TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 305, Number 2, February 1988

ON AITCHISON'S CONSTRUCTION BY ISOTOPY

DANIEL SILVER

ABSTRACT. We describe a method introduced by I. Aitchison for constructing doubly slice fibered n-knots. We prove that all high-dimensional simple doubly slice fibered n-knots can be obtained by this construction. (Even-dimensional n-knots are required to be Z-torsion-free.) We also show that any possible rational Seifert form can be realized by a doubly slice fibered classical knot.

Introduction. In his Ph.D. thesis [1] Iain R. Aitchison introduced a technique for obtaining doubly slice fibered $(n-2)$-knots in S^n called *construction by isotopy.* Aitchison proved that the doubly slice knot $k\sharp - k$ is constructible by isotopy for any fibered $(n-2)$-knot $k \subset S^n$, $n \geq 3$. He then made the provocative conjecture: Every doubly slice fibered knot is constructible by isotopy. This general conjecture remains unsettled.

We begin with a review of construction by isotopy. In §2 we study the odd-dimensional knots arising from the construction; in this case a Seifert matrix can be computed immediately from the initial data. Consequently, we verify Aitchison's conjecture for simple $(2q-1)$-knots, $q \geq 2$.

Construction by isotopy was explored for classical knots in [2]. We apply the main result of that paper in order to gain information about possible Seifert matrices for doubly slice fibered classical knots.

In the final section we combine construction by isotopy and Kojima's classification of simple Z-torsion-free fibered $2q$-knots, $q \geq 4$, in order to obtain necessary and sufficient conditions for such a knot to be doubly slice. Our characterization uses L.P. matrices and achieves a very simple form similar to Sumner's results for odd-dimensional knots [15]. Previously Kearton had obtained a criterion using F-forms. We verify Aitchison's conjecture also for this case.

We intend to examine construction by isotopy from a more geometric point of view in a future article.

All manifolds in this paper are smooth and oriented. Maps are smooth. Homology groups have integer coefficients, and homomorphisms are induced by inclusion unless otherwise indicated.

An $(n-2)$-knot, $n \geq 3$, is a submanifold k of the n-sphere S^n which is diffeomorphic to S^{n-2}. The knot is *trivial* if it bounds an embedded ball. We say that k is *fibered* if the closure of $S^n - \mathrm{nbd}(k)$ admits a locally trivial fibration over S^1 (we require that the fibration restricted to the boundary be trivial).

A ball pair (B^{n+1}, D^{n-1}) is *fibered* if the closure of $B^{n+1} - \mathrm{nbd}(D^{n-1})$ fibers over S^1, as above. Note that in this case the boundary of the ball pair is a fibered

Received by the editors September 30, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57M25, 57Q45.
Supported in part by the University of South Alabama Research and Public Service Council.

$(n-2)$-knot. A ball pair (B^{n+1}, D^{n-1}) is *invertible* if there exists another ball pair (B^{n+1}, D_0^{n-1}) such that $D^{n-1} = D_0^{n-1}$ and the $(n-1)$-knot created by the union $D^{n-1} \cup D_0^{n-1} \subset B^{n+1} \cup B^{n+1} = S^{n+1}$ is trivial. In this case, the $(n-2)$-knot $k = \partial D^{n-1}$ is said to be *doubly slice*, $n \geq 3$.

It is a pleasure to thank L. Richard Hitt and C. Kearton for helpful comments. I also wish to thank Iain Aitchison for introducing me to this topic.

1. Constructing knots by isotopy. Let $\mathcal{H}$ be a compact connected n-dimensional submanifold of S^n, $n \geq 3$. Choose a small n-ball B within $\mathcal{H}$ such that $B \cap \partial\mathcal{H}$ is an $(n-1)$-ball D.

Let $h_t \colon S^n \to S^n$, $0 \leq t \leq 1$, be an isotopy such that $h_0 =$ identity and h_1, restricted to $\mathcal{H}$, is a diffeomorphism $\phi \colon \mathcal{H} \to \mathcal{H}$. Assume that h_t preserves B, with h_1 restricting to the identity. The diffeomorphism ϕ induces an automorphism $\phi_* \colon H_r(\mathcal{H}) \to H_r(\mathcal{H})$ for each $r \geq 0$. We will assume that identity $-\phi_*$ is also an automorphism for each $r \geq 0$.

Construct the mapping torus $M_\phi = \mathcal{H} \times_\phi S^1$:

$$M_\phi = \mathcal{H} \times [0,1]/\{(x,0) \sim (\phi(x),1)\}.$$

We will identify $\mathcal{H}$ with $\mathcal{H} \times \{0\} \subset M_\phi$. Attach a 2-handle γ along $D \times S^1 \subset \partial M_\phi$ in order to obtain an $(n+1)$-manifold W. It is easy to verify that W is a homology ball.

PROPOSITION 1.1. *If $W = B^{n+1}$, then (W, D) is an invertible fibered ball pair.*

PROOF. It is obvious that (W, D) is fibered. To show that the ball pair is invertible we generalize an argument in [**2**, §5]: Let B' be a small n-ball in $S^n - \operatorname{int} \mathcal{H}$ such that $B' \cap \partial\mathcal{H} = D$. Recall that ϕ is the restriction of the diffeomorphism $h_1 \colon S^n \to S^n$. By further isotopy we may assume that h_1, restricted to B', is the identity. Consider the mapping torus $(S^n - \operatorname{int} B') \times_{h_1} S^1$, which is diffeomorphic to $B^n \times S^1$ (since h_1 is isotopic to the identity). Observe that $D \times S^1 \subset S^{n-1} \times S^1 = \partial B' \times S^1$. Regard $S^{n-1} \times D^2$ as $D_+^{n-1} \times D^2 \cup D_-^{n-1} \times D^2$, and glue $S^{n-1} \times D^2$ to $\partial B' \times S^1$ first by attaching the 2-handle $D_+^{n-1} \times D^2$ $(= \gamma)$ along $D \times S^1$, and then capping off with $D_-^{n-1} \times D^2$ to yield S^{n+1}. We obtain a trivial $(n-1)$-knot $S^{n-1} \times \{0\} \subset S^{n-1} \times D^2 \subset S^{n+1}$. Moreover, the intersection of $S^{n-1} \times \{0\}$ and $M_\phi \cup \gamma = W$ is D. Hence (W, D) is invertible. □

COROLLARY 1.2 [**1**, PROPOSITION 6.6]. *If $W = B^{n+1}$, then k is a doubly slice fibered $(n-2)$-knot in S^n. In particular, this is true when $n \geq 5$ and $\pi_1(\mathcal{H}) \cong \pi_1(\partial\mathcal{H}) \cong 1$.*

PROOF. The second assertion follows from the h-cobordism theorem. (See [**12**, p. 108].) □

When an $(n-2)$-knot k in S^n arises by the above construction we say that k is *constructible by isotopy*.

CONJECTURE (AITCHISON [**1**]). Every doubly slice fibered $(n-2)$-knot in S^n, $n \geq 3$, is constructible by isotopy.

2. Odd-dimensional case. Assume that $\mathcal{H} = \natural_{i=1}^d S^q \times D^{q+1} \subset S^{2q+1}$, where d, q are positive integers. In this case, when $W = B^{2q+1}$ construction by isotopy yields a simple doubly slice fibered $(2q-1)$-knot $k \subset S^{2q+1}$. (Recall that an

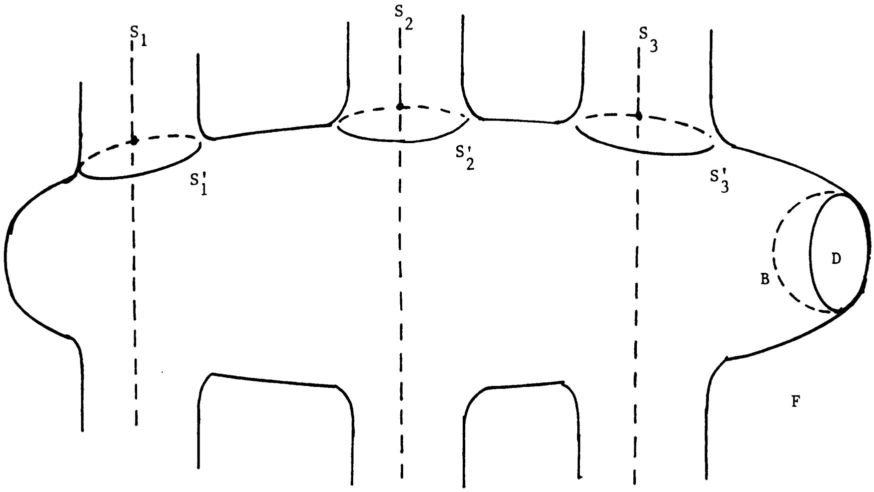

FIGURE 1

$(n-2)$-knot $k \subset S^n$ is *simple* if $\pi_i(S^n - k) \cong \pi_i(S^1)$ for $1 \leq i \leq (n-3)/2$.) We will compute a Seifert matrix for k:

Denote the boundary of $\mathcal{H}$ by F. Choose embedded q-spheres $S_1, \dots, S_d$, $S_1', \dots, S_d' \subset F - D$ (as in Figure 1) representing a symplectic basis $a_1, \dots, a_d$, $b_1, \dots, b_d$ for $H_q(F)$. Regarding $\mathcal{H}$ as a ball with q-handles $h_1^q, \dots, h_d^q$, S_i' is a belt sphere of the ith q-handle, and S_i is a dual sphere, $1 \leq i \leq d$.

Let $A = (\alpha_{ij})$ denote the $d \times d$ matrix representation of the induced automorphism $\phi_*\colon H_q(\mathcal{H}) \to H_q(\mathcal{H})$, with respect to the basis for $H_q(\mathcal{H})$ represented by $S_1, \dots, S_d$.

Notice that the punctured manifold $F_0 = F - \operatorname{int} D$ is a Seifert manifold (and a fiber) for the knot k.

PROPOSITION 2.1 [**1**, pp. 65–66]. *The monodromy $(\phi|_{F_0})_*\colon H_q(F_0) \to H_q(F_0)$ has matrix representation*

$$M = \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix},$$

with respect to the basis for $H_q(F_0)$ represented by $S_1, \dots, S_d$, $S_1', \dots, S_d'$.

COROLLARY 2.2. *The knot k has Seifert matrix*

$$\begin{pmatrix} 0 & (-1)^{q+1}A^t \\ I_d & 0 \end{pmatrix},$$

with respect to an appropriate basis for $H_q(F_0)$. ($I_d = d \times d$ identity matrix.)

PROOF OF PROPOSITION 2.1. It suffices to show that the automorphism $(\phi|_F)_*\colon H_q(F) \to H_q(F)$ has matrix representation M, with respect to $a_1, \dots, a_d$, $b_1, \dots, b_d$. Let $\hat{\mathcal{H}}$ denote the dual handlebody of $\mathcal{H}$, i.e. $\hat{\mathcal{H}} = S^{2q+1} - \operatorname{int} \mathcal{H}$. Each q-sphere S_i is null-homologous in $\mathcal{H}$. Since ϕ is the restriction of the final map of an ambient isotopy, the image $\phi(S_i)$ is also null-homologous in $\hat{\mathcal{H}}$. Hence

$$(\phi|_F)_*(a_i) = \sum_k \alpha_{ik} a_k, \qquad 1 \leq i \leq d.$$

Similarly, each q-sphere S_j' is null-homologous in $\mathcal{Y}$, so

$$(\phi|_F)_*(b_j) = \sum_k \beta_{jk} b_k, \qquad 1 \le j \le d,$$

for some integers β_{jk}. Since the intersection number $a_i \cdot b_j$ is δ_{ij}, we have

$$(\phi|_F)_*(a_i) \cdot (\phi|_F)_*(b_j) = \sum_k \alpha_{ik}\beta_{jk} = \delta_{ij}.$$

Hence $(\beta_{jk})^t = A^{-1}$. □

PROOF OF COROLLARY 2.2. The matrix

$$J = \begin{pmatrix} 0 & I_d \\ (-1)^q I_d & 0 \end{pmatrix}$$

describes the intersection pairing on F_0, with respect to the basis for $H_q(F_0)$ represented by $S_1, \ldots, S_d$, $S_1', \ldots, S_d'$. Let V denote the corresponding Seifert matrix. Then

$$M = (-1)^{q+1} V^{-1} V^t \quad \text{and} \quad J = V + (-1)^q V^t.$$

(See [8], for example.) Hence $J = V(I - M)$. Since $I - M$ is invertible, $V = J(I - M)^{-1}$. Computation reveals

$$V = \begin{pmatrix} 0 & (I - A^{-t})^{-1} \\ (-1)^q (I - A)^{-1} & 0 \end{pmatrix}.$$

Finally,

$$PVP^t = \begin{pmatrix} 0 & (-1)^{q+1} A^t \\ I_d & 0 \end{pmatrix},$$

where

$$P = \begin{pmatrix} I - A^{-t} & 0 \\ 0 & (-1)^{q+1} A \end{pmatrix}. \quad \square$$

The calculations above have several interesting applications:

By [15] any doubly slice fibered $(2q-1)$-knot in S^{2q+1} has a Seifert matrix of the form

$$(*) \qquad V = \begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix},$$

where R and S are integral unimodular matrices of the same size d. We can apply the congruence:

$$\begin{pmatrix} I_d & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix} \begin{pmatrix} I_d & 0 \\ 0 & S \end{pmatrix} = \begin{pmatrix} 0 & RS^{-t} \\ I_d & 0 \end{pmatrix}$$

to obtain an equivalent Seifert matrix V_0. Let $A = (-1)^{q+1} S^{-1} R^t$. Since $V_0 + (-1)^q V_0^t$ is unimodular, the matrix $I - A$ is also.

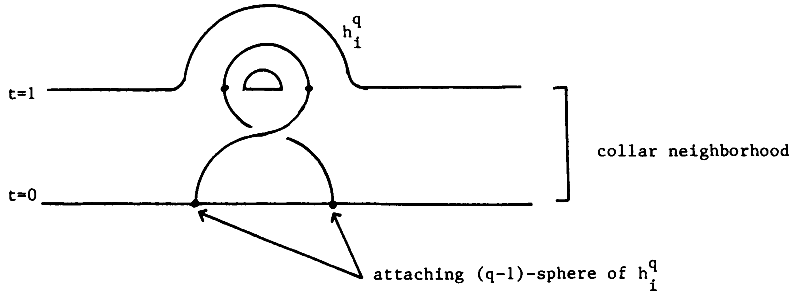

FIGURE 2a. Realizing U_i

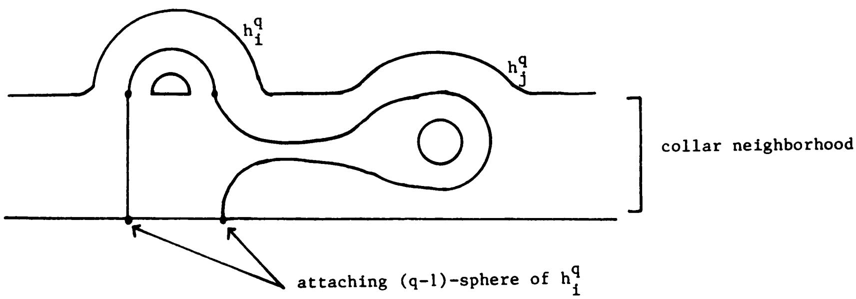

FIGURE 2b. Realizing T_{ij}

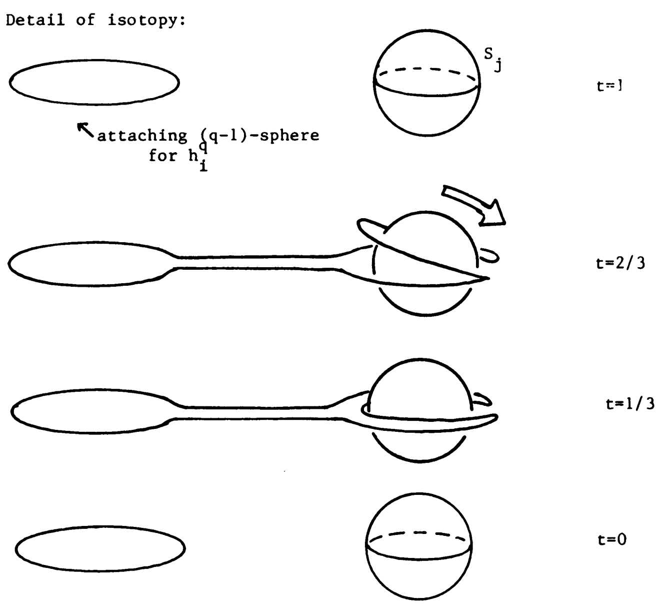

FIGURE 2c. Realizing T_{ij} (continued)

FIGURE 2

COROLLARY 2.3. *Any matrix of the form* $(*)$ *is a Seifert matrix of a simple* $(2q-1)$*-knot in* S^{2q+1}, $q \geq 2$, *constructible by isotopy.*

PROOF. Decompose A into elementary matrices of the following type:

$T_{ij} = d \times d$ matrix differing from I_d by $+1$ in the ij-position, $i \neq j$.

$U_i = d \times d$ matrix differing from I_d by -1 in the ii-position.

(See [**13**], for example.) Slide the q-handles of $\mathcal{H}$ according to the decomposition. By "sliding h_i^q" we mean isotoping the attaching sphere of h_i^q in a collar neighborhood of

$$\partial(B^{2q+1} \cup h_1^q \cup \cdots \cup h_{i-1}^q \cup h_{i+1}^q \cup \cdots \cup h_d^q),$$

and then extending the isotopy over the remainder of S^{2q+1}, as in Figure 2. (Handle slides should avoid B.) The restriction to $\mathcal{H}$ of the final map of the isotopy is a diffeomorphism $\phi\colon \mathcal{H} \to \mathcal{H}$. Moreover, the induced automorphism $\phi_*\colon H_q(\mathcal{H}) \to H_q(\mathcal{H})$ has matrix representation A, with respect to the basis for $H_q(\mathcal{H})$ represented by $S_1, \ldots, S_d$. Corollary 2.2 now completes the proof. □

Corollary 2.3 says that any "possible" Seifert matrix can be realized by a doubly slice fibered $(2q-1)$-knot in S^{2q+1}, $q \geq 2$. Recall that for a simple $(2q-1)$-knot in S^{2q+1}, $q \geq 2$, the S-equivalence class of any Seifert matrix is a complete isotopy invariant. (See [**10**].) Consequently, we obtain the following result which was stated (but incorrectly proved) in [**1**, Theorem 6.5].

COROLLARY 2.4. *Every simple doubly slice fibered* $(2q-1)$*-knot in* S^{2q+1}, $q \geq 2$, *is constructible by isotopy.*

Finally, let $(Q\alpha_V, [\ ])$ denote the rational Seifert form determined by $(*)$. (See [**16**].) We prove a result for classical knots.

COROLLARY 2.5. *There exists a doubly slice fibered ribbon knot* $k \subset S^3$ *with rational Seifert form* $(Q\alpha_V, [\ ])$.

PROOF. By the theory of rational canonical matrices we can find a matrix P, invertible over Q, such that

$$P^{-1}AP = \operatorname{diag}(C(f_1), \ldots, C(f_n)),$$

where $f_1, \ldots, f_n$ are the invariant factors of $A = S^{-1}R^t$, and $C(f_i)$ denotes the companion matrix of f_i. Denote $P^{-1}AP$ by D.

The characteristic polynomial of A is $f_1(t) \cdots f_n(t)$. Since $\det A = \pm 1$, the first and last coefficient of each $f_i(t)$ are ± 1. Also, since $\det(I - A) = \pm 1$, each $f_i(1) = \pm 1$. By the results of [**2**] together with Proposition 2.1, we can find a doubly slice fibered ribbon knot $k \subset S^3$ with monodromy

$$\begin{pmatrix} C(f_i) & 0 \\ 0 & (C(f_i))^{-t} \end{pmatrix}$$

and corresponding intersection pairing

$$\begin{pmatrix} 0 & I_{d_i} \\ -I_{d_i} & 0 \end{pmatrix},$$

with respect to an appropriate basis, $1 \leq i \leq n$. Here $d_i = \operatorname{degree} f_i$.

Consider the composite knot $k = k_1 \sharp \cdots \sharp k_n \subset S^3$. With respect to an appropriate basis, k has monodromy

$$\begin{pmatrix} D & 0 \\ 0 & D^{-t} \end{pmatrix}$$

and corresponding intersection pairing

$$\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

As in the proof of Corollary 2.2, k has Seifert matrix

$$V_1 = \begin{pmatrix} 0 & D^t \\ I_d & 0 \end{pmatrix}.$$

By [**16**] the rational Seifert forms determined by V and V_1, respectively, are isometric iff V and V_1 are rationally congruent. The latter is easy to see:

$$\begin{pmatrix} P^t & 0 \\ 0 & P^{-1}S^{-1} \end{pmatrix} \begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & S^{-t}P^{-t} \end{pmatrix} = \begin{pmatrix} 0 & D^t \\ I_d & 0 \end{pmatrix}. \quad \square$$

Question. Can we realize any possible (integral) Seifert form by a doubly slice fibered knot in S^3—perhaps by forming a connected sum of knots constructed in [**2**]? We offer the following partial result, the proof of which follows from [**11**, p. 21].

COROLLARY 2.6. *If the minimal polynomial of A has no repeated factors, then $(Q\alpha_V, [\ ])$ contains only finitely many integral Seifert forms.*

3. Even-dimensional case. A simple fibered $2q$-knot $k \subset S^{2q+2}$ is *Z-torsion-free* if $\pi_q(S^{2q+2} - k)$ has no Z-torsion. For such knots Kojima [**9**] has defined certain matrices A, B of linking numbers. In general, A is an integral unimodular matrix such that $A - I$ is also unimodular. (Such a matrix is called *s-unimodular.*) Also, B is a symmetric Z_2 matrix. The pair (A, B) is called an *L.P. matrix.* Kojima defines an equivalence relation for L.P. matrices, and proves that any two simple fibered Z-torsion-free $2q$-knots, $q \geq 4$, are isotopic iff their L.P. matrices are equivalent.

Let M_+ (resp. M_-) be an $m \times m$ (resp. $n \times n$) integral s-unimodular matrix. Let N be any $m \times n$ Z_2 matrix. We will say that the L.P. matrix (A, B) is *doubly-null-cobordant* if A and B have the forms

$$A = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix}.$$

THEOREM 3.1. *Let k be any simple Z-torsion-free fibered $2q$-knots in S^{2q+2}, $q \geq 4$. The following statements are equivalent:*

(1) *k is constructible by isotopy.*

(2) *k is doubly slice.*

(3) *k has a doubly-null-cobordant L.P. matrix.*

(Compare with results of Sumner [**15**, Theorems 2.3 and 3.1] for odd-dimensional knots.)

COROLLARY 3.2. *Every simple Z-torsion-free doubly slice fibered $2q$-knot in S^{2q+2}, $q \geq 4$, is constructible by isotopy.*

PROOF OF THEOREM 3.1. (1) implies (2) is a consequence of Corollary 1.2.

We prove that (2) implies (3): Suppose that k is double slice. Then $k = k_+ \cap k_-$, where $(B_\varepsilon, k_\varepsilon)$ is a ball pair, $\varepsilon = \pm$, and $k_+ \cup k_-$ is the unknot in $S^{2q+3} = B_+ \cup B_-$. Let K (resp. K_ε) denote the exterior of k in S^{2q+3} (resp. k_ε in B_ε). Then $K = K_+ \cap K_-$. By the van Kampen theorem, Hurewicz theorem and Mayer-Vietoris sequence it follows that K_ε has the $(q-1)$-homotopy type of S^1. Also, $H_r(\tilde{K}) \cong H_r(\tilde{K}_+) \oplus H_r(\tilde{K}_-)$ and $H_r(\tilde{K}) = \ker(H_r(\tilde{K}) \to H_r(\tilde{K}_{-\varepsilon}))$, $r = q, q+1$. Here $\tilde{}$ denotes infinite cyclic covering space. All other homology groups vanish. (These observations appear in [**7**].)

Since k is fibered, $H_q(\tilde{K})$ and $H_{q+1}(\tilde{K})$ are finitely generated over Z. (In fact, they are finitely generated free Z-modules of equal rank. See [**9**, p. 673].) Hence $H_*(\tilde{K}_\varepsilon)$ is finitely generated over Z in all dimensions. By Serre's theorem (see [**14**, p. 509]) $\pi_*(\tilde{K}_\varepsilon)$, and hence $\pi_*(K_\varepsilon)$, is finitely generated in all dimensions. Using the Browder-Levine fibration theorem [**3**] we can extend the fibration of k over k_ε. More precisely, the fiber map $K \to S^1$ extends naturally to a fiber map $\partial K_\varepsilon \to S^1$, regarding ∂K_ε as the result of 0-framed surgery on $k \subset S^{2q+3}$. By the Browder-Levine theorem the latter fiber map extends to K_ε.

Let F_ε denote the fiber of K_ε. Then $F_+ \cap F_-$ is the fiber F_0 of the knot k. Also, $F_+ \cup F_-$ must be a $(2q+1)$-ball in S^{2q+3} since it is the fiber of the unknot $k_+ \cup k_-$. Since $\tilde{K}_\varepsilon$ is diffeomorphic to $F_\varepsilon \times R$, we can identify $H_*(F_\varepsilon)$ (resp. $H_*(F_0)$) with $H_*(\tilde{K}_\varepsilon)$ (resp. $H_*(\tilde{K})$). Then by the remarks above, $H_r(F_0) \cong H_r(F_+) \oplus H_r(F_-)$ and $H_r(F_\varepsilon) = \ker(H_r(F_0) \to H_r(F_{-\varepsilon}))$, $r = q, q+1$.

If $x \in H_q(F_\varepsilon)$ and $y \in H_{q+1}(F_\varepsilon)$, then the intersection number $x \cdot y$ must vanish, since representative cycles for these classes are null-homologous in $F_{-\varepsilon}$. Consequently, the nonsingular intersection pairing

$$\bullet\colon H_q(F_0) \times H_{q+1}(F_0) \to Z$$

induces nonsingular pairings

$$\bullet\colon H_q(F_\varepsilon) \times H_{q+1}(F_{-\varepsilon}) \to Z.$$

Let $b_1, \ldots, b_m$ (resp. $b'_1, \ldots, b'_n$) be a basis for the free Z-module $H_{q+1}(F_+)$ (resp. $H_{q+1}(F_-)$). Let $a_1, \ldots, a_m$ (resp. $a'_1, \ldots, a'_n$) be a dual basis for $H_q(F_+)$ (resp. $H_q(F_-)$) with respect to intersection pairing.

By a result of Whitehead [**19**, p. 555] we have the short exact sequences

$$0 \to \bar{H}_q(F_0) \xrightarrow{\omega} \pi_{q+1}(F_0) \xrightarrow{h} H_{q+1}(F_0) \to 0$$
$$0 \to \bar{H}_q(F_\varepsilon) \xrightarrow{\omega_\varepsilon} \pi_{q+1}(F_\varepsilon) \xrightarrow{h_\varepsilon} H_{q+1}(F_\varepsilon) \to 0,$$

where $\bar{H}_*(X)$ denotes the quotient module $H_*(X)/2H_*(X)$. The maps h, h_ε are Hurewicz maps, while ω, ω_ε are obtained by composition with the nontrivial element of $\pi_{q+1}(S^q)$.

We can find elements $\beta_1, \ldots, \beta_m \in \pi_{q+1}(F_+)$ represented by disjoint embedded $(q+1)$-spheres in F_0, and such that $h_+(\beta_i) = b_i$, $1 \leq i \leq m$. We can do this as follows: Regard F_+ as $F_0 \times I \cup$(handles of index $\geq q+1$). (See [**17**, p. 24].) By the Basis Theorem [**12**, p. 92] and Whitney trick ([**12**, p. 71], for example) we can slide the attaching spheres of appropriate $(q+2)$-handles h_i^{q+2}, $1 \le i \le m$, past the belt sphere of each $(q+1)$-handle h_j^{q+1}, $1 \le j \le m$, so that the cores of these $(q+2)$-handles are disjoint properly embedded disks in F_+ representing the basis $b_1, \ldots, b_m$ for $H_{q+2}(F_+, F_0)$ ($\cong H_{q+1}(F_0)$). Let $\beta_1, \ldots, \beta_m$ be the homotopy classes corresponding to the boundaries of these disks.

Similarly find $\beta'_1, \ldots, \beta'_n \in \pi_{q+1}(F_-)$ represented by disjoint embedded $(q+1)$-spheres in F_0, and such that $h_-(\beta'_j) = b'_j$, $1 \le j \le n$.

Of course, the $(q+1)$-sphere representing β_i may intersect the $(q+1)$-sphere representing β'_j for various i and j. We will adjust $\beta_1, \ldots, \beta_m$ in order to obtain a "nice basis" $\beta_1, \ldots, \beta_m$, $\beta'_1, \ldots, \beta'_n$ for the torsion free part of $\pi_{q+1}(F_0)$. (See [**9**].) Fix i, $1 \le i \le m$. The correspondence

$$b'_j \mapsto \lambda(\beta_i, \beta'_j) \in Z_2, \qquad 1 \le j \le n,$$

induces a homomorphism $f\colon \bar{H}_{q+1}(F_-) \to Z_2$. Here λ denotes the generalized intersection pairing introduced by Wall [**18**]. Since the induced pairing $\bullet\colon \bar{H}_q(F_+) \times \bar{H}_{q+1}(F_-) \to Z_2$ is nonsingular, we can find an element $\bar{u}_i \in \bar{H}_q(F_+)$ such that

$$f(\bar{y}) = \bar{u}_i \cdot \bar{y}, \quad \text{for each } \bar{y} \in \bar{H}_{q+1}(F_-).$$

Let $u_i \in H_q(F_+)$ represent $\bar{u}_i$. Then the adjusted class $\beta_i + \omega_+(u_i)$ satisfies

$$\begin{aligned} \lambda(\beta_i + \omega_+(u_i), \beta'_j) &= \lambda(\beta_i, \beta'_j) + \lambda(\omega_+(u_i), \beta'_j) \\ &= \lambda(\beta_i, \beta'_j) + \bar{u}_i \cdot \bar{b}_j \quad \text{(see [18, p. 255])} \\ &= \lambda(\beta_i, \beta'_j) + f(b'_j) \\ &= 0, \qquad 1 \le j \le n. \end{aligned}$$

Similarly, since $u_i \cdot b_k = 0$,

$$\lambda(\beta_i + \omega_+(u_i), \beta_k + \omega_+(u_k)) = 0, \qquad 1 \le k \le m.$$

As before, the elements $\beta_1 + \omega_+(u_1), \ldots, \beta_m + \omega_+(u_m)$ can be represented by spheres that bound disjoint properly embedded $(q+2)$-disks in F_+: By [**6**, Corollary 1.1] we can find a properly embedded $(q+2)$-disk in h_j^{q+1} such that the boundary is a $(q+1)$-sphere in $S^q \times D^{q+2} \subset \partial h_j^{q+1}$ representing $\omega_+(a'_j)$, $1 \le j \le n$. Simply pipe disjoint copies of these $(q+2)$-disks together with the cores of $h_1^{q+2}, \ldots, h_m^{q+2}$, in the appropriate manner.

Replace each β_i by $\beta_i + \omega_+(u_i)$, $1 \le i \le m$. For convenience $\beta_1, \ldots, \beta_m$ will now denote these adjusted classes. Then $\beta_1, \ldots, \beta_m, \beta'_1, \ldots, \beta'_n$ is a nice basis, and we can use it to compute an L.P. matrix (A, B) for k. By definition

$$A = \begin{pmatrix} (\theta(a_i, b_k)) & (\theta(a_i, b'_j)) \\ (\theta(a'_j, b_i)) & (\theta(a'_j, b'_l)) \end{pmatrix}$$

and

$$B = \begin{pmatrix} (\theta'(\beta_i, \beta_k)) & (\theta'(\beta_i, \beta'_j)) \\ (\theta'(\beta'_j, \beta_i)) & (\theta'(\beta'_j, \beta'_l)) \end{pmatrix}.$$

As in [**9**], $\theta(a, b)$ is defined to be the linking number $L(S_a, i(S_b))$, where S_a (resp. S_b) is an embedded sphere in F_0 representing a (resp. b), and $i\colon F_0 \to S^{2q+2} - F_0$ is translation in the negative normal direction. Similarly, $\theta'(\beta, \beta')$ is the Z_2 linking number $L'(S, i(S'))$ in the sense of Haefliger [**5**], where S (resp. S') is an embedded sphere in F_0 representing β (resp. β').

Since $q \geq 4$, $\theta'(\beta_i, \beta'_j) = \theta'(\beta'_j, \beta_i)$. (See [**5**].) The following lemma completes the proof that (A, B) is a doubly-null-cobordant L.P. matrix.

LEMMA 3.3. $\theta(a_i, b'_j) = \theta(a'_j, b_i) = 0$, $\theta'(\beta_i, \beta_k) = \theta'(\beta'_j, \beta'_l) = 0$, $1 \leq i, k \leq m$, $1 \leq j, l \leq n$.

PROOF. Using the fibration of $k_+ \subset B_+$ it is easy to see that a_i and $i_* b'_j$ are represented by spheres that bound disjoint properly embedded disks D_i and D'_j, respectively, in B_+. Then $\theta(a_i, b'_j)$ is the element of $\pi_{q+1}(S^{2q+2} - \partial D_i) \cong Z$ represented by $\partial D'_j$. But the inclusion map induces an isomorphism

$$\pi_{q+1}(S^{2q+2} - \partial D_i) \xrightarrow{\cong} \pi_{q+1}(B - D)$$

since the ball pair (B_+, D_i) is unknotted when $q \geq 4$. (See [**6**, Corollary 4.1].) Hence $\theta(a_i, b'_j)$ is trivial.

The remaining assertions follow in a similar way. □

We prove that (3) implies (1): Let k be a simple Z-torsion-free fibered $2q$-knot in S^{2q+2}, $q \geq 4$, with doubly-null-cobordant L. P. matrix (A, B), where

$$A = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix}.$$

We will construct by isotopy another such knot k' with L.P. matrix (A, B). It will follow by [**9**, Theorem 3] that k and k' are isotopic, and hence k is constructible by isotopy.

Let

$$\mathcal{H} = \natural_{i=1}^{m} S^{q+1} \times D^{q+1} \natural_{j=1}^{n} S^q \times D^{q+2} \subset S^{2q+2}.$$

We will use the notation of §1. The q-spheres $(1 \times S^q)_1, \ldots, (1 \times S^q)_m$, $(S^q \times 0)_1, \ldots, (S^q \times 0)_n$ represent a basis $a_1, \ldots, a_m, a'_1, \ldots, a'_n$ for $H_q(F_0)$. As above, by Whitehead's theorem [**19**, p. 555] we have a short exact sequence

$$0 \to \bar{H}_q(F_0) \xrightarrow{\omega} \pi_{q+1}(F_0) \xrightarrow{h} H_{q+1}(F_0) \to 0.$$

Let $\beta_1, \ldots, \beta_m, \beta'_1, \ldots, \beta'_n$ denote the elements of $\pi_{q+1}(F_0)$ represented by

$$(S^{q+1} \times 0)_1, \ldots, (S^{q+1} \times 0)_m, (1 \times S^{q+1})_1, \ldots, (1 \times S^{q+1})_n.$$

Let $U_+ = (u_{ik})$ (resp. $U_- = (u'_{jl})$) denote the $m \times m$ matrix $I - M_+^{-t}$ (resp. the $n \times n$ matrix $I - M_-^{-t}$). Let $V_+ = (v_{il})$ (resp. $V_- = (v'_{jk})$) denote the $m \times n$ Z_2 matrix $M_+^{-t} N (I - M_-)^{-1}$ (resp. the $n \times m$ Z_2 matrix $M_-^{-t} N^t (I - M_+)^{-1}$).

LEMMA 3.4. *There exists an isotopy $h_t \colon S^{2q+2} \to S^{2q+2}$, $0 \leq t \leq 1$, such that $h_0 =$ identity, $h|_{\mathcal{H}}$ is a diffeomorphism $\phi \colon \mathcal{H} \to \mathcal{H}$, and ϕ induces an automorphism $(\phi|_{F_0})_* \colon \pi_{q+1}(F_0) \to \pi_{q+1}(F_0)$ such that*

$$(**) \qquad \begin{aligned} (\phi|_{F_0})_*(\beta_i) &= \sum_{k=1}^{m} u_{ik}\beta_k + \sum_{l=1}^{n} v_{il}\omega(a'_l), \\ (\phi|_{F_0})_*(\beta'_j) &= \sum_{l=1}^{n} u'_{jl}\beta'_l + \sum_{k=1}^{m} v'_{jk}\omega(a_k), \end{aligned}$$

$1 \leq i \leq m$, $1 \leq j \leq n$. *Moreover, each $h_t|_B =$ identity.*

PROOF OF LEMMA 3.4. Regard $\mathcal{H}$ as a ball with trivially attached handles $h_1^{q+1}, \ldots, h_m^{q+1}, h_1^q, \ldots, h_n^q$. Slide the $(q+1)$-handles of $\mathcal{H}$ over each other according to an elementary matrix decomposition of U_+, as in the proof of Corollary 2.3. Similarly slide the q-handles of $\mathcal{H}$ over each other according to a decomposition of U_-. The restriction of the final map of the isotopy (so far) induces an automorphism of $\pi_{q+1}(F_0)$ that takes β_i to $u_{i1}\beta_1 + \cdots + u_{im}\beta_m$, and β'_j to $u'_{j1}\beta'_1 + \cdots + u'_{jn}\beta'_n$, $1 \le i \le m$, $1 \le j \le n$.

Continue isotoping $\mathcal{H}$ as follows. By Haefliger's embedding theorem [**4**] we can represent $\omega(a'_1), \ldots, \omega(a'_n)$ by disjoint embedded $(q+1)$-spheres $S_1, \ldots, S_n \subset F_0$, respectively. Slide each $(q+1)$-handle h_i^{q+1} around these spheres according to the matrix V_+; i.e. if $v_{il} \neq 0$ then slide h_i^{q+1} around S_l, as in Figure 2C with h_i^q and S_j relabeled h_i^{q+1} and S_l, respectively. Extend the isotopy over the remainder of S^{2q+2}.

Let ϕ denote the restriction to $\mathcal{H}$ of the final map of the isotopy that we have described. Clearly ϕ satisfies (**) for $\beta_1, \ldots, \beta_m$. A straightforward calculation, using the fact that

$$(\phi|_{F_0})_*(\beta_i) \cdot (\phi|_{F_0})_*(\beta'_j) = 0, \qquad 1 \le i \le m,\ 1 \le j \le n,$$

shows that ϕ also has the desired effect on $\beta'_1, \ldots, \beta'_n$.

The last condition is easily satisfied since the handle slides can avoid B. □

Now construct by isotopy the knot k', using the diffeomorphism ϕ. It follows by [**9**, Lemma 4] that k' has L.P. matrix (A, B). □

REFERENCES

1. I. R. Aitchison, *Isotoping and twisting knots and ribbons*, Ph.D. Dissertation, Univ. of California at Berkeley, 1984.
2. I. R. Aitchison and D. S. Silver, *On certain fibred ribbon disc pairs*, Trans. Amer. Math. Soc. (to appear).
3. W. Browder and J. Levine, *Fibering manifolds over a circle*, Comment. Math. Helv. **40** (1966), 153–160.
4. A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47–82.
5. ——, *Differentiable links*, Topology **1** (1962), 241–244.
6. J. F. P. Hudson, *Embeddings of bounded manifolds*, Proc. Cambridge Philos. Soc. **72** (1972), 11–20.
7. C. Kearton, *Doubly-null-concordant simple even-dimensional knots*, Proc. Roy. Soc. Edinburgh **96A** (1984), 163–174.
8. M. Kervaire and C. Weber, *A survey of multidimensional knots*, Knot Theory, Lecture Notes in Math., vol. 685 (J. C. Hausmann, ed.), Springer-Verlag, Berlin and New York, 1978.
9. S. Kojima, *A classification of some even dimensional fibred knots*, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **24** (1977), 671–683.
10. J. Levine, *An algebraic classification of some knots of codimension two*, Comment. Math. Helv. **45** (1970), 185–198.
11. ——, *Finiteness of symplectic class number and an application to knot theory*, preprint.
12. J. Milnor, *Lectures on the h-cobordism theorem*, Princeton Univ. Press, Princeton, N. J., 1965.
13. J. J. Rotman, *The theory of groups: An Introduction*, Allyn and Bacon, Boston, Mass., 1965.
14. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
15. D. W. Sumners, *Invertible knot cobordisms*, Comment. Math. Helv. **46** (1971), 240–256.
16. H. Trotter, *On S-equivalence of Seifert matrices*, Invent. Math. **20** (1973), 173–207.

17. C. T. C. Wall, *Differential topology*, Part IV, *Theory of handle decompositions*, mimeographed, Cambridge Univ., 1964.
18. ——, *Classification problems in differential topology.* I, Topology **2** (1963), 253–261.
19. G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math., no. 61, Springer-Verlag, Berlin and New York, 1978.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SOUTH ALABAMA, MOBILE, ALABAMA 36688