

THE ACTION OF A SOLVABLE GROUP ON AN INFINITE SET NEVER HAS A UNIQUE INVARIANT MEAN

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ABSTRACT. Theorem 1 of the paper proves a conjecture of J. Rosenblatt on nonuniqueness of invariant means for the action of a solvable group G on an infinite set X . The same methods used in this proof yield even a more general result: Nonuniqueness still holds if G is an amenable group containing a solvable subgroup H such that $\text{card}(G/H) \leq \text{card}(H)$.

1. Introduction. Theorem 1 shows that for the action of a solvable group G on an infinite set X there can never exist a unique G -invariant mean on $l^\infty(X)$, where $l^\infty(X)$ denotes the space of all bounded real valued functions on X . A mean m on $l^\infty(X)$ is a positive linear functional on $l^\infty(X)$ with $m(1_X) = 1$, m is called a " G -invariant mean" if $m(gf) = m(f)$ for every $f \in l^\infty(X)$ and for every $g \in G$, where $gf(x) = f(g^{-1}(x))$. Theorem 2 is a generalization of Theorem 1 and proves nonuniqueness for the action of an amenable group G on X , where G contains a solvable subgroup H such that $\text{card}(G/H) \leq \text{card}(X)$. This paper is largely inspired by [4]. Proposition 6 of [4] shows that if G is nilpotent, then there will always exist more than one G -invariant mean on $l^\infty(X)$. Additionally it is stated in the same paper [4, p. 531]: "It seems to us very unlikely that Proposition 6 is false if G is solvable, but the same proof does not work". Nonuniqueness need not hold for the general amenable group action as Yang [6] recently showed. This solves a conjecture which is known as "Rosenblatt's problem" in the mathematical literature [5, p. 628]. The proof uses the continuum hypothesis. In Theorem 1 and Theorem 2 even more than nonuniqueness is proved: There exist two disjoint subsets Y_1, Y_2 of X such that $\bigcap_{i=1}^n g_i Y_k \neq \emptyset$ for $k = 1, 2$ and for every finite subset $\{g_1, \dots, g_n\}$ of G . If G does not act transitively on X , then there always exist many G -invariant means. The methods used in [4] show nonuniqueness if $\text{card}(G) \leq \text{card}(X)$. Obviously there exist examples of solvable groups G such that $\text{card}(G) > \text{card}(X)$ and such that G acts transitively on X . Example 3 of [6] even gives an example of a nilpotent group with this property.

EXAMPLE 1. We put $X = \bigoplus_{i=1}^{\infty} Z_i$ where $Z_i = \mathbf{Z}$ for every $i \in \mathbf{N}$. For $A \subset \mathbf{N}$ we define π_A as follows. If $x = (n_i)_{i \in \mathbf{N}}$ let $\pi_A(x) = (m_i)_{i \in \mathbf{N}}$ where $m_i = n_i$ if $i \notin A$ and $m_i = -n_i$ if $i \in A$. For $a \in X$ set $\tau_a(x) = a + x$ where addition is defined coordinatewise. Let G be the group generated by τ_a for all $a \in X$ and by π_A for every $A \subset \mathbf{N}$, then G is solvable and $\text{card}(G) > \text{card}(X)$.

In the proof, we use the following notations: $\text{Per}(X)$ will denote the group of all invertible functions from X onto X . For $g \in \text{Per}(X)$ and $Y \subset X$ we put $gY = \{g(y) : y \in Y\}$. For $C \subset \text{Per}(X)$ we put $\mathcal{O}_C = \{C_x : x \in X\}$ to be the set of

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all C -orbits in X . For $F \subset \text{Per}(X)$, and $Y \subset X$, we put $FY = \{fY : f \in F\}$. For $A, B \subset \text{Per}(X)$, AB denotes the complex product. \mathcal{C} will always denote the set of all cardinals c such that $c \leq \text{card}(X)$.

2. Proof of the nonuniqueness of the mean. It is the essential problem of the following proof that we have to consider group actions where $\text{card}(G) > \text{card}(X)$. The rough idea is to split up the action of G into the action of a finite nested sequence of subsets P_i of $\text{Per}(X)$ on P_i -orbits A_i such that $\text{card}(P_i) \leq \text{card}(A_i)$. Then we use the transfinite induction argument of Rosenblatt and Tala-grand [4]. Unfortunately, if $\text{card}(X)$ is an uncountable limit cardinal, some problems arise. In this case, we have to choose a family of P_i -orbits $A_m : m \in M$ with increasing cardinality where the index set M is an infinite subset of \mathcal{C} with supremum $\text{card}(X)$ such that $\text{card}(A_m) = m$ for every $m \in M$. To each orbit A_m there will correspond a subset Q_m of $\text{Per}(X)$ such that $\text{card}(Q_m) \leq \text{card}(A_m)$. For each A_m and Q_m we shall use the transfinite induction argument of [4]. Finally we make a transfinite induction on M to get the desired result. This does not work for an arbitrary M . However, if M is a discrete subset of \mathcal{C} , where \mathcal{C} is endowed with the order topology, the argument can be done. It is proved in Lemma 2 that there always exists a discrete subset M of \mathcal{C} , which is still large enough for our purposes.

Before starting with the proof of Lemma 1, we state the following two well-known equalities for cardinals, we are going to use throughout the proof (cf. [1, p. 128]):

(i) $\aleph_\alpha^n = \aleph_\alpha$ for every ordinal α and for every $n \in \mathbb{N}$.

(ii) $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_{\max(\alpha, \beta)}$ for all ordinals α, β . We also use standard results about amenable groups (cf. [2]). Throughout we assume that \mathcal{C} is endowed with the order topology. We start with a technical lemma.

LEMMA 1. (i) *Let M be a subset of \mathcal{C} with supremum s . Then there exists a discrete subset M' of M with supremum s .*

(ii) *Let M be a discrete subset of \mathcal{C} . For every $m \in M$ let $N(m)$ be a discrete subset of \mathcal{C} with supremum m . Then for every $m \in M$ there exists a cardinal $c_m \in \mathcal{C}$ such that all the sets $N'(m) = \{n \in N(m) : n \geq c_m\}$ are nonempty and pairwise disjoint and such that $\bigcup_{m \in M} N'(m)$ is discrete.*

PROOF. (i) We can assume that $s \notin M$ otherwise we would choose $M' = \{s\}$. We now construct M' via transfinite induction on $M - \{s\}$. Assume that we have constructed for every $c < a$, $c, a \in M - \{s\}$ a discrete subset $M(c)$ of M such that $M(c)$ contains an element b with $b \geq c$ and such that for $c, c' \in M$, $c < c' < a$ implies that $M(c) \subset M(c')$. We put $N = \bigcup_{c \in M, c < a} M(c)$. If there exists an element m of N such that $m \geq a$ then we put $M(a) = N$ otherwise we choose a cardinal $b \in M$ such that $a < b < s$ and put $M(a) = N \cup \{b\}$. In both cases $M(a)$ is discrete. We define $M' = \bigcup_{m \in M, m < s} M(m)$. M' is a discrete subset of M and has s as supremum.

(ii) We use again a transfinite induction: Assume that we have constructed such a net of cardinals c_m for every $m \in M$ with $m < a$ where $a \in M$. Let $N = \bigcup_{m \in M, m < a} N(m)$, then $\text{card}(N) < a$, because if $\text{card}(N) = a$ then $a \in \overline{N}$, this together with $a \notin N$ implies that there does not exist a neighborhood V of a such that $V \cap M = \{a\}$ which is a contradiction to the discreteness of M . If $a \in N(a)$ we put $c_a = a$; otherwise we choose c_a such that $\text{card}(N) < c_a < a$.

Definition 1 now defines a cardinality property we are going to need in Definition 2. It is the condition that allows us to use the transfinite induction argument (cf. [4]).

DEFINITION 1. Let $C \subset \text{Per}(X)$ and $Y \subset X$. We say that a set $C' \subset \text{Per}(X)$ is Y -complete with respect to C if C' is finite or $\text{card}(C') \leq \text{card}(Y)$ and if for every $c \in C$ there exists a $c' \in C'$ such that $c'(y) = c(y)$ for every $y \in Y$.

EXAMPLE 2. Let A be an abelian subgroup of $\text{Per}(X)$ and let Y be an A -invariant subset of X such that A acts transitively on Y . Then there exists a subset A' of A which is Y -complete with respect to A .

We now give a definition, which we shall explain below.

DEFINITION 2. Let G be a subgroup of $\text{Per}(X)$ and let $C_i: i \in \{1, \dots, n\}$, D and E be subsets of $\text{Per}(X)$. For every $i \in \{1, \dots, n\}$ we put $P_i = C_i C_{i+1} \cdots C_n D E$. Furthermore we put $P_{n+1} = D E$ and $P_{n+2} = E$. We say that $(C_i: i \in \{1, \dots, n\}, D, E)$ is a splitting of G if

- (i) $G \subset P_1$,
- (ii) for every $i \in \{1, \dots, n+2\}$ and for every $A_1, A_2 \in \mathcal{O}_{P_i}$, $A_1 \cap A_2 \neq \emptyset$ implies that $A_1 = A_2$,
- (iii) for every $i \in \{1, \dots, n+2\}$ and for every $A \in \mathcal{O}_{P_i}$, we have $pA \subset A$ for every $p \in P_i$,
- (iv) for every $i \in \{1, \dots, n\}$ and for every $A \in \mathcal{O}_{P_i}$, there exists a set which is A -complete with respect to C_i ,
- (v) for every $A \in \mathcal{O}_E$ and for every $d \in D$, we have $dA \in \mathcal{O}_E$; and if $dA = d'A$ for every $A \in \mathcal{O}_E$, then $d = d'$.

Definition 2 introduces the groups for which we shall prove the nonuniqueness of the mean. Property (i) says that the group action is fully described by the family of subsets of $\text{Per}(X)$, (ii) and (iii) guarantee that we have properly shaped orbits. (iv) is the cardinality property which is necessary for the transfinite induction argument. The sets D and E as well as property (v) will be used for the first time in the proof of Theorem 1, when we show that every solvable group admits a splitting where $D = E = \{e\}$. They are just defined in that way to keep the proof of Theorem 1 short.

LEMMA 2. Let $(C_i: i \in \{1, \dots, n\}, \{e\}, \{e\})$ be a splitting of G where G acts transitively on X . Then there exists a $j \in \{1, \dots, n\}$ and a net of P_j -orbits $A_m: m \in M$ where $M \neq \emptyset$ such that

- (i) M is a discrete subset of \mathcal{C} ,
- (ii) for every $m \in M$ we have $\text{card}(A_m) = m$,
- (iii) for every $i \in \{1, \dots, j-1\}$ and for every $B \in \mathcal{O}_{P_i}$, such that $M_B \neq \emptyset$ where $M_B = \{m \in M: A_m \subset B\}$, we have $\text{card}(\bigcup_{m \in M_B} A_m) = \text{card}(B)$,
- (iv) for every $m \in M$ there exists a cardinal $c \in \mathcal{C}$ such that for every P_{j+1} -orbit B which is contained in A_m , $\text{card}(B) \leq c < m$ holds.

PROOF. We reformulate Lemma 2 such that an induction argument can be applied.

Claim 1. Let $(C_i: i \in \{1, \dots, n\}, \{e\}, \{e\})$ and G be as above and let $t \in \{1, \dots, n\}$; then there exists a $j \leq t$ and a net of P_j -orbits $A_m: m \in M$ such that either $j \leq t$ and (i)-(iv) of Lemma 2 hold or $j = t$ and (i)-(iii) of Lemma 2 hold.

We prove Claim 1 by induction on t : For $t = 1$ there is nothing to prove. We assume therefore that Claim 1 holds for $t < n$ and give the proof for $t + 1$.

Let $A_m : m \in M$ be the net of P_j -orbits. We can assume that $j = t$, otherwise we would have the proof for $t + 1$. The essential idea of the following is the construction of a net of P_{t+1} -orbits $B_n : n \in N(m)$ in each A_m such that $\text{card}(B_n) = n$ for every $n \in N(m)$ and such that $\text{card}(\bigcup_{n \in N(m)} B_n) = m$. Lemma 1(i) and (ii) then complete the proof. The sets $B_n : n \in N(m)$ however can only be constructed if Claim 2 from below holds. Claim 3 proves that we can assume without loss of generality that Claim 2 holds.

Claim 2. For every $m \in M$ and for every cardinal c such that $c < m$ there exists a P_{t+1} -orbit B such that $B \subset A_m$ and $c < \text{card}(B) \leq m$.

Claim 3. Let $M_1 = \{m \in M : \text{for every cardinal } c < m \text{ there exists a } P_{t+1}\text{-orbit } B \subset A_m \text{ such that } c < \text{card}(B) \leq m\}$ and let $M_2 = M - M_1$. Then for every $k \in \{1, \dots, t\}$ and for every P_k -orbit B such that $M_B \neq \emptyset$ there exists a net $A_n : n \in N$ of P_t -orbits which are contained in B such that

- (i) either $N \subset M^1$ or $N \subset M^2$,
- (ii) for every $i \in \{k, \dots, t\}$ and for every P_i -orbit B' such that $N_{B'} \neq \emptyset$ we have $\text{card}(\bigcup_{n \in N_{B'}} A_n) = \text{card}(B')$.

We prove Claim 3 by induction on k . For $k = t$ there is nothing to prove. We now assume that Claim 3 holds for $k > 1$ and give the proof for $k - 1$.

Let B be a P_{k-1} -orbit such that $M_B \neq \emptyset$. Then there exists a net $B_b : b \in \beta$ of P_k -orbits such that $\text{card}(B_b) = b$, such that $M_{B_b} \neq \emptyset$ for every $b \in \beta$ and such that β has $\text{card}(B)$ as supremum. We fix $b \in \beta$; then there exists a net $A_n : n \in N(b)$ such that (ii) holds and such that either $N(b) \subset M^1$ or $N(b) \subset M^2$. Let β^1 be the subset of β where $N(b) \subset M^1$ and β^2 the subset of β where $N(b) \subset M^2$. Then because $\beta = \beta^1 \cup \beta^2$ either β^1 or β^2 has $\text{card}(B)$ as supremum in \mathcal{C} . Lemma 1(i) and (ii) applied to that β^i yields the existence of a net $A_n : n \in N$ such that (i) and (ii) of Claim 3 hold for B . This, however, implies Claim 3.

For $k = 1$ Claim 3 implies that we can assume for the proof of Claim 1 that Claim 2 holds: The net $A_n : n \in N$ fulfils (i)–(iii) of Lemma 2 and also (iv) of Lemma 2 if $N \subset M^2$ which would conclude the proof of Claim 1. Therefore we can assume that $N \subset M^1$ which means that Claim 2 holds.

We now fix a $m \in M$. Claim 2 and Lemma 1(i) imply the existence of a net of P_{t+1} -orbits $B_n : n \in N(m)$ such that $B_n \subset A_m$ and $\text{card}(B_n) = n$ for every $n \in N(m)$ and such that $N(m)$ is a discrete subset of \mathcal{C} with supremum m . Lemma 1(ii) now implies the existence of a net $B_n : n \in N$ of P_{t+1} -orbits such that (i)–(iii) of Lemma 2 hold. This proves Claim 1. For $t = n$ Claim 1 implies Lemma 2.

We are now able to show the nonuniqueness of the mean for amenable groups which have a splitting where $D = E = \{e\}$ by proving the existence of two disjoint subsets A_1, A_2 such that $\bigcap_{i=1}^n g_i A_k \neq \emptyset$ for $k = 1, 2$ and for every finite subset $\{g_1, \dots, g_n\}$ of G . Claim 4 contains the transfinite induction argument of [4]. It is the proof of the induction step of Claim 5. The proof of Claim 5 is the transfinite induction on M which together with Claim 6 yields the complete result.

LEMMA 3. *Let G be an amenable subgroup of $\text{Per}(X)$ and let $(C_i : i \in \{1, \dots, n\}, \{e\}, \{e\})$ be a splitting of G . If X is an infinite set then there exist many G -invariant means on $l^\infty(X)$.*

PROOF. We can assume that G acts transitively on X otherwise if Y_1 and Y_2 are two different G -orbits then there exist two G -invariant means m_1 and m_2 on $l^\infty(X)$ such that $m(1_{Y_k}) = 1$ for $k = 1, 2$ (cf. [2, pp. 16, 17] or [3, Lemma 1]) which implies that $m_1 \neq m_2$.

We can therefore take the net $A_m := m \in M$ of Lemma 2. We are now going to construct to each A_m a suitable subset Q_m of $\text{Per}(X)$ such that $\text{card}(Q_m) \leq \text{card}(A_m)$ which implies that for each $m \in M$ the transfinite induction argument can be applied and such that a subsequent induction on M yields the complete result.

We now consider the case where $\text{card}(X) > \aleph_0$. Let $A_m : m \in M$ be the net of P_j -orbits. We can assume that $m \geq \aleph_0$ for every $m \in M$ because the net $A_m : m \in M'$ where $M' = \{m \in M : m \geq \aleph_0\}$ also fulfils (i)-(iv) of Lemma 2.

Let B^k be the collection of all P_k -orbits which contain at least one A_m ; then $B^1 = \{X\}$ because G acts transitively. Let H_X be X -complete with respect to C_1 . For every $B \in B^2$ we now choose a subset H_X^B of H_X such that $\text{card}(H_X^B) \leq \text{card}(B)$, such that $H_X^{B_1} \subset H_X^{B_2}$ if $\text{card}(B_1) \leq \text{card}(B_2)$, and such that $\bigcup_{B \in B^2} H_X^B = H_X$. We can do that because of (iii) of Lemma 2. Let H_B be a B -complete subset of C_2 . We put $H_{X,B} = H_X^B H_B$; then $\text{card}(H_{X,B}) \leq \text{card}(B)$. For $B \in B^2$ let $B_B^3 = \{B' \in B^3 : B' \subset B\}$. Now we choose for every $B' \in B_B^3$ a subset $H_{X,B}^{B'}$ such that $\text{card}(H_{X,B}^{B'}) \leq \text{card}(B')$, such that $H_{X,B}^{B_1} \subset H_{X,B}^{B_2}$ if $\text{card}(B'_1) \leq \text{card}(B'_2)$, and such that $\bigcup_{B' \in B_B^3} H_{X,B}^{B'} = H_{X,B}$ and put $H_{X,B,B'} = H_{X,B}^{B'} H_{B'}$ where $H_{B'}$ is B' -complete with respect to C^3 . We continue this construction until we reach the P_j -orbits A_m . We put $Q_m = H_{X,B^2,B^3,\dots,B^{j-1},A_m}$. Then $\text{card}(Q_m) \leq m$.

If $\text{card}(X) = \aleph_0$ the construction is much easier. Instead of considering the net $A_m : m \in M$ of Lemma 2, we choose a P_j -orbit B where B is countable and where j is the largest index such that a countable orbit exists. We then put $A_{\aleph_0} = B$ and choose $A_m : m \in \{\aleph_0\}$ as net of P_j -orbits. Finally we choose for Q_{\aleph_0} a subset of $\text{Per}(X)$ which is A_{\aleph_0} -complete with respect to $C_1 C_2 \cdots C_j$.

Claim 4. Let $m \in M$ and $W \subset X$ such that $\text{card}(W) < m$. Let $F_m = \{F \subset Q_m : F \text{ finite}\}$ and let $F_\alpha : \alpha < \delta_m$ be an enumeration of F_m where δ_m is the least ordinal such that $\text{card}(\delta_m) = \text{card}(Q_m) \leq m$. Then there exists a net of P_{j+1} -orbits $P_\alpha^k : \alpha < \delta_m, k = 1, 2$, which are contained in A_m such that all sets $F_\alpha P_\alpha^k : \alpha < \delta_m, k = 1, 2$, together with W are pairwise disjoint.

We prove this with transfinite induction. Let $\beta < \delta_m$ and assume that we have constructed such a net for every $\alpha < \beta$. We now construct P_{j+1} -orbits P_β^1 and P_β^2 . Let $V = W \cup \bigcup_{\alpha < \beta} F_\alpha P_\alpha^1 \cup \bigcup_{\alpha < \beta} F_\alpha P_\alpha^2$. Then if $\text{card}(X) = \aleph_0$, we get $\text{card}(\bigcup_{\alpha < \beta} F_\alpha P_\alpha^k) < \aleph_0$ for $k = 1, 2$; and if $\text{card}(X) > \aleph_0$, there exists (because of (iv) of Lemma 2) a cardinal c such that $\text{card}(P_\alpha^k) \leq c < m$ for every $\alpha < \beta$ and for $k = 1, 2$, consequently we get $\text{card}(\bigcup_{\alpha < \beta} F_\alpha P_\alpha^k) \leq \max(c, \text{card}(\beta)) < m$ for $k = 1, 2$. Therefore we get in both cases $\text{card}(V) < m$. Let $F_\beta = \{q_1, \dots, q_r\}$, then $\text{card}(\bigcup_{i=1}^r q_i^{-1} V) < m$. If $\text{card}(X) = \aleph_0$, A obviously has to contain \aleph_0 -many P_{j+1} -orbits, if $\text{card}(X) > \aleph_0$ property (iv) of Lemma 2 implies the existence of m -many P_{j+1} -orbits in A_m . In both cases there therefore has to exist a P_{j+1} -orbit P_β^1 , such that $P_\beta^1 \subset A_m - \bigcup_{i=1}^r q_i^{-1} V$ which implies that $P_\beta^1 \subset A_m$ and that $F_\beta P_\beta^1, W$ together with all sets $F_\alpha P_\alpha^k : \alpha < \beta, k = 1, 2$, are pairwise disjoint. We now put

$V' = V \cup F_\beta P_\beta^1$ and repeat the construction from above to get a P_{j+1} -orbit P_β^2 . This proves Claim 4.

Claim 5. Let $A_m : m \in M$ and $Q_m : m \in M$ be as above; then there exist subsets Y_1, Y_2 of $X, Y_1 \cap Y_2 = \emptyset$ such that for every $m \in M$ and for every finite subset $\{q_1, \dots, q_r\}$ of Q_m there exist P_{j+1} -orbits Z_1, Z_2 such that $Z_1, Z_2 \subset A_m$ and $\bigcup_{i=1}^r q_i Z_k \subset Y_k$ for $k = 1, 2$.

We prove Claim 5 by induction on M . Let m_0 be the least cardinal of M . Let $W = \emptyset$ and $m = m_0$. Then Claim 4 implies Claim 5 for $m = m_0$ if we put $Y_k(m_0) = \bigcup_{\alpha < \delta_{m_0}} F_\alpha P_\alpha^k$ for $k = 1, 2$. We now assume that Claim 5 holds for every $n < m; n, m \in M$. That means for every $n < m$ there exists a subset $Y_k(n)$ of $X, \text{card}(Y_k(n)) \leq n$ such that Claim 5 holds for every finite subset of $Q_{n'}$, where $n' \leq n$ and that $Y_k(n) \subset Y_k(n')$ for $n \leq n' < m$ and for $k = 1, 2$. Let $W = \bigcup_{n < m} Y_1(n) \cup \bigcup_{n < m} Y_2(n)$; then $\text{card}(W) < m$ because $\text{card}(Y_k(n)) \leq n$ for $n \in M, n < m$ and for $k = 1, 2$ and because there exists a neighborhood V of m such that $V \cap M = \{m\}$ due to the discreteness of M . We now use Claim 4 to conclude the proof of Claim 5.

Claim 6. If there exist subsets Y_1, Y_2 of X such that for every $m \in M$ and for every finite subset $\{q_1, \dots, q_r\}$ of Q_m there exist P_{j+1} -orbits $Z_1, Z_2 \subset A_m$ such that $\bigcup_{i=1}^r q_i Z_k \subset Y_k$ for $k = 1, 2$ then for every finite subset $\{h_1, \dots, h_r\}$ of P_1 there exist $x_1, x_2 \in X$ such that $\bigcup_{i=1}^r h_i(x_k) \subset Y_k$ for $k = 1, 2$.

Let $h_i = c_{i,1} c_{i,2} \dots c_{i,j} b_i$ where $c_{i,k} \in C_k$ and $b_i \in P_{j+1}$. Then there exists a $c'_{i,1} \in H_X$ such that $c'_{i,1}(x) = c_{i,1}(x)$ for every $x \in X$. There exists a $B \in B^2$ such that $c'_{i,1} \in H_X^B$ for every $i \in \{1, \dots, r\}$. Let $c'_{i,2} \in H_B$ such that $c'_{i,2}(x) = c_{i,2}(x)$ for every $x \in B$. Then $c'_{i,1} c'_{i,2} \in H_{X,B}$. Therefore there exists a P_3 -orbit $B' \subset B$ such that $c'_{i,1} c'_{i,2} \in H_{X,B}^{B'}$ for every $i \in \{1, \dots, r\}$. We continue this construction until we reach a P_j -orbit A_m . Then there exists a subset $\{q_1, \dots, q_r\}$ of Q_m such that $h_i(x) = q_i b_i(x)$ for every $x \in A_m$ and for every $i \in \{1, \dots, r\}$. Let Z_1, Z_2 be P_{j+1} -orbits such that $Z_1, Z_2 \subset A_m$ and $\bigcup_{i=1}^r q_i Z_k \subset Y_k$ for $k = 1, 2$, then $q_i Z_k = c_{i,1} c_{i,2} \dots c_{i,j} Z_k$ for $k = 1, 2$ and $i \in \{1, \dots, r\}$; and because of (iii) of Definition 2, we get $b_i Z_k \subset Z_k$. This together implies Claim 6.

Let Y_1, Y_2 be the disjoint subsets of Claim 5 and let $\{g_1, \dots, g_r\}$ be an arbitrary finite subset of G . Then $\{g_1^{-1}, \dots, g_r^{-1}\} \subset P_1$ and because of Claim 5 and Claim 6 there exist $x_1, x_2 \in X$ such that $\bigcup_{i=1}^r g_i^{-1}(x_k) \subset Y_k$ for $k = 1, 2$. This however implies that $\bigcap_{i=1}^r g_i Y_k \neq \emptyset$ for $k = 1, 2$, which implies the existence of two G -invariant means m_1 and m_2 on $l^\infty(X)$ such that $m_k(1_{Y_k}) = 1$ for $k = 1, 2$ (cf. [3, Lemma 2]). As Y_1 and Y_2 are disjoint m_1 and m_2 have to be two different G -invariant means.

THEOREM 1. *Let G be a solvable group which acts on the infinite set X ; then there exist many G -invariant means on $l^\infty(X)$.*

PROOF. It is well known that every solvable group is amenable (cf. [2]). Because of Lemma 1 it now remains to prove that there exists a splitting $(C_i : i \in \{1, \dots, n\}, \{e\}, \{e\})$ of G .

Claim 7. Let $(C_i : i \in \{1, \dots, n\}, D, E)$ be a splitting of G . Because of (ii) and (v) of Definition 2 we can regard D to be a subset of $\text{Per}(O_E)$. We assume that D is a solvable subgroup of $\text{Per}(O_E)$. Then there exists a splitting $(C_i : i \in \{1, \dots, n'\}, \{e\}, E)$ of G .

We prove Claim 7 by induction on the length of the commutator chain of D . For $D = \{e\}$ there is nothing to prove. Assume that Claim 7 holds for every D with a commutator chain of length $k \leq n$ and assume that D has a commutator chain of length $n + 1$ that means $D = D_0 \supseteq D_1 \supseteq \dots \supseteq D_n \supseteq D_{n+1} = \{e\}$. Then $(C_i: i \in \{i, \dots, n\}, D/D_n, [\bigcap_{A \in \mathcal{O}_E} T_D(D_n A)]E)$ is a splitting of G where D/D_n is considered to be a subgroup of $\text{Per}(\mathcal{O}_{D_n E})$ and where $T_D(D_n A) = \{d \in D: d(D_n A) = D_n A\}$. As Claim 7 holds for n we therefore get a splitting $(C_i: i \in \{1, \dots, n'\}, \{e\}, [\bigcap_{A \in \mathcal{O}_E} T_D(D_n A)]E)$ of G which implies that $(C_i: i \in \{1, \dots, n'\}, \bigcap_{A \in \mathcal{O}_E} T_D(D_n A), E)$ is a splitting of G . We define $H_1 = \{h \in \text{Per}(X): \text{for every } A \in \mathcal{O}_{D_n E} \text{ there exists } d_A \in D_n \text{ such that } d_A(x) = h(x), \text{ for every } x \in A\}$. Let $A_i: i \in I$ be a collection of E -orbits such that each $D_n E$ -orbit contains exactly one of the sets A_i . For every $i \in I$ let S_i be a system of representatives for the cosets of $T_D(A_i)/(T_D(A_i) \cap D_n)$ where $T_D(A_i) = \{d \in D: d(A_i) = A_i\}$. S_i acts as a solvable group with a commutator chain of length $k \leq n$ on the E -orbits contained in $D_n A_i$. We now define $H_2 = \{h \in \text{Per}(X): \text{for every } i \in I \text{ there exists a } s_i \in S_i \text{ such that } s_i(x) = h(x) \text{ for every } x \in D_n A_i\}$, then $\bigcap_{A \in \mathcal{O}_E} T_D(D_n A) \subset H_1 H_2$. H_2 can be embedded into the group $\prod_{i \in I} S_i$ which is solvable with a commutator chain of length $k \leq n$. H_1 is an abelian group which acts transitively on every set $D_n A_i$; therefore Example 2 implies that for every $A \in \mathcal{O}_E$ there exists a set which is $D_n A$ -complete with respect to H_1 . We put $C_{n'+1} = H_1$ then $(C_i: i \in \{1, \dots, n' + 1\}, H_2, E)$ is a splitting of G , as H_2 has a commutator chain of length $k \leq n$ there has to exist a splitting $(C_i: i \in \{1, \dots, n''\}, \{e\}, E)$ of G which implies Claim 7.

Obviously $(\{e\}, G, \{e\})$ is a splitting of G . Claim 7 implies the existence of a splitting $(C_i: i \in \{1, \dots, n\}, \{e\}, \{e\})$ of G which yields because of Lemma 3 the nonuniqueness of the mean.

REMARK. We could simplify the proof of the result from above a lot, if we assume that $\text{card}(X)$ is no limit cardinal, because then all the nets $A_m: m \in M$ could be reduced to nets which contain only one element. Such an assumption, however, would not be justified as it is easy to construct examples where $\text{card}(X)$ is a limit cardinal, G is solvable and $\text{card}(G) > \text{card}(X)$ by similar methods as in Example 1.

We can even prove a slight generalization of Theorem 1.

THEOREM 2. *Let G be an amenable group which acts on the infinite set X . Let H be a solvable subgroup of G such that $\text{card}(G/H) \leq \text{card}(X)$. Then there exist many G -invariant means on $l^\infty(X)$.*

PROOF. Let C be a system of representatives for the cosets of G/H . Then $(C, H, \{e\})$ is a splitting of G . Claim 7 and Lemma 3 imply the nonuniqueness.

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REFERENCES

1. H. Bachmann, *Transfinite Zahlen*, *Ergeb. Math. Grenzgeb.*, Bd. 1, Springer-Verlag, Berlin and New York, 1967.
2. F. Greenleaf, *Invariant means on topological groups*, Van Nostrand, New York and London, 1969.

3. S. Krasa, *Non uniqueness of invariant means for amenable group actions*, *Monatsh. Math.* **100** (1985), 121–125.
4. J. Rosenblatt and M. Talagrand, *Different types of invariant means*, *J. London Math. Soc.* (3) **24** (1981), 525–532.
5. J. Rosenblatt, *Uniqueness of invariant means for measure preserving transformations*, *Trans. Amer. Math. Soc.* **265** (1981), 623–636.
6. Z. Yang, *Action of amenable groups and uniqueness of invariant means*, preprint.

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