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UNIVALENT FUNCTIONS IN $H \cdot \overline{H}(D)$

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ABSTRACT. Functions in $H \cdot \overline{H}(D)$ are sense-preserving of the form $f = h \cdot \bar{g}$ where h and g are in $H(D)$. Such functions are solutions of an elliptic nonlinear P.D.E. that is studied in detail especially for its univalent solutions.

1. Introduction. Let D be a domain of $\mathbf{C}$, and $H(D)$ the set of all analytic functions defined on D endowed with the topology of normal (locally uniform) convergence. Denote by $H \cdot \overline{H}(D)$ the set of all complex-valued mappings f defined on D of the form

$$f = \mathrm{H} \cdot \overline{\mathrm{G}}; \quad \mathrm{H} \text{ and } \mathrm{G} \text{ are locally in } H(D) \tag{1.1}$$

which are open and preserve the orientation. Such a mapping satisfies the nonlinear elliptic differential equation

$$\bar{f}_{\bar{z}} = [a \cdot \bar{f}/f] f_z \tag{1.2}$$

where

$$a \in H(D) \quad \text{and} \quad a(D) \subset U = \{\zeta; |\zeta| < 1\}. \tag{1.3}$$

The motivation behind the study of such a class comes from the fact that for any sense-preserving harmonic function $u = \mathrm{H}_1 + \overline{\mathrm{G}}_1$, H_1 and G_1 in $H(D)$, e^u is a nonvanishing function of $H \cdot \overline{H}(D)$. Thus, of particular interest are those functions of $H \cdot \overline{H}(D)$ that vanish in D, as their zeros correspond to some singularities of harmonic functions.

In §2 we study solutions of (1.2) with a as in (1.3). By a solution we mean a complex-valued function in the Sobolev space $W^{1,2}_{\mathrm{loc}}$ which satisfies (1.2) almost everywhere. For $a \equiv 0$ we are led to the set of nonconstant function in $H(D)$. However, for other functions, a, satisfying (1.3) we may have solutions which are not in $H \cdot \overline{H}(D)$. For instance $f(z) = z|z|^{2\alpha}$, $\mathrm{Re}\{\alpha\} > -\frac{1}{2}$ and $f(1) = 1$ is a solution of (1.2) in $\mathbf{C}$ with $a \equiv \bar{\alpha}/(1+\alpha)$. We then denote by $\mathscr{F}(a, D)$ the set of all nonconstant solutions of (1.2) in D, where the given function a always satisfies (1.3). The relation between $\mathscr{F}(a, D)$ and $H \cdot \overline{H}(D)$ is finally established.

§3 is concerned with the univalent solutions of (1.2) with a as in (1.3). It includes the characterization of the univalent functions of $\mathscr{F}(a, \mathbf{C})$.

§4 contains an example showing that in general, the Riemann Mapping Theorem fails in our case. Instead, we establish the Mapping Theorem from the unit disk into a bounded simply connected domain of $\mathbf{C}$, the boundary of which is locally connected.

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2. Representation theorems. Let D be a domain of $\mathbf{C}$ and $f \in \mathscr{F}(a, D)$. Then f is a nonconstant locally quasiregular mapping and therefore it is open and sense preserving. Denote by $Z(f)$ the zero set of f, i.e.,

$$Z(f) = \{z \in D;\ f(z) = 0\}.$$

For $z_0 \in D \setminus Z(f)$, let $B(z_0, \rho) \subseteq D \setminus Z(f)$, where

$$B(z_0, \rho) = \{z;\ |z - z_0| < \rho\}.$$

Since

$$\overline{(\log f)_{\bar z}} = a(\log f)_z; \qquad z \in D \setminus Z(F),$$

we may choose a branch of $\log f$ which is harmonic in $B(z_0, \rho)$ [4]. Observe that f_z/f and $\overline{f_{\bar z}/f}$ are in $H(B(z_0, \rho))$. Put

$$\mathrm{H}(z) = f(z_0) \exp\left(\int_{z_0}^{z} f_s(s)/f(s)\, ds\right)$$

and

$$\mathrm{G}(z) = \exp\left(\overline{\int_{z_0}^{z} f_{\bar s}(s)/f(s)\, d\bar s}\right),$$

for $z \in B(z_0, \rho)$. Then $f = \mathrm{H} \cdot \overline{\mathrm{G}} \in H \cdot \overline{H}(B(z_0, \rho))$. Note that although f_z/f and $\overline{f_{\bar z}/f}$ are in $H((D \setminus Z(f))$, yet H and G can be multivalued locally analytic functions.

Conversely, let f be in $H \cdot \overline{H}(D)$ and $0 \in f(D)$. Then $f = \mathrm{H} \cdot \overline{\mathrm{G}}$, H and G in $H(D)$, is open and preserves orientation. Therefore, $a = (\overline{f_{\bar z}/f})/(f_z/f) = (\mathrm{G}'/\mathrm{G})/(\mathrm{H}'/\mathrm{H})$ is in $H((D \setminus Z(\mathrm{H}' \cdot \mathrm{G}))$ and $a(D \setminus Z(h')) \subset \overline{U}$. Since f is sense preserving, H is not a constant, which implies that $Z(\mathrm{H}'\mathrm{G})$ is a discrete set in D. Therefore, $a \in H(D)$ and $a(D) \subset U$.

Summarizing, we have the following lemma.

LEMMA 2.1. *Let D be a simply connected domain of* $\mathbf{C}$. *A nonvanishing function f is in $H \cdot \overline{H}(D)$ if and only if f is in $\mathscr{F}(a, D)$ for some function a satisfying* (1.3).

Next, we shall investigate the behavior of a solution f in $\mathscr{F}(a, D)$ at a point $z_0 \in D$ where f vanishes. We start by noting that $Z(f)$ is discrete in D. Indeed, f is a nonconstant locally quasiregular mapping and therefore it is continuous, open and light. By a theorem of Stoiloff it follows that f can be represented as a composition of two functions

$$f = E \circ \chi \tag{2.1}$$

where χ is a locally quasiconformal homeomorphism on D and $E \in H(\chi(D))$. The result follows.

LEMMA 2.2. *Let f be in $\mathscr{F}(a, D)$. Suppose that $f(z_0) = 0$ and that $B(z_0, \rho) \setminus \{z_0\} \subset D \setminus Z(f)$. Then f admits the representation*

$$f(z) = (z - z_0)^n |z - z_n|^{2\beta} \cdot h(z) \cdot \overline{g(z)}; \qquad z \in B(z_0, \rho), \tag{2.2}$$

where $n \in N$, $\beta = n\overline{a(z_0)}(1 + a(z_0))/(1 - |a(z_0)|^2)$ *and therefore* $\operatorname{Re}\{\beta\} > -n/2$. h *and* g *are in* $H(B(z_0, \rho))$, $h(z_0) \neq 0$ *and* $g(z_0) = 1$.

PROOF. Since f_z/f and $\overline{f_{\bar z}/f}$ are in $H(B(z_0, \rho)\setminus\{z_0\})$, they admit a Laurent series expansion at z_0. Therefore

$$\begin{aligned} f(z) &= \text{const}\exp\left[\int^z f_z/f\,dz + \int^z f_{\bar z}/f\,d\bar z\right] \\ &= \text{const}\, h(z)\cdot\overline{g(z)}\,(z - z_0)^{\alpha}\overline{(z - z_0)}^{\beta} h_1(z)\,\overline{g_1(z)}, \end{aligned}$$

where h and g are in $H(B(z_0, \rho))$, $h(z_0)\cdot g(z_0) \neq 0$, and

$$h_1(z) = \exp\left(\sum_{n=1}^{\infty} c_n/(z - z_0)^n\right)$$

and

$$g_1(z) = \exp\left(\sum_{n=1}^{\infty} d_n/(z - z_0)^n\right)$$

are in $H(\mathbf{C}\setminus\{z_0\})$. Since f is single-valued in $B(z_0, \rho)$, $(z - z_0)^{\alpha}\overline{(z - z_0)}^{\beta}$ is also single-valued there and therefore $\alpha = \beta + n$ for some $n \in \mathbf{Z}$. By the generalized Schwarz Lemma for the quasiregular mapping f we have $|f| = O(|z - z_0|^{\gamma})$ as z tends to z_0 for some $\gamma > 0$ and therefore $|h_1(z)g_1(z)| = O(1/|z - z_0|^{\delta})$ for some positive δ. We then conclude that $h_1 \cdot g_1$ is a polynomial p of $1/(z - z_0)$. Put $R(z) = (z - z_0)[h_1'(z)/h_1(z) + g_1'(z)/g_1(z)]$. Then $\lim_{z\to z_0} R(z)$ exists and is finite. On the other hand we have

$$\begin{aligned} (2.3)\quad a(z_0) = \lim_{z\to z_0} a(z) &= \lim_{z\to z_0}\frac{(z - z_0)\left[g'(z)/g(z) + g_1'(z)/g_1(z)\right] + \bar\beta}{(z - z_0)\left[h'(z)/h(z) + h_1'(z)/h_1(z)\right] + \beta + n} \\ &= \lim_{z\to z_0}\frac{R(z) - (z - z_0)h_1'(z)/h_1(z) + \bar\beta}{(z - z_0)h_1'(z)/h_1(z) + \beta + n} \end{aligned}$$

from which we deduce that $\lim_{z\to z_0}(z - z_0)h_1'(z)/h_1(z)$ exists. Since $|a(z_0)| < 1$, this last limit is finite and therefore $(z - z_0)h_1'(z)/h_1(z) = -\sum_{n=1}^{\infty} nc_n/(z - z_0)^n$ is in $H(B(z_0, \rho))$. This implies that h_1 is the constant one. Similarly one shows that g_1 is also identically one. Now (2.3) reduces to $a(z_0) = \bar\beta/(\beta + n)$ which, if solved for β, gives $\beta = n\overline{a(z_0)}(1 + a(z_0))/(1 - |a(z_0)|^2)$. Since $|a(z_0)| < 1$ and $f(0) = 0$ we get that $n \in \mathbf{N}$ and $\operatorname{Re}\{\beta\} > -n/2$. This concludes the proof of the lemma. ■

The main result of this section is

THEOREM 2.3. *Let D be a simply connected domain of* $\mathbf{C}$. *If f is in $H\cdot\overline{H}(D)$ then f is in $\mathscr{F}(a, D)$ for some function a satisfying* (1.3) *such that $a(z)$ is a rational number in* $[0, 1)$ *whenever $z \in Z(f)$. Conversely, let f be in $\mathscr{F}(a, D)$ and suppose that for each $z_0 \in Z(f)$ we have $a(z_0) = p(z_0)/q(z_0) \in [0, 1)$ where $p(z_0) \in \mathbf{N}\cup\{0\}$, $q(z_0) \in \mathbf{N}$, and $q(z_0) - p(z_0)$ is a divisor of $p(z_0)$. Then f is in $H\cdot\overline{H}(D)$.*

PROOF. Let

$$f \in H\cdot\overline{H}(D).$$

Then f is not a constant and belongs to $\mathcal{F}(a, D \setminus Z(f))$ for some a as in (1.3). Since $Z(f)$ is discrete in D, the function a has an analytic continuation in D and $a(D) \subset U$. Let $z_0 \in Z(f)$. Then by (1.1) we have

$$a(z_0) = \lim_{z \to z_0} \left[(z - z_0)\mathrm{G}'(z)/\mathrm{G}(z)\right] / \left[(z - z_0)\mathrm{H}'(z)/\mathrm{H}(z)\right]$$
$$= p/q \in \mathbf{Q} \cap [0,1)$$

where p and q are the zero order of H and G at z_0, respectively. Conversely, let $f \in \mathcal{F}(a, D)$ and suppose that for each $z \in Z(f)$ we have $a(z) = p(z)/q(z) \in [0,\ 1)$ where $p(z) \in \mathbf{N} \cup \{0\}$, $q(z) \in \mathbf{N}$, and $q(z) - p(z)$ is a divisor of $p(z)$. Fix $\zeta \in D$. If $f(\zeta) \neq 0$, then by Lemma 2.1 $f \in H \cdot \overline{H}(B(\zeta, \rho))$ whenever $B(\zeta, \rho) \subset D$. If $f(\zeta) = 0$, then by Lemma 2.2, (2.2) holds with $\beta = p/(q - p) \in \mathbf{N}$ and again: $f \in H \cdot \overline{H}(B(\zeta, \rho))$ whenever $B(\zeta, \rho) \subset D$. Observe that if $f = \mathrm{H}_1 \cdot \overline{\mathrm{G}}_1 = \mathrm{H}_2 \cdot \overline{\mathrm{G}}_2$ on a disk $B(\zeta, \rho) \subset D$ and $\mathrm{G}_1(z_0) = \mathrm{G}_2(z_0)$ then $\mathrm{H}_1 = \mathrm{H}_2$ and $\mathrm{G}_1 = \mathrm{G}_2$. D being simply connected, there are H and G in $H(D)$ such that $f = \mathrm{H} \cdot \overline{\mathrm{G}} \in H \cdot \overline{H}(D)$.

3. Univalent functions in $H \cdot \overline{H}(D)$. Let D be a simply connected domain of $\mathbf{C}$, and $z_0 \in D$. Then the following characterization follows from Theorem 2.3.

THEOREM 3.1. *Let f be a univalent mapping defined on D such that $f(z_0) = 0$. Then f is in $H \cdot \overline{H}(D)$ if and only if $f \in \mathcal{F}(a, D)$ for some a satisfying* (1.3) *such that $a(z_0) = m/(1 + m)$; $m \in \mathbf{N} \cup \{0\}$.*

PROOF. If $f \in H \cdot \overline{H}(D)$ is univalent, then the exponent n in (2.2) is one and $a(z_0) = m/(1 + m)$ where m is a nonnegative integer. The converse is covered by Theorem 2.3. ■

LEMMA 3.2. *Let D be a simply connected domain of $\mathbf{C}$ and f a univalent function in $\mathcal{F}(a, D)$. Then we have*

(a) *$f_z(z) \neq 0$ for all $z \in D$ whenever $f(z) \neq 0$, and*

(b) *If $f(z_0) = 0$ then $\lim_{z \to z_0}(z - z_0)f_z(z)/f(z)$ exists and is in $\mathbf{C} \setminus \{0\}$.*

Therefore $(z - z_0)f_z/f$ is a nonvanishing function in $H(D)$.

PROOF. (a) Let $f(z) \neq 0$. Then $\log f$ can be defined as a univalent harmonic mapping in a small disk around z. It follows that $(\log f)_z(z) = f_z(z)/f(z) \neq 0$ and therefore $f_z(z) \neq 0$.

(b) Suppose that $f(z_0) = 0$. Then by Lemma 2.2 and the univalence of f we have

$$f(z) = (z - z_0)|z - z_0|^{2\beta}h(z) \cdot \overline{g(z)}, \qquad z \in B(z_0, \rho) \subset D, \tag{3.1}$$

where h and g are as in Lemma 2.2 and $\operatorname{Re}\beta > -\frac{1}{2}$. Therefore we have

$$\lim_{z \to z_0} (z - z_0)f_z(z)/f(z) = 1 + \beta \neq 0. \quad ■$$

LEMMA 3.3. *Let $f_0 \in \mathcal{F}(a_0, D)$ be univalent and $\alpha \in \{\alpha \in \mathbf{C};\ \operatorname{Re}\{\alpha\} > -\frac{1}{2}\}$. Then $f = f_0 \cdot |f_0|^{2\alpha}$ is univalent and belongs to $\mathcal{F}(a, D)$ where*

$$a = \frac{1 + \bar{\alpha}}{1 + \alpha}\left[\frac{a_0 + \bar{\alpha}/(1 + \bar{\alpha})}{1 + a_0\alpha/(1 + \alpha)}\right]$$

satisfies (1.3).

PROOF. Direct calculations show that $\overline{f_{\bar{z}}} = a(\bar{f}/f)f_z$ in D. Since $\operatorname{Re}\{\alpha\} > -\frac{1}{2}$ we have $|\bar{\alpha}/(1+\alpha)| < 1$ and therefore a satisfies (1.3). Next, f is not a constant since f_0 is not a constant and therefore $f \in \mathcal{F}(a, D)$. The univalence of f follows from the fact that $w|w|f^{2\alpha}$, $\operatorname{Re}\alpha > -\frac{1}{2}$, is univalent in $\mathbf{C}$. ∎

In our next result we consider univalent solutions in $\mathcal{F}(a, \mathbf{C})$. By Liouville's Theorem we know that $a(z) \equiv a \in U$.

THEOREM 3.4. *A function $f \in \mathcal{F}(a, \mathbf{C})$ is univalent in $\mathbf{C}$ if and only if*

$$f(z) = \operatorname{const}(z - z_0)|z - z_0|^{2\beta}; \quad \beta = \bar{a}(1 + a)/(1 - |a|^2) \tag{3.2}$$

and $z_0 \in \mathbf{C}$.

PROOF. Let f be of the form (3.2). In Lemma 3.2 put $D = \mathbf{C}$, $f_0(z) = (z - z_0)$, $a_0(z) = 0$, and $\alpha = \beta$. Then we get that $f \in \mathcal{F}(\bar{\beta}/(1+\beta), \mathbf{C})$ and is univalent in $\mathbf{C}$. Conversely, let f be univalent and in $\mathcal{F}(a, \mathbf{C})$, $a(z) \equiv a \in U$. Put $\hat{f} = f|f|^{-2\bar{a}/(1+\bar{a})}$. Then by Lemma 3.3 $\hat{f}$ is an entire univalent function and therefore $\hat{f}(z) = \operatorname{const}(z - z_0)$. Solving for f we get that $f = \operatorname{const}(z - z_0)|z - z_0|^{2\beta}$, $\beta = \bar{a}(1 + a)/(1 - |a|^2)$. ∎

Let now D be a simply connected domain of $\mathbf{C}$, $D \neq \mathbf{C}$, and $f \in \mathcal{F}(a, D)$. If $0 \notin f(D)$, then $\log f$ can be defined as a univalent harmonic mapping on D. Since such mappings have been extensively studied **[1–4]**, we assume that $0 \in f(D)$. Denote by ϕ a conformal mapping from the unit disk U onto D. If $f \in \mathcal{F}(a, D)$ then $f \circ \phi \in \mathcal{F}(\hat{a}, U)$ where $\hat{a} = a \circ \phi$. Therefore we may assume that $D = U$ and $f(0) = 0$. Furthermore, by applying the postmapping $cw|w|^{2\alpha}$, $\alpha = \overline{-a(0)}/(1 + \overline{a(0)})$ and c an appropriate constant, we may assume that $a(0) = 0$ and $f_z(0) = 1$. We then denote

$$S_M = \bigcup_{a \in A} \{f \text{ univalent in } \mathcal{F}(a, U);\ f(0) = 0,\ f_z(0) = 1\},$$

where A denotes the set of all functions $a \in H(U)$ such that $a(U) \subset U$ and $a(0) = 0$. As a direct consequence of Theorem 2.3 we get that

$$S_M = \{f = z \cdot h \cdot \bar{g} \in H \cdot \overline{H}(U);\ f \text{ univalent and } h(0) = g(0) = 1\}. \tag{3.3}$$

Our first result concerning S_M is

THEOREM 3.5. *S_M is compact in the topology of normal convergence.*

PROOF. Let f_n, $n \in \mathbf{N}$, be in S_M. Then by considering an approprite subsequence of $\{f_n\}_{n=1}^{\infty}$ we may assume that the corresponding $\{a_n\}_{n=1}^{\infty}$ converges to some function a in A. By Schwarz' Lemma for a_n we know that each f_n is a K_r-quasiconformal mapping in rU for all $r < 1$. By a well-known result on quasiconformal mappings we know that f_n converges normally in rU to a K_r-quasiconformal function $f \in \mathcal{F}(a, rU)$ for all $r < 1$. Therefore f is in S_M. ∎

The following lemmas are needed later on.

LEMMA 3.6. *For $f \in S_M$ we have*

$$1/16 \leqslant \operatorname{dist}(0, \partial f(U)) \leqslant 1.$$

PROOF. Since $a(0) = 0$ we have $|f_{\bar{z}}(z)| \leqslant |z||f_z(z)|$ for all $z \in U$ and from (3.3) we deduce that $f(z) = z + O(|z|^2)$ near zero. By Lemma 3.3 in [3] we conclude that

$$|f(z)| \geqslant |z|/4(1+|z|)^2$$

for all $z \in U$. In particular the disk $\{w;\ |w| < 1/16\}$ is in $f(U)$.

On the other hand

$$\operatorname{dist}(0, \partial f(U)) = \varliminf_{|z|\to 1} |f(z)| = \varliminf_{|z|\to 1} |h(z)g(z)| \leqslant |h(0)g(0)| = 1. \qquad \blacksquare$$

LEMMA 3.7. *Let $f = zh\bar{g}$ be in S_M. Then $s = zh/g$ is locally univalent in U.*

PROOF. By Lemma 3.2 we know that zf_z/f is a nonvanishing function in $H(U)$. Since $zs'/s = (1-a)zf_z/f$ for some $a \in A$, zs'/s does not vanish in U. But $s'(0) = 1$; therefore $O \notin s'(U)$ and the result follows. ■

4. Mapping theorem. In this section we look for an analogue of the Riemann Mapping Theorem. Let $\Omega \neq \mathbf{C}$ be a simply connected domain in $\mathbf{C}$ and let $a \in H(U)$, $a(U) \subset U$ be given. Fix a $z_0 \in U$ and $w_0 \in \Omega$. We are interested in the existence of a univalent function $f \in \mathscr{F}(a, U)$, $f(U) = \Omega$, normalized by $f(z_0) = w_0$ and $f_z(z_0) > 0$. Let us start with an example which will show that this problem is not solvable in general.

Suppose that we want to find a univalent mapping $f \in \mathscr{F}(-z, U)$ normalized by $f(0) = 0$ and $f_z(0) > 0$ such that f maps U onto $\Omega = \mathbf{C} \setminus (-\infty, -1]$. Assume that such a function exists. Then $f = zh\bar{g} \equiv s|g|^2 \in H \cdot \overline{H}(U)$, $s'(0) > 0$, and $g(0) = 1$. Furthermore, we have

(i) $s \in H(U)$ and s is locally univalent in U (Lemma 3.6), and

(ii) $\arg f/z = \arg s/z$ is a bounded harmonic function on U.

We will show that $s(z)/s'(0) = k(z) \equiv z/(1-z)^2$. First, observe that s is univalent in U. Indeed, $s \circ f^{-1}(w) = w/|g \circ f^{-1}(w)|^2 \equiv w \cdot p(w)$, where $p(w) > 0$ on $f(U)$, is a continuous locally univalent function in $f(U)$ and therefore maps each radial line segment $\{w = Re^{it},\ 0 \leqslant R < R_0\}$ in $f(U)$ injectively onto $\{w = \rho e^{it},\ 0 \leqslant \rho < \rho_0 \leqslant \infty\}$. Since $f(U)$ is a starlike domain with respect to the origin, we conclude that $s \circ f^{-1}$ is univalent to $f(U)$. Hence s is univalent in U. Now $\lim_{r\to 1} s(re^{it}) = \hat{s}(e^{it})$ exists almost everywhere on ∂U and by (ii) we know that $\hat{s}(e^{it})$ lies on the negative real axis almost everywhere. Therefore $s(z)/s'(0) = k(z)$. Next, we shall determine the function g such that $f \in \mathscr{F}(-z, U)$. We need to solve

$$\overline{f_{\bar{z}}/f} = g'/g = -zf_z/f = -zk'/k - zg'/g, \quad g(0) = 1. \tag{4.1}$$

The unique solution of (4.1) is $g(z) = (1-z)$ and therefore we get that

$$f = \operatorname{const} z(1-\bar{z})/(1-z). \tag{4.2}$$

Observe that f is univalent in U, but maps U onto a disk and not Ω. In other words, there is no univalent mapping in $\mathscr{F}(-z, U)$ such that $f(0) = 0$, $f_z(0) > 0$, and $f(U) = \Omega$. However, we have the following Mapping Theorem.

THEOREM 4.1. *Let Ω be a bounded simply connected domain of* $\mathbf{C}$ *whose boundary is locally connected. Fix $0 \in \Omega$ and let $a \in H(U)$ such that $a(U) \subset U$ be given. Then there is a univalent function $f \in \mathscr{F}(a, \Omega)$ having the following properties*:

(i) *$f(U) \subset \Omega$, normalized at the origin by $f(z) = cz|z|^{2\beta}(1 + o(1))$, where $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$ and $c > 0$.*

(ii) *$\lim_{z \to e^{it}} f(z) = \hat{f}(e^{it})$ exists and is in $\partial\Omega$ for all $t \in \partial U \setminus E$, where E is a countable set.*

(iii) *For each $e^{it_0} \in \partial U$, we have that*

$$f_*(e^{it_0}) = \operatorname*{ess\,lim}_{t \uparrow t_0} \hat{f}(e^{it}) \quad and \quad f^*(e^{it_0}) = \operatorname*{ess\,lim}_{t \downarrow t_0} \hat{f}(e^{it})$$

exist and are in $\partial\Omega$.

(iv) *For $e^{it_0} \in E$, the cluster set of f at e^{it_0} lies on a helix joining the point $f^*(e^{it_0})$ to the point $f_*(e^{it_0})$.*

REMARKS. (1) If $a(0) = m/1 + m$, $m \in N \cup \{0\}$, then f is in $H \cdot \overline{H}(U)$.

(2) In the case where $\|a\| = \operatorname{Sup}_{z \in U} |a(z)| < 1$, properties (i) and (ii) imply that $f(U) = \Omega$.

(3) If $e^{it_0} \in E$ and $f_*(e^{it_0}) = f^*(e^{it_0})$ then the cluster set at e^{it_0} is a circle centered at the origin of radius $|f^*(e^{it_0})|$.

(4) Suppose that $A = f^*(e^{it_0}) \neq f_*(e^{it_0}) = B$. Then there are infinitely many helices joining A and B. Our claim is that the cluster set of f at e^{it_0} lies on one of them. Thus, for example, the cluster set of

$$f(z) = \frac{z(1 - \bar{z})}{(1 - z)} \exp\left(-2 \arg\left(\frac{1 - iz}{1 - z}\right)\right)$$

at $z = 1$ lies on the helix $\gamma(\tau) = \exp[-\tau + i(\pi/2 + \tau)]$ joining the points $f^*(1) = -e^{-\pi/2}$ and $f_*(1) = -e^{3\pi/2}$, whereas the cluster set of f at $z = -i$ is the straight line segment from $f^*(-i) = -e^{-\pi/2}$ to $f_*(-i) = -e^{3\pi/2}$.

PROOF. Assume first that $a(0) = 0$. Let ϕ be the conformal mapping from U onto Ω normalized by $\phi(0) = 0$, $\phi'(0) > 0$. Denote by $\Omega_n = \{w = \phi(z);\ |z| < r_n\}$, $r_n = n/(n + 1)$, $n \in \mathbf{N}$. Then, there exists a univalent function $f_n \in \mathscr{F}(a_n \equiv a(r_n z), U)$, mapping U onto Ω_n such that $f_n(0) = 0$ and $(f_n)_z(0) > 0$. Indeed, consider $F_n = (1/r_n)\phi^{-1} \circ f_n$. Then F_n has to satisfy the nonlinear elliptic equation

$$\overline{(F_n)_{\bar{z}}} = a_n \frac{\bar{f}}{f} \cdot \frac{\overline{\phi' \circ F_n}}{\phi' \circ F_n} \cdot (F_n)_z; \quad F(0) = 0, \quad (F_n)_z(0) > 0$$

and map U onto U univalently. This has a solution (see for example the proof of Theorem 5.1 in **[4]**) and therefore the existence of f_n follows. Next, we show the existence of a mapping f having the properties of the theorem.

Since Ω is bounded, then by applying the diagonal procedure on the exhaustion of U, we conclude that there is a subsequence f_{n_k} which converges normally to a function f satisfying (1.2) with the given $a \in A$ and $f(0) = 0$. By Lemma 3.6, we have

$$\operatorname{dist}(0, \partial\Omega_1) \leqslant (f_n)_z(0) \leqslant 16 \operatorname{dist}(0, \partial\Omega).$$

Therefore $f_z(0) > 0$ and f is univalent. Furthermore, by the argument principle for quasiconformal mappings we have that $f(U) \subset \Omega$. Now, since each prime end of $\partial\Omega$ is singleton, ϕ has a uniformly continuous extension to $\overline{U}$ and $0 \notin \phi(\partial U)$. Observe that the branch of $\log(\phi/z)$, $\log \phi'(0) \in \mathbf{R}$, is harmonic in U and continuous on $\overline{U}$. Therefore $\log(\phi/z)$ is bounded in $\overline{U}$. Likewise we claim that the branch of $g = \log(f/z)$, $\operatorname{Im} g(0) = 0$ is bounded in $\overline{U}$. To see this, let $g_n = \log(f_n/z)$, $\operatorname{Im} g_n(0) = 0$ be defined as continuous harmonic functions in U. We shall show that g_n are uniformly bounded. Indeed, each f_n is a K_{r_n}-quasiconformal mapping on U with $K_{r_n} = (1 + r_n)/(1 - r_n)$ and $f_n(U) = \Omega_n$ is bounded by an analytic Jordan curve. Hence f_n has a continuous univalent extension to $\overline{U}$ and therefore g_n admits a continuous extension to $\overline{U}$. Evidently $\operatorname{Re}\{g_n\} = \log|f_n/z|$ are uniformly bounded since Ω_n are uniformly bounded. As of $\operatorname{Im}\{g_n\}$, there are nondecreasing continuous functions $\tau_n(t)$ defined on $\mathbf{R}$ by

$$\arg\left[f_n(e^{it})/e^{it}\right] = \arg\left[\phi\left(r_n e^{i\tau_n(t)}\right)/e^{i\tau_n(t)}\right] + \tau_n(t) - t \tag{4.3}$$

which satisfy $\tau_n(t + 2\pi) = \tau_n(t) + 2\pi$ for all $t \in \mathbf{R}$. Therefore there are $k_n \in \mathbf{Z}$ such that

$$|\tau_n(t) - t - 2k_n\pi| \leqslant 2\pi$$

or

$$2(|k_n| - 1)\pi \leqslant |\tau_n(t) - t| \leqslant 2(|k_n| + 1)\pi. \tag{4.4}$$

On the other hand, $\int_0^{2\pi} \arg[f_n(e^{it})/e^{it}]\,dt = 0$, which implies that there is a t_n such that $f_n(e^{it_n})e^{-it_n} > 0$ and therefore

$$\begin{aligned} 2\left(|k_n| - 1\right)\pi &\leqslant |\tau_n(t_n) - t_n| = \left|\arg\left[\phi\left(r_n e^{i\tau_n(t_n)}\right)/e^{i\tau_n(t_n)}\right]\right| \\ &\leqslant 2\left(|k_n| + 1\right)\pi. \end{aligned}$$

But $\sup_{|z|=1} |\arg \phi(z)/z| = M < \infty$ implies that $|k_n| \leqslant 1 + M/2\pi$. Finally from (4.3) and (4.4) we get that

$$\operatorname{Im}\{g_n(z)\} = \arg\left[f_n(z)/z\right] \leqslant 2M + 4\pi.$$

This concludes the proof of our claim.

Now, $\lim_{r\to 1} \log[f(re^{it})/re^{it}]$ and therefore $\tilde{f}(e^{it}) = \lim_{r\to 1} f(re^{it})$ exists almost everywhere. In fact $\tilde{f}(e^{it}) \subset \partial\Omega$, since f_n is quasiconformal on U and therefore extends to a homemorphism from $\overline{U}$ onto $\overline{\Omega}_n$. Fix ε, $0 < \varepsilon < 1$, and consider a finite covering $\bigcup_j B(e^{it_j}, \varepsilon)$ of ∂U. Let γ_j be a conformal mapping from U onto $C_j = U \cap B(e^{it_j}, \varepsilon)$. Then $0 \notin f(B(e^{it_j}, \varepsilon))$ and therefore $F_j = \log f \circ \gamma_j$ can be defined as a univalent harmonic function from U onto $K_j \subset \Omega$. By Theorem 5.3 in **[4]** we conclude that except for at most a countable set E_j the unrestricted limit $\hat{F}_j(e^{it}) = \lim_{z\to e^{it}} F_j(z)$ exists, is continuous and belongs to $\overline{K_j}$. Let $E = \bigcup_j E_j$; then since each γ_j can be extended to a homeomorphism to $\overline{U}$ we conclude that $\hat{f}(e^{it}) = \lim_{z\to e^{it}} f(z)$ exists, is continuous, and belongs to $\partial\Omega$ for $e^{it} \in \partial U \setminus E$. By the same theorem, at the points $e^{i\theta}$ of E, the one-sided essential limits of $\log \hat{f}(e^{it})$ exist, are different, and belong to $\partial\Omega$; and finally, the cluster set at $e^{i\theta}$ of E is a straight line segment joining $(\log \hat{f})^*(e^{it})$ and $(\log \hat{f})_*(e^{it})$. Therefore $A_0 = \hat{f}(e^{i\theta})$ and $B_0 = \hat{f}_*(e^{i\theta})$ exist and belong to $\partial\Omega$ for $e^{i\theta} \in E$. The cluster set of f at such a point lies on a single helix $\exp(\lambda \log A_0 + (1 - \lambda)\log B_0)$, $0 < \lambda < 1$, joining A_0 and B_0 (depending on the corresponding values of $\log A_0$ and $\log B_0$). If for some point $e^{i\theta} \in E$, $\hat{f}^*(e^{it_0}) = \hat{f}_*(e^{it_0})$, then $\log A_0 = \log B_0 + 2\pi i$ and therefore the cluster set of f at $e^{i\theta_0}$ is $B_0 \exp[(1 - \lambda)2\pi i]$, $0 < \lambda < 1$, i.e., a circle centered at the origin of radius $|f^*(e^{i\theta_0})|$.

To remove the assumption $a(0) = 0$, we apply what has been proved to the domain

$$\tilde{D} = \left\{ w|w|^{-2\overline{(a(0))}/(1+\overline{a(0)})};\ w \in D \right\}$$

with

$$\tilde{a}(z) = (a(z) - a(0))/(1 - \overline{a(0)}\,a(z))$$

to obtain a mapping $\tilde{f}\colon U \to \tilde{D}$. Then

$$f = \tilde{f}|\tilde{f}|^{2\overline{a(0)}(1+a(0))/(1-|a(0)|^2)}$$

will be the desired solution. ∎

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