all $n, m \in \mathbf{N}$ results in $\mu_1(s + t) \geq \mu_1(s) + \mu_1(t)$. To establish the reverse inequality, note first that for any $x, y \in W$ such that $x \leq y$ we have

$$\begin{aligned} x + (s + t + x) \wedge y &= (s + t + 2x) \wedge (x + y) \\ &\leq ((s + t + 2x) \wedge (s + x + y)) \wedge ((t + x + y) \wedge 2y) \\ &= (s + x + (t + x) \wedge y) \wedge (y + (t + x) \wedge y) \\ &= (s + x) \wedge y + (t + x) \wedge y. \end{aligned}$$

Then setting $x = n(u \wedge v)$ and $y = nv$, we obtain $\mu_1(s + t) \leq \mu_1(s) + \mu_1(t)$. Hence μ_1 is additive. Using the left order-continuity of μ and property (H4) of W, we easily see that μ_1 is also left order-continuous. Consequently, μ_1 is a hyperharmonic morphism.

Let W_{μ_1} be as in (3.1) and define the mapping $\mu_0\colon W_{\mu_1} \to \overline{\mathbf{R}}_+$ by $\mu_0(s) = \mu(s) - \mu_1(s)$ $(s \in W_{\mu_1})$. Assuming that $s, t \in W_{\mu_1}$ and $s \leq t$, we conclude from

$$\begin{aligned} s + (t + n(u \wedge v)) \wedge nv &= (s + t + n(u \wedge v)) \wedge (s + nv) \\ &\leq (s + t + n(u \wedge v)) \wedge (t + nv) \\ &= t + (s + n(u \wedge v)) \wedge nv \end{aligned}$$

that $\mu(s) + \mu_1(t) \leq \mu(t) + \mu_1(s)$. This ensures that μ_0 is increasing. Since the equality $\mu = \mu_1 + \mu_0$ holds in W_{μ_1}, we have $\sup \mu_1(f) + \sup \mu_0(f) = \sup \mu(f) = \mu(\bigvee F) = \mu_1(\bigvee F) + \mu_0(\bigvee F)$ for any upward directed family $F \subset W_{\mu_1}$ with $\bigvee F \in W_{\mu_1}$. This yields $\sup_{f \in F} \mu_0(f) = \mu_0(\bigvee F)$, because μ_1 is left order-continuous and $\bigvee F$ belongs to W_{μ_1}. By Theorem 3.1, the mapping μ_0 can be extended to the hyperharmonic morphism $\mu_2\colon W \to \overline{\mathbf{R}}_+$ given by

$$\mu_2(s) = \sup\{\mu_0(t) | t \leq s, t \in W_{\mu_1}\}.$$

Assuming that $t \in W_{\mu_1}$, we have $\mu_2(t) + \mu_1(t) = \mu(t)$. Hence the equality $\mu_2 + \mu_1 = \mu$ holds in W_{μ_1}. Since μ, μ_1, and μ_2 are left order-continuous, it also holds in $\overline{W}_{\mu_1}$. Assuming $\mu_1(s) = \infty$, we see from the inequality

$$\mu((s + n(u \wedge v)) \wedge nv) \leq \mu(s) + n\mu(u \wedge v), \qquad n \in \mathbf{N},$$

that $\infty = \mu_1(s) \leq \mu(s)$ and so $\mu_1(s) = \mu(s) = \infty$. Thus $\mu_2 + \mu_1 = \mu$ holds everywhere.

Finally, we have to show that $\mu_1(u) = \mu_1(u \wedge v)$ and $\mu_2(v) = \mu_2(u \wedge v)$. The first relation follows from

$$(u + n(u \wedge v)) \wedge nv = ((n+1)u) \wedge nv = (n+1)(u \wedge v) \wedge nv.$$

To verify the second, let $t \in W_{\mu_1}$ with $t \leq v$. Then

$$\begin{aligned}\mu_2(t) &= \mu(t) - \sup_{n \in \mathbf{N}} (\mu((t + n(u \wedge v)) \wedge nv) - n\mu(u \wedge v)) \\ &\leq \mu(t) - \sup_{n \in \mathbf{N}} (\mu(t) + \mu(n(u \wedge v) \wedge (n-1)v) - n\mu(u \wedge v)) = \mu_2(u \wedge v).\end{aligned}$$

Since $\mu_1(u \wedge v) \leq \mu(u \wedge v) < \infty$, this yields

$$\mu_2(v) = \sup\{\mu_2(t) | t \leq v,\ t \in W_{\mu_1}\} = \mu_2(u \wedge v).$$

Consequently,

$$\mu(u \wedge v) = \mu_1(u \wedge v) + \mu_2(u \wedge v) = \mu_1(u) + \mu_2(v).$$

THEOREM 3.3. *Let W be a hyperharmonic cone satisfying* (H4). *Then $\tilde{u} \wedge \tilde{v} = \widetilde{u \wedge v}$ for all $u, v \in W$.*

PROOF. The following proof, based on Proposition 3.2, is the same as that given for H-cones by N. Boboc, Gh. Bucur, and A. Cornea (see [**3**, Theorem 2.3.7(d)]). Obviously, $\tilde{u} \wedge \tilde{v} \geq \widetilde{u \wedge v}$. If $(\widetilde{u \wedge v})(\mu) = \infty$ for $\mu \in W^*$, evidently the equality $(\tilde{u} \wedge \tilde{v})(\mu) = (\widetilde{u \wedge v})(\mu)$ holds. If $(\widetilde{u \wedge v})(\mu) < \infty$ for $\mu \in W^*$, then the preceding proposition ensures that there exist $\mu_1, \mu_2 \in W^*$ such that $\mu = \mu_1 + \mu_2$ and $\mu(u \wedge v) = \mu_1(u) + \mu_2(v)$. Hence we have

$$\begin{aligned}\mu(u \wedge v) &= \mu_1(u) + \mu_2(v) = \tilde{u}(\mu_1) + \tilde{v}(\mu_2) \\ &\geq (\tilde{u} \wedge \tilde{v})(\mu_1) + (\tilde{u} \wedge \tilde{v})(\mu_2) = (\tilde{u} \wedge \tilde{v})(\mu).\end{aligned}$$

Consequently, $\tilde{u} \wedge \tilde{v} = \widetilde{u \wedge v}$, completing the proof.

Next we consider whether $S_{\tilde{v}}\tilde{u}$ is equal to $\widetilde{S_v u}$ for all $u, v \in W$. We recall that by [**8**, Proposition 6.13]

$$(S_{\tilde{v}}\tilde{u})(\mu) = \sup\{\lambda(u) - \lambda(v) | \lambda \in W^*, \lambda \leq \mu, \lambda(v) < \infty\} \qquad (\mu \in W^*). \tag{3.2}$$

It is easy to see that $S_v u = S_{u \wedge v}$ for all $u, v \in W$. Hence we obtain $S_{\tilde{v}}\tilde{u} = S_{\widetilde{u \wedge v}}\tilde{u} = S_{\widetilde{u \wedge v}}\tilde{u}$ for all $u, v \in W$. This allows us to restrict our attention to the case where $v \leq u$.

Suppose that u and v are two distinct elements of W such that $v \leq u$ and $\bigvee(u \wedge nv) = u$. Then every hyperharmonic morphism $\varphi\colon W \to \overline{\mathbf{R}}_+$ with $\varphi(v) = 0$ also satisfies $\varphi(u) = 0$. Assume that there does not exist a hyperharmonic morphism $\lambda\colon W \to \overline{\mathbf{R}}_+$ satisfying $0 < \lambda(v) < \lambda(u)$. Then by (3.2) we see that $(S_{\tilde{v}}\tilde{u})(\mu) = 0$ for all $\mu \in W^*$. If we choose a mapping μ such that $\mu = \infty$ on $W \backslash \{0\}$ and $\mu(0) = 0$, we have $\mu(S_v u) = \infty \neq (S_{\tilde{v}}\tilde{u})(\mu)$. Hence without some separation property $S_{\tilde{v}}\tilde{u}$ is not always equal to $\widetilde{S_v u}$.

The following separation property is sufficient:

(S) for any $u, v \in W$ such that $u \neq \underline{u}, u < v$ there exists a hyperharmonic morphism $\mu\colon W \to \overline{\mathbf{R}}_+$ satisfying $0 < \mu(u) < \mu(v)$.

Note that $u < v$ means $u \leq v$ and $u \neq v$.

THEOREM 3.4. *Let W be a hyperharmonic cone satisfying* (H4) *and* (S). *Then, for any $u, v \in W$, $S_{\tilde{v}}\tilde{u} = \widetilde{S_v u}$.*

PROOF. The case $v = \underline{v}$ is simple. For any $\mu \in W^*$, take $\lambda(u) = \mu(S_{\underline{v}}u)$. Then [**8**, Lemma 6.3], assures us that λ is a hyperharmonic morphism. Since $\lambda \leq \mu$ we have

$$\mu(S_{\underline{v}}u) = \lambda(u) - \lambda(\underline{v}) \leq (S_{\widetilde{(v)}}\tilde{u})(\mu) \leq (\widetilde{S_{\underline{v}}u})(\mu) = \mu(S_{\underline{v}}u).$$

Assume now that $v \leq u$ and $\underline{v} \neq v \neq u$. Let $\mu\colon W \to \overline{\mathbf{R}}_+$ be an arbitrary hyperharmonic morphism and let $W_\mu, \overline{W}_\mu$ be defined as before in (3.1). Suppose that $u \in \overline{W}_\mu$. By (S), there exists a hyperharmonic morphism $\lambda\colon W \to \overline{\mathbf{R}}_+$ satisfying $\lambda(u) > \lambda(v) > 0$. Define the mapping $\lambda_n\colon W \to \overline{\mathbf{R}}_+$ $(n \in \mathbf{N})$ by

$$\lambda_n(s) = \begin{cases} n\lambda(s), & \text{if } s \notin \overline{W}_\mu, \\ 0, & \text{if } s \in \overline{W}_\mu. \end{cases}$$

Since $\overline{W}_\mu$ is a solid convex subcone of W and $\bigvee F$ belongs to $\overline{W}_\mu$ for any upward directed family $F \subset \overline{W}_\mu$, we easily see that λ_n is a hyperharmonic morphism satisfying $\lambda_n \leq \mu$ for all $n \in \mathbf{N}$. Hence we have

$$\widetilde{S_v u}(\mu) \geq (S_{\tilde{v}}\tilde{u})(\mu) \geq n(\lambda(u) - \lambda(v)) > 0$$

for all $n \in \mathbf{N}$. This yields $\widetilde{S_v u}(\mu) = (S_{\tilde{v}}\tilde{u})(\mu)$.

Next assume that $u \in W_\mu$. In this case we have to use the operator $S\colon U \times U \to U$

$$S_t s = \min\{w \in U | s \leq w + t\}$$

in several different hyperharmonic cones U. To avoid misunderstanding, we shall therefore denote by S^U the operator S with respect to a hyperharmonic cone U. Without losing generality, we may assume that $\mu(u) \neq 0$ so that $u \neq \underline{u}$. Using (1.5) and (1.6), we easily see that the set

$$V = \{s \in W | \underline{u} \leq s \leq \bar{u}\} \qquad \left(\bar{u} = \bigvee_{n \in \mathbf{N}} nu\right)$$

is a solid convex subcone of $\underline{u} + W$. Moreover, if $F \subset V$, the $\bigvee F \in V$. Thus V is a hyperharmonic cone, and by Proposition 1.2 u is cancellable in V. Since the set $V_u = \{s \in W | s \leq nu \text{ for some } n \in \mathbf{N}\}$ is dense in V, the hyperharmonic morphism $\mu|V$ is cancellable by [**8**, Proposition 6.23]. Moreover, $\mu|V_c$ (V_c the set of cancellable elements of V) is an H-integral. Hence, we obtain

$$(S^{V^*}_{\widetilde{v+\underline{u}}}\tilde{u})(\mu) = \mu(S^V_{v+\underline{u}}u)$$

by virtue of Proposition 3 in [**4**]. It is easy to see that $S^V_{v+\underline{u}}u = S^{\underline{u}+W}_{v+\underline{u}}$. In view of Lemma 4.11, we note that $S^{\underline{u}+W}_{v+\underline{u}}u = \underline{u} + S^W_v u$ implying

$$(S^{V^*}_{\widetilde{v+\underline{u}}}\tilde{u})(\mu) = \mu(S^V_{v+\underline{u}}u) = \mu(\underline{u} + S^W_v u) = \mu(S_v u).$$

Since every $\lambda \in V^*$ can be extended to the hyperharmonic morphism $\lambda^*\colon W \to \overline{\mathbf{R}}_+$ given by $\lambda^*(s) = \lambda(\underline{u} + s \wedge \bar{u})$ $(s \in W)$, we conclude that

$$\begin{aligned} \mu(S_v u) &= (S^{V^*}_{\widetilde{v+\underline{u}}}\tilde{u})(\mu) = \sup\{\lambda(u) - \lambda(v + \underline{u}) | \lambda \leq \mu, \lambda \in V^*\} \\ &= \sup\{\lambda^*(u) - \lambda^*(v) | \lambda^* \leq \mu, \lambda^* \in W^*\} \leq (S_{\tilde{v}}\tilde{u})(\mu) \leq \mu(S_v u). \end{aligned}$$

Hence we have $\mu(S_v u) = (S_{\tilde{v}}\tilde{u})(\mu)$.

Finally, the last case $u \in \overline{W}_\mu \backslash W_\mu$ follows from

$$\begin{aligned}(S_{\tilde{v}}\tilde{u})(\mu) &= \sup\{(S_{\tilde{v}}\tilde{t})(\mu) | t \in W_\mu, t \leq u\} \\ &= \sup\{\widetilde{S_v t}(\mu) | t \in W_\mu, t \leq u\} = \widetilde{S_v u}(\mu),\end{aligned}$$

completing the proof.

In order to show that $\widetilde{(\underline{u})} = \underline{(\tilde{u})}$ we need the result:

PROPOSITION 3.5. *Let W be a hyperharmonic cone satisfying* (H4) *and* (S). *Suppose that u is an element of W such that $\underline{u} \neq u$. If a hyperharmonic morphism $\mu\colon W \to \overline{\mathbf{R}}_+$ satisfies $\mu(\underline{u}) = 0$, then there exists an upward directed family of hyperharmonic morphisms $(\mu_\alpha)_{\alpha \in I}$ having the properties $\bigvee_{\alpha \in I} \mu_\alpha = \mu$ and $\mu_\alpha(u) < \infty$ for all $\alpha \in I$.*

PROOF. Let u be an arbitrary element of W such that $\underline{u} \neq u$, and let $\mu\colon W \to \overline{\mathbf{R}}_+$ be a hyperharmonic morphism satisfying $\mu(\underline{u}) = 0$. We have to consider only the case $\mu(u) = \infty$. Denote by F the family of all hyperharmonic morphisms $\varphi\colon W \to \overline{\mathbf{R}}_+$ such that $0 < \varphi(u) < \infty$. By virtue of (S), the family F is nonempty. Our aim is to show that $\bigvee_{\varphi \in F}(\mu \wedge \varphi) = \mu$. According to Theorem 2.7, this is equivalent to

$$\mu \wedge \left(\bigvee_{\varphi \in F} \varphi \right) = \mu, \tag{3.3}$$

because the family F is upward directed. To simplify the notation, put $\bigvee_{\varphi \in F} \varphi = \psi$. If $(\mu \wedge \psi)(v)$ is infinite for $v \in W$, then obviously $\infty = (\mu \wedge \psi)(v) \leq \mu(v)$, implying (3.3).

First assume that $u \in W$ is cancellable. Here we have to consider only the case $(\mu \wedge \psi)(v) < \infty$, $v \in W$. Then Proposition 2.3 ensures that there exist v_1 and v_2 having the properties $v = v_1 + v_2$, $\mu(v_1) = (\mu \wedge \psi)(v_1)$, $\psi(v_2) = (\mu \wedge \psi)(v_2)$. Our task is to show that v_2 must be zero. If v_2 is zero, then $v_1 = v$ and $\mu(v) = \mu(v_1) = (\mu \wedge \psi)(v_1) = (\mu \wedge \psi)(v)$ verifying (3.3).

Suppose that v_2 is not zero. We first note that $\psi(v_2) = 0$. Indeed, the case $\psi(v_2) \neq 0$ is impossible, since $n\varphi \in F$ for any $\varphi \in F$ and

$$n\varphi(v_2) \leq \psi(v_2) = (\mu \wedge \psi)(v_2) \leq (\mu \wedge \psi)(v) < \infty.$$

We have to consider two cases: (1) $v_2 \neq v_2 \wedge \bar{u}$ and (2) $v_2 = v_2 \wedge \bar{u}$. In case (1), for any $\varphi \in F$,

$$\varphi^*(x) = \begin{cases} \varphi(x), & \text{if } x \leq \bar{u}; \\ \infty, & \text{if } x \nleq \bar{u} \end{cases}$$

is a hyperharmonic morphism belonging to F. This leads to a contradiction, since $\infty = \varphi^*(v_2) \leq \psi(v_2)$. In case (2) we have $v_2 \leq \bar{u}$ and therefore $v_2 \wedge u \neq 0$. Then by (S), there exists a hyperharmonic morphism $\lambda\colon W \to \overline{\mathbf{R}}_+$ such that $0 < \lambda(v_2 \wedge u) < \infty$. Denote by E the set of cancellable elements $x \in W$ satisfying $x \leq \overline{v_2} = \overline{v_2 \wedge u}$. Then E is an H-cone. Moreover, for any $x \in E$, we have $\bigvee(x \wedge n(v_2 \wedge u)) = x \wedge \bar{v}_2 = x$, and therefore $\lambda|E$ is an H-integral. By virtue of Theorem 2.3.6 in [**3**], there exists an upward directed family $(\lambda_\alpha)_{\alpha \in I}$ of H-integrals satisfying the conditions $\lambda = \sup_{\alpha \in I} \lambda_\alpha$ and $\lambda_\alpha(u \wedge \bar{v}_2) < \infty$. From

$0 < \lambda(u \wedge v_2) \leq \lambda(u \wedge \bar{v}_2)$, it follows that $\lambda_{\alpha_0}(u \wedge \bar{v}_2) > 0$ for some $\alpha_0 \in I$. We may extend λ_{α_0} to the whole hyperharmonic cone W by setting

$$\lambda^*_{\alpha_0}(x) = \sup_{t \in E} \lambda_{\alpha_0}(x \wedge t) \qquad (x \in W).$$

Then $\lambda^*_{\alpha_0}$ is a hyperharmonic morphism satisfying $0 < \lambda_{\alpha_0}(u \wedge \bar{v}_2) = \lambda^*_{\alpha_0}(u) < \infty$ and $\lambda^*_{\alpha_0}(v_2) > \lambda_{\alpha_0}(u \wedge v_2) > 0$. This contradicts the fact that $\psi(v_2) = 0$, completing the proof that $v_2 = 0$ whenever u is cancellable.

Next, assume that u is not cancellable. Since $\mu(\underline{u}) = 0$ by hypothesis, we have $\bigvee_{\varphi \in F}(\mu \wedge \varphi)(\underline{u}) = \mu(\underline{u}) = 0$, yielding

$$\bigvee_{\varphi \in F} (\mu \wedge \varphi)(x) = \bigvee_{\varphi \in F} (\mu \wedge \varphi)(x + \underline{u}) \qquad (x \in W). \tag{3.4}$$

Taking account of this result, we consider (3.3) in the hyperharmonic cone $\underline{u} + W$, where u is cancellable by Proposition 1.2. Note first that the set $F' = \{\varphi|\underline{u}+W|\varphi \in F\}$ is the family of all hyperharmonic morphisms φ in $\underline{u} + W$ with $0 < \varphi(u) < \infty$. Indeed, every hyperharmonic morphism $\varphi\colon \underline{u} + W \to \overline{\mathbf{R}}_+$ can be extended to the hyperharmonic morphism $\varphi^*\colon W \to \overline{\mathbf{R}}_+$ given by $\varphi^*(x) = \varphi(x+\underline{u})$ $(x \in W)$. Using the same argument, we also see that

$$(\mu \wedge \varphi)|\underline{u} + W = (\mu|\underline{u} + W) \wedge (\varphi|\underline{u} + W) \tag{3.5}$$

for any $\varphi \in F$. Indeed, if λ is a hyperharmonic morphism satisfying the conditions $\lambda \leq \mu$, $\lambda \leq \varphi$, then $\lambda^*(x) = \lambda(x+\underline{u}) \leq \mu(x+\underline{u}) = \mu(x)$, and similarly $\lambda^*(x) \leq \varphi(x)$. Hence $\lambda^* \leq \mu \wedge \varphi$ and so $\lambda \leq (\mu \wedge \varphi)|\underline{u} + W$. Since $(\mu \wedge \varphi)|\underline{u} + W \leq (\mu|\underline{u} + W) \wedge (\varphi|\underline{u} + W)$, the equality (3.5) holds. Using the cancellable case, treated above, we obtain

$$(\mu|\underline{u} + W) \wedge (\psi|\underline{u} + W) = \mu|\underline{u} + W.$$

This yields (3.3), finishing the proof.

LEMMA 3.6. *Let W be a hyperharmonic cone. Then $s \vee t = S_{s \wedge t}(s + t) + \underline{s} + \underline{t}$ for any $s, t \in W$.*

PROOF. We first note that

$$s + s \wedge t \leq s + t \leq S_{s \wedge t}(s + t) + \underline{s} + \underline{t} + s \wedge t.$$

By (1.8) we have $s \leq s + \underline{s \wedge t} \leq S_{s \wedge t}(s + t) + \underline{s} + \underline{t}$. Similarly, we obtain $t \leq S_{s \wedge t}(s + t) + \underline{s} + \underline{t}$. Assume that $w \in W$ is an arbitrary element such that $w \geq s$ and $w \geq t$. Then $w + s \wedge t = (w + s) \wedge (w + t) \geq s + t$, which yields $w \geq S_{s \wedge t}(s + t)$. By (1.7) we have $w \geq S_{s \wedge t}(s + t) + \underline{s} + \underline{t}$, completing the proof.

THEOREM 3.7. *Let W be a hyperharmonic cone satisfying* (H4) *and* (S). *Then the evaluation mapping has the following properties*:

(a) $S_{\tilde{t}}\tilde{s} = \widetilde{S_t s}$,

(b) $\tilde{s} \wedge \tilde{t} = \widetilde{s \wedge t}$, $\tilde{s} \vee \tilde{t} = \widetilde{s \vee t}$,

(c) $s \leq t \Leftrightarrow \tilde{s} \leq \tilde{t}$,

(d) $\widetilde{(\underline{s})} = \underline{(\tilde{s})}$,

(e) $\widetilde{\bigvee F} = \bigvee_{f \in F} \tilde{f}$,

for any $s, t \in W$ and $F \subset W$.

PROOF. The assertion (a) and the first part of (b) have already been proved. Using these, the preceding lemma and (d), we arrive at the second part of (b), and this implies (e). For (c), assuming first that $\tilde{s} \leq \tilde{t}$ $(s \neq t)$, we have $0 = S_{\tilde{t}}\tilde{s} = \widetilde{S_t s}$, which leads to $(S_t s)(\mu) = 0$ for all $\mu \in W^*$. If $S_t s$ is nonzero, then considering the mapping $\mu \in W^*$ such that $\mu = \infty$ on $W \backslash \{0\}$ and $\mu(0) = 0$ one easily obtains a contradiction. Hence $S_t s = 0$ and $s \leq t$. This establishes $(\Leftarrow)$, and the converse is trivial.

In order to prove (d), it suffices to consider just the case $s \neq \underline{s}$. Plainly, we have $\widetilde{(\underline{s})} \leq \bigwedge_{n \in \mathbf{N}} (\tilde{s}/n)$. If $\mu(\underline{s}) = \infty$ for $\mu \in W^*$, then $\widetilde{(\underline{s})}(\mu) = \underline{(\tilde{s})}(\mu)$. Assuming $\mu(\underline{s}) = 0$ for $\mu \in W^*$, we apply Proposition 3.5 to obtain an upward directed family $(\mu_\alpha)_{\alpha \in I}$ satisfying $(\mu_\alpha)(s) < \infty$ and $\mu = \sup \mu_\alpha$. Then $\underline{(\tilde{s})}(\mu_\alpha) = 0 = \widetilde{(\underline{s})}(\mu_\alpha)$ for all $\alpha \in I$, which leads to $\underline{(\tilde{s})}(\mu) = \sup \underline{(\tilde{s})}(\mu_\alpha) = 0$. Hence $\underline{(\tilde{s})}(\mu) = \widetilde{(\underline{s})}(\mu)$.

4. Minimal hyperharmonic morphisms. In the sequel let W be a hyperharmonic cone satisfying (H4) and (S). We use the same definition of minimality as in axiomatic potential theory (see e.g. [**5**]), but here we need the convention $0 \cdot \infty = 0$.

DEFINITION. A hyperharmonic morphism $\mu \colon W \to \overline{\mathbf{R}}_+$ is called *minimal* provided that for all $\varphi \in W^*$

$$\varphi \preceq \mu \Rightarrow \varphi = \alpha\mu \quad \text{for some } \alpha \geq 0. \tag{4.1}$$

PROPOSITION 4.1. *A minimal hyperharmonic morphism $\mu \colon W \to \overline{\mathbf{R}}_+$ is cancellable and therefore finite on a dense subset of W.*

PROOF. Suppose that $\mu \colon W \to \overline{\mathbf{R}}_+$ is minimal. By virtue of (1.5) the mapping $\underline{\mu}$ $(= \bigwedge(\mu/n))$ satisfies the equality $\underline{\mu} + \mu = \mu$. Hence from (4.1), it follows $\underline{\mu} = \alpha\mu$ for some $\alpha \geq 0$. The case $\alpha = 0$ results in $\underline{\mu} = 0$, and thus μ is cancellable by (1.4). Assume next that $\alpha > 0$. Then one easily sees that $\underline{\mu} = \mu$. We recall that the mapping $\underline{\mu}$ is given by

$$\underline{\mu}(s) = \begin{cases} 0, & \text{if } s \in \overline{W}_\mu; \\ \infty, & \text{if } s \notin \overline{W}_\mu, \end{cases}$$

where $\overline{W}_\mu$ is defined in (3.1). Suppose that there exists $u \notin \overline{W}_\mu$ such that $\underline{u} \neq u$. By (S), there exists a hyperharmonic morphism $\varphi \colon W \to \overline{\mathbf{R}}_+$ satisfying $0 < \varphi(u) < \varphi(2u) < \infty$. The mapping $\varphi^* \colon W \to \overline{\mathbf{R}}_+$ given by

$$\varphi^*(s) = \begin{cases} 0, & \text{if } s \in \overline{W}_\mu; \\ \varphi, & \text{if } s \notin \overline{W}_\mu \end{cases}$$

is a hyperharmonic morphism satisfying $\varphi^* \preceq \mu$, since $\varphi^* + \mu = \mu$ by (1.6). However, there does not exist $\alpha \in \mathbf{R}_+$ such that $\varphi^* = \alpha\mu$. Hence $u = \underline{u}$ for all $u \notin \overline{W}_\mu$. This leads to a contradiction, since $\varphi^* \colon W \to \overline{\mathbf{R}}_+$ defined by

$$\varphi^*(s) = \begin{cases} 0, & \text{if } s \leq \underline{u}; \\ \infty, & \text{if } s \nleq \underline{u} \end{cases}$$

is a hyperharmonic morphism having the properties $\varphi^* \preceq \mu$ and $\varphi^* \neq \alpha\mu$ for $\alpha \geq 0$. Consequently, we have $\overline{W}_\mu = W$, forcing $\mu = \underline{\mu} = 0$, which is obviously cancellable. Finally, cancellability makes μ finite on a dense set in view of Proposition 6.23 in [**8**].

COROLLARY 4.2. *For any nonzero minimal hyperharmonic morphism* $\mu\colon W \to \overline{\mathbf{R}}_+$ *there exists an element* $u \in W$ *satisfying* $0 < \varphi(u) < \infty$.

In vector lattices minimal real-valued additive functions are exactly those additive functions φ having the property $\varphi(s \wedge t) = \inf(\varphi(s), \varphi(t))$ for all s and t (see e.g. [**12**, p. 74]). A similar result for H-integrals has been proved by N. Boboc, Gh. Bucur and A. Cornea [**3**, p. 59]. We show now that minimal hyperharmonic morphisms also have this characterization. We recall that $x \in B$ is an *extreme point* of a convex set B if the equality $x = \lambda u + (1-\lambda)v$ for some λ $(0 < \lambda < 1)$ implies $u = v = x$.

THEOREM 4.3. *Let* W *be a hyperharmonic cone satisfying* (H4) *and* (S). *Then the nonzero minimal hyperharmonic morphisms* $\mu\colon W \to \overline{\mathbf{R}}_+$ *can be characterized as those* μ *in* $W^*\backslash\{0\}$ *such that* $0 < \mu(u) < \infty$ *for some* $u \in W$ *and either* (*hence both*) *of the following conditions holds*:

(i) $\mu(s \wedge t) = \inf(\mu(s), \mu(t))$ *for all* $s, t \in W$,

(ii) $\mu/\mu(u)$ *is an extreme point of the convex set* $\{\varphi \in W^* | \varphi(u) \leq 1\}$.

PROOF. Assume that $\mu\colon W \to \overline{\mathbf{R}}_+$ is a mapping satisfying $0 < \mu(u) < \infty$ for some $u \in W$ and the condition (ii). We show that (i) holds. The condition $\mu(s \wedge t) = \infty$ for $s, t \in W$ directly implies (i). Assuming $\mu(s \wedge t) < \infty$ for $s, t \in W$, we see from Proposition 2.3 that there exist hyperharmonic morphisms $\mu_1, \mu_2\colon W \to \overline{\mathbf{R}}_+$ satisfying $\mu = \mu_1 + \mu_2$ and $\mu(s \wedge t) = \mu_1(s) + \mu_2(t)$. Setting $\alpha = \mu_1(u)/\mu(u)$, we have

$$\frac{\mu}{\mu(u)} = \alpha \frac{\mu_1}{\mu_1(u)} + (1-\alpha)\frac{\mu_2}{\mu_2(u)},$$

and the extremality of μ implies $\mu_1/\mu_1(u) = \mu/\mu(u)$ and $\mu_2/\mu_2(u) = \mu/\mu(u)$. Consequently,

$$\begin{aligned}\inf(\mu(s), \mu(t)) &\geq \mu(s \wedge t) = \alpha\mu(s) + (1-\alpha)\mu(t) \\ &\geq \alpha \inf(\mu(s), \mu(t)) + (1-\alpha)\inf(\mu(s), \mu(t)) = \inf(\mu(s), \mu(t)).\end{aligned}$$

Hence (i) holds.

Assume that $\mu\colon W \to \overline{\mathbf{R}}_+$ is a hyperharmonic morphism satisfying (i) and that there exists $u \in W$ such that $0 < \mu(u) < \infty$. Let $\mu = \varphi_1 + \varphi_2$ for $\varphi_1, \varphi_2 \in W^*$. The set $W_1 = \{s \in W | \mu(s) = 1\}$ is nonempty, since $u/\mu(u)$ belongs to W_1. From (i) it follows that $s \wedge t \in W_1$ for all $s, t \in W_1$. Let us fix $s \in W_1$. For any $t \in W_1$, we have

$$\varphi_1(s \wedge t) + \varphi_2(s \wedge t) = \mu(s \wedge t) = 1 = \mu(s) = \varphi_1(s) + \varphi_2(s)$$

and

$$\varphi_1(s \wedge t) + \varphi_2(s \wedge t) = \mu(s \wedge t) = 1 = \mu(t) = \varphi_1(t) + \varphi_2(t).$$

These equalities imply $\varphi_i(t) = \varphi_i(s) = \varphi_i(s \wedge t)$ for all $t \in W_1$ and $i = 1, 2$. Setting $\alpha_i = \varphi_i(s)$, we easily see that $\varphi_i(t) = \alpha_i\mu(t)$ for all $t \in \overline{W}_\mu$. If $\mu(t) = \infty$ and $t \in W$, then $\mu(t \wedge nu) = n\mu(u) > 0$ and

$$\varphi_i(t) \geq \sup_{n \in \mathbf{N}} \varphi_i(t \wedge nu) = \infty.$$

Hence $\varphi_i = \alpha_i\mu$.

Finally, assume that $\mu\colon W \to \overline{\mathbf{R}}_+$ is a nonzero hyperharmonic morphism. We show that (ii) holds. By Corollary 4.2, there exists $u \in W$ such that $0 < \mu(u) < \infty$.

Suppose that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for some α $(0 < \alpha < 1)$ and $\mu_i(u) \leq \mu(u)$ $(i = 1, 2)$. Minimality of μ ensures that $\mu_1 = \beta\mu$ and $\mu_2 = \gamma\mu$ $(\beta, \gamma \geq 0)$. Since $0 \leq \mu_i(u) \leq \mu(u) < \infty$ we obtain

$$\alpha\mu_1(u) + (1-\alpha)\mu_2(u) = \alpha\mu(u) + (1-\alpha)\mu(u) \geq \alpha\mu_1(u) + (1-\alpha)\mu(u).$$

Hence $\mu_1(u) = \mu(u) = \mu_2(u)$. This yields $\beta = \gamma = 1$ and therefore $\mu_1 = \mu = \mu_2$, completing the proof.

We say that a function cone $\mathfrak{F}$ on W *separates points of* W if for any $s, t \in W$ $(s \neq t)$, there exists $f \in \mathfrak{F}$ such that $f(s) \neq f(t)$.

THEOREM 4.4. *Let W be a hyperharmonic cone satisfying* (H4) *and* (S). *If minimal hyperharmonic morphisms separate points of W, then W is isomorphic to a hyperharmonic cone $\mathfrak{F}$ of extended real-valued positive functions on a set X having the properties*:

(4.2) *addition, scalar multiplication and order relation are pointwise*;

(4.3) *for any $x \in X$ there exists $f \in \mathfrak{F}$ such that* $0 < f(x) < \infty$;

(4.4) $f \wedge g = \inf(f, g)$ *for all* $f, g \in \mathfrak{F}$;

(4.5) $\bigvee F = \sup_{f \in F} f$ *for any nonempty upward directed family* $F \subset \mathfrak{F}$.

Conversely, if W admits such an isomorphism, then the minimal hyperharmonic morphisms on W separate points.

PROOF. Assume first that minimal hyperharmonic morphisms separate points of W, and take $X = \{\mu \in W^* | \mu \text{ is minimal}\}$. Then property (4.4) follows from Theorem 4.3, and the rest of conditions are trivial.

Conversely, assume that a hyperharmonic cone is isomorphic to a hyperharmonic cone $\mathfrak{F}$ of extended real-valued positive functions on a set X having the properties (4.2)–(4.5). Let φ be an isomorphism from W onto $\mathfrak{F}$, i.e., φ is additive and $\varphi(u) \leq \varphi(v) \Leftrightarrow u \leq v$. For any $x \in X$, we define a mapping $\hat{x}\colon W \to \overline{\mathbf{R}}_+$ by

$$\hat{x}(u) = \varphi(u)(x), \qquad u \in W.$$

This mapping satisfies

$$\hat{x}(u \wedge v) = \varphi(u \wedge v)(x) = \inf(\varphi(u)(x), \varphi(v)(x)) = \inf(\hat{x}(u), \hat{x}(v))$$

for all $u, v \in W$. Moreover, there exists $u \in W$ such that $\varphi(u)(x) > 0$. Hence, by Theorem 4.3, $\hat{x}$ is a minimal hyperharmonic morphism. Obviously, the mappings $\hat{x}, x \in X$, separate points of W.

EXAMPLE. Let X be a $\mathcal{S}$-harmonic space and $\mathcal{U}$ a hyperharmonic sheaf on X in the sense of H. Bauer [**2**] or C. Constantinescu and A. Cornea [**6**]. The cone $\mathcal{U}^+(X)$ of positive hyperharmonic functions on X is a hyperharmonic cone satisfying (H4) and (S), and the minimal hyperharmonic morphisms separate points of $\mathcal{U}^+(X)$. Indeed, the cone $\mathcal{U}^+(X)$ is a hyperharmonic cone by [**8**, Example 2.7], and property (H4) is obvious. In order to show (S), let u and v be elements of $\mathcal{U}^+(X)$ such that $u \neq \underline{u}$, $u < v$. Then there exists a point $x \in X$ such that $0 \leq u(x) < v(x)$. Since $u \neq \underline{u}$, there also exists a point $y \in X$ with $0 < u(y) < \infty$. For $z \in X$, define a mapping $\hat{z}\colon \mathcal{U}^+(X) \to \overline{\mathbf{R}}_+$ by $\hat{z}(w) = w(z)$ $(w \in \mathcal{U}^+(X))$. Then $\hat{z}$ is a hyperharmonic morphism. Hence $\hat{x} + \hat{y}$ is a hyperharmonic morphism satisfying

$$0 < (\hat{x} + \hat{y})(u) = u(x) + u(y) < v(x) + v(y) = (\hat{x} + \hat{y})(v).$$

Consequently, (S) holds in $\mathcal{U}^+(X)$. To establish minimality of $\hat{x}, x \in X$, we have to prove that there exists a positive hyperharmonic morphism with $0 < u(x) < \infty$. By the axiom of positivity (see [**6**, p. 30]) there exists a relatively compact neighborhood U_x of x and a positive harmonic function h on U_x such that $0 < h(y)$ for all $y \in U_x$. Let V be a relatively compact neighborhood of x with $\overline{V}_x \subset U_x$. Using Tietze's Extension Theorem, we can find a continuous function f (with compact support) such that $f = h$ on $\overline{V}_x$. Hence, by [**6**, Proposition 2.2.3], the function Rf is harmonic on V_x and therefore finite on V_x. Since $Rf \geq f$, we have $0 < Rf(x) < \infty$. Theorem 4.3 ensures that $\hat{x}$ is a minimal hyperharmonic morphism for any $x \in X$, and the functions $\hat{x}, x \in X$, obviously separate points of $\mathcal{U}^+(X)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JOENSUU, SF-80101 JOENSUU 10, FINLAND

Current address: Department of Mathematics, University of Washington, Seattle, Washington 98195