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THE CONTINUOUS (α, β)-JACOBI TRANSFORM AND ITS INVERSE WHEN $\alpha + \beta + 1$ IS A POSITIVE INTEGER

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ABSTRACT. The continuous (α, β)-Jacobi transform is introduced as an extension of the discrete Jacobi transform by replacing the polynomial kernel by a continuous one. An inverse transform is found for both the standard and a modified normalization and applied to a version of the sampling theorem. An orthogonal system forming a basis for the range is shown to have some unusual properties, and is used to obtain the inverse.

1. Introduction. Recently a number of works have been devoted to continuous analogs of discrete finite integral transforms. The continuous Legendre transform and its inverse were studied by Butzer, Stens, and Wehrens [**1**]. Their results were extended to continuous Jacobi transforms for values of the parameters satisfying $\alpha + \beta = 0$ by Deeba and Koh [**3**]. They were extended in a different direction by Walter[**10**] who studied Gegenbauer transforms ($\alpha = \beta$) when 2α was a positive integer.

These continuous transforms are not related directly to those studied by Koornwinder [**8**] which have considerably different behavior. They have, as an application, versions of a sampling theorem or cardinal series [**2**].

In this work we consider a number of cases of a continuous Jacobi transform in which $\alpha + \beta + 1$ is a positive integer. These include as special cases all of the transforms mentioned above. We first obtain an inverse transform for the standard Jacobi normalization. This inverse has a kernel defined by an infinite series. Unfortunately with this standard normalization the transformed function is not an entire function as in the other cases. To rectify this, we introduce a renormalized transform whose range is a set of entire functions. Its inverse is then given in closed form.

2. Preliminaries. In this section we recall some of the basic background material necessary for our investigation.

2.1. *Jacobi functions and transforms.* For any real numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the hypergeometric function $F(a, b; c; z) = {}_2F_1(a, b; c; z)$ is given by

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \qquad |z| < 1, \tag{2.1}$$

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where the series converges at $z = -1$ and $z = 1$ provided that $c-a-b+1>0$ and $c-a-b>0$ respectively.

The Jacobi function $P_\lambda^{(\alpha,\beta)}(x)$ of the first kind is defined by

$$P_\lambda^{(\alpha,\beta)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)} F\left(-\lambda, \lambda+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right), \quad x \in (-1,1], \tag{2.2}$$

where $\alpha, \beta > -1$, $\lambda \in R$ and $\lambda+\alpha+1 \neq 0, -1, 2, \ldots$. Since

$$P_{-\lambda}^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha-\lambda+1)\Gamma(\lambda-\alpha-\beta)}{\Gamma(1-\lambda)\Gamma(\lambda-\beta)} P_{\lambda-\alpha-\beta-1}^{(\alpha,\beta)}(x)$$

we may restrict ourselves to the case $\lambda \geq -((\alpha+\beta+1)/2)$. The function $P_\lambda^{(\alpha,\beta)}(x)$ satisfies the differential equation

$$(1-x^2)y'' + (\beta-\alpha-(\alpha+\beta+2)x)y' + \lambda(\lambda+\alpha+\beta+1)y = 0. \tag{2.3}$$

Let

$$L_x^{(\alpha,\beta)} = (1-x^2)\frac{d^2}{dx^2} + [\beta-\alpha-(\alpha+\beta+2)x]\frac{d}{dx} \tag{2.4}$$

be the differential operator associated with this equation. Then (2.3) becomes

$$L_x^{(\alpha,\beta)} P_\lambda^{(\alpha,\beta)}(x) = -\lambda(\lambda+\alpha+\beta+1)P_\lambda^{(\alpha,\beta)}(x). \tag{2.5}$$

For integer values of λ, $P_\lambda^{(\alpha,\beta)}(x)$ reduces to the usual Jacobi polynomial as defined in [**9**].

It can be shown [**3**] that for $\lambda, \nu \geq -((\alpha+\beta+1)/2)$, $\lambda \neq \nu$, $\lambda \neq -(\nu+\alpha+\beta+1)$ and $-1 < \alpha, \beta$,

$$\begin{aligned} &\frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_\lambda^{(\alpha,\beta)}(x) P_\nu^{(\beta,\alpha)}(-x)\, dx \\ &\quad = \frac{\Gamma(\lambda+\alpha+1)\Gamma(\nu+\beta+1)}{\pi(\lambda-\nu)(\lambda+\nu+\alpha+\beta+1)} \\ &\quad \times \left\{ \frac{\sin \pi\lambda}{\Gamma(\nu+1)\Gamma(\lambda+\alpha+\beta+1)} - \frac{\sin \pi\nu}{\Gamma(\lambda+1)\Gamma(\nu+\alpha+\beta+1)} \right\}. \end{aligned} \tag{2.6}$$

If we denote the Jacobi polynomial of degree n by $P_n^{(\alpha,\beta)}(x)$, then

$$\frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\, dx = \delta_{nm} h_n^{(\alpha,\beta)}, \tag{2.7}$$

where

$$h_n^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}. \tag{2.8}$$

Since $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ it follows that

$$\begin{aligned} \hat{P}_\lambda^{(\alpha,\beta)}(n) &= \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_\lambda^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, dx \\ &= \frac{(-1)^n \Gamma(\lambda+\alpha+1)\Gamma(n+\beta+1)\sin \pi\lambda}{\pi(\lambda-n)(\lambda+n+\alpha+\beta+1)n!\Gamma(\lambda+\alpha+\beta+1)}, \qquad \lambda \neq n, \end{aligned} \tag{2.9}$$

and hence

$$P_\lambda^{(\alpha,\beta)}(x) = \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{P}_n^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x), \qquad x \in (-1,1), \tag{2.10}$$

where the series converges

(i) absolutely and uniformly on any compact subset of $(-1,1)$ if $-1 < \beta < \frac{1}{2}$,

(ii) in $L^2((1-x)^\alpha(1+x)^\beta)$ if $-1 < \beta < 1$,

(iii) in the sense of generalized functions for any $-1 < \beta$.

Here we have used the fact that

$$P_\lambda^{(\alpha,\beta)}(x) = O(1/\sqrt{\lambda}) \quad \text{as } \lambda \to \infty \text{ uniformly in } x \in [a,b] \subset (-1,1). \tag{2.11}$$

We also note that

$$P_\lambda^{(\alpha,\beta)}(x) = O(\lambda^{\max(\alpha,\beta)}) \quad \text{as } \lambda \to \infty. \tag{2.12}$$

Let $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ and $f(x) \in L^p\{(-1,1), w^{(\alpha,\beta)}(x)\}$, $p \geq 1$. Then the discrete Jacobi transform $\hat{f}^{(\alpha,\beta)}(n)$ of $f(x)$ is defined by

$$\hat{f}^{(\alpha,\beta)}(n) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) f(x)\, dx \tag{2.13}$$

and the series expansion of $f(x)$ in terms of the Jacobi polynomials is given by

$$f(x) \sim \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x), \qquad x \in (-1,1). \tag{2.14}$$

Analogously, if $f(x)(1+x)^{-\beta} \in L^1\{(-1,1), w^{(\alpha,\beta)}(x)\}$, then the continuous Jacobi transform $\hat{f}^{(\alpha,\beta)}(\lambda)$ of $f(x)$ will be defined by

(2.15)

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_\lambda^{(\alpha,\beta)}(x) f(x)\, dx, \qquad \lambda > -\frac{\alpha+\beta+1}{2}.$$

The following proposition gives a series representation for the continuous Jacobi transform $\hat{f}^{(\alpha,\beta)}(\lambda)$.

PROPOSITION 2.1. *Let $f(x)$ be $2p$ times continuously differentiable with support in $(-1,1)$, $2p > \max(\alpha,\beta) + \frac{3}{2}$; then*

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) \hat{P}_\lambda^{(\alpha,\beta)}(n) \tag{2.16}$$

where the series converges uniformly on any compact subset of $[0,\infty)$.

PROOF. By substituting the uniformly convergent series (2.14) in (2.15), we obtain

$$\begin{aligned}
\hat{f}^{(\alpha,\beta)}(\lambda) &= \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} \left(\sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x) \right) P_\lambda^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)\, dx \\
&= \frac{1}{2^{\alpha+\beta+1}} \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_\lambda^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)\, dx \\
&= \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) \hat{P}_\lambda^{(\alpha,\beta)}(n)
\end{aligned}$$

by (2.9). The resulting series converges uniformly on any compact subset of $[0,\infty)$. This follows from the relations

$$1/h_n^{(\alpha,\beta)} = O(n), \quad \hat{f}^{(\alpha,\beta)}(n) = O(n^{-2p-1/2}) \quad \text{and} \quad \hat{P}_\lambda^{(\alpha,\beta)}(n) = O(n^{\beta-2})$$

uniformly on compact subsets of $[0,\infty)$ as $n\to\infty$.

Let $q = (\alpha+\beta+1)/2$ and define

$$d\sigma(\lambda) = \begin{cases} \dfrac{\Gamma^2(\lambda+q)\lambda\sin\pi\lambda\, d\lambda}{\Gamma(\lambda+\alpha-q+1)\Gamma(\lambda+\beta-q+1)} & \text{if } q \text{ is a half-integer,} \\ \dfrac{\Gamma^2(\lambda+q)\cos\pi\lambda\, d\lambda}{\Gamma(\lambda+\alpha-q+1)\Gamma(\lambda+\beta-q+1)} & \text{if } q \text{ is an integer.} \end{cases} \tag{2.17}$$

We shall show in the next section that $\{\hat{P}_\lambda^{(\alpha,\beta)}(n)\}$ and $\{\hat{P}_\lambda^{(\beta,\alpha)}(n)\}$ form a bi-orthogonal sequence with respect to $d\sigma$.

2.2. *An associated orthonormal system.* We begin with the complete orthonormal system on $[-\pi,\pi]$ given by

$$\left\{\frac{1}{\sqrt{\pi}}\cos\left(n+\frac{1}{2}\right)w, \frac{1}{\sqrt{\pi}}\sin\left(n+\frac{1}{2}\right)w\right\}_{n=0}^{\infty}.$$

Their Fourier transforms will be denoted by c_n^0 and r_n^0 respectively; i.e.,

$$\begin{Bmatrix} c_n^0(\lambda) \\ r_n^0(\lambda) \end{Bmatrix} = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{iw\lambda}\begin{Bmatrix} \cos\left(n+\frac{1}{2}\right)w \\ \sin\left(n+\frac{1}{2}\right)w. \end{Bmatrix} dw, \qquad n=0,1,2,\ldots, \tag{2.18}$$

By Plancherel's identity $\{c_n^0, r_n^0\}$ are a complete orthonormal system in $L^2(\mathbf{R})$. This system will be used with $q=(\alpha+\beta+1)/2$ a half odd integer. For q an integer we shall use the system

$$\begin{aligned} \begin{Bmatrix} c_n^e(\lambda) \\ r_n^e(\lambda) \end{Bmatrix} &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{iw\lambda}\begin{Bmatrix} \cos nw \\ \sin nw \end{Bmatrix} dw, \qquad n=1,2,\ldots, \\ c_0^e(\lambda) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iw\lambda}\, dw \end{aligned} \tag{2.19}$$

which again is clearly an orthonormal system. We shall show that these systems are related to $\hat{P}_\lambda^{(\alpha,\beta)}(n)$ given in (2.9) which may be expressed as

$$\hat{P}_{\lambda-q}^{(\alpha,\beta)}(n) = \frac{\sin\pi(\lambda-n-q)(2n+2q)}{\pi(\lambda^2-(n+q)^2)2(n+q)}\,\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)}\,\frac{\Gamma(\lambda+\alpha+1-q)}{\Gamma(\lambda+q)}. \tag{2.20}$$

By a striaghtforward calculation we see that

$$s_n^q(\lambda-q) = \frac{\sin\pi(\lambda-n-q)}{\pi(\lambda^2-(n+q)^2)}(2n+2q) = \sqrt{2}\begin{cases} c_{n+[q]}^0(\lambda), & q \text{ half-integer,} \\ r_{n+q}^e(\lambda), & q \text{ integer.} \end{cases} \tag{2.21}$$

PROPOSITION 2.2. *The functions given by*

$$\frac{\Gamma(\lambda+q)}{\Gamma(\lambda+\alpha+1-q)}2(n+q)\frac{\Gamma(n+1)}{\Gamma(n+\beta+1)}\hat{P}_{\lambda-q}^{(\alpha,\beta)}(n) = s_n^q(\lambda-q) \tag{2.22}$$

are orthonormal on $(0,\infty)$ *with respect to the standard measure.*

This is a consequence of the orthonormality of $s_n^q(\lambda-q)/\sqrt{2}$ on $(-\infty,\infty)$ together with its evenness when q is a half-integer and oddness when q is an integer.

PROPOSITION 2.3. *The functions given by* (2.22) *satisfy*

$$\begin{aligned}\int_0^\infty s_n^q(\lambda-q)s_m^q(\lambda-q)\frac{\Gamma(\lambda+\alpha+1-q)}{\Gamma^2(\lambda+q)}\Gamma(\lambda+\beta+1-q)\,d\sigma(\lambda)\\ =\frac{(-1)^{m+[q]}}{2}\gamma_{q,m}\delta_{mn},\end{aligned}\tag{2.23}$$

where

$$\gamma_{q,m}=\begin{cases}1, & q \text{ an integer},\\ (m+q), & q \text{ a half-integer},\end{cases}$$

and $d\sigma$ *is the measure given by* (2.17).

This follows from the fact that the integral in (2.23) for q a half-integer is just

$$\begin{aligned}&\int_0^\infty s_n^q(\lambda-q)s_m^q(\lambda-q)\lambda\sin\pi\lambda\,d\lambda\\ &=2\int_0^\infty c_{n+[q]}^0(\lambda)c_{m+[q]}^0(\lambda)\lambda\sin\pi\lambda\,d\lambda.\end{aligned}\tag{2.24}$$

But

(2.25)

$$\begin{aligned}c_k^0(\lambda)\lambda\sin\pi\lambda&=\frac{\sin\pi\lambda}{\sqrt{2\pi}}\int_{-\pi}^{\pi}-iD(e^{iw\lambda})\cdot\cos\left(k+\frac12\right)w\,dw\\
&=\frac{\sin\pi\lambda}{\sqrt{2\pi}}\int_{-\pi}^{\pi}(-i)e^{iw\lambda}\left(k+\frac12\right)\sin\left(k+\frac12\right)w\,dw\\
&=\frac{1}{2\sqrt{2\pi}}\int_{-\pi}^{\pi}(e^{i(w-\pi)\lambda}-e^{i(w+\pi)\lambda})\left(k+\frac12\right)\sin\left(k+\frac12\right)w\,dw\\
&=\frac{(k+\frac12)}{2\sqrt{2\pi}}\left\{\int_{-2\pi}^{0}e^{iw\lambda}\sin\left(k+\frac12\right)(w+\pi)\,dw\right.\\
&\qquad\left.-\int_0^{2\pi}e^{iw\lambda}\sin\left(k+\frac12\right)(w-\pi)\,dw\right\}\\
&=\frac{(k+\frac12)}{2\sqrt{2\pi}}(-1)^k\int_{-2\pi}^{2\pi}e^{iw\lambda}\cos\left(k+\frac12\right)w\,dw.\end{aligned}$$

Hence it is the Fourier transform of the function

$$\frac{(k+\frac12)}{2\sqrt{\pi}}(-1)^k\cos(k+\tfrac12)w\chi_{[-2\pi,2\pi]}(w)$$

where $\chi_{[-2\pi,2\pi]}$ is the characteristic function of the interval $[-2\pi,2\pi]$. Again by Plancherel's identity, (2.24) is given by

$$\begin{aligned}&\int_{-\pi}^{\pi}\frac{\cos(n+q)(w)}{\sqrt{\pi}}\,\frac{(m+q)(-1)^{m+[q]}}{2\sqrt{\pi}}\cos(m+q)(w)\,dw\\ &=\frac{(m+q)(-1)^{m+[q]}}{2}\delta_{mn}.\end{aligned}\tag{2.26}$$

For q an integer (2.23) becomes

$$
\begin{aligned}
(2.27)\qquad & 2\int_0^\infty r^e_{n+q}(\lambda) r^e_{m+q}(\lambda)\cos\pi\lambda\,d\lambda \\
& = \int_{-\pi}^{\pi} \frac{\sin(n+q)w}{\sqrt{\pi}}\,\frac{(-1)^{m+q}}{2\sqrt{\pi}}\sin(m+q)w\,dw = \frac{1}{2}(-1)^{m+q}\delta_{mn}
\end{aligned}
$$

since

$$
r^e_n(\lambda)\cos\pi\lambda = \frac{1}{\sqrt{2\pi}}\int_{-2\pi}^{2\pi}\frac{e^{iw\lambda}(-1)^n}{2\sqrt{\pi}}\sin nw\,dw.
$$

REMARK. There are many other measures that could be used. For example the calculation in (2.25) could be iterated to obtain

$$
c^0_k(\lambda)\lambda^2\sin^2\pi\lambda = \frac{(k+\frac{1}{2})^2}{4\pi}\left\{\int_{-3\pi}^{\pi} + \int_{-\pi}^{3\pi}\right\} e^{iw\lambda}\cos\left(k+\frac{1}{2}\right)w\,dw
$$

which is again orthogonal to $c^0_n(\lambda)$.

3. An inverse transform given by a series. In this section we use the biorthogonality of $\{\hat{P}^{(\alpha,\beta)}_{\lambda-q}(n)\}$ with $\{\hat{P}^{(\beta,\alpha)}_{\lambda-q}(n)\}$ to find an inverse transform to the continuous Jacobi transform when $\alpha+\beta$ is an integer ≥ 0. The expression in Proposition 2.3 may be written as

$$
\begin{aligned}
(3.1)\qquad & \int_0^\infty \hat{P}^{(\alpha,\beta)}_{\lambda-q}(n)\hat{P}^{(\beta,\alpha)}_{\lambda-q}(m)\,d\sigma(\lambda) \\
& = \frac{(-1)^{n+[q]}(n+1)_{2q-1}}{2(n+q)} h^{(\alpha,\beta)}_n \delta_{n,m}\gamma_{q,n}
\end{aligned}
$$

where $\gamma_{q,n} = 1$ if q is an integer and $\gamma_{q,n} = (n+q)$ if q is a half-integer. Let

$$
(3.2)\qquad R(x,\lambda) = (-1)^{[q]}\sum_{n=0}^{\infty}\frac{2(n+q)}{\gamma_{q,n}h^{(\alpha,\beta)}_n(n+1)_{2q-1}}\hat{P}^{(\beta,\alpha)}_{\lambda-q}(n)P^{(\beta,\alpha)}_n(-x).
$$

The series defining $R(x,\lambda)$ converges absolutely and uniformly on any compact subset of $(-1,1)\times[0,\infty)$ provided that

(i) $\beta > -\frac{1}{2}$ for $q = \frac{1}{2}, \frac{3}{2}, \ldots$,

(ii) $\beta > \frac{1}{2}$ for $q = 1, 2, 3, \ldots$.

Moreover it is dominated by $(\lambda+1)^{-\alpha}$. We shall always assume that this is indeed the case. In the next theorem we derive an inversion formula for the continuous Jacobi transform given by (2.15).

THEOREM 3.1. *Let* $\alpha > -1$, $\beta > -\frac{1}{2}$ *except that* $\beta > \frac{1}{2}$ *is* $\alpha+\beta = 1, 3, 5, \ldots$; $f(x)$ *be such that its continuous Jacobi transform has the representation* (2.16) *and is dominated by* $\lambda^{-\alpha-2}$. *Then*

$$
(3.3)\qquad f(x) = \int_0^\infty \hat{f}^{(\alpha,\beta)}(\lambda-q)R(x,\lambda)\,d\sigma(\lambda)
$$

where $R(x,\lambda)$ *and* $d\sigma(\lambda)$ *are given by* (3.2) *and* (2.17) *respectively.*

PROOF. Let q be a half positive odd integer. Then by substituting (2.16) and (3.2) into (3.3), we obtain

$$\begin{aligned}\int_0^\infty & \hat{f}^{(\alpha,\beta)}(\lambda-q)R(x,\lambda)\,d\sigma(\lambda) \\ &= \int_0^\infty \sum_{n=0}^\infty \frac{1}{h_n^{(\alpha,\beta)}}\hat{f}^{(\alpha,\beta)}(n)\hat{P}_{(\lambda-q)}^{(\alpha,\beta)}(n)2(-1)^{[q]} \\ &\quad\times \sum_{m=0}^\infty \frac{1}{h_m^{(\alpha,\beta)}}\hat{P}_{\lambda-q}^{(\beta,\alpha)}(m)\frac{P_m^{(\beta,\alpha)}(-x)}{(m+1)_{2q-1}}\,d\sigma(\lambda).\end{aligned}$$

By interchanging the summation and the integration signs, which is permissable because of the dominated uniform convergence of the series involved, we obtain

$$\begin{aligned}\int_0^\infty & \hat{f}^{(\alpha,\beta)}(\lambda-q)R(x,\lambda)\,d\sigma(\lambda) \\ &= \sum_{n,m=0}^\infty \frac{2(-1)^{[q]}}{h_n^{(\alpha,\beta)}h_m^{(\alpha,\beta)}}\hat{f}^{(\alpha,\beta)}(n)\frac{P_m^{(\beta,\alpha)}(-x)}{(m+1)_{2q-1}} \\ &\quad\times \int_0^\infty \hat{P}_{\lambda-q}^{(\alpha,\beta)}(n)\hat{P}_{\lambda-q}^{(\beta,\alpha)}(m)\,d\sigma.\end{aligned}$$

Upon using (3.1), we immediately obtain

$$\int_0^\infty \hat{f}^{(\alpha,\beta)}(\lambda-q)R(x,\lambda)\,d\sigma(\lambda) = \sum_{n=0}^\infty \frac{1}{h_n^{(\alpha,\beta)}}\hat{f}^{(\alpha,\beta)}(n)P_n^{(\alpha,\beta)}(x) = f(x)$$

since $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$.

The proof for $q = 1, 2, 3, \ldots$ is similar. Q.E.D.

4. A renormalized Jacobi transform. The standard normalization (2.2) of the Jacobi function does not give us an entire function in λ unless α is an integer. This may be rectified by adopting a normalization similar to that of the Gegenbauer polynomials when $\alpha = \beta$. Accordingly, we define

$$\begin{aligned}\varphi_\lambda^{(\alpha,\beta)}(x) &= \binom{\lambda+\alpha+\beta}{\beta}P_\lambda^{(\alpha,\beta)}(x) \\ &= \frac{\Gamma(\lambda+2q)}{\Gamma(\beta+1)\Gamma(\lambda+1)\Gamma(\alpha+1)}F\left(-\lambda,\lambda+2q;\alpha+1;\frac{1-x}{2}\right),\end{aligned} \tag{4.1}$$

where $2q = \alpha+\beta+1$ is an integer as before. Then $\varphi_\lambda^{(\alpha,\beta)}$ as the product of a polynomial and an entire function is itself entire. Moreover, $\varphi_{\lambda-q}^{(\alpha,\beta)}(x)$ is either an even or odd function of λ about 0 according to whether $2q$ is respectively odd or even.

This will enable us to devise somewhat simpler formulas for the modified transform. We can rewrite formulas (2.7), (2.10) and (2.15) in terms of $\varphi_\lambda^{(\alpha,\beta)}$. They are

$$2^{-2q}\int_{-1}^{1}\varphi_n^{(\alpha,\beta)}(x)\varphi_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx = \frac{\Gamma(n+2q)\Gamma(n+\beta+1)}{\Gamma^2(\beta+1)\Gamma(n+\alpha+1)n!2(n+q)}\delta_{mn}; \tag{4.2}$$

$$\begin{aligned}\varphi_\lambda^{(\alpha,\beta)}(x) &\sim \sum_{n=0}^{\infty}\frac{2^{2q}}{\|\varphi_n\|^2}\varphi_n^{(\alpha,\beta)}(x)\cdot 2^{-2q} \\ &\quad\times\int_{-1}^{1}\varphi_\lambda^{(\alpha,\beta)}(t)\varphi_n^{(\alpha,\beta)}(t)(1-t)^\alpha(1+t)^\beta\,dt \\ &= \sum_{n=0}^{\infty}s_n^q(\lambda)\varphi_n^{(\alpha,\beta)}(x)\end{aligned} \tag{4.3}$$

where s_n^q is given by (2.21); and

$$\begin{aligned}F(\lambda) = F^{(\alpha,\beta)}(\lambda) &= \frac{1}{2^{2q}}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\varphi_\lambda^{(\alpha,\beta)}(x)f(x)\,dx \\ &= \binom{\lambda+2q-1}{\beta}\hat{f}^{(\alpha,\beta)}(\lambda)\end{aligned} \tag{4.4}$$

will be the modified Jacobi transform.

PROPOSITION 4.1. *Let F be the modified Jacobi transform of a C^∞ function f with support in $(-1,1)$, $\alpha+\beta+1=2q$, an integer, $\beta>-1$. Then*

(i) *F is an entire function in λ.*

(ii) $F(-\lambda-q)=F(\lambda-q)(-1)^{2q+1}$.

(iii)

$$F(\lambda)=\sum_{n=0}^{\infty}F(n)s_n^q(\lambda), \tag{4.5}$$

where $s_n^q(\lambda)$ is the orthonormal sequence given by (2.21) and the convergence is uniform on compact subsets of $(-\infty,\infty)$.

(iv) *$F(\lambda)=O(\lambda^{-p})$ as $\lambda\to\infty$ for all $p\ge 0$.*

Conclusion (iii) is a restatement of Proposition 2.1, while (ii) follows immediately from the symmetry of $\varphi_\lambda^{(\alpha,\beta)}$. Conclusion (i) can be shown by taking the complex derivative of $F^{(\alpha,\beta)}(\lambda)$ with respect to λ under the integral in (4.4) and then using the fact that $\varphi_\lambda^{(\alpha,\beta)}$ is entire. Conclusion (iv) is based on integration by parts with the operator $L_x^{(\alpha,\beta)}$ of (2.4) applied to $f(x)$ repeatedly.

In order to obtain an inverse formula analogous to (3.3) we use the fact that

$$\frac{\hat{P}_{\lambda-q}^{(\alpha,\beta)}(n)}{h_n^{(\alpha,\beta)}} = s_n^q(\lambda-q)\frac{\binom{n+2q-1}{\beta}}{\binom{\lambda+q-1}{\beta}} \tag{4.6}$$

which follows from (4.3) and (2.10). From (3.2) we have

$$R(x,\lambda)=(-1)^q\sum_{n=0}^{\infty}\frac{2}{\gamma_{q,n}}\frac{(n+q)}{(n+1)_{2q-1}}\frac{s_n^q(\lambda-q)}{\binom{\lambda+q-1}{\alpha}}\varphi_n^{(\beta,\alpha)}(-x). \tag{4.7}$$

Hence we have

$$f(x) = \int_0^\infty \hat{f}^{(\alpha,\beta)}(\lambda - q)R(x,\lambda)\,d\sigma(\lambda) = \int_0^\infty F(\lambda - q)\frac{R(x,\lambda)}{\binom{\lambda+q-1}{\beta}}\,d\sigma(\lambda). \tag{4.8}$$

In the next section we shall devise another inverse given by a kernel with a closed form expression.

5. An inverse operator with a closed form kernel. In order to derive the closed form we shall proceed inductively beginning with the two cases $q = \frac{1}{2}$ and $q = 1$. In both cases the inverse operator is known or is easy to derive.

5.1. *The case* $q = \frac{1}{2}$. In this case the inverse is given by [**3**]

$$f(x) = 4\Gamma(\alpha+1)\Gamma(\beta+1)\int_0^\infty F\left(\lambda - \frac{1}{2}\right)\varphi_{\lambda-1/2}^{(\beta,\alpha)}(-x)\lambda\sin\pi\lambda\,d\lambda. \tag{5.1}$$

This integral converges for $F(\lambda - \frac{1}{2}) = O(\lambda^{-3/2-\alpha-\varepsilon})$; $\varepsilon > 0$ uniformly for x in interior subintervals of $(-1,1)$.

5.2. *The case* $q = 1$. This can be derived from (4.8) but we shall do it directly. We first observe that

$$\int_0^\infty s_n^1(\lambda-1)s_m^1(\lambda-1)\cos\pi\lambda\,d\lambda = \frac{(-1)^{n+1}}{2}\delta_{m,n} \tag{5.2}$$

by (2.23). Hence formally we have

$$\begin{aligned}
&\int_0^\infty \varphi_{\lambda-1}^{(\beta,\alpha)}(-x)F(\lambda-1)\cos\pi\lambda\,d\lambda\\
&\quad= \int_0^\infty \sum_{n=0}^\infty s_n^1(\lambda-1)\varphi_n^{(\beta,\alpha)}(-x)\sum_{n=0}^\infty F(n)s_n^1(\lambda-1)\cos\pi\lambda\,d\lambda\\
&\quad= \sum_{n=0}^\infty F(n)\frac{(-1)^{n+1}}{2}\varphi_n^{(\beta,\alpha)}(-x)\\
&\quad= -\sum_{n=0}^\infty F(n)\varphi_n^{(\alpha,\beta)}(x)\frac{\Gamma(n+\alpha+1)\Gamma(\beta+1)}{2\Gamma(n+\beta+1)\Gamma(\alpha+1)}\\
&\quad= -\sum_{n=0}^\infty F(n)\frac{\varphi_n^{(\alpha,\beta)}(x)}{\|\varphi_n^{(\alpha,\beta)}\|^2\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&\quad= \frac{f(x)}{(-4)\Gamma(\alpha+1)\Gamma(\beta+1)}
\end{aligned} \tag{5.3}$$

and the inverse is given by

$$f(x) = (-4)\Gamma(\alpha+1)\Gamma(\beta+1)\int_0^\infty F(\lambda-1)\varphi_{\lambda-1}^{(\beta,\alpha)}(-x)\cos\pi\lambda\,d\lambda. \tag{5.4}$$

This integral converges for $F(\lambda-1) = O(\lambda^{-1/2-\alpha-\varepsilon})$. Although the series in (5.3) may not always converge pointwise in this case, once the formula (5.4) is known, various alternate proofs can be used (e.g. both sides represent the same element of $\mathcal{D}'(-1,1)$; since both are continuous functions, they are equal).

5.3. *The case $2q$ an odd integer.* We begin with (5.1) for

$$f(x) = \varphi_n^{(\alpha,\beta)}(x)/\|\varphi_n^{(\alpha,\beta)}\|^2.$$

Since, by (4.3), the expansion coefficient of $\varphi_n^{(\alpha,\beta)}$ is $s_n^q(\lambda)$, it follows that

$$\frac{2\varphi_n^{(\alpha,\beta)}(x)}{\|\varphi_n^{(\alpha,\beta)}\|^2} = 4\Gamma(\alpha+1)\Gamma(\beta+1)\int_0^\infty s_n^{1/2}\left(\lambda-\frac{1}{2}\right)\varphi_{\lambda-1/2}^{(\beta,\alpha)}(-x)\lambda\sin\pi\lambda\,d\lambda \tag{5.5}$$

holds for $2q = 1$. By (2.26) we have in this case that

$$\int_0^\infty s_n^q(\lambda-q)s_m^q(\lambda-q)\lambda\sin\pi\lambda\,d\lambda = \frac{(q+n)(-1)^{n+[q]}}{2}\delta_{mn}. \tag{5.6}$$

We shall use the fact that $s_n^q(\lambda-q) = s_{n-1}^{q+1}(\lambda-q-1)$ which is used to show that [10]

$$\frac{s_n^q(\lambda-q)}{(\lambda^2-(q-1)^2)(\lambda^2-(\frac{1}{2})^2)} = \frac{s_n^q(\lambda-q)n!}{\Gamma(n+2q)} + \delta_{n,q}^0(\lambda) \tag{5.7}$$

where $\delta_{n,q}^0(\lambda)$ is orthogonal to all $s_m^q(\lambda-q)$, $m = 0, 1, \dots,$ with respect to $\lambda\sin\pi\lambda\,d\lambda$. Hence we can write

$$\begin{aligned}
&\int_0^\infty s_n^q(\lambda-q)\varphi_{\lambda-q}^{(\beta,\alpha)}(-x)\frac{\lambda\sin\pi\lambda}{(\lambda^2-(\frac{1}{2})^2)\cdots(\lambda^2-(q-1)^2)}\,d\lambda \\
&= \int_0^\infty \frac{s_n^q(\lambda-q)n!}{\Gamma(n+2q)}\sum_{m=0}^\infty s_m^q(\lambda-q)\varphi_m^{(\beta,\alpha)}(-x)\lambda\sin\pi\lambda\,d\lambda \\
&= \frac{(n+q)(-1)^{n+[q]}n!}{2\Gamma(n+2q)}\varphi_n^{(\beta,\alpha)}(-x) \\
&= \frac{(-1)^{[q]}}{2}\,\frac{(n+q)n!}{\Gamma(n+2q)}\,\frac{\Gamma(n+\alpha+1)\Gamma(\beta+1)}{\Gamma(n+\beta+1)\Gamma(\alpha+1)}\varphi_n^{(\alpha,\beta)}(x) \\
&= \frac{(-1)^{[q]}}{2}\,\frac{\varphi_n^{(\alpha,\beta)}(x)}{\|\varphi_n^{(\alpha,\beta)}\|^2}\,\frac{2^{2q-1}}{\Gamma(\beta+1)\Gamma(\alpha+1)}.
\end{aligned}$$

Hence by dividing by the appropriate factors, we find that

$$\begin{aligned}
\frac{\varphi_n^{(\alpha,\beta)}(x)}{\|\varphi_n^{(\alpha,\beta)}\|^2} &= (-1)^{q-1/2}2^{2-2q}\Gamma(\alpha+1)\Gamma(\beta+1) \\
&\quad\times\int_0^\infty s_n^q(\lambda-q)\varphi_{\lambda-q}^{(\beta,\alpha)}(-x)\frac{\lambda\sin\pi\lambda}{(\lambda^2-(\frac{1}{2})^2)\cdots(\lambda^2-(q-1)^2)}\,d\lambda.
\end{aligned} \tag{5.8}$$

Hence the inverse operator is given by using the fact that

$$F(\lambda) = \sum_{n=0}^\infty F(n)s_n^\lambda(\lambda-q)$$

to deduce that

$$\begin{aligned}
2^{2q}\sum_{n=0}^\infty F(n)\frac{\varphi_n^{(\alpha,\beta)}(x)}{\|\varphi_n^{(\alpha,\beta)}\|^2} &= f(x) = (-1)^{q-1/2}4\Gamma(\alpha+1)\Gamma(\beta+1) \\
&\quad\times\int_0^\infty F(\lambda-q)\varphi_{\lambda-q}^{(\beta,\alpha)}(-x)\frac{\lambda\sin\pi\lambda}{(\lambda^2-(\frac{1}{2})^2)\cdots(\lambda^2-(q-1)^2)}\,d\lambda.
\end{aligned} \tag{5.9}$$

5.4. *The case in which q is an integer.* In this case we have by (2.27)

$$\int_0^\infty s_n^q(\lambda-q)s_m^q(\lambda-q)\cos\pi\lambda\,d\lambda = \frac{(-1)^{n+q}}{2}\delta_{mn}.$$

The expression corresponding to (5.7) is

$$\frac{s_n^q(\lambda-q)}{(\lambda^2-(q-1)^2)\cdots(\lambda^2-1)} = \frac{s_n^q(\lambda-q)n!(n+q)}{\Gamma(n+2q)} + \delta_{n,q}^e(\lambda) \tag{5.10}$$

where again $\delta_{n,q}^e$ is orthogonal to $s_m^q(\lambda-q)$ with respect to $\cos\pi\lambda\,d\lambda$. Then we have

$$\int_0^\infty s_n^q(\lambda-q)\varphi_{\lambda-q}^{(\beta,\alpha)}(-x)\frac{\cos\pi\lambda\,d\lambda}{(\lambda^2-1)\cdots(\lambda^2-(q-1)^2)}$$
$$= \frac{n!(n+q)(-1)^{n+q}}{[\Gamma(n+2q)]^2}\varphi_n^{(\beta,\alpha)}(-x) = (-1)^q\frac{\varphi_n^{(\alpha,\beta)}(x)}{\|\varphi_n^{(\alpha,\beta)}\|^2}\,\frac{2^{2q-2}}{\Gamma(\beta+1)\Gamma(\alpha+1)}.$$

Hence the inverse operator is given by

$$\begin{aligned} f(x) &= (-1)^q 4\Gamma(\alpha+1)\Gamma(\beta+1) \\ &\times \int_0^\infty F(\lambda-q)\varphi_{\lambda-q}^{(\beta,\alpha)}(-x)\frac{\cos\pi\lambda}{(\lambda^2-1)\cdots(\lambda^2-(q-1)^2)}\,d\lambda. \end{aligned} \tag{5.11}$$

THEOREM 5.1. *Let $f\in\mathcal{D}(-1,1)$ and let $F(\lambda)$ be its (α,β)-Jacobi transform, given by* (4.4) *where $\alpha+\beta+1=2q$, a positive integer. Then the inverse transform is given by* (5.11) *when $2q$ is even and by* (5.9) *when $2q$ is odd where the integrals converge uniformly for x in interior intervals of* $(-1,1)$.

PROOF. We have already shown that (5.11) and (5.9) are the formal inverses. Since $F(\lambda)$ is a rapidly decreasing entire function by Proposition 4.1 and $\varphi_{\lambda-1}^{(\beta,\alpha)}(x) = O(\lambda^{\alpha-1/2})$ uniformly on interior intervals of $(-1,1)$ we need only show that the integrand has no finite singularities. Since

$$F(\lambda-q) = \sum_{n=0}^\infty F(n)s_n^q(\lambda-q)$$

and $s_n^q(k)=0$ for k an integer $\neq n$, it follows that when $2q$ is even, $F(1-q) = F(2-q) = \cdots = F(-1) = 0$ and when $2q$ is odd $F(\frac{1}{2}-q) = F(\frac{3}{2}-q) = \cdots = F(-1) = 0$. Hence the zeros of F match those of the denominator and the conclusion follows.

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