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CORRECTION TO "CARTAN SUBALGEBRAS OF SIMPLE LIE ALGEBRAS"

ROBERT LEE WILSON

Richard Block has pointed out that the argument of (5.5) in [**5**] is incorrect. In particular, the inequality "$n_{\beta+\alpha} + n_{\beta-\alpha} \geq n(p-1)p$" is unjustified. Thus the proof of Theorem 2.1 is incomplete. However, the theorem is correct as stated. We give a corrected proof here.

We will use the notation of [**5**], will continue the numbering of sections from [**5**] and will refer to results from [**5**] and from this note by their numbers without further identification. Thus Propositon $m.n$ is to be found in §m of [**5**] if $m \leq 5$ and in §m of this note if $m \geq 6$.

We begin by noting that if $\dim T = 1$ and $\overline{H} \neq T + I$ then Γ generates a cyclic group and so Proposition 3.3 shows we must have $(\alpha, \alpha) \in S$ for some $\alpha \in \Gamma$. However, the results of §4 show that this is impossible. Thus Theorem 2.1 holds when $\dim T = 1$. This is the only case of Theorem 2.1 used in [**6**] and hence the classification of the simple Lie algebras of toral rank one given in [**6**] is valid. We will use this classification below.

We will now assume that (3.3.2) holds and derive a contradiction, thus proving Theorem 2.1.

6. A module for $\sum L_{i\alpha}$.

(6.1) Let α and β be as in (3.3.2). For any $0 \neq \gamma$, $\delta \in \mathbf{Z}\alpha + \mathbf{Z}\beta$ define

$$(\cdot,\cdot)_\delta \colon L_\gamma \times L_{-\gamma} \to F$$

by

$$(u,v)_\delta = \delta([b,[u,v]]) \tag{6.1.1}$$

for all $u \in L_\gamma$, $v \in L_{-\gamma}$. For $0 \neq \gamma \in \mathbf{Z}\alpha + \mathbf{Z}\beta$ define $K_\gamma = (L_{-\gamma})^\perp$ where the complement is taken relative to the form $(\cdot,\cdot)_\delta$ for any $\delta \notin \mathbf{Z}\gamma$. Since $(u,v)_\delta$ is a linear function of δ and

$$(L_\gamma, L_{-\gamma})_\gamma = (0) \tag{6.1.2}$$

by (3.3.2) this definition is independent of the choice of δ.

Let $m_\gamma = \dim(L_\gamma/K_\gamma)$. Then by (3.3.2) we have m_α, $m_\beta \neq 0$. As is shown in §5.2, $p|m_\gamma$. We may, without loss of generality, assume that $m_\alpha \geq m_\gamma$ for all $\gamma \in \mathbf{Z}\alpha + \mathbf{Z}\beta$, $\gamma \neq 0$.

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(6.2) LEMMA. *Let* $0 \neq \gamma$, $\delta \in \mathbf{Z}\alpha + \mathbf{Z}\beta$, $\gamma \notin \mathbf{Z}\delta$. *Then* $[\overline{H}, K_\gamma] \subseteq K_\gamma$ *and* $[L_\delta, K_\gamma] \subseteq K_{\gamma+\delta}$.

PROOF. From §3.5 we have that $([h,x],y)_\delta = -(x,[h,y])_\delta$ for all $x \in L_\gamma$, $y \in L_{-\gamma}$, $h \in \overline{H}$. Hence $[\overline{H}, K_\gamma] \subseteq K_\gamma$. Also

$$\begin{aligned}\delta([b,[[L_\delta,K_\gamma],L_{-\delta-\gamma}]]) &\subseteq \delta([b,[[L_\delta,L_{-\delta-\gamma}],K_\gamma]]) + \delta([b,[L_\delta,[K_\gamma,L_{-\delta-\gamma}]]]) \\ &\subseteq \delta([b,[L_{-\gamma},K_\gamma]]) + \delta([b,[L_\delta,L_{-\delta}]]) \\ &= (L_{-\gamma},K_\gamma)_\delta + (L_\delta,L_{-\delta})_\delta = (0)\end{aligned}$$

by the definition of K_γ and (6.1.2). As $\delta \notin \mathbf{Z}(\gamma+\delta)$ we have $[L_\delta, K_\gamma] \subseteq K_{\delta+\gamma}$.

(6.3) Let $\overline{L}^{(\alpha)} = \overline{H} + \sum_{i=1}^{p-1} L_{i\alpha}$.

COROLLARY. *The adjoint representation induces a representation* θ *of* $\overline{L}^{(\alpha)}$ *on* $\sum_{i=0}^{p-1} L_{\beta+i\alpha}/K_{\beta+i\alpha}$. *Furthermore* $L_{i\alpha} \cap \ker\theta = K_{i\alpha}$ *for* $1 \leq i \leq p-1$.

PROOF. By the lemma $K_{i\alpha} \subseteq (L_{i\alpha} \cap \ker\theta)$. Now $\ker\theta$ is an ideal in $\overline{L}^{(\alpha)}$ so $[b,[L_{-i\alpha}, L_{i\alpha} \cap \ker\theta]] \subseteq (\ker\theta) \cap (T+I)$. Then if $u \in [b,[L_{-i\alpha}, L_{i\alpha} \cap \ker\theta]]$ we have $[u, L_\beta] \subseteq K_\beta$ and so $\beta(u) = 0$. Therefore $(L_{-i\alpha}, L_{i\alpha} \cap \ker\theta)_\beta = (0)$ and so we have equality.

7. Solvable subalgebras of $\overline{L}^{(\alpha)}$.

(7.1) LEMMA. *Let* A *be an* $(\operatorname{ad} T)$*-invariant subalgebra of* $\overline{L}^{(\alpha)}$. *Assume that* $(\operatorname{ad} x)|_A$ *is nilpotent for every* $x \in A_0$. *Then* A *is solvable.*

PROOF. We will proceed by induction on $\dim A$. Since $[A,A]$ is $(\operatorname{ad} T)$-invariant and $[A,A]_0 \subseteq A_0$ the result holds if $[A,A] \neq A$. Let Q be a maximal subalgebra of A containing A_0. Then Q induces a Lie algebra of linear transformations of A/Q. As $(A/Q)_0 = (0)$, and as the elements of $A_0 = Q_0$ act nilpotently on A/Q by hypothesis, the Engel-Jacobson theorem implies that this algebra is nil, hence annihilates a subspace of A/Q. By the maximality of Q this implies that $\dim(A/Q) = 1$ and $[A,A] \subseteq Q$. Thus $[A,A] \neq A$, as required.

(7.2) Let $U = \sum_{i=1}^{p-1}[L_{i\alpha}, L_{-i\alpha}]$ and $M = U + \sum_{i=1}^{p-1} L_{i\alpha}$. Let τ denote the canonical homomorphism of M onto $M/(\operatorname{solv} M)$.

LEMMA. *If* M *is not solvable then* $M/(\operatorname{solv} M)$ *is simple and has toral rank one with respect to* $\tau(U)$. *Consequently, if* $\dim\tau(M) < p^2 - 2$ *then* M *contains an* $(\operatorname{ad} T)$*-invariant solvable subalgebra* $S \supseteq U$ *with* $\dim(L_\alpha/S_\alpha) + \dim(L_{-\alpha}/S_{-\alpha}) \leq 1$.

PROOF. Lemma 7.1 shows that any proper ideal of M is solvable. Thus if M is not solvable, $M/(\operatorname{solv} M)$ is simple. Furthermore, since M is not solvable $(\operatorname{ad} U)|_M = (\operatorname{ad} M_0)|_M$ is not nil so U is a Cartan subalgebra of M. Hence $\tau(U)$ is a Cartan subalgebra of $M/(\operatorname{solv} M)$ and $M/(\operatorname{solv} M)$ obviously has toral rank one with respect to $\tau(U)$. Since $\ker(\theta|_M)$ is a proper ideal in M it is contained in $\operatorname{solv} M$. Thus $\dim\theta(M) \geq \dim(M/(\operatorname{solv} M))$. Since $M/(\operatorname{solv} M) \cong \mathfrak{sl}(2)$, some $W(1:\mathbf{n})$ or some $H(2:\mathbf{n}:\Phi)^{(2)}$ by Theorem 1.4 of [**6**] the inequality $\dim\theta(M) < p^2 - 2$ forces $M/(\operatorname{solv} M) \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$. Then taking S to be the preimage in M of the solvable subalgebra $\tau(M_\alpha) + \tau(U)$ in $M/(\operatorname{solv} M)$ gives the result.

(7.3) Let $\widetilde{U} = \overline{Fb} + \overline{U}$ and $\widetilde{M} = \widetilde{U} + M$.

LEMMA. *$\widetilde{M}$ contains an $(\operatorname{ad} T)$-invariant solvable subalgebra $S \supseteq \widetilde{U}$ such that $\dim(L_\alpha/S_\alpha) + \dim(L_{-\alpha}/S_{-\alpha}) \leq 3$.*

PROOF. If M is solvable we may take $\widetilde{M} = S$. Now assume M is not solvable. By Lemma 7.2 $M/(\operatorname{solv} M)$ is simple and has toral rank one with respect to $\tau(U)$ so by the previously established (§4) rank one case of Theorem 2.1 $[U,U]/([U,U] \cap \operatorname{solv} M)$ is nil. Since $\operatorname{solv} M$ is a proper ideal in M $\operatorname{ad}((\operatorname{solv} M)_0)|_M$ is nil, and hence $(\operatorname{ad}[U,U])|_M$ is nil. Now

$$[\widetilde{U}, \widetilde{U}] = [\overline{Fb} + \overline{U}, \overline{Fb} + \overline{U}] = [Fb + U, Fb + U] = [b, U] + [U, U].$$

But $\alpha([b,U]) = (0)$ by §4. Thus $\operatorname{ad}([b,U])|_M$ is nil and so $(\operatorname{ad}[\widetilde{U},\widetilde{U}])|_M$ is nil. Thus setting $V = (\operatorname{ad} \widetilde{U})|_M$ we see that $V = T_2 + J_2$ where T_2 is a one-dimensional torus in V and J_2 is a nil ideal in V. Thus the algebra $(\operatorname{ad} \widetilde{M})|_M$ with Cartan subalgebra V satisfies the hypotheses of §2 of [**6**]. Taking (P,Q) to be a pair of subspaces of $(\operatorname{ad} \widetilde{M})|_M$ satisfying (2.1.1)–(2.1.4) of [**6**] and with P of maximal dimension among all such pairs we see by Corollary 2.10 of [**6**] that $P = (\operatorname{ad} \widetilde{M})|_M$. Then Lemma 2.8 of [**6**] shows that each weight of P/Q has multiplicity one and that there exists a solvable subalgebra S' of Q which contains V and satisfies $\dim(Q_\alpha/S'_\alpha) + \dim(Q_{-\alpha}/S'_{-\alpha}) \leq 1$. Letting $S \subseteq \widetilde{M}$ be the inverse image of S' gives the result.

8. Conclusion.

(8.1) LEMMA. *Let S be an $(\operatorname{ad} T)$-invariant solvable subalgebra of $\widetilde{M}$. Let $s_i = \dim(L_{i\alpha}/S_{i\alpha})$ for $1 \leq i \leq p-1$. Then $s_1 + s_{-1} \geq 2$ and if $b \in S$ then $s_1 + s_{-1} \geq 3$. Furthermore, if $S \supseteq \widetilde{U}$ and $s_1 + s_{-1} = 3$ then $\dim \theta(\widetilde{M}) \leq p^2 - p + 6$.*

PROOF. By Corollary 6.3 S acts on $C = \sum L_{\beta+i\alpha}/K_{\beta+i\alpha}$. Let W be an irreducible S-submodule of C. Then

$$\dim W \leq \sum_{i=0}^{p-1} m_{\beta+i\alpha} \leq p m_\alpha. \tag{8.1.1}$$

Define $\Lambda \colon S \to \operatorname{End} W$ by $\Lambda(x) = \theta(x)|_W$ for $x \in S$.

The representation theory of solvable restricted Lie algebras (Schue [**2**], Strade [**3**], Weisfieler and Kac [**4**]; cf. Theorem 1.13.1 of [**1**]) shows that $\dim W = p^m$ and that the restricted subalgebra S_1 of $\operatorname{End} W$ generated by $\Lambda(S)$ contains a restricted subalgebra Q_1 which preserves a one-dimensional subspace $N \subseteq W$ and satisfies $\dim(S_1/Q_1) = m$. Then setting $Q = \Lambda^{-1}(\Lambda(S) \cap Q_1)$ we have $\dim(S/Q) \leq m$.

As usual, write $Q_{i\alpha} = Q \cap S_{i\alpha}$ and set $P_{i\alpha} = \{x \in S_{i\alpha} | [b,x] \in Q_{i\alpha}\}$. Write $q_i = \dim S_{i\alpha}/Q_{i\alpha}$ and $p_i = \dim Q_{i\alpha}/(Q_{i\alpha} \cap P_{i\alpha})$. Clearly $q_i \leq m$. Also if $\lambda \colon Q_{i\alpha} \to L_{i\alpha}/Q_{i\alpha}$ is defined by $\lambda(x) = [b,x] + Q_{i\alpha}$ then $p_i = \operatorname{rank} \lambda \leq \dim L_{i\alpha}/Q_{i\alpha} = q_i + s_i$. However, if $b \in S$ then $\lambda \colon Q_{i\alpha} \to S_{i\alpha}/Q_{i\alpha}$ so $p_i \leq q_i$.

Suppose $x \in P_{i\alpha} \cap Q_{i\alpha}$, $y \in P_{-i\alpha} \cap Q_{-i\alpha}$. Then $[b,[x,y]] = [[b,x],y] + [x,[b,y]] \in [Q,Q]$ and hence $[b,[x,y]]$ acts trivially on the one-dimensional Q-module N. But this implies $(x,y)_\beta = \beta([b,[x,y]]) = 0$. Since $\dim L_{i\alpha}/(P_{i\alpha} \cap Q_{i\alpha}) \leq s_i + p_i + q_i$ Lemma 2.5.1 of [**1**] shows that

$$m_{i\alpha} = \operatorname{rank}(\cdot,\cdot)_\beta|_{L_{i\alpha} \times L_{-i\alpha}} \leq s_i + s_{-i} + p_i + p_{-i} + q_i + q_{-i} \tag{8.1.2}$$

and so

$$m_{i\alpha} \le 4m + 2s_i + 2s_{-i}.$$

As $p|m_\alpha$ this implies $s_1 + s_{-1} \ge (p - 4m)/2$. If $m = 1$ (as $p \ge 11$) this implies $s_1 + s_{-1} > 3$. Also by (8.1.1) we have $m_\alpha \ge p^{m-1}$ and so

$$p^{m-1} \le 4m + 2s_1 + 2s_{-1}.$$

If $s_1 + s_{-1} < 2$ this implies

(8.1.3) $$p^{m-1} \le 4m + 2.$$

As $p \ge 11$ this is impossible for $m \ge 2$. Thus $s_1 + s_{-1} \ge 2$. Note that this implies $\widetilde{M} \ne S$ so M is not solvable and hence there exists $t \in \widetilde{U}$ with $t^p = t$ and $\alpha(t) = 1$.

Now suppose $b \in S$. Then we have seen that $p_i \le q_i \le m$ and so (8.1.2) gives

$$m_{i\alpha} \le 4m + s_i + s_{-i}.$$

If $s_1 + s_{-1} < 3$ we see that (8.1.3) again holds. Thus $m \ge 2$ is impossible (and $m = 1$ was previously shown to imply $s_1 + s_{-1} > 3$). Thus $s_1 + s_{-1} \ge 3$ when $b \in S$. Furthermore $s_1 + s_{-1} = 3$ implies that $p^{m-1} \le 4m + 3$ so $m = 2$, $p = 11$, $m_\alpha = p$ and $W = C$.

Now let $t \in \widetilde{U}$ be as above ($t^p = t$, $\alpha(t) = 1$). Then (letting $z_Q(t)$ denote the centralizer of t in Q) $(\operatorname{ad} z_Q(t))|_{\widetilde{M}}$ is nil, since otherwise $Q = z_Q(t) + \sum_{i=1}^{p-1} Q_{i\alpha}$ and so $q_1 + q_{-1} \le 2$ which (as $p_i \le q_i$) implies $m_\alpha \le 7$, a contradiction. Also $b \in Q$ implies $Q_i \subseteq P_i$ and so $p_i = 0$ for all i. Then $m_\alpha \le q_1 + q_{-1} + s_1 + s_{-1} \le 7$, again a contradiction. Thus (as $\dim S_1/Q_1 = 2$) we have $S_1 = Ft + Fb + Q_1$ and so $z_S(t) = Ft + Fb + z_Q(t)$. Suppose $y \in z_Q(t)$. Then $[y, b] \equiv a_0 t + a_1 b \bmod z_Q(t)$ and so $(\operatorname{ad} y)^k b \equiv a_0 a_1^{k-1} t + a_1^k b \bmod z_Q(t)$. As $z_{\widetilde{M}}(t) \subseteq \overline{H}$ is nilpotent we have $a_1 = 0$. Thus $[z_Q(t), b] \subseteq Ft + z_Q(t)$ and so $(\operatorname{ad} b)^k z_Q(t) \subseteq Ft + z_Q(t)$ for all k. But $(\operatorname{ad} b)^2 \overline{H} \subseteq I$ by Lemma 3.1 and so (since $(\operatorname{ad} z_Q(t))|_{\widetilde{M}}$ is nil) we have $(\operatorname{ad} b)^k z_Q(t) \subseteq z_Q(t) \cap I$ for all $k \ge 2$. Of course, this implies $((\operatorname{ad} b)^k z_Q(t))N = (0)$ for all $k \ge 2$. Now let $X = \{z \in z_Q(t) | zN = (0)\}$. Then $\dim z_Q(t)/X \le 1$.

Let $\mu\colon X \to \widetilde{U}/X$ be defined by $\mu(x) = [b, x] + X$. Then if $x \in \ker \mu$ we have $((\operatorname{ad} b)^k x)N = (0)$ for all $k \ge 0$ and so (since $[t, x] = 0$) we have $\theta(x) = 0$. Thus $\ker \mu \subseteq \ker \theta$. Now $\operatorname{rank} \mu \le \dim \widetilde{U}/X \le 3$ so $\dim \theta(\widetilde{U}) \le 6$. Since Corollary 6.3 shows $\dim \theta(M_{i\alpha}) = \dim L_{i\alpha}/K_{i\alpha} = m_{i\alpha} \le m_\alpha = p$ for all i, $1 \le i \le p - 1$, this implies $\dim \theta(\widetilde{M}) \le p^2 - p + 6$, as required.

(8.2) Let S be the subalgebra given by Lemma 7.3. Applying Lemma 8.1 to S gives $\dim \theta(\widetilde{M}) \le p^2 - p + 6$. Thus (as $\dim \theta(\widetilde{M}) \ge \dim \theta(M) \ge \dim \tau(M)$) Lemma 7.2 shows that M contains an $(\operatorname{ad} T)$-invariant solvable subalgebra $S' \supseteq U$ with $\dim(L_\alpha/S'_\alpha) + \dim(L_{-\alpha}/S'_{-\alpha}) \le 1$. However, applying Lemma 8.1 to S' yields a contradiction. Thus (3.3.2) is impossible and the proof of Theorem 2.1 is complete.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903