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TESTING ANALYTICITY ON ROTATION INVARIANT FAMILIES OF CURVES

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Dedicated to Professor Ivan Vidav on the occastion of his seventieth birthday, January 17, 1988

ABSTRACT. Let $\Gamma \subset C$ be a piecewise smooth Jordan curve, symmetric with respect to the real axis, which contains the origin in its interior and which is not a circle centered at the origin. Let Ω be the annulus obtained by rotating Γ around the origin. We characterize the curves Γ with the property that if $f \in C(\Omega)$ is analytic on $s\Gamma$ for every s, $|s| = 1$, then f is analytic in Int Ω.

1. Introduction. Throughout the paper we assume that $\Gamma \subset C$ is a piecewise smooth Jordan curve which is symmetric with respect to the real axis, does not contain the origin and is not a circle centered at the origin. We denote by D the bounded domain with boundary Γ. We denote by Ω the closed annulus obtained by rotating Γ around the origin: $\Omega = \{sz\colon z \in \Gamma,\ |s| = 1\}$. We denote by a, b the inner and the outer radius of Ω, respectively.

We call the curve Γ *regular* if every continuous function on Ω which is analytic on each curve $s\Gamma$, $|s| = 1$, is analytic in $\operatorname{Int}\Omega$, that is, if $f \in C(\Omega)$ and if

(1) for each $s \in C$, $|s| = 1$, the function $f|(s\Gamma)$ has a continuous extension to $s\bar{D}$ which is analytic in sD

then f is analytic in $\operatorname{Int}\Omega$. We call Γ *singular* if it is not regular.

When studying the conditions which imply the regularity of Γ one has to distinguish two cases:

(i) 0 is in the exterior of Γ, i.e. $0 \in C\backslash\bar{D}$,

(ii) 0 is in the interior of Γ, i.e. $0 \in D$.

In the first case the situation is simple.

THEOREM 0 [**1**]. *If* 0 *is in the exterior of* Γ *then* Γ *is regular.*

In the present paper we study the second case and from now on we assume that 0 *is in the interior of* Γ. Now the situation is more complicated. We illustrate this with two examples.

EXAMPLE 1. Suppose that Γ contains an arc of a circle centered at the origin. If $|s_1| = |s_2| = 1$ and if s_1 is close to s_2 then $b(s_1D) \cap b(s_2D)$ contains an arc. This implies that Γ is regular [**1**].

EXAMPLE 2. Let Γ be a circle whose center is different from the origin and which contains the origin in its interior. The function $f(z) = 1/\bar{z}$ shows that Γ is singular.

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2. A characterization of singular curves. If $f \in C(\Omega)$, $a \le r \le b$ and $n \in Z$ define

$$A_n(f,r) = r^{-n}\frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi} f(re^{i\varphi})\,d\varphi.$$

Note that if f is analytic in $\operatorname{Int}\Omega$ then for each r, $a \le r \le b$, $A_n(f,r)$ is equal to the nth coefficient in the Laurent series of f. Note also that f is analytic in $\operatorname{Int}\Omega$ if and only if for each $n \in Z$ the function $r \mapsto A_n(f,r)$ is constant on $[a,b]$ [**1**].

LEMMA 1. *Suppose that $f \in C(\Omega)$ satisfies* (1). *Then for every $n \in Z$ the function $z \mapsto z^n A_n(f,|z|)$ has a continuous extension from Γ to $\bar{D}$ which is analytic in D.*

Note that in [**1**] the lemma is stated for smooth curves Γ. However, its proof works equally well for piecewise smooth curves Γ.

THEOREM 1. *Γ is singular if and only if there are $n \in N$ and a function G, continuous on $\bar{D}$ and analytic in D such that the function $w \mapsto G(w)/w^n$ is nonconstant and depends only on $|w|$ on Γ, that is, if $w_1, w_2 \in \Gamma$, $|w_1| = |w_2|$, then $G(w_1)/w_1^n = G(w_2)/w_2^n$.*

PROOF. Suppose that Γ is singular. This means that there is an $f \in C(\Omega)$ which satisfies (1) and which is not holomorphic in $\operatorname{Int}\Omega$. By Lemma 1, for each $n \in Z$ there is a continuous function G_n on $\bar{D}$, analytic in D and such that $A_n(f,|z|) = G_n(z)/z^n$ $(z \in \Gamma)$. It follows that $A_n(f,r) = 0$ $(a \le r \le b)$ and that $A_0(f,r) = \text{const}$ $(a \le r \le b)$ [**1**]. Since f is not analytic in $\operatorname{Int}\Omega$ there is some $n > 0$ such that $r \mapsto A_n(f,r)$ is nonconstant. Put $G = G_n$. Clearly G has the required properties.

Conversely, suppose that there are $n \in N$ and a continuous function G on $\bar{D}$, analytic in D and such that $w \mapsto G(w)/w^n$ is nonconstant and depends only on $|w|$ on Γ. Define the function g on Ω by

$$g(|z|e^{i\alpha}) = G(z)/z^n \qquad (z \in \Gamma,\ 0 \le \alpha \le 2\pi).$$

Then g is well defined and continuous on Ω, depends only on $|z|$ and is not a constant. Put $f(z) = z^n g(z)$ $(z \in \Omega)$. If $|s| = 1$ and $z \in \Gamma$ then $f(sz) = (sz)^n g(|z|) = (sz)^n G(z)/z^n = s^n G(z)$. This shows that f satisfies (1). Since $f \in C(\Omega)$ and since f is not analytic in $\operatorname{Int}\Omega$ it follows that Γ is singular. This completes the proof.

3. Singular curves and symmetry. By our assumption, Γ is symmetric with respect to the real axis. A singular curve may have no other lines of symmetry [**1**, Example 5]. However, once it contains two arcs whose union is symmetric with respect to a line L through 0 then it is symmetric with respect to L. This is a consequence of the following

THEOREM 2. *Let Γ be a singular curve. Suppose that there are an arc $\Lambda \subset \Gamma$ and an α, $0 < \alpha < 2\pi$, such that $e^{i\alpha}\Lambda \subset \Gamma$. Then $\Gamma = e^{i\alpha}\Gamma$ and consequently Γ is symmetric with respect to the lines through* 0 *and $e^{in\alpha/2}$, $n \in N$. In particular α/π must be rational.*

For a set $E \subset C$ write $E^* = \{\bar{\varsigma}\colon \varsigma \in E\}$. To prove Theorem 2 we need the following lemma.

LEMMA 2 [1]. *Let $P \subset C$ be an open set with piecewise smooth boundary. Let f be a continuous function on $\bar{P}$ which is analytic in P and let g be a continuous function on $\bar{P}^*$ which is analytic in P^*. Suppose that $f(w) = g(\bar{w})$ $(w \in bP)$. Then f is a constant.*

PROOF OF THEOREM 2. By Theorem 1 there are $n \in N$ and a function F, continuous on $\bar{D}$, analytic in D such that $w \mapsto G(w) = F(w)/w^n$ is nonconstant and depends only on $|w|$ on Γ. So there is a function $\varphi\colon [a,b] \to C$ such that $G(w) = \varphi(|w|)$ $(w \in \Gamma)$. Note that G is continuous on $\bar{D}\backslash\{0\}$ and analytic on $D\backslash\{0\}$.

Let $P = D \cap (e^{i\alpha}D)$. Denote by P_1 the component of P which contains $e^{i\alpha}\Lambda$ in its boundary. Assume for a moment that $\bar{P}_1$ does not contain 0. Then G is continuous on $\bar{P}_1$ and analytic in P_1. If $w \in (bP_1) \cap \Gamma$ then $G(w) = \varphi(|w|)$. Further, if $w \in e^{i\alpha}\Lambda$ then $e^{-i\alpha}w \in \Lambda$ so $G(w) = \varphi(|w|) = \varphi(|e^{-i\alpha}w|) = G(e^{-i\alpha}w)$. This implies that $G(w) = G(e^{-i\alpha}w)$ $(w \in \bar{P}_1)$. In particular, if $w \in (bP_1) \cap (e^{i\alpha}\Gamma)$ then $G(w) = G(e^{-i\alpha}w) = \varphi(|e^{-i\alpha}w|) = \varphi(|w|)$ so $G(w) = \varphi(|w|)$ $(w \in (bP_1) \cap (e^{i\alpha}\Gamma))$.

Let $Q = D \cap e^{-i\alpha}D$. Denote by Q_1 the component of Q which contains $(e^{i\alpha}\Lambda)^* = e^{-i\alpha}\Lambda^*$ in its boundary. Note that $Q_1^* = P_1$. So $\bar{Q}_1$ does not contain 0 and consequently G is continuous on $\bar{Q}_1$ and analytic in Q_1. If $w \in (bQ_1) \cap \Gamma$ then $G(w) = \varphi(|w|)$. Further, if $w \in (e^{i\alpha}\Lambda)^*$ then $e^{i\alpha}w \in \Lambda^*$ so $G(w) = \varphi(|w|) = \varphi(|e^{-i\alpha}w|) = G(e^{i\alpha}w)$. This implies that $G(w) = G(e^{i\alpha}w)$ $(w \in \bar{Q}_1)$. In particular, if $w \in (bQ_1) \cap (e^{-i\alpha}\Gamma)$ then $G(w) = G(e^{i\alpha}w) = \varphi(|e^{i\alpha}w|) = \varphi(|w|)$ so $G(w) = \varphi(|w|)$ $(w \in (bQ_1) \cap (e^{-i\alpha}\Gamma))$.

We have proved that G is continuous on $\bar{P}_1$, analytic in P_1, continuous on $\bar{P}_1^*$, analytic in P_1^* and satisfies $G(w) = \varphi(|w|)$ $(w \in bP_1, w \in bP_1^*)$. So $G(w) = G(\bar{w})$ $(w \in bP_1)$. By Lemma 2 it follows that G is constant on $\bar{P}_1$, a contradiction.

Thus we proved that $0 \in \bar{P}_1$. Since P contains a neighborhood of 0 it follows that $0 \in P_1$.

Let $B = D\backslash\bar{P}_1$ and assume that B is not empty. If we repeat the above argument we see that $G(w) = \varphi(|w|)$ $(w \in bP_1)$. Further, since $G(w) = \varphi(|w|)$ $(w \in \Gamma = bD)$ it follows that G is continuous on $\bar{B}$, analytic in B and satisfies $G(w) = \varphi(|w|)$ $(w \in bB)$. In the same way, considering $B^* = D\backslash\bar{Q}_1$ instead of B we prove that G is continuous on $\bar{B}^*$, analytic in B^* and satisfies $G(w) = \varphi(|w|)$ $(w \in bB^*)$. So $G(\bar{w}) = G(w)$ $(w \in bB)$ and by Lemma 2 G is a constant, a contradiction. Consequently $B = \varnothing$ so $D \subset \bar{P}_1 \subset \bar{D} \subset e^{i\alpha}D$ which implies that $D = e^{i\alpha}D$ and $\Gamma = e^{i\alpha}\Gamma$. Further, since $e^{in\alpha}D = D$ $(n \in N)$ and since $D^* = D$ it follows that $(e^{-in\alpha/2}D)^* = e^{in\alpha/2}D = e^{-in\alpha/2}D$ which proves that Γ is symmetric with respect to the lines through 0 and $e^{in\alpha/2}$, $n \in N$. Since Γ is not a circle centered at 0 it follows that α/π must be rational. This completes the proof.

COROLLARY 1. *Let Γ be a singular curve. Suppose that $0 < \beta < \pi$ and that Γ contains two arcs whose union is symmetric with respect to the line L through 0 and $e^{i\beta}$. Then Γ is symmetric with respect to L. In particular, β/π must be rational.*

PROOF. By the assumption there are arcs Λ_1, Λ_2 such that $(e^{-i\beta}\Lambda_1)^* = e^{-i\beta}\Lambda_2$ which implies that $e^{2i\beta}\Lambda_1^* = \Lambda_2$. Since $\Gamma^* = \Gamma$ it follows that $\Lambda_1^* \subset \Gamma$ and Theorem 2 implies that Γ is symmetric with respect to L. This completes the proof.

4. Two examples. We denote by Δ the open unit disc in C.

PROPOSITION 1. *Let Γ be a triangle. Then Γ is singular if and only if Γ is an equilateral triangle centered at the origin.*

Recall that we are assuming that $\Gamma = \Gamma^*$.

PROOF. Suppose that Γ is singular. By Corollary 1 the lines through 0 which are perpendicular to the sides of Γ are the lines of symmetry for Γ which proves that Γ is an equilateral triangle centered at 0. Conversely, suppose that Γ is an equilateral triangle centered at 0. Let $\Psi\colon D \to \Delta$ be the conformal map which satisfies $\Psi(0) = 0$, $\Psi'(0) > 0$. Define

$$G(w)/w^3 = \Psi(w)^3 + 1/\Psi(w)^3.$$

Then G is continuous on $\bar{D}$, analytic in D and it is easy to see that $G(w)/w^3$ depends only on $|w|$ on Γ. By Theorem 1 Γ is singular. This completes the proof.

PROPOSITION 2. *Let Γ be a rectangle. Then Γ is singular if and only if Γ is a square centered at the origin.*

PROOF. Suppose that Γ is a square centered at the origin. Let $\Psi\colon D \to \Delta$ be the conformal map which satisfies $\Psi(0) = 0$, $\Psi'(0) > 0$. Then

$$G(w)/w^4 = \Psi(w)^4 + 1/\Psi(w)^4$$

depends only on $|w|$ on Γ. In the same way as above, Theorem 1 implies that Γ is singular.

Conversely, assume that Γ is singular. Corollary 1 implies that Γ is symmetric with respect to the imaginary axis. With no loss of generality assume that $a + i$, $a - i$, $-a + i$, $-a - i$ are the vertices of Γ where $a > 0$. We have to prove that $a = 1$.

It suffices to prove the following: Suppose that there is a nonconstant function G, continuous on $\bar{D}\backslash\{0\}$ and analytic in $D\backslash\{0\}$ which depends only on $|w|$ on Γ. Then $a = 1$. Let G be as above. With no loss of generality assume that $G(a+i) = 0$. Since $G(w)$ depends only on $|w|$ on Γ it follows that we can extend G to the rectangle Q with vertices $3a - i$, $3a + 3i$, $-a - i$, $-a + 3i$ by

$$\left.\begin{array}{l} G(2i + w) = G(w), \\ G(2a + w) = G(w), \\ G(2a + 2i + w) = G(w) \end{array}\right\} \qquad (w \in \bar{D}\backslash\{0\}),$$

to get a continuous function on $\bar{Q}\backslash\{0, 2a, 2i, 2a+2i\}$. In particular, G is analytic in a neighborhood of $a+i$. Put $f(w) = G(w - (a+i))$. Then f is analytic in an open disc U centered at 0. By the properties of G we have $f(w) = f(-w)$ $(w \in U \cap R)$ so $f(w) = f(-w)$ $(w \in U)$. Further, whenever $z \in R_-$ and $w \in iR_-$ have the same distance from $-a - i$ we have $f(w) = f(z)$.

Put $w(z) = i((z^2 + 2az + 1)^{1/2} - 1)$ where $1^{1/2} = 1$. Passing to a smaller U if necessary we may assume that w is analytic in U. We have

$$(z + a)^2 + 1 = (w(z)/i + 1)^2 + a^2 \qquad (z \in U)$$

which means that if $z \in U \cap R_-$ then $w \in iR_-$ and w and z have the same distance from $-a - i$.

There is a disc U' centered at 0 such that $w(U') \subset U$. For every $z \in U' \cap R_-$ we have $f(w(z)) = f(z)$ which implies that

$$f(w(z)) = f(z) \qquad (z \in U'). \tag{2}$$

Recall that $f(0) = 0$. Since f is not a constant there is some $k \in N$ such that $f'(0) = \cdots = f^{(k)}(0) = 0$, $f^{(k+1)}(0) \neq 0$. By (2) we have

$$f^{(k+1)}(z) = f^{(k+1)}(w(z))w'(z)^{k+1}$$
$$+ \text{terms containing } f^{(k)}(w(z)), \ldots, f'(w(z)) \text{ as factors,}$$

so $f^{(k+1)}(0) = f^{(k+1)}(0)w'(0)^{k+1}$ which implies that $w'(0)^{k+1} = 1$. On the other hand, $w'(0) = ia$ by the definition of $w(z)$. Consequently $a = 1$. This completes the proof.

REMARK. There is no similar result when Γ is a pentagon. There are pentagons Γ which are singular and which are not equilateral. To see this observe that there are pentagons of the same form as [**1**, Example 5].

5. Functions analytic on $s\Gamma$, $|s| = 1$. We now proceed to obtain a more detailed description of singular curves. The first step in this direction is a characterization of functions f which satisfy (1).

From now on we denote by Φ the conformal map from Δ to D which satisfies $\Phi(0) = 0$, $\Phi'(0) > 0$.

THEOREM 3. *A function $f \in C(\Omega)$ satisfies* (1) *if and only if the following two conditions are satisfied:*

(i) *for each $n < 0$, $A_n(f, r) = 0$ $(a \leq r \leq b)$,*

(ii) *for each $n \geq 0$ there is a polynomial P_n satisfying $P_n(1/\varsigma) = \varsigma^{-2n}P_n(\varsigma)$ $(\varsigma \in C)$ such that*

$$A_n(f, |\Phi(\varsigma)|) = \varsigma^{-n}P_n(\varsigma) \qquad (\varsigma \in b\Delta).$$

REMARK. Note that the degree of P_n is at most $2n$. Note also that $\varsigma^{-n}P_n(\varsigma)$ has the form $b_{n0} + b_{n1}(\varsigma + 1/\varsigma) + \cdots + b_{nn}(\varsigma^n + 1/\varsigma^n)$ so that (ii) is equivalent to

(ii') for each $n \geq 0$ there are $a_{n0}, a_{n1}, \ldots, a_{nn}$ such that

$$A_n(f, |\Phi(e^{i\varphi})|) = a_{n0} + a_{n1}\cos\varphi + \cdots + a_{nn}\cos n\varphi.$$

PROOF. *The only if part.* Observe first that by the symmetry of Γ with respect to the real axis we have $\Phi(\bar{\varsigma}) = \overline{\Phi(\varsigma)}$ $(\varsigma \in b\Delta)$.

Assume that $f \in C(\Omega)$ satisfies (1). By Lemma 1 for every $n \in Z$ the function $z \mapsto z^n A_n(|z|)$ has a continuous extension from Γ to $\bar{D}$ which is analytic in D. It follows that for each $n \in Z$ there is a function F_n, continuous on $\bar{\Delta}$, analytic in Δ, such that

$$\Phi(\varsigma)^n A_n(f, |\Phi(\varsigma)|) = F_n(\varsigma) \qquad (\varsigma \in b\Delta). \tag{3}$$

If $n < 0$ it follows that $\varsigma \mapsto A_n(f, |\Phi(\varsigma)|)$ is the boundary function of a function G, continuous on $\bar{\Delta}$, analytic in Δ, which has a zero at $\varsigma = 0$. Since $G(\varsigma) = G(\bar{\varsigma})$ $(\varsigma \in b\Delta)$ it follows that $G = 0$ which proves (i).

Let $n \geq 0$. We have $\Phi(\varsigma) = \varsigma\Psi(\varsigma)$ $(\varsigma \in \bar{\Delta})$ where both Ψ and $1/\Psi$ are continuous on $\bar{\Delta}$ and analytic in Δ. By (3) we have

$$F_n(\varsigma)/\Psi(\varsigma)^n = \varsigma^{2n}F_n(1/\varsigma)/\Psi(1/\varsigma)^n \qquad (\varsigma \in b\Delta) \tag{4}$$

which implies that

$$\varsigma \mapsto \begin{cases} F_n(\varsigma)/\Psi(\varsigma)^n & (\varsigma \in \bar{\Delta}), \\ F_n(1/\varsigma)\cdot \varsigma^{2n}/\Psi(1/\varsigma)^n & (\varsigma \in C\backslash\Delta), \end{cases}$$

is an entire function of polynomial growth at infinity so it is a polynomial that we denote by P_n. By (4) we have $P_n(\varsigma) = \varsigma^{2n}P_n(1/\varsigma)$ $(\varsigma \in C)$. Further, by (3), $A_n(f, |\Phi(\varsigma)|) = F_n(\varsigma)/\Phi(\varsigma)^n = \varsigma^{-n}P_n(\varsigma)$ $(\varsigma \in b\Delta)$ which proves (ii). This completes the proof of the only if part.

The if part. Suppose that $f \in C(\Omega)$ satisfies (i) and (ii). We first show that for each $n \in Z$ and for each s, $|s| = 1$, the function $z \mapsto z^n A_n(f, |z|)$ has a continuous extension from $s\Gamma$ to $s\bar{D}$ which is analytic in sD, that is, $\varsigma \mapsto (s\Phi(\varsigma))^n A_n(f, |\Phi(\varsigma)|)$ is the boundary function of a function continuous on $\bar{\Delta}$, analytic in Δ. If $n < 0$ this is so since $A_n(r) = 0$ $(a \leq r \leq b)$. If $n > 0$ then

$$(s\Phi(\varsigma))^n A_n(f, |\Phi(\varsigma)|) = s^n\Psi(\varsigma)^n\varsigma^n\varsigma^{-n}P_n(\varsigma) = s^n\Psi(\varsigma)^n P_n(\varsigma)$$

which, on $b\Delta$, is the boundary function of $s^n\Psi P_n$ which is continuous on $\bar{\Delta}$ and analytic in Δ.

For each r, $a \leq r \leq b$, $\sum_{-\infty}^{\infty} A_n(f, r)r^n e^{in\varphi}$ is the Fourier series of $\varphi \mapsto f(re^{i\varphi})$. Let

$$\sigma_m(f, r, e^{i\varphi}) = m^{-1}\left(A_0(f, r) + \sum_{-1}^{1} A_k(f, r)r^k e^{ik\varphi} + \cdots + \sum_{-(m-1)}^{m-1} A_k(f, r)r^k e^{ik\varphi}\right)$$

be its mth Cezàro mean. By the uniform continuity of f on Ω the family $\{\theta \mapsto f(re^{i\theta})\colon a \leq r \leq b\}$ is uniformly equicontinuous on $[0, 2\pi]$. The usual proof of Féjer's theorem [**2**] applied to $\sum_{-\infty}^{\infty} A_n(f, r)r^n e^{in\varphi}$ shows that $\sigma_m(f, r, e^{i\varphi})$ converges to $f(re^{i\varphi})$ uniformly for r and φ, $a \leq r \leq b$, $0 \leq \varphi \leq 2\pi$. Consequently, on Ω, $f(z)$ is the uniform limit of the sequence

$$f_m(z) = m^{-1}\left(A_0(f, |z|) + \sum_{-1}^{1} A_k(f, |z|)z^k + \cdots + \sum_{-(m-1)}^{m-1} A_k(f, |z|)z^k\right).$$

By the preceding discussion each f_m satisfies (1) so the same holds for f. This completes the proof of Theorem 3.

6. Another characterization of singular curves.

THEOREM 4. *Let Γ and Φ be as in Theorem 3. Γ is singular if and only if there are $n > 0$ and real numbers $a_1, a_2, \ldots, a_n$, not all equal to zero, and a function F on $[a, b]$ such that*

$$(5)\quad F(|\Phi(\varsigma)|) = a_1(\varsigma + 1/\varsigma) + a_2(\varsigma^2 + 1/\varsigma^2) + \cdots + a_n(\varsigma^n + 1/\varsigma^n) \qquad (\varsigma \in b\Delta).$$

REMARK. (5) should be understood as an incidence relation, that is, if

$$h(\varsigma) = a_1(\varsigma + 1/\varsigma) + \cdots + a_n(\varsigma^n + 1/\varsigma^n)$$

then $h(\varsigma_1) = h(\varsigma_2)$ whenever $\varsigma_1, \varsigma_2 \in b\Delta$ and $|\Phi(\varsigma_1)| = |\Phi(\varsigma_2)|$. Note also that $h(e^{i\varphi})$ is a trigonometric polynomial

$$h(e^{i\varphi}) = c_1\cos\varphi + \cdots + c_n\cos n\varphi.$$

PROOF. *The only if part.* Suppose that Γ is singular. This means that there is an $f \in C(\Omega)$ which satisfies (1) and which is not holomorphic in $\operatorname{Int}\Omega$. It follows that there is some $n \in Z$ such that $r \mapsto A_n(f,r)$ is nonconstant on $[a,b]$ [**1**]. By Theorem 3 it follows that there is some $n > 0$ such that

$$A_n(f, |\Phi(\varsigma)|) = \varsigma^{-n} P_n(\varsigma) \qquad (\varsigma \in b\Delta)$$

where $\varsigma \mapsto \varsigma^{-n}P_n(\varsigma)$ is not a constant and P_n is a polynomial satisfying $P_n(1/\varsigma) = \varsigma^{-2n}P_n(\varsigma)$ $(\varsigma \in C)$, that is,

$$A_n(f, |\Phi(\varsigma)|) = b_0 + b_1(\varsigma + 1/\varsigma) + \cdots + b_n(\varsigma^n + 1/\varsigma^n) \qquad (\varsigma \in b\Delta)$$

where at least one of $b_1, \ldots, b_n$ is different from zero. If all nonzero b_k are pure imaginary let

$$F(|\Phi(\varsigma)|) = i^{-1}(A_n(f, |\Phi(\varsigma)|) - b_0) \qquad (\varsigma \in b\Delta),$$

and otherwise let

$$F(|\Phi(\varsigma)|) = \operatorname{Re}(A_n(f, |\Phi(\varsigma)|) - b_0) \qquad (\varsigma \in b\Delta).$$

In either case $F(|\Phi(\varsigma)|)$ is of the form (5). This completes the proof of the only if part.

The if part. Suppose that there are $n > 0$ and real numbers $a_1, \ldots, a_n$, not all equal to zero, and a function F on $[a,b]$ such that (5) holds. Let h be as in the Remark. By (5), $F(|w|) = h(\Phi^{-1}(w))$ $(w \in \Gamma)$. Suppose for a moment that F is not continuous on $[a,b]$. This implies that there are r, $a \le r \le b$, and a sequence $w_n \in \Gamma$, $|w_n| \to r$, such that $|F(|w_n|) - F(r)| \ge \delta > 0$ $(n \in N)$. By the compactness of Γ we may, passing to a subsequence if necessary, assume that $w_n \to w \in \Gamma$. It follows that $|(h \circ \Phi^{-1})(w_n) - (h \circ \Phi^{-1})(w)| \ge \delta$ $(n \in N)$, a contradiction. This proves that F is continuous on $[a,b]$. Define

$$f(z) = z^n F(|z|) \qquad (z \in \Omega).$$

Clearly f is continuous on Ω. Let $s \in b\Delta$. We have $f(s\Phi(\varsigma)) = s^n\Phi(\varsigma)^n h(\varsigma)$ $(\varsigma \in b\Delta)$. Since h has a pole of at most order n at the origin and Φ has a zero at the origin it follows that $f|(s\Gamma)$ has a continuous extension to $s\bar{D}$ which is analytic in sD. It follows that f satisfies (1). Since F is not a constant it follows that f is not analytic in $\operatorname{Int}\Omega$. This proves that Γ is singular. The proof is complete.

7. Properties of singular curves.

PROPOSITION 3. *Suppose that Γ intersects a circle centered at the origin in an infinite set of points. Then Γ is regular.*

PROOF. Suppose that Γ is singular. By Theorem 4 there are $n > 0$ and numbers $a_1, \ldots, a_n$, not all equal to zero, and a function F on $[a,b]$ such that $F(|\Phi(\varsigma)|) = h(\varsigma)$ $(\varsigma \in b\Delta)$ where $h(\varsigma) = a_1(\varsigma + 1/\varsigma) + \cdots + a_n(\varsigma^n + 1/\varsigma^n)$. By our assumption h has the same value at an infinite sequence of points on $b\Delta$, a contradiction. This completes the proof.

PROPOSITION 4. *Let Γ be singular. Then $|w|$ has only a finite number of local maxima or minima on Γ, that is, $\varphi \mapsto |\Phi(e^{i\varphi})|$ has only a finite number of local maxima or minima on $(0,\pi)$.*

PROOF. By Theorem 4 there is a trigonometric polynomial $h(\varphi) = b_1\cos\varphi + \cdots + b_n\cos n\varphi$, not identically zero, such that $|\Phi(e^{i\varphi})| = |\Phi(e^{i\psi})|$ implies that

$h(\varphi) = h(\psi)$. Suppose now that $\varphi \mapsto |\Phi(e^{i\varphi})|$ has a local maximum at φ_0, $0 < \varphi_0 < \pi$. By Proposition 3 there is a neighborhood U of φ_0 such that $|\Phi(e^{i\varphi})| < |\Phi(e^{i\varphi_0})|$ $(\varphi \in U\backslash\{\varphi_0\})$. Continuity of Φ implies that there are sequences $\varphi_n' \nearrow \varphi_0$, $\varphi_n'' \searrow \varphi_0$ such that $|\Phi(e^{i\varphi_n'})| = |\Phi(e^{i\varphi_n''})|$ $(n \in N)$ and consequently $h(\varphi_n') = h(\varphi_n'')$ $(n \in N)$. It follows that $h'(\varphi_0) = 0$. Since h' has only finitely many zeros on $(0,\pi)$ it follows that $\varphi \mapsto |\Phi(e^{i\varphi})|$ has only finitely many local maxima or minima on $(0,\pi)$. This completes the proof.

PROPOSITION 5. *Let Γ be such that $\theta \mapsto |\Phi(e^{i\theta})|$ has only a finite number of local extrema on $(0,\pi)$. Moreover, assume that whenever φ is the point of local extremum then Γ is symmetric with respect to the line through* 0 *and $\Phi(e^{i\varphi})$. Then Γ is singular.*

PROOF. Let φ_i, $1 \le i \le n$, be the points of local extrema of $\theta \mapsto |\Phi(e^{i\theta})|$ on $(0,\pi)$, $\varphi_1 < \varphi_2 < \cdots < \varphi_n$. Because of the symmetry we have $\varphi_k = k\pi/(n+1)$. Write $\varphi_0 = 0$, $\varphi_{n+1} = \pi$ and observe that between φ_k and φ_{k+1}, $\theta \mapsto |\Phi(e^{i\theta})|$ is either strictly increasing or strictly decreasing, $0 \le k \le n$; in particular, it is one-to-one. It is periodic with period $2\pi/(n+1)$. Now it is easy to see that there is a function F on $[a,b]$ such that $F(|\Phi(e^{i\varphi})|) = \cos(n+1)\varphi$. By Theorem 4 the curve Γ is singular. This completes the proof.

PROPOSITION 6. *Every curve Γ is the limit of a sequence of singular curves.*

PROOF. We first prove that a curve Γ is singular if the conformal map Φ is a polynomial. Let $\Phi(\varsigma) = \sum_{k=1}^{m} a_k\varsigma^k$. Since $\Phi(\bar\varsigma) = \overline{\Phi(\varsigma)}$ $(\varsigma \in \bar\Delta)$, $a_1,\dots,a_m$ are real. We have

$$\begin{aligned}
|\Phi(e^{i\theta})|^2 &= \sum_{k=1}^{m}\sum_{l=1}^{m} a_k a_l e^{i(k-l)\theta} \\
&= a_m a_1 e^{i(m-1)\theta} + (a_{m-1}a_1 + a_m a_2)e^{i(m-2)\theta} \\
&\quad + \cdots + (a_2a_1 + a_3a_2 + \cdots + a_m a_{m-1})e^{i\theta} \\
&\quad + (a_1^2 + a_2^2 + \cdots + a_m^2) \\
&\quad + (a_1a_2 + a_2a_3 + \cdots + a_{m-1}a_m)e^{-i\theta} \\
&\quad + \cdots + (a_1a_{m-1} + a_2a_m)e^{-i(m-2)\theta} + a_1a_m e^{-i(m-1)\theta}
\end{aligned}$$

which shows that there are real numbers $A_1,\dots,A_m$, not all equal to zero, such that $|\Phi(\varsigma)|^2 = A_1(\varsigma + 1/\varsigma) + \cdots + A_m(\varsigma^m + 1/\varsigma^m)$ $(\varsigma \in b\Delta)$. By Theorem 4 it follows that Γ is singular.

Now, let Γ be a curve and let $\Phi\colon \Delta \to D$ be the usual conformal map. Let $\Phi(\varsigma) = p_1\varsigma + p_2\varsigma^2 + \cdots$ $(\varsigma \in \Delta)$. Since $\Phi(\bar\varsigma) = \overline{\Phi(\varsigma)}$ $(\varsigma \in \Delta)$, p_k are real, $k \in N$. Let $\varepsilon > 0$. Since Φ is continuous on $\bar\Delta$ for each $n \in N$ there is an r_n, $0 < r_n < 1$, such that

$$(6) \qquad |\Phi(r_n\varsigma) - \Phi(\varsigma)| < \varepsilon/n \qquad (\varsigma \in b\Delta).$$

With no loss of generality assume that $r_n \nearrow 1$. Further, by the uniform convergence of the Taylor series on compact subsets of Δ it follows that for each $n \in N$ there is an $m \in N$ such that if $g_n(\varsigma) = p_1\varsigma + \cdots + p_m\varsigma^m$ then g_n maps $r_n\Delta$ conformally onto a domain D_n bounded by a smooth curve Γ_n, and moreover that

$$(7) \qquad |g_n(r_n\varsigma) - \Phi(r_n\varsigma)| < \varepsilon/n \qquad (\varsigma \in b\Delta).$$

Since p_k are real we have $\Gamma_n^* = \Gamma_n$. Further, if ε is chosen small enough at the beginning then each Γ_n contains 0 in its interior. Since g_n are polynomials the curves Γ_n are singular. By (6) and (7) it follows that Γ is the limit of the curves Γ_n. This completes the proof.

Every curve Γ is the limit of a sequence Γ_n of the curves such that for each n, Γ_n meets a circle centered at the origin in an arc. By Proposition 3 such curves are regular so we have

PROPOSITION 7. *Every curve Γ is the limit of a sequence of regular curves.*

8. Functions analytic on circles. Let Γ be a circle. We know from Example 2 in §1 that Γ is singular and that the function $f(z) = 1/\bar{z}$ satisfies (1). Clearly $g(z) = z$ also satisfies (1) and so does every uniform limit on Ω of a sequence of polynomials in f and g. It turns out that there are no other functions which satisfy (1).

THEOREM 5. *Let Γ be a circle. The $f \in C(\Omega)$ satisfies* (1) *if and only if f is the uniform limit of a sequence of polynomials in z and $1/\bar{z}$. Consequently, if f satisfies* (1) *then f is analytic on every circle contained in Ω and containing* 0 *in its interior.*

PROOF. The proof is similar to the proof of Theorem 3 but with Φ replaced by the linear map mapping Δ onto D (thus dropping the requirement that $\Phi(0) = 0$).

Let $\Gamma = \{\rho + \varsigma R \colon \varsigma \in b\Delta\}$ where $0 < \rho < R$. Suppose that $f \in C(\Omega)$ satisfies (1). By Lemma 1 it follows that for each $n \in Z$ there is a function G_n, continuous on $\bar{\Delta}$, analytic in Δ such that $(\rho + \varsigma R)^n A_n(f, |\rho + \varsigma R|) = G_n(\varsigma)$ $(\varsigma \in b\Delta)$. As in the proof of Theorem 3 we see that $A_n(r) = 0$ $(a \leq r \leq b,\ n < 0)$. Let $n \geq 0$. We have $A_n(f, |\rho + \varsigma R|) = G_n(\varsigma)/(\rho + \varsigma R)^n$ $(\varsigma \in b\Delta)$ which implies that

$$G_n(\varsigma)/(\rho + \varsigma R)^n = G_n(\bar{\varsigma})/(\rho + \bar{\varsigma} R)^n = \varsigma^n G_n(1/\varsigma)(R + \varsigma\rho)^n \qquad (\varsigma \in b\Delta)$$

so

$$\varsigma \mapsto \begin{cases} (R + \varsigma\rho)^n G_n(\varsigma) & (\varsigma \in \bar{\Delta}), \\ (\rho + \varsigma R)^n \varsigma^n G_n(1/\varsigma) & (\varsigma \in C \backslash \Delta), \end{cases}$$

is a polynomial that we denote by P_n. We have $P_n(\varsigma) = \varsigma^{2n} P_n(1/\varsigma)$ $(\varsigma \in C)$. Now,

$$\begin{aligned} A_n(f, |\rho + \varsigma R|) &= \frac{P_n(\varsigma)}{(\rho + \varsigma R)^n (R + \varsigma\rho)^n} \\ &= \frac{P_n(\varsigma)/\varsigma^n}{|\rho + \varsigma R|^{2n}} \qquad (\varsigma \in b\Delta), \end{aligned}$$

so there are $a_0, a_1, \ldots, a_n$ such that

$$|\rho + Re^{i\varphi}|^{2n} A_n(|\rho + Re^{i\varphi}|) = a_0 + a_1 \cos\varphi + \cdots + a_n \cos n\varphi.$$

For every $m \in N$, $\cos m\varphi$ is a polynomial in $\cos\varphi$ of degree m, so

$$|\rho + Re^{i\varphi}|^{2n} A_n(|\rho + Re^{i\varphi}|) = Q_n(\cos\varphi)$$

where Q_n is a polynomial of degree $\leq n$. Write $|\rho + Re^{i\varphi}| = t$. Clearly $a \leq t \leq b$ $(0 \leq \varphi \leq \pi)$. We have $(\rho + R\cos\varphi)^2 + R^2 \sin^2\varphi = t^2$ which implies that $\cos\varphi = (t^2 - R^2 - \rho^2)/2\rho R$, so

$$t^{2n} A_n(t) = Q_n((t^2 - R^2 - \rho^2)/2\rho R) = S_n(t^2)$$

where S_n is a polynomial of degree $\leq n$ so there are $c_0, c_1, \ldots, c_n$ such that

$$A_n(r) = c_0 + c_1/r^2 + \cdots + c_n/r^{2n} \qquad (a \leq r \leq b).$$

If $z = re^{i\varphi}$ it follows that

$$\begin{aligned} z^n A_n(|z|) &= z^n(c_0 + c_1/(z\bar{z}) + \cdots + c_n/(z^n\bar{z}^n)) \\ &= c_0 z^n + c_1 z^{n-1}/\bar{z} + \cdots + c_n/\bar{z}^n. \end{aligned}$$

Recall that on Ω, $f(z)$ is the uniform limit of the sequence

$$f_m(z) = m^{-1}\left(A_0(f, |z|) + \sum_{-1}^{1} A_k(f, |z|)z^k + \cdots + \sum_{-(m-1)}^{m-1} A_k(f, |z|)z^k\right)$$

which implies that it is the uniform limit of a sequence of polynomials in z and $1/\bar{z}$. This completes the proof.

Thus, if Γ is a circle then $f \in C(\Omega)$ satisfies (1) if and only if f belongs to the function algebra on the closed annulus Ω generated by the functions z and $1/\bar{z}$. It might be interesting to study this algebra in the theory of function algebras.

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