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CONVERGENCE ACCELERATION FOR GENERALIZED CONTINUED FRACTIONS

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ABSTRACT. The main result in this paper is the proof of convergence acceleration for a suitable modification (as defined by de Bruin and Jacobsen) in the case of an n-fraction for which the underlying recurrence relation is of Perron-Kreuser type. It is assumed that the characteristic equations for this recurrence relation have only simple roots with differing absolute values.

1. Introduction and notation. Modifications of continued fractions have been studied extensively during the past few years by different authors [**6**, **10**, **11**, **12**, **13**]. Recently L. Jacobsen and M. G. de Bruin generalized this concept of modification to n-fractions or generalized continued fractions (GCF's). (For detail the reader is referred to [**3**]; the notation used will be that from [**3**].)

Let n be a fixed natural number ($n \geq 1$). Consider a sequence of complex $(n+1)$-tuples $(b_k, a_k^{(1)}, \ldots, a_k^{(n)})$, $a_k^{(1)} \neq 0$. Then the n-fraction associated with this sequence, written as

$$K_{k=1}^{\infty} \begin{pmatrix} a_k^{(1)} \\ \vdots \\ a_k^{(n)} \\ b_k \end{pmatrix} \tag{1a}$$

is given by the sequence of approximants $\{A_k^{(1)}/B_k, \ldots, A_k^{(n)}/B_k\}_{k=1}^{\infty}$ (if they exist), where the numerators and denominators satisfy the recurrence relation

$$X_k = b_k X_{k-1} + a_k^{(n)} X_{k-2} + \cdots + a_k^{(1)} X_{k-n-1}, \qquad k = 1, 2, \ldots, \tag{1b}$$

with initial values

$$A_{-j}^{(i)} = \delta_{i+j,n+1}, \quad B_{-j} = \delta_{n+1+j,n+1} \qquad (j = 0, \ldots, n) \tag{2}$$

for $i = 1, \ldots, n$. The n-fraction is said to converge in $\hat{\mathbf{C}}^n$ if the following limit exists in $\hat{\mathbf{C}}^n$:

$$\{\xi_0^{(1)}, \ldots, \xi_0^{(n)}\} = \lim_{k \to \infty} \{A_k^{(1)}/B_k, \ldots, A_k^{(n)}/B_k\}. \tag{3}$$

For $k \geq 1$ we introduce the Moebius transforms

$$s_k^{(1)}(w^{(1)}, \ldots, w^{(n)}) = \frac{a_k^{(1)}}{b_k + w^{(n)}},$$

$$s_k^{(i)}(w^{(1)}, \ldots, w^{(n)}) = \frac{a_k^{(i)} + w^{(i-1)}}{b_k + w^{(n)}} \qquad (i = 2, \ldots, n)$$

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and for $i = 1, \dots, n$

$$S_1^{(i)}(w^{(1)}, \dots, w^{(n)}) = s_1^{(i)}(w^{(1)}, \dots, w^{(n)}),\tag{4}$$
$$S_k^{(i)}(w^{(1)}, \dots, w^{(n)}) = S_{k-1}^{(i)}(s_k^{(1)}, \dots, s_k^{(n)}) \qquad (k \geq 2) \text{ (if they exist).}$$

Then we have the connection formula (cf. [**1**])

$$S_k^{(i)}(w^{(1)}, \dots, w^{(n)}) = \frac{A_k^{(i)} + A_{k-1}^{(i)} w^{(n)} + A_{k-2}^{(i)} w^{(n-1)} + \cdots + A_{k-n}^{(i)} w^{(1)}}{B_k + B_{k-1} w^{(n)} + B_{k-2} w^{(n-1)} + \cdots + B_{k-n} w^{(1)}}.\tag{5}$$

From this formula it follows that

$$A_k^{(i)}/B_k = S_k^{(i)}(0, \dots, 0) \quad \text{for } i = 1, \dots, n,\tag{6}$$

and we see that the sequence of approximants of the n-fraction (1) may be expressed by the Moebius transforms $S_k^{(i)}$. So as in the case of ordinary continued fractions, the approximants of the n-fraction (1) may be evaluated by replacing its tails, i.e.

$$K_{k=m+1}^{\infty} \begin{pmatrix} a_k^{(1)} \\ \vdots \\ a_k^{(n)} \\ b_k \end{pmatrix}\tag{7}$$

by zero for successive values of m. If the tail (7) converges, we let $(\xi_m^{(1)}, \dots, \xi_m^{(n)})$ denote its value. Since the choice of zeros for the tails is a rather arbitrary one (the values of convergent tails converge to zero in exceptional cases only), the concept of modification of an n-fraction is introduced.

Given a sequence of n-tuples $\{(w_k^{(1)}, \dots, w_k^{(n)})\}_{k=1}^{\infty}$ of numbers from $\mathbf{C}$, a modification of (1) is given by the sequence of n-tuples

$$\{S_k^{(1)}(w_k^{(1)}, \dots, w_k^{(n)}), \dots, S_k^{(n)}(w_k^{(1)}, \dots, w_k^{(n)})\}_{k=1}^{\infty}.\tag{8}$$

We assume $(w_k^{(1)}, \dots, w_k^{(n)})$ to be chosen such that (8) is well defined. It can be seen from our main theorem that for convergent n-fractions and an appropriate choice of the $w_k^{(i)}$ the sequence of modified approximants (8) converges to the value of the n-fraction faster than the sequence of ordinary approximants (6) (convergence acceleration is characterized by

$$\left.\lim_{k \to \infty} \frac{\xi_0^{(i)} - S_k^{(i)}(w^{(1)}, \dots, w^{(n)})}{\xi_0^{(i)} - S_k^{(i)}(0, \dots, 0)} = 0 \qquad (i = 1, \dots, n)\right).\tag{9}$$

For the sequel we will restrict ourselves to the case that the n-fraction converges in $\mathbf{C}^n$. First we shall state a very important theorem in the theory of linear recurrence relations, i.e. the Perron-Kreuser theorem (cf. [**4**, **7**, **8**, **9**]), which we shall use in the proof of Theorem I.

The Perron-Kreuser theorem. The recurrence relation

$$X_k = b_k X_{k-1} + a_k^{(n)} X_{k-2} + \cdots + a_k^{(1)} X_{k-n-1}, \qquad k = 1, 2, \dots,\tag{1b}$$

is said to be of Perron-Kreuser type if its coefficients satisfy

$$b_k = -s_{n+1} k^{m_{n+1}} (1 + o(1)) \qquad (k \to \infty),\tag{10}$$
$$a_k^{(i)} = -s_i k^{m_i} (1 + o(1)) \qquad (k \to \infty), \; i = 1, \dots, n,$$

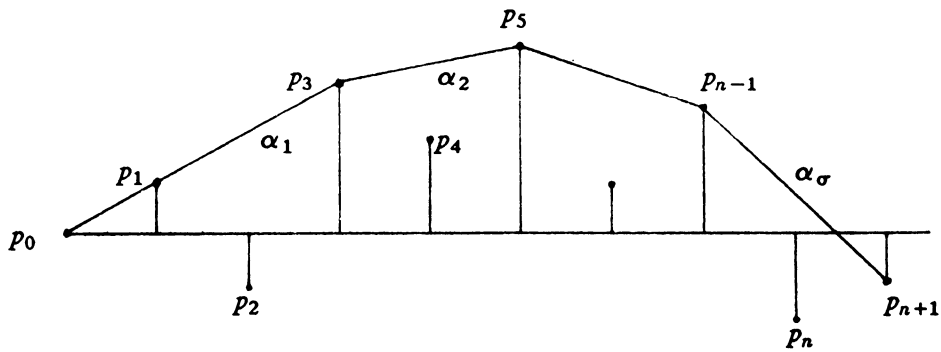

FIGURE 1

where the s_i are real or complex numbers and s_{n+2} is defined to be one. The m_i are real or $-\infty$ with $m_1 > -\infty$; m_{n+2} is defined to be zero. With such a recurrence relation we can associate a uniquely defined Newton-Puiseux polygon in a rectangular (x, y)-coordinate system (see Figure 1).

Let the points $p_0, p_1, \ldots, p_{n+1}$ be defined by $x = i$, $y = m_{n+2-i}$ $(i = 0, \ldots, n+1)$. Then some of the points $p_0, \ldots, p_{n+1}$ are connected with linear segments in such a way that the resulting polygon is concave downwards, each point p_i $(i = 1, \ldots, n)$ being on or below the resulting figure, p_0 and p_{n+1} being the endpoints of the figure. If the polygon so constructed has σ distinct linear segments, their respective slopes are denoted by $\alpha_1, \ldots, \alpha_\sigma$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_\sigma$, and their abscissas are denoted by $0 = e_0 < e_1 < \cdots < e_\sigma = n+1$. It follows that

$$\alpha_\lambda = \frac{m_{n+2-e_\lambda} - m_{n+2-e_{\lambda-1}}}{e_\lambda - e_{\lambda-1}}, \qquad \lambda = 1, \ldots, \sigma, \tag{11}$$

and the Perron-Kreuser theorem states

THEOREM. *A linear recurrence relation with the above Newton-Puiseux polygon has a fundamental system of $n+1$ solutions which fall into σ classes. Each of these classes is further broken into subclasses, the λth class $(\lambda = 1, 2, \ldots, \sigma)$ containing β_λ subclasses. Let $q_\gamma^{(\lambda)}$ $(\gamma = 1, 2, \ldots, \beta_\lambda)$ denote the number of linearly independent solutions in the γth subclass of the λth class. Then each of the $q_\gamma^{(\lambda)}$ solutions X_k and their nonzero linear combinations satisfy*

$$\limsup_{k\to\infty} (|X_k|/(k!)^{\alpha_\lambda})^{1/k} = t_\gamma^{(\lambda)}. \tag{12}$$

Here the $t_\gamma^{(\lambda)}$ are distinct positive numbers which are the moduli of the roots of the following characteristic equation corresponding to the λth class:

$$r_0^{(\lambda)} x^{e_\lambda - e_{\lambda-1}} + r_1^{(\lambda)} x^{e_\lambda - e_{\lambda-1}-1} + \cdots + r_{e_\lambda - e_{\lambda-1}}^{(\lambda)} = 0, \tag{13}$$

where

$$r_i^{(\lambda)} = s_{n+2-e_{\lambda-1}-i} \quad \text{or} \quad 0$$

depending on whether the point $(e_{\lambda-1} + i, m_{n+2-e_{\lambda-1}-i})$ falls, respectively, on or below the λth side of the Newton-Puiseux polygon. The number $q_\gamma^{(\lambda)}$ is equal to *the number of roots, counting their multiplicities, of* (13) *with absolute value* $t_\gamma^{(\lambda)}$. *Thus, it follows that*

$$q_1^{(\lambda)} + q_2^{(\lambda)} + \cdots + q_{\beta_\gamma}^{(\lambda)} = e_\lambda - e_{\lambda-1}. \tag{14}$$

Further, to each simple root u *of* (13) *whose absolute value is distinct from the absolute values of the other roots, there corresponds a solution* X_k *in the* λ*th class which satisfies*

$$\lim_{k\to\infty} (X_{k+1}/k^{\alpha_\lambda} X_k) = u. \tag{15}$$

REMARKS. If $e_\lambda > e_{\lambda-1} + 1$, but none of the points p_i, $e_{\lambda-1} < i < e_\lambda$, lies on the Newton-Puiseux polygon, then all the $e_\lambda - e_{\lambda-1}$ roots of (13) are distinct but have coinciding absolute values

$$t_1^{(\lambda)} = |s_{n+2-e_\lambda}/s_{n+2-e_{\lambda-1}}|^{1/(e_\lambda - e_{\lambda-1})}.$$

(That is, $\beta_\lambda = 1$ and $q_1^{(\lambda)} = e_\lambda - e_{\lambda-1}$.)

If $e_\lambda = e_{\lambda-1} + 1$, then there is only one linearly independent solution of (1b) from the λth class. It behaves according to (15) with $u = -s_{n+2-e_\lambda}/s_{n+2-e_{\lambda-1}}$. If in addition $\lambda = 1$, then this is a dominant solution of (1b) and thus the GCF (1a) converges under mild conditions. Indeed, since then $a_k^{(i)}/b_k b_{k-1} \cdots b_{k-1-n+i} \to 0$ it follows that all the tails (7) converge from some m on ([**1**], see also Theorem 1).

From now on we will assume that for every λ the roots of (13) belonging to the λth class are all different in modulus. Let u_i ($i = e_{\lambda-1} + 1, \ldots, e_\lambda$) be the roots of the characteristic equation for the λth class, with

$$|u_{e_{\lambda-1}+1}| > \cdots > |u_{e_\lambda}|. \tag{16}$$

Since $r_{e_\lambda - e_{\lambda-1}}^{(\lambda)} = s_{n+2-e_\lambda} \neq 0$ we also have all $|u_i| > 0$.

It is easy to prove that in this case the recurrence relation (1b) has a basis ordered by domination: using (15) we are able to choose for each $i = 1, \ldots, n+1$ a solution $\{D_k^{(i)}\}$ of (1b) which satisfies

$$\lim_{k\to\infty} (D_{k+1}^{(i)}/k^{\alpha_\lambda} D_k^{(i)}) = u_i \quad \text{for } i = e_{\lambda-1} + 1, \ldots, e_\lambda \tag{17}$$

and using (16) we then have

$$\lim_{k\to\infty} (D_k^{(i+1)}/D_k^{(i)}) = 0 \quad \text{for } i = 1, \ldots, n. \tag{18}$$

Hence the $\{D_k^{(i)}\}$ form a basis which is ordered by domination. From [**2**] follows therefore that the n-fraction connected with these recurrence relations, and all its tails, converge in $\hat{\mathbf{C}}^n$. From (17) we have

$$\begin{aligned} D_{k+1}^{(i)}/D_k^{(i)} &= k^{\alpha_\lambda} u_i (1 + o(1)) \qquad (k\to\infty) \\ &= v_i(k)(1 + o(1)) \qquad (k\to\infty) \end{aligned} \tag{19}$$

where we have put $v_i(k) = k^{\alpha_\lambda} u_i$.

Later on we shall also need the following:

Elementary symmetric functions. The elementary symmetric functions for a set of m arbitrary complex numbers $x_1, \ldots, x_m$ are defined as

$$\begin{aligned} f_1(x_1, \ldots, x_m) &= x_1 + \cdots + x_m, \\ f_2(x_1, \ldots, x_m) &= x_1x_2 + x_1x_3 + \cdots + x_{m-1}x_m, \\ &\cdots \\ f_m(x_1, \ldots, x_m) &= x_1x_2 \cdots x_m. \end{aligned} \tag{20}$$

If we construct from these f_i the polynomial

$$P_{x_1,\ldots,x_m}(x) = x^m - f_1x^{m-1} + f_2x^{m-2} + \cdots + (-1)^m f_m, \tag{21}$$

then this polynomial has the roots $x_1, \ldots, x_m$, and so we have

$$P_{x_1,\ldots,x_m}(x_i) = x_i^m - f_1x_i^{m-1} + f_2x_i^{m-2} + \cdots + (-1)^m f_m = 0, \qquad i = 1, \ldots, m. \tag{22}$$

This can be seen as a system of m equations in the m unknowns f_i $(i = 1, \ldots, m)$. Solving it using Cramer's rule gives us

$$f_{m-i+1} = \det V^{(i)} / \det V^{(m+1)}, \qquad i = 1, \ldots, m, \tag{23}$$

where $V^{(i)}$ is the square matrix arising from the matrix V with

$$V = \begin{pmatrix} 1 & 1 & \cdot & 1 \\ x_1 & x_2 & \cdot & x_m \\ \vdots & \vdots & \vdots & \vdots \\ x_1^m & x_2^m & \cdot & x_m^m \end{pmatrix}$$

by removing the ith row.

2. Convergence of the values of the tails of an n-fraction. In Theorem I we shall give a result concerning the limit of the tails of an n-fraction, and this for two special types of Perron-Kreuser recurrence relations:

$$\text{I.} \quad e_1 > 1, \quad \alpha_1 = 0, \tag{24}$$

$$\text{II.} \quad e_1 = 1, \quad \alpha_1 > \alpha_2 = 0. \tag{25}$$

(Note that using an equivalence transformation (see e.g. [**1**, **4**]) every recurrence relation of Perron-Kreuser type can be brought into one of these two forms without changing the character of the Newton-Puiseux diagram or the equations (13).)

In our situation these two types of recurrence relations correspond to GCF's with the following properties.

Type I. $m_i \leq 0$ for all i, where $m_i = 0$ for $i = n + 2 - e_1$ and at least one $i > n + 2 - e_1$, but for no $i < n + 2 - e_1$.

Case (i). $m_{n+1} = 0$ (which is always the case if $e_1 = 2$). Then

$$c_k^{(i)} = a_k^{(i)} / b_k b_{k-1} \cdots b_{k-1-n+i} = (-1)^{n-i+1} (s_i / s_{n+1}^{n-i+2}) k^{m_i} (1 + o(1))$$

and thus $a_k^{(i)} \to 0$ and $c_k^{(i)} \to 0$ for all $i < n + 2 - e_1$. If in particular $e_1 = 2$, then (13) has the two solutions

$$u_{1,2} = -(s_{n+1}/2)(1 \pm \sqrt{1 - 4s_n/s_{n+1}^2})$$

for $\lambda = 1$, that is, $-s_n/s_{n+1}^2 \notin (-\infty, -\frac{1}{4}]$.

Case (ii). $m_{n+1} < 0$ (which can only happen if $e_1 > 2$ and thus $n > 1$). Then

$$c_k^{(i)} = (-1)^{n-i+1}(s_i/s_{n+1}^{n-i+2})k^{m_i-(n-i+2)m_{n+1}}(1+o(1))$$

and thus $c_k^{(i)} \to \infty$ for $i = n+2-e_1$ and at least one $i > n+2-e_1$, but possibly also for other values of i. If in particular $e_1 = 3$, then (13) has the three solutions

$$u_{1,2,3} = (s_{n-1}/2)^{1/3}\left\{\rho^p\left(-1+\sqrt{1+4s_n^3/27s_{n-1}^2}\right)^{1/3}\right.$$
$$\left.+\rho^q\left(-1-\sqrt{1+4s_n^3/27s_{n-1}^2}\right)^{1/3}\right\}$$

for $\lambda = 1$, where $\rho = e^{i2\pi/3}$ and $(p,q) = (0,0), (1,2)$ and $(2,1)$. That is $-s_{n-1}^2/s_n^3 \notin (-\infty, 0] \cup [\frac{4}{27}, +\infty)$.

Type II. $m_{n+1} \geq m_i$ for all i with equality for $i = n+2-e_2$, $m_{n+1} > 0$. Then $c_k^{(i)} \to 0$ for all i.

THEOREM I. *The values of the tails, $\xi_k^{(i)}$ $(i = 1, \ldots, n)$, of the GCF associated with a recurrence relation* (1b) *of Type* I *for which* (16) *holds for all λ, satisfy*

$$\text{(26)}\quad \text{(a)}\quad \lim_{k\to\infty} \xi_k^{(n-i+1)} = (-1)^i f_i(u_2, \ldots, u_{e_1}) \quad \textit{if } i = 1, \ldots, e_1 - 1,$$

$$\text{(27)}\quad \text{(b)}\quad \lim_{k\to\infty} \frac{\xi_k^{(n-i+1)}}{k^{\alpha(i)}} = (-1)^{e_1-1} f_{e_1-1}(u_2, \ldots, u_{e_1}) \frac{r_{i-e_{\lambda-1}+1}^{(\lambda)}}{r_0^{(2)}} \quad \textit{if } i = e_{\lambda-1}, \ldots, e_\lambda - 1,$$

for $\lambda = 2, \ldots, \sigma$ with

$$\alpha(i) = (e_2 - e_1)\alpha_2 + (e_3 - e_2)\alpha_3 + \cdots + (i - e_{\lambda-1} + 1)\alpha_\lambda.$$

For a recurrence relation of Type II *we have*

$$\text{(28)}\qquad \lim_{k\to\infty} \frac{\xi_k^{(n-i+1)}}{k^{\alpha(i)}} = \frac{r_{i-e_{\lambda-1}+1}^{(\lambda)}}{r_0^{(2)}} \quad \textit{if } i = e_{\lambda-1}, \ldots, e_\lambda - 1,\ \lambda = 2, \ldots, \sigma.$$

REMARK. In particular this means that

$$\lim_{k\to\infty} \xi_k^{(i)} = \begin{cases} 0 & \text{if } i \leq n+1-e_1 \\ (-1)^{n-i+1} f_{n-i+1}(u_2, \ldots, u_{e_1}) & \text{if } i > n+1-e_1 \end{cases} \quad \text{for Type I}$$

and

$$\lim_{k\to\infty} \xi_k^{(i)} = \begin{cases} 0 & \text{if } i \leq n+1-e_2 \\ r_{n-i+1}^{(2)}/r_0^{(2)} & \text{if } i > n+1-e_2 \end{cases} \quad \text{for Type II.}$$

PROOF. Let us consider the shifted recurrence relation

$$\text{(29)}\qquad X_m = b_{m+k}X_{m-1} + a_{m+k}^{(n)}X_{m-2} + \cdots + a_{m+k}^{(1)}X_{m-n-1} \qquad (m \geq 1).$$

The tails $\xi_k^{(i)}$ then satisfy $\xi_k^{(i)} = \lim_{m\to\infty} A_{k,m}^{(i)}/B_{k,m}$ where the $A_{k,m}^{(i)}$ and $B_{k,m}$ follow from (1b) and the initial values (2) as for the nonshifted case. Because the $D_k^{(i)}$'s form a basis for (1b) the shifted $D_k^{(i)}$'s form a basis for (29):

$$\begin{aligned} A_{k,m}^{(i)} &= \phi_1^{(k,i)} D_{m+k}^{(1)} + \phi_2^{(k,i)} D_{m+k}^{(2)} + \cdots + \phi_{n+1}^{(k,i)} D_{m+k}^{(n+1)}, \qquad i = 1,\ldots,n, \\ B_{k,m} &= \psi_1^{(k)} D_{m+k}^{(1)} + \psi_2^{(k)} D_{m+k}^{(2)} + \cdots + \psi_{n+1}^{(k)} D_{m+k}^{(n+1)}. \end{aligned} \tag{30}$$

If $l^{(i)} = (l_1^{(i)}, \ldots, l_{n+1}^{(i)})^T$ $(i = 1, \ldots, n+1)$ with $l_j^{(i)} = \delta_{i,j}$ $(j = 1, \ldots, n+1)$ and if $\mathbf{D}_k$ is defined by

$$\mathbf{D}_k = \begin{pmatrix} D_{k-n}^{(1)} & D_{k-n}^{(2)} & \cdots & D_{k-n}^{(n+1)} \\ D_{k-n+1}^{(1)} & D_{k-n+1}^{(2)} & \cdots & D_{k-n+1}^{(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ D_k^{(1)} & D_k^{(2)} & \cdots & D_k^{(n+1)} \end{pmatrix} \tag{31}$$

then $\phi_1^{(k,i)}$ and $\psi_1^{(k)}$ may be calculated from

$$\begin{aligned} &\mathbf{D}_k(\phi_1^{(k,i)}, \ldots, \phi_{n+1}^{(k,i)})^T = l^{(i)} \quad \text{for } i = 1, \ldots, n, \\ &\mathbf{D}_k(\psi_1^{(k)}, \ldots, \psi_{n+1}^{(k)})^T = l^{(n+1)}. \end{aligned}$$

Since the $\{D_k^{(i)}\}$'s form a basis, we have $\det \mathbf{D}_k \neq 0$ and Cramer's rule gives us

$$\phi_1^{(k,i)} = (-1)^{i+1} \frac{\det \mathbf{D}_k^{(i,1)}}{\det \mathbf{D}_k}, \quad \psi_1^{(k)} = (-1)^n \frac{\det \mathbf{D}_k^{(n+1,1)}}{\det \mathbf{D}_k}$$

where $\mathbf{D}_k^{(p,1)}$ is the matrix arising from $\mathbf{D}_k$ by removing the pth row and the first column. Since the basis of the $\{D_k^{(i)}\}$'s is ordered by domination, it follows that $\det \mathbf{D}_k^{(n+1,1)} \neq 0$ from some k on. Hence

$$\xi_k^{(n-i+1)} = \frac{\phi_1^{(k,n-i+1)}}{\psi_1^{(k)}} = (-1)^i \frac{\det \mathbf{D}_k^{(n-i+1,1)}}{\det \mathbf{D}_k^{(n+1,1)}}, \qquad i = 1, \ldots, n,$$

is well defined and finite. If we divide the lth column of $\mathbf{D}_k^{(n-i+1,1)}$ and $\mathbf{D}_k^{(n+1,1)}$ by $D_{k-n}^{(l+1)}$ for $l = 1, \ldots, n$, then, using (19) and (23) we find

$$\frac{\phi_1^{(k,n-i+1)}}{\psi_1^{(k)}} = (-1)^i f_i(v_2(k), \ldots, v_{n+1}(k))(1 + o(1)) \qquad (k \to \infty)$$

since

$$\frac{D_{k-n+j}^{(i)}}{D_{k-n}^{(i)}} = \frac{D_{k-n+j}^{(i)}}{D_{k-n+j-1}^{(i)}} \cdots \frac{D_{k-n+1}^{(i)}}{D_{k-n}^{(i)}} = (v_i(k))^j(1 + o(1)) \qquad (k \to \infty), j = 0, \ldots, n.$$

Now if the recurrence relation (1b) is of Type I, we have

$$\begin{aligned} \phi_1^{(k,n-i+1)}/\psi_1^{(k)} = (-1)^i f_i(u_2, \ldots, u_{e_1}, k^{\alpha_2} u_{e_1+1}, \ldots, k^{\alpha_2} u_{e_2}, \ldots, \\ k^{\alpha_\sigma} u_{e_{\sigma-1}+1}, \ldots, k^{\alpha_\sigma} u_{n+1})(1 + o(1)). \end{aligned}$$

If $i = 1, \ldots, e_1 - 1$ it is easy to see that

$$\phi_1^{(k,n-i+1)}/\psi_1^{(k)} = (-1)^i f_i(u_2, \ldots, u_{e_1})(1 + o(1)) \qquad (k \to \infty)$$

since $0 > \alpha_2 > \alpha_3 > \cdots > \alpha_\sigma$. Hence

$$\begin{aligned}\lim_{k\to\infty} \xi_k^{(n-i+1)} &= \lim_{k\to\infty} \lim_{m\to\infty} \frac{A_{k,m}^{(n-i+1)}}{B_{k,m}} = \lim_{k\to\infty} \frac{\phi_1^{(k,n-i+1)}}{\psi_1^{(k)}} \\ &= (-1)^i f_i(u_2, \ldots, u_{e_1})\end{aligned}$$

using (30) and the fact that the basis of the $\{D_k^{(i)}\}$'s is ordered by domination. This proves part (a) of the theorem. If $i = e_{\lambda-1}, \ldots, e_\lambda - 1$ $(\lambda = 2, \ldots, \sigma)$ we find, using the properties of the f_i:

$$\begin{aligned}\frac{\phi_1^{(k,n-i+1)}}{\psi_1^{(k)}} &= (-1)^i f_{e_1-1}(u_2, \ldots, u_{e_1}) \left(\prod_{j=2}^{\lambda-1} k^{(e_j - e_{j-1})\alpha_j} f_{e_j - e_{j-1}}(u_{e_{j-1}+1}, \ldots, u_{e_j}) \right) \\ &\quad \times k^{(i - e_{\lambda-1}+1)\alpha_\lambda} f_{i-e_{\lambda-1}+1}(u_{e_{\lambda-1}+1}, \ldots, u_{e_\lambda})(1 + o(1)) \\ &= (-1)^i k^{\alpha(i)} f_{e_1-1}(u_2, \ldots, u_{e_1}) \left(\prod_{j=2}^{\lambda-1} f_{e_j - e_{j-1}}(u_{e_{j-1}+1}, \ldots, u_{e_j}) \right) \\ &\quad \times f_{i-e_{\lambda-1}+1}(u_{e_{\lambda-1}+1}, \ldots, u_{e_\lambda})(1 + o(1)) \\ &= (-1)^i k^{\alpha(i)} f_{e_1-1}(u_2, \ldots, u_{e_1}) \left(\prod_{j=2}^{\lambda-1} (-1)^{e_j - e_{j-1}} \frac{r_{e_j - e_{j-1}}^{(j)}}{r_0^{(j)}} \right) \\ &\quad \times (-1)^{i-e_{\lambda-1}+1} \frac{r_{i-e_{\lambda-1}+1}^{(\lambda)}}{r_0^{(\lambda)}} (1 + o(1)) \quad \text{using (13), (21)} \\ &= (-1)^i k^{\alpha(i)} f_{e_1-1}(u_2, \ldots, u_{e_1}) (-1)^{i-e_1+1} \left(\prod_{j=2}^{\lambda-1} \frac{r_{e_j - e_{j-1}}^{(j)}}{r_0^{(j)}} \right) \\ &\quad \times \frac{r_{i-e_{\lambda-1}+1}^{(\lambda)}}{r_0^{(\lambda)}} (1 + o(1)) \\ &= (-1)^{e_1-1} k^{\alpha(i)} f_{e_1-1}(u_2, \ldots, u_{e_1}) \frac{r_{i-e_{\lambda-1}+1}^{(\lambda)}}{r_0^{(2)}} (1 + o(1)) \qquad (k \to \infty)\end{aligned}$$

since $r_0^{(j)} = r_{e_{j-1}-e_{j-2}}^{(j-1)}$ for $j = 2, \ldots, n+1$. Hence,

$$\begin{aligned}\lim_{k\to\infty} \frac{\xi_k^{(n-i+1)}}{k^{\alpha(i)}} &= \lim_{k\to\infty} \lim_{m\to\infty} \frac{A_{k,m}^{(n-i+1)}}{k^{\alpha(i)} B_{k,m}} = \lim_{k\to\infty} \frac{\phi_1^{(k,n-i+1)}}{k^{\alpha(i)} \psi_1^{(k)}} \\ &= (-1)^{e_1-1} f_{e_1-1}(u_2, \ldots, u_{e_1}) \frac{r_{i-e_{\lambda-1}+1}^{(\lambda)}}{r_0^{(2)}}.\end{aligned}$$

This proves part (b). To prove the second part of the theorem a similar argument may be used.

3. The main theorem. Let us define modifying factors $w_k^{(i)} = w^{(i)}$ for $i = 1, \ldots, n$; $k \geq 1$ by

$$w^{(i)} = \lim_{k \to \infty} \xi_k^{(i)} \tag{32}$$

using Theorem I.

THEOREM II. *Given the n-fraction*

$$K_{k=1}^{\infty} \begin{pmatrix} a_k^{(1)} \\ \vdots \\ a_k^{(n)} \\ b_k \end{pmatrix} \tag{1a}$$

satisfying the assumptions (10), (16) *and* (24) *or* (25) *and such that for* $j = 1, \ldots, n$

$$\{A_k^{(j)}\} \notin \operatorname{vect}\left\{\{D_k^{(i)}\} | i = e_\rho + 1, \ldots, n+1\right\} \quad \text{with } \rho = \begin{cases} 1 & \text{for Type I,} \\ 2 & \text{for Type II} \end{cases} \tag{33}$$

(*for a set* V vect V *denotes the vector space spanned by the elements of* V). *If the n-fraction converges to a finite value* $\{\xi_0^{(1)}, \ldots, \xi_0^{(n)}\} \in \mathbf{C}^n$, *we have*

$$\lim_{k \to \infty} \frac{\xi_0^{(i)} - S_k^{(i)}(w^{(1)}, \ldots, w^{(n)})}{\xi_0^{(i)} - S_k^{(i)}(0, \ldots, 0)} = 0 \qquad (i = 1, \ldots, n). \tag{34}$$

REMARK. In [**3**] de Bruin and Jacobsen proved this theorem for limit-1-periodic n-fractions. As seen from our point of view limit-1-periodic n-fractions are n-fractions associated with a Perron-Kreuser recurrence relation of Type I with $\rho = \sigma = 1$.

EXAMPLE. Consider the 3-fraction

$$a_k^{(1)} = 51k + 2500 + \frac{1}{k+1} + \frac{1}{k(k+1)},$$
$$a_k^{(2)} = -100k - 5051 + \frac{1}{k+1} + \frac{1}{k(k+1)},$$
$$b_k = k + 150 + \frac{1}{k+1} + \frac{1}{k(k+1)}$$

with $w^{(1)} = 51$, $w^{(2)} = -100$. The exact value of the GCF (all digits correct) is

$$\xi_0^{(1)} = 25.849821113137, \quad \xi_0^{(2)} = -51.896713180952.$$

In the tables below values of $S_k^{(i)}(0,0)$ and $S_k^{(i)}(w^{(1)}, w^{(2)})$ (Table 1) and of the acceleration coefficients

$$\eta_k^{(i)} = (\xi_0^{(i)} - S_k^{(i)}(w^{(1)}, w^{(2)}))/(\xi_0^{(i)} - S_k^{(i)}(0,0))$$

(Table 2) are given. (The acceleration shows up rather late since the constant terms are so large.)

PROOF OF THE MAIN THEOREM. The proof given in [**3**] for the special case mentioned above is still valid, only a few changes have to be made. The proof consists of three steps:

(35) (a) $S_{p+k}^{(i)}(0,\ldots,0) - S_k^{(i)}(0,\ldots,0) \neq 0 \quad \text{for } k \geq k_0,\ p \geq p_0.$

(36) (b) $$\lim_{p\to\infty} \frac{S_{p+k}^{(i)}(0,\ldots,0) - S_k^{(i)}(w^{(1)},\ldots,w^{(n)})}{S_{p+k}^{(i)}(0,\ldots,0) - S_k^{(i)}(0,\ldots,0)} \in \mathbf{C}.$$

(c) The proof of (34).

Table 1

ν	$S_\nu^{(1)}(0,0)$	$S_\nu^{(1)}(w^{(1)},w^{(2)})$	$S_\nu^{(2)}(0,0)$	$S_\nu^{(2)}(w^{(1)},w^{(2)})$
1	16.78947368	49.07692308	−33.88157895	−98.05769231
10	25.79789811	33.00217711	−51.79348166	−66.11677994
20	25.84765251	26.10655259	−51.89240164	−52.40713782
30	25.84878915	25.86690369	−51.89466146	−51.93067617
40	25.84890247	25.85223504	−51.89488677	−51.90151247
50	25.84894320	25.85029284	−51.89496774	−51.89765105
60	25.84898775	25.84990368	−51.89505631	−51.89687734
70	25.84910613	25.84983028	−51.89529167	−51.89673140
75	25.84925570	25.84982352	−51.89558906	−51.89671797
80	25.84948770	25.84982165	−51.89605030	−51.89671424
85	25.84969579	25.84982121	−51.89646401	−51.89671338
90	25.84979017	25.84982113	−51.89665165	−51.89671321
95	25.84981540	25.84982112	−51.89670181	−51.89671319
100	25.84982026	25.84982111	−51.89671148	−51.89671318
105	25.84982101	25.84982111	−51.89671297	−51.89671318
110	25.84982110	25.84982111	−51.89671316	−51.89671318
115	25.84982111	25.84982111	−51.89671318	−51.89671318

Table 2

ν	$\eta_\nu^{(1)}$	$\eta_\nu^{(2)}$
1	− 2.56359948	− 2.56234444
10	− 137.74927241	− 137.74927241
20	− 118.38567573	− 118.38567573
30	− 16.55343491	− 16.55343491
40	− 2.62770954	− 2.62770954
50	− 0.53732460	− 0.53732460
60	− 0.09907743	− 0.09907743
70	− 0.01281565	− 0.01281565
75	− 0.00425901	− 0.00425901
80	− 0.00159917	− 0.00159917
85	− 0.00079294	− 0.00079294
90	− 0.00050498	− 0.00050498
95	− 0.00036394	− 0.00036394
100	− 0.00027561	− 0.00027561
105	− 0.00021376	− 0.00021376
110	− 0.00016843	− 0.00016843
115	− 0.00013477	− 0.00013468

The fact that all limits $\xi_0^{(i)}$ are finite and the fact that the basis $\{D_k^{(i)}\}$ is ordered by domination together imply that $\{B_k, D_k^{(2)}, \dots, D_k^{(n+1)}\}$ is also a basis for (1b) with

$$\text{(37)} \qquad \lim_{k\to\infty} \frac{B_k}{k^{\alpha_1} B_{k-1}} = u_1 \quad (\neq 0), \quad \lim_{k\to\infty} \frac{D_k^{(i)}}{B_k} = 0 \quad (i = 2, \dots, n).$$

Using the same argument as in [**3**], we have

$$\text{(38)} \qquad A_k^{(i)} = \delta_1^{(i)} B_k + \sum_{q=2}^{n+1} \delta_q^{(i)} D_k^{(q)} \qquad (i = 1, \dots, n; k \geq 1)$$

for some $\delta_q^{(i)}$ with $\delta_1^{(i)} = \xi_0^{(i)}$ and

$$\text{(39)} \qquad \forall i \in \{1, \dots, n\} \exists t \in \{2, \dots, e_\rho\} (\delta_q^{(i)} = 0 \ (q = 2, \dots, t-1) \& \delta_t^{(i)} \neq 0)$$

with ρ as in (33).

The proof of (a) is exactly the same as in [**3**]: we write

$$S_{p+k}^{(i)}(0, \dots, 0) - S_k^{(i)}(0, \dots, 0) = \frac{A_{p+k}^{(i)}}{B_{p+k}} - \frac{A_k^{(i)}}{B_k} = \frac{A_{p+k}^{(i)} B_k - A_k^{(i)} B_{p+k}}{B_{p+k} B_k}.$$

From (37) it follows that the denominator is different from zero for $k \geq k_1$, $p \geq 1$. Clearly it is also different from ∞. Using (38) the numerator becomes

$$A_{p+k}^{(i)} B_k - A_k^{(i)} B_{p+k} = B_{p+k} D_k^{(t)} \sum_{q=t}^{n+1} \delta_q^{(i)} \left(\frac{D_{p+k}^{(q)} B_k}{B_{p+k} D_k^{(t)}} - \frac{D_k^{(q)}}{D_k^{(t)}} \right)$$

where the factor before the summation sign is different from zero for $k \geq k_2 \geq k_1$, $p \geq 1$ by (37) and (17) since $u_i \neq 0$ for all i, and

$$\begin{aligned} \left| \sum_{q=t}^{n+1} \delta_q^{(i)} \left(\frac{D_{p+k}^{(q)} B_k}{B_{p+k} D_k^{(t)}} - \frac{D_k^{(q)}}{D_k^{(t)}} \right) \right| &= \left| \sum_{q=t}^{n+1} \delta_q^{(i)} \frac{D_k^{(q)}}{D_k^{(t)}} \left(\frac{D_{p+k}^{(q)} B_k}{D_k^{(q)} B_{p+k}} - 1 \right) \right| \\ &\geq \frac{1}{2} \left| \delta_t^{(i)} \left(\frac{D_{p+k}^{(t)} B_k}{D_k^{(t)} B_{p+k}} - 1 \right) \right| \geq \frac{1}{4} |\delta_t^{(i)}| > 0 \end{aligned}$$

for sufficiently large k and p. (The first inequality follows since $D_k^{(q)}/D_k^{(t)} \to 0$ for all $q > t$ and since $D_{p+k}^{(q)} B_k / D_k^{(q)} B_{p+k} \to 0$ as $p \to \infty$ and is thus bounded from some p and k on. The second inequality comes from using $D_{p+k}^{(q)} B_k / D_k^{(q)} B_{p+k} \to 0$ again for $q = t$.) This proves (a).

Using (5) we rewrite the expression in (b) as

$$\text{(40)} \qquad \begin{aligned} &\frac{S_{p+k}^{(i)}(0, \dots, 0) - S_k^{(i)}(w^{(1)}, \dots, w^{(n)})}{S_{p+k}^{(i)}(0, \dots, 0) - S_k^{(i)}(0, \dots, 0)} \\ &\quad = \left(1 + \sum_{j=0}^{n-1} w^{(n-j)} \frac{B_{k-j-1}}{B_k} \right)^{-1} \sum_{j=-1}^{n-1} w^{(n-j)} q_{p,k,j}^{(i)} \end{aligned}$$

where $w^{(n+1)} = 1$ and

$$q_{p,k,j}^{(i)} = \frac{A_{p+k}^{(i)} B_{k-j-1} - A_{k-j-1}^{(i)} B_{p+k}}{A_{p+k}^{(i)} B_k - A_k^{(i)} B_{p+k}}$$
$$= \sum_{q=2}^{n+1} \delta_q^{(i)} \left(\frac{D_{p+k}^{(q)}}{B_{p+k}} B_{k-j-1} - D_{k-j-1}^{(q)} \right) \Bigg/ \sum_{q=2}^{n+1} \delta_q^{(i)} \left(\frac{D_{p+k}^{(q)}}{B_{p+k}} B_k - D_k^{(q)} \right).$$

The first factor in (40) satisfies (use (37))

(41)
$$\lim_{k\to\infty} \left(1 + \sum_{j=0}^{n-1} w^{(n-j)} \frac{B_{k-j-1}}{B_k} \right) = 1 \quad \text{if the recurrence relation is of Type II,}$$
$$= \frac{P_{u_2,\dots,u_{e_1}}(u_1)}{u_1^{e_1-1}} \neq 0 \quad \text{for Type I (use (21)).}$$

It is therefore finite for sufficiently large k. For the second factor in (40) we have

$$\lim_{p\to\infty} q_{p,k,j}^{(i)} = \frac{\sum_{q=t}^{n+1} \delta_q^{(i)} D_{k-j-1}^{(q)}}{\sum_{q=t}^{n+1} \delta_q^{(i)} D_k^{(q)}} = \frac{\sum_{q=t}^{n+1} \delta_q^{(i)} D_{k-j-1}^{(q)} / D_k^{(t)}}{\sum_{q=t}^{n+1} \delta_q^{(i)} D_k^{(q)} / D_k^{(t)}}$$

where

$$\left| \sum_{q=t}^{n+1} \delta_q^{(i)} D_k^{(q)} / D_k^{(t)} \right| \geq \frac{1}{2} |\delta_t^{(i)}| > 0$$

for $k \geq k_0$. This proves (b).

(c) Since

$$\lim_{p\to\infty} S_{p+k}^{(i)}(0,\dots,0) = \xi_0^{(i)} \qquad (k \geq 1)$$

we have for the acceleration coefficients

$$\lim_{k\to\infty} \frac{\xi_0^{(i)} - S_k^{(i)}(w^{(1)},\dots,w^{(n)})}{\xi_0^{(i)} - S_k^{(i)}(0,\dots,0)}$$
$$= \lim_{k\to\infty} \lim_{p\to\infty} \frac{S_{p+k}^{(i)}(0,\dots,0) - S_k^{(i)}(w^{(1)},\dots,w^{(n)})}{S_{p+k}^{(i)}(0,\dots,0) - S_k^{(i)}(0,\dots,0)}$$
$$= \lim_{k\to\infty} \left\{ \left(1 + \sum_{j=0}^{n-1} w^{(n-j)} \frac{B_{k-j-1}}{B_k} \right)^{-1} \right.$$
$$\left. \times \sum_{j=-1}^{n-1} w^{(n-j)} \frac{\sum_{q=t}^{n+1} \delta_q^{(i)} D_{k-j-1}^{(q)} / D_{k-j-1}^{(t)}}{\sum_{q=t}^{n+1} \delta_q^{(i)} D_k^{(q)} / D_{k-j-1}^{(t)}} \right\} \quad \text{with } t \leq e_\rho$$
$$= \frac{u_1^{e_1-1}}{P_{u_2,\dots,u_{e_1}}(u_1)} \sum_{j=-1}^{n-1} w^{(n-j)} \frac{\delta_t^{(i)}}{\delta_t^{(i)} u_t^{j+1}} = \frac{u_1^{e_1-1}}{P_{u_2,\dots,u_{e_1}}(u_1)} \frac{P_{u_2,\dots,u_{e_1}}(u_t)}{u_t^{e_1-1}} = 0$$

if the recurrence relation is of Type I, and

$$\sum_{j=-1}^{n-1} w^{(n-j)} \frac{\delta_t^{(i)}}{\delta_t^{(i)} u_t^{j+1}} = \frac{P_{u_2,\ldots,u_{e_2}}(u_t)}{u_t^{e_2-1}} = 0$$

if the recurrence relation is of Type II, where we have used (22) and (33).

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