TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 306, Number 1, March 1988

FIXED POINTS OF ARC-COMPONENT-PRESERVING MAPS

CHARLES L. HAGOPIAN

ABSTRACT. The following classical problem remains unsolved:

If M is a plane continuum that does not separate the plane and f is a map of M into M, must f have a fixed point?

We prove that the answer is yes if f maps each arc-component of M into itself. Since every deformation of a space preserves its arc-components, this result establishes the fixed-point property for deformations of nonseparating plane continua. It also generalizes the author's theorem **[10]** that every arcwise connected nonseparating plane continuum has the fixed-point property. Our proof shows that every arc-component-preserving map of an indecomposable plane continuum has a fixed point. We also prove that every tree-like continuum that does not contain uncountably many disjoint triods has the fixed-point property for arc-component-preserving maps.

1. Introduction. According to the Lefschetz fixed-point theorem, every deformation of a polyhedron with nonzero Euler characteristic has a fixed point. A variety of concepts have been used to extend this result **[6, 7, 8, 9, 21, 27]**. Recently, the author **[13]** used the dog-chases-rabbit principle to prove that every deformation of a uniquely arcwise connected continuum has a fixed point. Young's example **[30]** of a uniquely arcwise connected continuum without the fixed-point property shows that the author's theorem **[13]** does not generalize to arc-component-preserving maps. However, every arc-component-preserving map of a uniquely arcwise connected plane continuum has a fixed point **[12]**. Here we establish the analogous theorem for nonseparating plane continua. Once again, our proof is based on the dog-chases-rabbit principle.

2. Definitions. A space S has the *fixed-point property* if for each map f of S into S, there exists a point p of S such that $f(p) = p$.

A map f of S is an *arc-component-preserving map* if f maps each arc-component of S into itself.

A map f of S is a *deformation* if there exists a map h of $S \times [0,1]$ onto S such that $h(p,0) = p$ and $h(p,1) = f(p)$ for each point p of S.

A *continuum* is a nondegenerate compact connected metric space.

A continuum is *uniquely arcwise connected* if it is arcwise connected and does not contain a simple closed curve.

A continuum is *indecomposable* if it is not the union of two of its proper subcontinua.

Received by the editors February 4, 1986 and, in revised form, February 16, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54F20, 54H25.

Key words and phrases. Fixed-point property, deformation, arc-component-preserving map, nonseparating plane continua, indecomposable continua, internal composant, tree-like continua, uncountably many disjoint triods, free chain, Borsuk ray, dog-chases-rabbit principle.

The author was partially supported by NSF Grant MCS-8205282.

A continuum T is a *triod* if T has a subcontinuum Z such that $T \sim Z$ is the union of three nonempty disjoint open sets.

A *tree* is a finite graph that does not contain a simple closed curve.

A continuum M is *tree-like* if for each positive number ε, there is a cover of M with mesh less than ε whose nerve is a tree.

In [**2**], Bellamy constructed a tree-like continuum that admits a fixed-point-free map. Whether or not this example can be modified to solve the classical plane fixed-point problem remains to be seen [**2**, p. 12; **20**, **25**, **26**].

3. Preliminaries. Henceforth, M is a continuum with metric ρ.

A *chain* is a finite collection $\mathcal{W} = \{W_i : 1 \leq i \leq n\}$ of open subsets of M such that $W_i \cap W_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

If $n > 2$ and W_1 also intersects W_n, then $\mathcal{W}$ is a *circular chain.*

If the mesh of $\mathcal{W}$ is less than ε, then $\mathcal{W}$ is an *ε-chain.*

If $\mathrm{Bd}(\bigcup(W_i : 1 \leq i \leq n\}) \subset \mathrm{Bd}(W_1 \cup W_n)$, then $\mathcal{W}$ is a *free chain.*

Let x be a point of M. Let X be the arc-component of M that contains x.

Assume

$$X \text{ does not contain a simple closed curve.} \tag{3.1}$$

Let z be a point of $X \sim \{x\}$. The arc, half-open arc, and the arc-segment (open arc) in M with endpoints x and z are denoted by $[x, z]$, $[x, z)$, and (x, z), respectively. We define $[x, x]$ to be $\{x\}$.

A chain $\mathcal{W} = \{W_i : 1 \leq i \leq n\}$ *follows* $[x, z]$ if $[x, z] \subset \bigcup \mathcal{W}$, $x \in W_1 \sim \mathrm{Cl}\, W_2$, and $z \in W_n \sim \mathrm{Cl}\, W_{n-1}$.

Assume that every subcontinuum of M that intersects (x, z) and $M \sim [x, z]$ also intersects $\{x, z\}$.

Then for each positive number ε,

$$\text{there is a free } \varepsilon\text{-chain that follows } [x, z]. \tag{3.2}$$

To see this, let $\mathcal{V} = \{V_i : 1 \leq i \leq n\}$ be an ε-chain that follows $[x, z]$. Since $[x, z]$ is a component of $[x, z] \cup (M \sim (V_1 \cup V_n))$, there exist disjoint open sets P and Q in M such that $[x, z] \subset P$, $M \sim \bigcup \mathcal{V} \subset Q$, and $M \sim (V_1 \cup V_n) \subset P \cup Q$ [**24**, Theorem 49, p. 17]. Let $W_1 = V_1$ and $W_n = V_n$. For each i $(1 < i < n)$, let $W_i = P \cap V_i$. Then $\{W_i : 1 \leq i \leq n\}$ is a free ε-chain that follows $[x, z]$. Hence (3.2) is true.

Assume

$$\begin{gathered} f \text{ is a fixed-point-free map of } M \text{ into } M \\ \text{and there is an arc in } M \text{ from } x \text{ to } f(x). \end{gathered} \tag{3.3}$$

Since the continuous image of an arc is arcwise connected [**17**, Theorem 1, p. 254 and Theorem 2, p. 256], for every point p of X, the arc $[p, f(p)]$ is in X.

By the compactness of M and the continuity of f, there is a positive number τ such that for every point p of M,

$$\rho(p, f(p)) > \tau. \tag{3.4}$$

Using assumptions (3.1) and (3.3), Borsuk [**5**] proved there exists a unique sequence $a_1, a_2, \dots$ of points of X such that $a_1 = x$ and for each positive integer n,

(3.5) $$\rho(a_n, a_{n+1}) = \tau/2 \qquad [\mathbf{5}, \text{p. } 19, (4_n)],$$

(3.6) $$\text{if } p \in [a_n, a_{n+1}), \text{ then } \rho(a_n, p) < \tau/2 \qquad [\mathbf{5}, \text{p. } 19\ (5_n)],$$

(3.7) $$[x, a_n] \cap [a_n, a_{n+1}] = \{a_n\} \qquad [\mathbf{5}, \text{p. } 19, (11)], \text{ and}$$

(3.8) $$a_n \in [x, f(a_n)] \qquad [\mathbf{5}, \text{p. } 19, (7_n)].$$

For each positive integer n, let ψ_n be a homeomorphism of the half-open real line interval $[n-1, n)$ onto $[a_n, a_{n+1})$. For each nonnegative real number r, let $\psi(r) = \psi_n(r)$ if $n - 1 \leq r < n$.

Let $P_x = \bigcup\{[x, a_n] : n = 2, 3, \dots\}$. By (3.7), ψ is a one-to-one map of the nonnegative real line $[0, +\infty)$ onto P_x. The map ψ determines a linear ordering $\ll$ of P_x with x as the first point.

The set P_x is called a *Borsuk ray.*

In [**4**, p. 123], Bing described the restriction of a fixed-point-free map to a Borsuk ray in terms of a dog chasing a rabbit. To continue in this spirit, one might think of our free chain as an open-ended hollow log through which the dog and rabbit run.

For each point p of P_x, let $P_x(p)$ denote $\{q \in P_x : p = q \text{ or } p \ll q\}$.

Let $L_x = \bigcap\{\operatorname{Cl} P_x(p) : p \in P_x\}$. By (3.5), L_x is not degenerate. Hence L_x is a subcontinuum of $\operatorname{Cl} P_x$.

The Borsuk ray P_x is *perfect* if $L_x = \operatorname{Cl} P_x$ and x belongs to every subcontinuum of M that intersects P_x and $M \sim P_x$.

For each point p of P_x, by [**12**, p. 98, (6)], $p \in [x, f(p)]$.

If P_x is perfect, it follows from (3.5) that for each point p of P_x,

(3.9) $$f(p) \in P_x(p).$$

Suppose M is in the plane E^2.

Assume there exist disjoint open sets Π and Σ in M such that $x \in \Pi$, $P_x \cap \Sigma \neq \varnothing$, and for each point p of $P_x \cap \Pi$,

(3.10) $$[p, f(p)] \cap \Sigma = \varnothing.$$

The remainder of this section is devoted to proving

(3.11) $$P_x \text{ is not perfect.}$$

Assume P_x is perfect. Since $L_x = \operatorname{Cl} P_x$, there exist points s, t, y, and z of P_x such that $\{s, t\} \subset \Sigma$, $\{y, z\} \subset \Pi$, and $s \ll y \ll t \ll z$.

By (3.9), $f(y) \in P_x(y)$ and $f(z) \in P_x(z)$. By (3.10), $f(y) \in (y, t)$. Let Y and Z be open subsets of Π such that $y \in Y$, $z \in Z$, $[x, y] \cap f(\operatorname{Cl} Y) = \varnothing$, and $[x, z] \cap f(\operatorname{Cl} Z) = \varnothing$.

Let ε be a positive number less than $\rho([x, y], \{t\} \cup f(Y))$, $\rho([y, z], \{s\} \cup f(Z))$, $\rho(\{s, t\}, M \sim \Sigma)$, $\rho(y, M \sim Y)$, and one-half of $\rho(z, M \sim Z)$.

By the argument for (3.2), there exist disjoint disks B and D in E^2 and a free ε-chain $\mathcal{W} = \{W_i : 1 \leq i \leq n\}$ that follows $[x, z]$ such that $\operatorname{Cl} W_1 = M \cap B$ and $\operatorname{Cl} W_n = M \cap D$.

Let W_m be an element of $\mathcal{W}$ that contains y.

Since $W_m \subset Y$, $\rho([x,y], f(Y)) > \varepsilon$, and $[x,y]$ intersects each element of $\{W_i : 1 \leq i \leq m\}$,

$$f(W_m) \cap \bigcup\{W_i : 1 \leq i \leq m\} = \varnothing. \tag{3.12}$$

Since $W_{n-1} \subset Z$, $\rho([y,z], f(Z)) > \varepsilon$, and $[y,z]$ intersects each element of $\{W_i : m \leq i \leq n\}$,

$$f(W_{n-1}) \cap \bigcup\{W_i : m \leq i \leq n\} = \varnothing. \tag{3.13}$$

We say that an arc $[u,v]$ in P_x is *ordered* from W_j to W_i in $\mathcal{W}$ if $u \ll v$, $u \in W_j$, $v \in W_i$, and $[u,v] \subset \bigcup\{W_k : i \leq k \leq j\}$.

Note that

(3.14) no arc in P_x is ordered from W_m to W_1 in $\mathcal{W}$.

To see this, assume there is an arc $[u,v]$ in $P_x \cap \bigcup\{W_i : 1 \leq i \leq m\}$ such that $u \ll v$, $u \in W_m$, and $v \in W_1$. Let W_α be an element of $\mathcal{W}$ that contains s. Since $\rho(s, [y,z]) > \varepsilon$ and $[y,z]$ intersects each element of $\{W_i : m \leq i \leq n\}$, it follows that $\alpha < m$. Therefore $[u,v] \cap W_\alpha \neq \varnothing$. Since $W_\alpha \subset \Sigma$ and $W_m \subset \Pi$, by (3.9) and (3.10), $f(u) \in [u,v]$, and this contradicts (3.12). Hence (3.14) is true.

Furthermore,

(3.15) no arc in P_x is ordered from W_{n-1} to W_m in $\mathcal{W}$.

To see this, assume there is an arc $[u,v]$ in $P_x \cap \bigcup\{W_i : m \leq i \leq n\}$ such that $u \ll v$, $u \in W_{n-1}$, and $v \in W_m$. Let W_β be an element of $\mathcal{W}$ that contains t. Note that $W_\beta \subset \Sigma$ and $W_{n-1} \cup W_n \subset \Pi$. Since $\rho(t, [x,y]) > \varepsilon$ and $[x,y]$ intersects each element of $\{W_i : 1 \leq i \leq m\}$, it follows that $m < \beta < n-1$. Therefore $[u,v] \cap W_\beta \neq \varnothing$. By (3.9) and (3.10), $f(u) \in [u,v]$, and this contradicts (3.13). Hence (3.15) is true.

Let C be an open disk in E^2 containing y such that $M \cap \operatorname{Cl} C \subset W_m$ and $(B \cup D) \cap \operatorname{Cl} C = \varnothing$. Let c_1 be the first point of $[x,y] \cap \operatorname{Cl} C$ with respect to $\ll$. Let b_1 be the last point of $[x, c_1] \cap B$. Let d_1 be the first point of $[y,z] \cap D$. Let c_2 be the last point of $[y, d_1] \cap \operatorname{Cl} C$.

Since $L_x = \operatorname{Cl} P_x$, it follows that $P_x(z) \cap C \neq \varnothing$. Let c_3 be the first point of $P_x(z) \cap \operatorname{Cl} C$.

Since $\mathcal{W}$ is free, by (3.15), $[z, c_3] \cap B \neq \varnothing$. Let b_2 be the last point of $[z, c_3] \cap B$.

Since $L_x = \operatorname{Cl} P_x$, by (3.14), there exists a point d of $P_x(c_3) \cap W_n$ such that $[c_3, d] \subset \bigcup\{W_i : 2 \leq i \leq n\}$. Let c_4 be the last point of $[c_3, d] \cap \operatorname{Cl} C$. If necessary, adjust C so that $c_3 \neq c_4$. Let d_2 be the first point of $[c_4, d] \cap D$.

By (3.14) and (3.15),

$$\left(\bigcup\{W_i : 1 < i < n\}\right) \sim (W_1 \cup W_n) \text{ contains } [b_1, d_1] \cup [b_2, d_2]. \tag{3.16}$$

Let H and I be two arc-segments in $E^2 \sim (B \cup D \cup [b_1, d_1] \cup [b_2, d_2] \cup \operatorname{Cl} C)$ with disjoint closures that go from B to D. Let Ω be the complementary domain of $B \cup D \cup H \cup I$ that contains C.

By (3.16), $[b_1, d_1]$ and $[b_2, d_2]$ are disjoint arcs in $\operatorname{Cl} \Omega$. Hence $\{b_1, d_1\}$ does not separate b_2 from d_2 in the simple closed curve $\operatorname{Bd} \Omega$ [**24**, Theorem 7, p. 144]. Since $\{b_1, b_2\} \subset \operatorname{Bd} B$ and $\{d_1, d_2\} \subset \operatorname{Bd} D$, the set $\{b_1, b_2\}$ does not separate d_1 from d_2

in $\mathrm{Bd}\,\Omega$. Since $[b_1, c_1], [c_2, d_1], [b_2, c_3]$, and $[c_4, d_2]$ are disjoint arcs in $(\mathrm{Cl}\,\Omega) \sim C$, it follows that $\{c_1, c_2\}$ does not separate c_3 from c_4 and $\{c_1, c_3\}$ does not separate c_2 from c_4 in $\mathrm{Bd}\,C$. Thus $\{c_1, c_4\}$ separates c_2 from c_3 in $\mathrm{Bd}\,C$.

Let J be an arc-segment in C that goes from c_2 to c_3. Let K be the arc in $\mathrm{Bd}\,C$ with endpoints c_1 and c_4 that is crossed by the arc $[b_2, c_3] \cup \mathrm{Cl}\,J$ at c_3. Then K crosses the simple closed curve $[c_2, c_3] \cup J$ only at c_3. Hence $[c_2, c_3] \cup J$ separates c_1 from c_4 in E^2. Since $[x, c_1] \cup [c_4, d]$ and $[c_2, c_3] \cup J$ are disjoint, $[c_2, c_3] \cup J$ separates x from d in E^2.

Let Δ be the complementary domain of $[c_2, c_3] \cup J$ that contains d. Since $L_x = \mathrm{Cl}\,P_x$ and $x \in E^2 \sim \mathrm{Cl}(\Delta \cup \Omega)$, it follows that $P_x(d)$ intersects $E^2 \sim (\Delta \cup \Omega)$. Let w be the first point of $P_x(d)$ in $E^2 \sim (\Delta \cup \Omega)$. Since $J \subset \Omega$ and $P_x(d) \cap [c_2, c_3] = \varnothing$, the point w is in $(\mathrm{Bd}\,\Omega) \sim \mathrm{Cl}\,\Delta$. Note that $d \in \Delta \sim \Omega$. Let u be the last point of $[d, w)$ in $E^2 \sim \Omega$. Then $u \in \Delta \cap \mathrm{Bd}\,\Omega$ and $(u, w) \subset \Omega$.

Since $u \in \Delta$, $w \in E^2 \sim \Delta$, and $[u, w] \cap [c_2, c_3] = \varnothing$, it follows that $[u, w] \cap C \neq \varnothing$. Let v be the first point of $[u, w] \cap \mathrm{Bd}\,C$. Since $[d_2, d]$ and $[c_2, c_3] \cup J$ are disjoint, $d_2 \in \Delta$. Thus $[c_4, d_2] \cup [u, v]$ is in Δ. Therefore $\{c_2, c_3\}$ does not separate c_4 from v in $\mathrm{Bd}\,C$. Hence $[u, v] \cup [c_4, d_2] \cup \mathrm{Bd}\,C$ contains an arc A that goes from u to d_2 in $\Delta \sim C$. Note that A is in $(\mathrm{Cl}\,\Omega) \sim ([b_2, c_3] \cup J \cup [c_2, d_1])$.

It follows from [**24**, Theorem 7, p. 144] that

(3.17) $\quad u$ is in the component of $(\mathrm{Bd}\,\Omega) \sim \{b_2, d_1\}$ that contains d_2.

By a similar argument,

(3.18) $\quad w$ is in the component of $(\mathrm{Bd}\,\Omega) \sim \{b_2, d_1\}$

that contains b_1.

Since $[u, w]$ and $[b_2, d_2]$ are disjoint arcs in $\mathrm{Cl}\,\Omega$, the set $\{b_2, d_2\}$ does not separate u from w in $\mathrm{Bd}\,\Omega$.

Therefore, by (3.17) and (3.18),

(3.19) $\quad u$ belongs to the arc in $(\mathrm{Bd}\,\Omega) \cap \mathrm{Bd}\,D$ that goes from d_1 to d_2.

Since $M \cap \mathrm{Cl}\,C \subset W_m$, the point v belongs to W_m. Since $\mathcal{W}$ is free and $(u, w) \subset E^2 \sim (B \cup D)$, it follows that $[u, w] \subset \bigcup\{W_i\colon 1 < i < n\}$. Hence, by (3.19), $u \in W_{n-1}$. Thus $[u, v]$ contains an arc that is ordered from W_{n-1} to W_m in $\mathcal{W}$, and this contradicts (3.15). Therefore (3.11) is true.

4. Results.

THEOREM 4.1. *If M is a nonseparating plane continuum and f is an arc-component-preserving map of M, then f has a fixed point.*

PROOF. Assume f moves each point of M. According to a theorem of Bell [**1**] and Sieklucki [**28**], there exists an indecomposable continuum Q in $\mathrm{Bd}\,M$ such that $Q = f(Q)$.

Following Krasinkiewicz [**16**], we define a composant C of Q to be *internal* if every continuum in the plane that intersects C and is not contained in Q intersects every composant of Q.

By [**16**, Theorem 2.3], Q has uncountably many internal composants. Since the composants of Q are disjoint [**24**, Theorem 138, p. 59], only countably many composants contain triods [**23**]. Moreover, since each composant is dense in Q

[**24**, Theorem 135, p. 58], only countably many composants contain continua that separate the plane. Hence there is an internal composant C of Q that does not contain a triod or a continuum that separates the plane.

Since M does not separate the plane, by [**15**, Theorem 2.1],

(4.2) every subcontinuum of M that intersects C and $M \sim C$ contains Q.

Let R be a subcontinuum of Q that intersects C such that

(4.3) $$R = f(R) \text{ and}$$

(4.4) no proper subcontinuum of R is mapped into itself by f [**29**, Theorem 11.1, p. 17].

Note that R may be Q.

Let $\{\Sigma_i : i = 1, 2, \dots\}$ be the set of elements of a countable open base for M that intersects R. For each positive integer i, let R_i be the set consisting of all points p in R such that p and $f(p)$ are the endpoints of an arc in $M \sim \mathrm{Cl}\,\Sigma_i$. Since f is fixed-point free, it follows from (4.3) that R is not an arc. Hence $R = \bigcup\{R_i : i = 1, 2, \dots\}$.

By the Baire category theorem, there is an integer j such that $\mathrm{Cl}\,R_j$ contains a nonempty open subset E of R. Since $R_j \cap \mathrm{Cl}\,\Sigma_j = \varnothing$, it follows that $E \cap (R \sim \mathrm{Cl}\,\Sigma_j) \neq \varnothing$. Let Π be an open subset of $M \sim \Sigma_j$ such that $\Pi \cap R$ is a nonempty subset of E.

Let x be a point of $C \cap R \cap \Pi$. Let X be the arc-component of M that contains x. By (4.2), $X \subset C$. Hence X does not contain a simple closed curve. Since $f(X) \subset X$, there is an arc in M from x to $f(x)$.

As in §3 (above), define the Borsuk ray P_x in X.

For each point p of P_x, by [**12**, p. 98, (6)], $p \in [x, f(p)]$.

Since X does not contain a triod, by (3.5), for each point p of P_x,

(4.5) $$f(p) \in P_x(p).$$

Note that

(4.6) $$P_x \subset R.$$

To see this, assume the contrary. Since $P_x \subset X \subset Q$, it follows that $R \neq Q$. Hence $R \subset C$. Define p to be the last point of P_x with the property that $[x, p] \subset R \cap P_x$. Since $X \subset C$, the continuum $[p, f(p)] \cup R$ is in C. Thus $[p, f(p)] \cup R$ does not separate the plane. Therefore $[p, f(p)] \cap R$ is connected [**24**, Theorem 22, p. 175]. By (4.3), $f(p) \in R$. Consequently $[p, f(p)] \subset R$. By (4.5), $[p, f(p)] \subset P_x(p)$. Therefore $[p, f(p)] \subset R \cap P_x$, and this contradicts the definition of p. Hence (4.6) is true.

Note that

(4.7) $$f(L_x) \subset L_x.$$

To see this, let q be a point of L_x and let $p_1, p_2, \dots$ be a sequence of points of P_x converging to q such that $p_1 \ll p_2 \ll \cdots$. It follows from (4.5) and the continuity of f that $f(p_1), f(p_2), \dots$ converges to a point of L_x. Hence $f(q) \in L_x$ and (4.7) is true.

By (4.6), L_x is a subcontinuum of R. In fact, by (4.4) and (4.7), $L_x = R$.

Hence

$$P_x \cap \Sigma_j \neq \emptyset. \tag{4.8}$$

For each point p of $P_x \cap \Pi$,

$$[p, f(p)] \cap \Sigma_j = \emptyset. \tag{4.9}$$

To establish (4.9), let $r_1, r_2, \ldots$ be a sequence of points of R_j that converges to p. For each positive integer i, let A_i be an arc in $M \sim \Sigma_j$ from r_i to $f(r_i)$. The limiting set T of the sequence $A_1, A_2, \ldots$ is a continuum in $M \sim \Sigma_j$ that contains $\{p, f(p)\}$ [**24**, Theorem 58, p. 23]. By (4.2), $T \cup [p, f(p)] \subset C$. Thus $T \cup [p, f(p)]$ does not separate the plane. Therefore $T \cap [p, f(p)]$ is connected. Since $\{p, f(p)\} \subset T$, it follows that $[p, f(p)] \subset T$. Hence (4.9) is true.

By (3.11), (4.8), and (4.9), P_x is not perfect. Since $L_x = R$, by (4.6), $L_x = \operatorname{Cl} P_x$. Hence there exist an open set G and a continuum H in M such that $x \in G \subset M \sim H$, $H \cap P_x \neq \emptyset$, and $H \cap (M \sim P_x) \neq \emptyset$. Since $L_x = \operatorname{Cl} P_x$, there is an arc $[x, z]$ in P_x such that $z \in G$ and $H \cap [x, z] \neq \emptyset$. By (4.2), $H \cup [x, z] \subset C$, and this contradicts the fact that C does not contain a triod. Therefore f has a fixed point.

COROLLARY 4.10. *Every arcwise connected nonseparating plane continuum has the fixed-point property* [**10, 11, 22**].

Suppose M is a nonseparating plane continuum and f is a map of M into M. If only countably many arc-components of M are permuted by f, then the proof of Theorem 4.1 can be modified to show that f has a fixed point.

QUESTION 4.11. If M is a nonseparating plane continuum and f is a map of M into M that maps one arc-component of M into itself, must f have a fixed point?

In the proof of Theorem 4.1, we used the assumption that M does not separate the plane only to establish the existence of Q and (4.2). If M is indecomposable and $Q = M$, then (4.2) is obviously true (even when M has infinitely many complementary domains). Hence we have also proved the following theorem:

THEOREM 4.12. *If M is an indecomposable plane continuum and f is an arc-component-preserving map of M, then f has a fixed point.*

A continuum is a *solenoid* if it is homeomorphic to an inverse limit of circles with covering maps as the bonding maps.

Every solenoid admits a fixed-point-free deformation. If the degree of each bonding map is greater than 1, the solenoid is indecomposable [**3**, Corollary, p. 118; **14**, Theorem 8, p. 249]. Hence the assumption that M is planar in Theorem 4.12 is necessary.

In [**18**, Problem 27, p. 369], Bellamy asked the following question:

If M is a tree-like continuum and f is a deformation of M, must f have a fixed point?

Theorem 4.1 shows that the answer to Bellamy's question is yes if M is planar. Our next theorem generalizes this result [**23**]:

THEOREM 4.13. *If H is a tree-like continuum that does not contain uncountably many disjoint triods and f is an arc-component-preserving map of H, then f has a fixed point.*

PROOF. Assume f moves each point of H.

Let M be a subcontinuum of H such that

(4.14) $$M = f(M) \text{ and}$$

(4.15) no proper subcontinuum of M is mapped into itself by f.

Since H is tree-like, by (4.14), for every point p of M,

(4.16) there is a unique arc $[p, f(p)]$ in M.

Hence the restriction of f to M is a fixed-point-free arc-component-preserving map of M.

Note that M is tree-like.

By a theorem of Manka [**19**], there exists an indecomposable continuum Q in M. Since M is tree-like, no arc in M intersects more than one composant of Q. Therefore M has uncountably many arc-components.

Let X be an arc-component of M that does not contain a triod. Since M is tree-like, X does not contain a simple closed curve.

Let x be a point of X. Define the Borsuk ray P_x in X.

Since X does not contain a triod, it follows from (3.5) and [**12**, p. 98, (6)] that for each point p of P_x,

(4.17) $$f(p) \in P_x(p).$$

Note that

(4.18) $$P_x \subset M.$$

To see this, assume the contrary. Let p be the last point of P_x with the property that $[x,p] \subset M \cap P_x$. By (4.16), $[p, f(p)] \subset M$. By (4.17), $[p, f(p)] \subset P_x(p)$. Therefore $[p, f(p)] \subset M \cap P_x$, and this contradicts the definition of p. Hence (4.18) is true.

By the argument for (4.7),

(4.19) $$f(L_x) \subset L_x.$$

By (4.15), (4.18), and (4.19)

(4.20) $$L_x = M.$$

The continuum M is indecomposable. For suppose M is the union of two proper subcontinua J and K. Then, by (4.18) and (4.20), there is an arc in P_x with both endpoints in J that intersects $K \sim J$, and this contradicts the fact that M is tree-like.

Let $\{\Sigma_i : i = 1, 2, \dots\}$ be a countable open base for M. For each positive integer i, let $M_i = \{p \in M : [p, f(p)] \cap \Sigma_i = \varnothing\}$. Since M is not an arc, $M = \bigcup\{M_i : i = 1, 2, \dots\}$. Hence there is an integer j such that $\mathrm{Cl}\, M_j$ contains a nonempty open subset Π of M.

Let C be a composant of M that does not contain a triod.

Assume without loss of generality that x belongs to $C \cap \Pi$.

By (4.20), $P_x \cap \Sigma_j \neq \varnothing$. By an argument similar to the one for (4.9), for each point p of $P_x \cap \Pi$, the arc $[p, f(p)]$ misses Σ_j.

By (4.18) and (4.20), $L_x = \mathrm{Cl}\, P_x$. Therefore, since C does not contain a triod, P_x is a perfect Borsuk ray in M.

As in the proof of (3.11), define a free chain $\mathcal{W} = \{W_1, W_2, \ldots, W_m, \ldots, W_n\}$ in M that follows an arc $[x, z]$ in P_x and has the property that

(4.21) no arc in P_x is ordered from W_m to W_1 in $\mathcal{W}$.

Let μ be a positive number less than $\rho(x, M \sim W_1)$ and $\rho([x, z], M \sim \bigcup \mathcal{W})$.

Let $\mathcal{T}$ be a cover of M with mesh less than μ whose nerve is a tree. Let E be an element of $\mathcal{T}$ that contains x. By (4.20), $E \cap P_x(z) \neq \varnothing$. Note that $E \subset W_1$. Since $\mathcal{W}$ is free and $\mathcal{T}$ does not contain a circular chain, $P_x(z)$ contains an arc that is ordered from W_m to W_1 in $\mathcal{W}$, and this contradicts (4.21). Hence f has a fixed point.

COROLLARY 4.22. *Bellamy's tree-like continuum without the fixed-point property* [**2**] *does not admit an arc-component-preserving map that is fixed-point free.*

QUESTION 4.23. Does every tree-like continuum have the fixed-point property for arc-component-preserving maps?

An affirmative answer to the following question would generalize the author's theorem [**12**] that every uniquely arcwise connected plane continuum has the fixed-point property.

QUESTION 4.24. If M is a plane continuum that does not contain a simple closed curve and f is an arc-component-preserving map of M, must f have a fixed point?

REFERENCES

1. H. Bell, *On fixed point properties of plane continua*, Trans. Amer. Math. Soc. **128** (1967), 539–548.
2. D. P. Bellamy, *A tree-like continuum without the fixed point property*, Houston J. Math. **6** (1979), 1–13.
3. R. H. Bing, *Embedding circle-like continua in the plane*, Canad. J. Math. **14** (1962), 113–128.
4. ——, *The elusive fixed point property*, Amer. Math. Monthly **76** (1969), 119–132.
5. K. Borsuk, *A theorem on fixed points*, Bull. Acad. Polon Sci. **2** (1954), 17–20.
6. R. F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman, Illinois, 1971.
7. E. Dyer, *A fixed point theorem*, Proc. Amer. Math. Soc. **7** (1956), 662–672.
8. E. Fadell, *Recent results in the fixed point theory of continuous maps*, Bull. Amer. Math. Soc. **76** (1970), 10–29.
9. F. B. Fuller, *The existence of periodic points*, Ann. of Math. (2) **57** (1953), 229–230.
10. C. L. Hagopian, *A fixed point theorem for plane continua*, Bull. Amer. Math. Soc. **77** (1971), 351–354; *Addendum*, Bull. Amer. Math. Soc. **78** (1972), 289.
11. ——, *Another fixed point theorem for plane continua*, Proc. Amer. Math. Soc. **31** (1972), 627–628.
12. ——, *Uniquely arcwise connected plane continua have the fixed-point property*, Trans. Amer. Math. Soc. **248** (1979), 85–104.
13. ——, *The fixed-point property for deformations of uniquely arcwise connected continua*, Topology Appl. **24** (1979), 207–212.
14. W. T. Ingram, *Concerning non-planar circle-like continua*, Canad. J. Math. **19** (1967), 242–250.
15. J. Krasinkiewicz, *Concerning the boundaries of plane continua and the fixed point property*, Bull. Acad. Polon. Sci. **21** (1973), 427–431.
16. ——, *Concerning the accessibility of composants of indecomposable plane continua*, Bull. Acad. Polon. Sci. **21** (1973), 621–628.
17. K. Kuratowski, *Topology*, Vol. II, Academic Press, New York, 1968.
18. I. W. Lewis, *Continuum theory problems*, Topology Proc. **8** (1983), 361–394.
19. R. Manka, *Association and fixed points*, Fund. Math. **91** (1976), 105–121.

20. M. M. Marsh, *A fixed point theorem for inverse limits of fans*, Proc. Amer. Math. Soc. **91** (1984), 139–142.
21. P. Minc, *Generalized retracts and the Lefschetz fixed point theorem*, Bull. Acad. Polon. Sci. **25** (1977), 291–299.
22. ——, *A fixed point theorem for weakly chainable plane continua*, preprint.
23. R. L. Moore, *Concerning triodic continua in the plane*, Fund. Math. **13** (1929), 261–263.
24. ——, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, R. I., 1962.
25. L. G. Oversteegen and J. T. Rogers, Jr., *An inverse limit description of an atriodic tree-like continuum and an induced map without a fixed point*, Houston J. Math. **6** (1980), 549–564.
26. ——, *Fixed-point-free maps on tree-like continua*, Topology Appl. **13** (1982), 85–95.
27. H. W. Siegbery and G. Skordev, *Fixed point index and chain approximations*, Pacific J. Math. **102** (1982), 455–486.
28. K. Sieklucki, *On a class of plane acyclic continua with the fixed point property*, Fund. Math. **63** (1968), 257–278.
29. G. T. Whyburn, *Analytic topology*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1963.
30. G. S. Young, *Fixed-point theorems for arcwise connected continua*, Proc. Amer. Math. Soc. **11** (1960), 880–884.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, SACRAMENTO, CALIFORNIA 95819