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FINITE CODIMENSIONAL INVARIANT SUBSPACES OF BERGMAN SPACES

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ABSTRACT. For a large class of bounded domains in $\mathbf{C}$, we describe those finite codimensional subspaces of the Bergman space that are invariant under multiplication by z. Using different techniques for certain domains in $\mathbf{C}^N$, we describe those finite codimensional subspaces of the Bergman space that are invariant under multiplication by all the coordinate functions.

Fix a positive integer N, and let V denote Lebesgue volume measure on $\mathbf{C}^N$ (so that if $N = 1$, then V is just area measure). Let $\Omega \subset \mathbf{C}^N$ be a domain, which, as usual, means that Ω is a nonempty open connected subset of $\mathbf{C}^N$. For f an analytic function from Ω to $\mathbf{C}$ and $1 \leq p < \infty$, the norm $\|f\|_{\Omega,p}$ is defined by

$$\|f\|_{\Omega,p} = \left(\int_\Omega |f|^p \, dV\right)^{1/p}.$$

The Bergman space $L_a^p(\Omega)$ is defined to be the set of analytic functions from Ω to $\mathbf{C}$ such that $\|f\|_{\Omega,p} < \infty$.

Our goal in this paper is to describe the closed finite codimensional subspaces of $L_a^p(\Omega)$ that are invariant under multiplication by the coordinate functions $z_1, \ldots, z_N$. The first section of the paper deals with planar domains; the second section in the paper concerns domains in $\mathbf{C}^N$. The concluding section of the paper presents some questions and areas for possible further research.

Planar domains. In this section, for a large class of bounded domains $\Omega \subset \mathbf{C}$ we characterize the closed finite codimensional subspaces of $L_a^p(\Omega)$ that are invariant under multiplication by z. The main result in this section is Theorem 5.

We begin with the following easy proposition. Our goal (Theorem 5) is to prove the converse of this proposition for a large class of domains.

PROPOSITION 1. *Let $1 \leq p < \infty$, and let Ω be a bounded domain of $\mathbf{C}$. Let q be a polynomial that has all its roots in Ω. Then $qL_a^p(\Omega)$ is a closed subspace of $L_a^p(\Omega)$ that is invariant under multiplication by z. Furthermore,* $\dim L_a^p(\Omega)/qL_a^p(\Omega)$ *equals* degree q.

PROOF. Let $\lambda_1, \ldots, \lambda_{\text{degree } q}$ denote the zeros of q, repeated according to multiplicity. It is easy to verify that

$$qL_a^p(\Omega) = \{f \in L_a^p(\Omega) \colon f(\lambda_1) = \cdots = f(\lambda_{\text{degree } q}) = 0\},$$

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where in the case of zeros of q of multiplicity larger than one we require derivatives of appropriate orders to equal zero. Since point evaluations (and point evaluations of derivatives of all orders) are continuous linear functionals on $L^p_a(\Omega)$, we conclude that $qL^p_a(\Omega)$ is closed in $L^p_a(\Omega)$.

Since $qL^p_a(\Omega)$ is the intersection of the kernels of a set of degree q linearly independent linear functionals, the codimension of $qL^p_a(\Omega)$ in $L^p_a(\Omega)$ must equal degree q.

Finally, it is obvious that $qL^p_a(\Omega)$ is invariant under multiplication by z, completing the proof. □

The following theorem is our first step in proving a converse to Proposition 1. Theorem 2 will be used in the proofs of Theorems 3 and 5. The proof of Theorem 2 is based upon ideas used by Gellar in [**8**, §9] in dealing with weighted shifts.

THEOREM 2. *Let $1 \leq p < \infty$, and let $\Omega \subset \mathbb{C}$ be a bounded domain such that $(z-\lambda)L^p_a(\Omega)$ is dense in $L^p_a(\Omega)$ for every $\lambda \in \partial\Omega$. Let E be a closed finite codimensional subspace of $L^p_a(\Omega)$ that is invariant under multiplication by z. Then there is a polynomial q whose roots lie in Ω such that $E = qL^p_a(\Omega)$.*

PROOF. Define an operator $T\colon L^p_a(\Omega)/E \to L^p_a(\Omega)/E$ by $T(g+E) = zg+E$; note that the verification that T is well defined uses the invariance of E under multiplication by z. If f is a polynomial, then

$$f(T)(g+E) = fg+E \quad \text{for every } g \in L^p_a(\Omega).$$

Since T is an operator on a finite dimensional space, there is a nonzero polynomial f, with degree at most $\dim L^p_a(\Omega)/E$, such that $f(T) = 0$. The equation above implies that $fL^p_a(\Omega) \subset E$.

Factor f as $f = qh$, where q is a polynomial whose roots lie in Ω and h is a polynomial whose roots lie in $\mathbb{C}\backslash\Omega$. If λ is a complex number not in the closure of Ω, then $(z-\lambda)L^p_a(\Omega)$ equals $L^p_a(\Omega)$; if $\lambda \in \partial\Omega$, then by hypothesis $(z-\lambda)L^p_a(\Omega)$ is dense in $L^p_a(\Omega)$. Thus $(z-\lambda)L^p_a(\Omega)$ is dense in $L^p_a(\Omega)$ for every zero of h, and so $hL^p_a(\Omega)$ is dense in $L^p_a(\Omega)$, and hence $qL^p_a(\Omega) \subset (fL^p_a(\Omega))^- \subset E$. Thus

$$\dim L^p_a(\Omega)/E \leq \dim L^p_a(\Omega)/qL^p_a(\Omega) = \text{degree } q \leq \text{degree } f \leq \dim L^p_a(\Omega)/E,$$

where the equality above comes from Proposition 1. The above inequalities imply that $\dim L^p_a(\Omega)/E = \dim L^p_a(\Omega)/qL^p_a(\Omega)$, and since $qL^p_a(\Omega) \subset E$, this implies that $qL^p_a(\Omega) = E$, completing the proof. □

To apply Theorem 2, we need to find conditions on a domain Ω that imply that $(z-\lambda)L^p_a(\Omega)$ is dense in $L^p_a(\Omega)$ for every λ in $\partial\Omega$. We begin this process by finding a simple geometric condition in the following theorem. A much deeper condition will be exhibited in Theorem 5.

By a wedge in $\mathbb{C}$, we mean the convex hull of a point (called the vertex of the wedge) and an arc of a circle centered at the point.

THEOREM 3. *Let $1 \leq p < \infty$, let $\Omega \subset \mathbb{C}$ be a bounded domain, and suppose that for each point $\lambda \in \partial\Omega$ there exists a wedge W in $\mathbb{C}\backslash\Omega$ with vertex λ. If E is a closed finite codimensional subspace of $L^p_a(\Omega)$ that is invariant under multiplication by z, then there is a polynomial q, all of whose roots lie in Ω, such that $E = qL^p_a(\Omega)$.*

PROOF. By Theorem 2, we need only show that $(z-\lambda)L^p_a(\Omega)$ is dense in $L^p_a(\Omega)$ for every λ in $\partial\Omega$. To do this, let $\lambda \in \partial\Omega$. By hypothesis, there is a wedge W in

$\mathbf{C}\backslash\Omega$ with vertex λ. Without loss of generality, we may assume that $\lambda = 0$ and that

$$W = \{z\colon |z| \le 2a \text{ and } -\delta \le \arg z \le \delta\}$$

for some positive constants $a > 0$ and $\delta \in (0, \pi/2)$.

Note that if $t \in (0, a)$ and $z \in \Omega$ (which implies that z is not in the wedge W), then

$$\left|\frac{z}{z-t} - 1\right| = \frac{t}{|z-t|} \le \frac{1}{\sin\delta}$$

and

$$\lim_{t \to 0^+} \left|\frac{z}{z-t} - 1\right| = 0.$$

Let $f \in L^p_a(\Omega)$. Then $f/(z-t) \in L^p_a(\Omega)$ for every $t \in (0, a)$. The conditions above, combined with the dominated convergence theorem, show that

$$\lim_{t \to 0^+} \int_\Omega \left| z\left[\frac{f(z)}{z-t}\right] - f(z)\right|^p dV(z) = 0.$$

It follows that $zL^p_a(\Omega)$ is dense in $L^p_a(\Omega)$, completing the proof. □

For $z \in \mathbf{C}$ and $r > 0$, let $B(z, r)$ denote the open ball centered at z with radius r. For a set $F \subset \mathbf{C}$, let $\gamma(F)$ denote the analytic capacity of F. As usual, $H^\infty(\Omega)$ denotes the set of bounded analytic functions on Ω.

To prove Theorem 5, we will need the following lemma, whose proof relies upon results of Hedberg and Lindberg. We would like to thank Lars Hedberg for bringing Lindberg's work to our attention.

LEMMA 4. *Let $1 \le p < \infty$. Let $\Omega \subset \mathbf{C}$ be a bounded domain such that no connected component of $\partial\Omega$ is equal to a point. Then $H^\infty(\Omega)$ is dense in $L^p_a(\Omega)$.*

PROOF. In [**10**, pp. 112–114], an argument due to Hedberg is used to show that $H^\infty(\Omega)$ is dense in $L^2_a(\Omega)$ if Ω is a simply connected domain with finite area. By localizing Hedberg's argument, Lindberg was able to establish the following result, which appears (in a slightly stronger form) as Theorem 4.3 of [**9**].

LINDBERG'S THEOREM. *Let $1 \le p < \infty$ and let $\Omega \subset \mathbf{C}$ be a bounded domain. If*

$$\overline{\lim_{r \to 0}} \frac{\gamma[B(\varsigma, r)^- \backslash \Omega]}{r} > 0$$

for every $\varsigma \in \partial\Omega$, then $H^\infty(\Omega)$ is dense in $L^p_a(\Omega)$.

We will prove Lemma 4 by verifying that a domain satisfying the hypothesis of Lemma 4 also satisfies the hypothesis of Lindberg's Theorem.

Fix a point $\varsigma \in \partial\Omega$, and let C_ς be the connected component of $\partial\Omega$ containing ς. By hypothesis, the diameter of C_ς is positive. Fix a number $r <$ (diameter $C_\varsigma)/2$, and let K be the connected component of $C_\varsigma \cap B(\varsigma, r)^-$ containing ς. We claim that K meets $\partial B(\varsigma, r)$, and hence (diameter $K) \ge r$. To verify this claim, first note that since (diameter $C_\varsigma) > 2r$, we have $C_\varsigma \not\subset B(\varsigma, r)$. Let $B = B(\varsigma, r)$ and $X = C_\varsigma \cap B^-$. Suppose that $K \cap \partial B = \varnothing$, so that $K \subset (C_\varsigma \cap B)$. Since $C_\varsigma \cap B$ is an open subset of X and since K is a connected component of X, there is a clopen subset R of X such that $K \subset R \subset (C_\varsigma \cap B)$ (see the corollary on p. 205 of [**7**]). Now, $C_\varsigma \cap R$ and $C_\varsigma \cap (\mathbf{C}\backslash R)$ (which equals $[C_\varsigma \cap (\mathbf{C}\backslash B)] \cup [X \backslash R]$) are disjoint, nonempty, closed subsets of $\mathbb{C}$ whose union is C_ς. This contradicts the connectedness of C_ς; hence K must meet ∂B.

Since $(B(\varsigma, r)^- \backslash \Omega) \supset K$ and since K is connected,

$$\frac{\gamma[B(\varsigma, r)^- \backslash \Omega]}{r} \geq \frac{\gamma(K)}{r} \geq \frac{\text{diameter } K}{4r} \geq \frac{1}{4}.$$

Hence, for a domain Ω satisfying the hypothesis of Lemma 4, we have

$$\overline{\lim_{r \to 0}} \frac{\gamma[B(\varsigma, r)^- \backslash \Omega]}{r} \geq \frac{1}{4}$$

for every $\varsigma \in \partial\Omega$. Lindberg's Theorem now gives the desired conclusion, completing the proof. □

We are now ready to prove the main result of this section. In Theorem 7, we will show that without the hypothesis that no connected component of $\partial\Omega$ is equal to a point, it is not necessarily true that every closed finite codimensional invariant subspace of $L^p_a(\Omega)$ is of the form $qL^p_a(\Omega)$.

THEOREM 5. *Let $1 \leq p < \infty$, and let $\Omega \subset \mathbb{C}$ be a bounded domain such that no connected component of $\partial\Omega$ is equal to a point. If E is a closed finite codimensional subspace of $L^p_a(\Omega)$ that is invariant under multiplication by z, then there is a polynomial q, all of whose roots lie in Ω, such that $E = qL^p_a(\Omega)$.*

PROOF. By Theorem 2, we need only show that $(z - \lambda)L^p_a(\Omega)$ is dense in $L^p_a(\Omega)$ for every λ in $\partial\Omega$.

By Lemma 4, we know that $H^\infty(\Omega)$ is dense in $L^s_a(\Omega)$ for $1 \leq s < \infty$. Hence, it will suffice to prove the theorem for p an integer, since if $1 \leq s \leq p$, then $L^p_a(\Omega)$ is contained in $L^s_a(\Omega)$ and convergence in $L^p_a(\Omega)$ implies convergence in $L^s_a(\Omega)$.

Let $\lambda \in \partial\Omega$.

First suppose that $p = 1$. If $h \in H^\infty(\Omega)$, then $h/(z - \lambda) \in L^1_a(\Omega)$, so $H^\infty(\Omega) \subset (z - \lambda)L^1_a(\Omega)$, and the density of $(z - \lambda)L^1_a(\Omega)$ in $L^1_a(\Omega)$ follows from the density of $H^\infty(\Omega)$ in $L^1_a(\Omega)$.

Now suppose that $p \geq 2$ is an integer. Let F be the connected component of $\partial\Omega$ containing λ and let $\lambda_1 \neq \lambda$ belong to F. Define a map φ of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ to itself by

$$\varphi(z) = (z - \lambda_1)/(z - \lambda).$$

Note that $G = \mathbb{C} \cup \{\infty\} \backslash \varphi(F)$ is a simply connected open set that does not contain 0 or ∞. It follows that there is an analytic pth root function on G. Thus

$$[(z - \lambda_1)/(z - \lambda)]^{1/p} \in L^p_a(\Omega)$$

since φ maps Ω into G.

Let Y denote the closure of $(z - \lambda)L^p_a(\Omega)$ in $L^p_a(\Omega)$. Our goal is, of course, to show that $Y = L^p_a(\Omega)$. Note that to accomplish this goal it suffices to show that $H^\infty(\Omega) \subset Y$.

Observe that Y is invariant under multiplication by every function in $H^\infty(\Omega)$ and that $z - \lambda \in Y$ (since $1 \in L^p_a(\Omega)$). Suppose that in addition to $z - \lambda$, the function $z - \lambda_1$ belongs to Y. We would then have $\lambda - \lambda_1 \in Y$; and by the invariance of Y under multiplication by $H^\infty(\Omega)$, we would have $H^\infty(\Omega) \subset Y$. Thus we may prove that $Y = L^p_a(\Omega)$ by establishing that $z - \lambda_1 \in Y$.

Let $\{h_n\} \subset H^\infty(\Omega)$ satisfy $\|h_n - [(z-\lambda_1)/(z-\lambda)]^{1/p}\|_{\Omega,p} \to 0$. Since $(z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{1/p} \in H^\infty(\Omega)$, we have

$$\|(z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{1/p}h_n - (z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{2/p}\|_{\Omega,p} \to 0.$$

It follows that $(z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{2/p} \in Y$. If $p=2$, then $z-\lambda_1 \in Y$ and we are through. Otherwise, we use the invariance of Y under multiplication by an $H^\infty(\Omega)$ function to see that $(z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{2/p}h_n \in Y$ for every n. Since $(z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{2/p} \in H^\infty(\Omega)$ (we are assuming that $p \geq 3$), we have

$$\|(z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{2/p}h_n - (z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{3/p}\|_{\Omega,p} \to 0,$$

and hence $(z-\lambda)[(z-\lambda_1)/(z-\lambda)]^{3/p} \in Y$. Continuing this way (if necessary), we see that $z-\lambda_1 \in Y$, completing the proof. □

As an example of an infinitely connected domain to which Theorem 5 applies, suppose that Ω is the interior of the unit disk with the segments

$$\bigcup_{n=2}^{\infty}\left\{\left(1-\frac{1}{n}, y\right) : 0 \leq y \leq \frac{1}{n}\right\}$$

removed. Theorem 5 shows that the closed, finite codimensional, invariant subspaces of $L_a^p(\Omega)$ are of the form $qL_a^p(\Omega)$, where q is a polynomial whose roots all lie in Ω.

Theorem 5 can be used to obtain a description of the Fredholm multiplication operators on $L_a^p(\Omega)$ for domains Ω such that no connected component of $\partial\Omega$ is a point. The Fredholm multiplication operators on $L_a^p(\Omega)$ for arbitrary domains $\Omega \subset \mathbf{C}$ already have been characterized by Axler [**3**, Theorem 23], but the proof given below in Corollary 6 presents a new (and simpler) method for solving this problem for a large class of domains.

For $1 \leq p < \infty$ and $h \in H^\infty(\Omega)$, let M_h denote the multiplication operator from $L_a^p(\Omega)$ to $L_a^p(\Omega)$ defined by $M_h f = hf$.

COROLLARY 6. *Let $1 \leq p < \infty$. Let Ω be a bounded domain such that no connected component of $\partial\Omega$ is equal to a point, and let $h \in H^\infty(\Omega)$. Then the multiplication operator $M_h\colon L_a^p(\Omega) \to L_a^p(\Omega)$ is Fredholm if and only if h is bounded away from 0 near $\partial\Omega$.*

PROOF. First suppose that M_h is Fredholm. Then the range of M_h, which equals $hL_a^p(\Omega)$, is a closed finite codimensional subspace of $L_a^p(\Omega)$ and is clearly invariant under multiplication by z. By Theorem 5, there is a polynomial q, whose roots all lie in Ω, such that $hL_a^p(\Omega) = qL_a^p(\Omega)$. In particular, $h = qg$ for some $g \in L_a^p(\Omega)$. Now $qgL_a^p(\Omega) = hL_a^p(\Omega) = qL_a^p(\Omega)$, so $gL_a^p(\Omega) = L_a^p(\Omega)$. Thus the operator of multiplication by g is invertible on $L_a^p(\Omega)$. Clearly, $g(\Omega)$ is contained in the spectrum of multiplication by g, so we can conclude that g is bounded away from 0 on Ω. This implies that h is bounded away from 0 near $\partial\Omega$, as desired.

To prove the converse, suppose that h is bounded away from 0 near $\partial\Omega$. Write $h = qf$, where q is a polynomial with all its roots in Ω and f is a function in $H^\infty(\Omega)$ that is bounded away from 0 on Ω. It is clear that $M_h = M_q M_f$, the operator M_f is invertible, the operator M_h is injective, and the range of M_h equals $qL_a^p(\Omega)$. Thus, to show that M_h is Fredholm, we need only show that $qL_a^p(\Omega)$ is a closed finite codimensional subspace of $L_a^p(\Omega)$. This follows from Proposition 1, completing the proof. □

It is natural to ask whether the conclusion of Theorem 5 holds without the assumption that no connected component of $\partial\Omega$ is equal to a point. So that we do not need to worry about what are called removable boundary points (for example, if Ω equals the unit disk with the origin deleted, then every function in $L_a^2(\Omega)$ extends to be analytic at the origin), the question should be phrased as follows: If $1 \leq p < \infty$ and $\Omega \subset \mathbb{C}$ is a bounded domain, can every closed finite codimensional subspace of $L_a^p(\Omega)$ that is invariant under multiplication by z be written in the form $qL_a^p(\Omega)$, where q is a polynomial?

The following theorem shows that the above question has a negative answer, even when $p = 2$ and even when we allow q to be an arbitrary function in $H^\infty(\Omega)$ rather than just a polynomial.

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$ and let $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$ denote the punctured unit disk. A domain Ω is said to be of type L provided

$$\Omega = \mathbb{D}^*\backslash\left(\bigcup_{n=1}^{\infty} B(x_n, r_n)^-\right),$$

where $\{x_n\}$, $\{r_n\}$ are sequences of positive numbers converging to zero in such a way that the disks $\{B(x_n, r_n)^-\}$ are pairwise disjoint and contained in $\mathbb{D}$. Domains of type L were studied by Zalcman in [**11**], where he showed that if the radii $\{r_n\}$ of an L-domain Ω converge to 0 sufficiently faster than the $\{x_n\}$ (specifically, if $\sum_{n=1}^{\infty} r_n/x_n < \infty$), then $\lim_{x\to 0-} f(x)$ exists for every $f \in H^\infty(\Omega)$ (this limit notation means that the limit is taken as x approaches 0 through the negative real numbers).

In the following proof, we show that if the radii $\{r_n\}$ of an L-domain Ω converge to 0 much faster than required by Zalcman, then $\lim_{x\to 0-} f(x)$ exists for every f in the Bergman space $L_a^2(\Omega)$.

THEOREM 7. *There exists a bounded domain $\Omega \subset \mathbb{C}$ and a closed finite codimensional subspace E of $L_a^2(\Omega)$ that is invariant under multiplication by z such that E cannot be written in the form $qL_a^2(\Omega)$ for any $q \in H^\infty(\Omega)$.*

PROOF. Let

$$\Omega = \mathbb{D}^*\backslash\left(\bigcup_{n=1}^{\infty} B(x_n, r_n)^-\right),$$

where $x_n = e^{-n}$ and $r_n = e^{-n}e^{-n^4 \exp(2n)}$.

We claim that if $f \in L_a^2(\Omega)$, then $\lim_{x\to 0-} f(x)$ exists as a finite complex number. In addition, we claim that there is a constant c such that

$$\left|\lim_{x\to 0-} f(x)\right| \leq c\|f\|_{\Omega,2} \qquad \text{for every } f \in L_a^2(\Omega).$$

Suppose that the above claims are valid. To see how this would finish the proof, let

$$E = \left\{f \in L_a^2(\Omega)\colon \lim_{x\to 0-} f(x) = 0\right\},$$

and note that E is invariant under multiplication by z. Since E is the kernel of a continuous linear functional on $L_a^2(\Omega)$, we know that E is closed and

$\dim(L^2_a(\Omega)/E) = 1$. Suppose that E could be written in the form $qL^2_a(\Omega)$ for some $q \in H^\infty(\Omega)$. Then the operator M_q of multiplication by q on $L^2_a(\Omega)$ would be a Fredholm operator (since M_q is injective and has a closed, finite codimensional range equal to E). Now Axler's characterization of Fredholm multiplication operators [**3**, Theorem 23] shows that q must be bounded away from 0 near $\partial\Omega$ (Proposition 14 of [**4**], along with the fact that the 2-essential boundary is closed, has been used here to conclude that the 2-essential boundary of Ω equals $\partial\Omega$). In particular, q must be bounded away from 0 near the origin. But $q \in qL^2_a(\Omega) = E$, which contradicts the definition of E. Thus our proof will be completed once we have verified the claims made in the previous paragraph.

To prove our claims, let $G = B(0, 1/2) \cap \{z \in \mathbf{C}\colon \operatorname{Re} z < 0\}$, let Γ_0 denote the boundary of $B(0, 3/4)$, oriented counterclockwise, and for $n = 1, 2, 3, \dots$ let Γ_n equal the boundary of $B(x_n, 2r_n)$, oriented clockwise.

The proof will consist of two steps:

Step 1. We will show that

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{\Gamma_n}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma$$

for every $f \in L^2_a(\Omega)$ and every $z \in G$.

Step 2. We will show that there is an absolute constant c such that

$$\left|\int_{\Gamma_n}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma\right| \le cn^{-2}\|f\|_{\Omega,2}$$

for every $f \in L^2_a(\Omega)$, every $z \in G^-$, and every positive integer n.

Suppose that Steps 1 and 2 can be carried out. For each positive integer m, let

$$f_m(z) = \frac{1}{2\pi i}\sum_{n=0}^{m}\int_{\Gamma_n}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma.$$

The sequence $\{f_m\}$ converges uniformly on G^- to a function $\tilde{f}$ continuous on G^- (by Step 2, since f_m is continuous on G^- for each m). The function $\tilde{f}$ is a continuous extension of f to G^- (by Step 1). Thus $\tilde{f}(0) = \lim_{x\to 0-} f(x)$ and

$$\left|\lim_{x\to 0-} f(x)\right| \le c(2\pi)^{-1}\left(\sum_{n=1}^{\infty} n^{-2}\right)\|f\|_{\Omega,2} \le c\|f\|_{\Omega,2}.$$

Hence, if Steps 1 and 2 are carried out, the claim follows and the proof will be complete.

To prove Step 1, let $f \in L^2_a(\Omega)$ and $z \in G$. For each positive integer n, let $B_n = B(0, (x_n + x_{n+1})/2)$ and let γ_n equal the boundary of B_n, oriented clockwise. Choose a positive integer N such that $z \notin B_N^-$.

Given $\varepsilon > 0$, choose $M \ge N$ such that $\int_{\Omega\cap 2B_M}|f|^2\,dV < \varepsilon^2$. Let $n > M$. For every domain Q, every $f \in L^2_a(Q)$, and every $\varsigma \in Q$, it is easy to see that

$$|f(\varsigma)| \le [\operatorname{dist}(\varsigma, \partial Q)]^{-1}\|f\|_{Q,2}.$$

If $\varsigma \in \gamma_n$, then applying the above estimate with $Q = \Omega \cap 2B_M$ shows that

$$|f(\varsigma)| \le 4\varepsilon/x_n.$$

Now use the Cauchy integral formula, to obtain

$$\begin{aligned}\left|f(z) - (2\pi i)^{-1}\sum_{j=0}^{n}\int_{\Gamma_j}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma\right| &= (2\pi)^{-1}\left|\int_{\gamma_n}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma\right| \le (2\pi)^{-1}\int_{\gamma_n}\frac{|f(\varsigma)|}{\operatorname{dist}(z,\gamma_n)}|d\varsigma| \\ &\le 2[\operatorname{dist}(z,\gamma_n)]^{-1}\frac{\varepsilon(x_n + x_{n+1})}{x_n} \le 4[\operatorname{dist}(z,\gamma_n)]^{-1}\varepsilon.\end{aligned}$$

Thus

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{\Gamma_n}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma$$

and the proof of Step 1 is complete.

To prove Step 2, let $f \in L_a^2(\Omega)$, let $z \in G^-$, and let n be a positive integer. Note that if $t \in (r_n, x_n - x_{n+1} - r_{n+1})$, then

$$\begin{aligned}\left|\int_{\Gamma_n}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma\right| &= \left|\int_0^{2\pi} f(x_n + te^{i\theta})te^{i\theta}[x_n + te^{i\theta} - z]^{-1}\,d\theta\right| \\ &\le \int_0^{2\pi}\left|f(x_n + te^{i\theta})\right|\left|x_n + te^{i\theta} - z\right|^{-1} t\,d\theta \\ &\le \frac{1}{x_{n+1}}\int_0^{2\pi}|f(x_n + te^{i\theta})|t\,d\theta \quad (\text{since } \operatorname{Re} z \le 0).\end{aligned}$$

Let $a = r_n$ and $b = x_n - x_{n+1} - r_{n+1}$. Multiplying both sides of the above inequality by $1/t$ and integrating with respect to t (from a to b), we get

$$\begin{aligned}\left|\int_{\Gamma_n}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma\right|\log\left(\frac{b}{a}\right) &\le \frac{1}{x_{n+1}}\int_0^{2\pi}\int_a^b\left|f(x_n + te^{i\theta})\right|\left(\frac{1}{t}\right)t\,dt\,d\theta \\ &\le \frac{1}{x_{n+1}}\left[\int_0^{2\pi}\int_a^b|f(x_n + te^{i\theta})|^2 t\,dt\,d\theta\right]^{1/2}\left[2\pi\int_a^b\left(\frac{1}{t^2}\right)t\,dt\right]^{1/2} \\ &\le (2\pi)^{1/2}\left(\frac{1}{x_{n+1}}\right)\|f\|_{\Omega,2}\left[\log\left(\frac{b}{a}\right)\right]^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\int_{\Gamma_n}\frac{f(\varsigma)}{\varsigma - z}\,d\varsigma\right| &\le (2\pi)^{1/2}\left(\frac{1}{x_{n+1}}\right)\|f\|_{\Omega,2}\left[\log\left(\frac{b}{a}\right)\right]^{-1/2} \\ &= (2\pi)^{1/2}e^{n+1}\|f\|_{\Omega,2}[n^4e^{2n} + \log(1 - e^{-1} - e^{-1}e^{-(n+1)^4\exp(2n+2)})]^{-1/2} \\ &\le cn^{-2}\|f\|_{\Omega,2}.\end{aligned}$$

Thus Step 2 has been verified, completing the proof of Theorem 7.

Domains in $\mathbb{C}^N$. We now turn to the situation in several complex variables. Fix a positive integer N, and let Ω be a bounded domain in $\mathbb{C}^N$.

A subspace E of $L_a^p(\Omega)$ is said to be invariant under multiplication by the coordinate functions if z_jE is contained in E for each j from 1 to N. We are interested in describing those closed subspaces E of $L_a^p(\Omega)$ that are invariant under multiplication by the coordinate functions and have finite codimension in $L_a^p(\Omega)$. The techniques used in the previous section do not work if $N > 1$.

Rather than strive for the greatest generality, we want to concentrate on illustrating the ideas, so from now on we will assume that the polynomials are dense in $L_a^p(\Omega)$ and that whenever λ is a point in $\mathbf{C}^N\setminus\Omega$, then there is a polynomial g such that $g(\lambda) = 1$ and $|g(z)| < 1$ for all z in Ω. For example, every ball and every polydisk satisfies these requirements on Ω.

If f is an analytic function on Ω, then by a partial derivative of f of order k we mean a function on Ω of the form $\partial^k f/\partial z_1^{k_1}\cdots\partial z_N^{k_N}$, where $k_1,\ldots,k_N$ are nonnegative integers and $k = k_1 + \cdots + k_N$. If $k = 0$, then the above expression should be interpreted to denote just f.

Let λ be a point in Ω. Then

$$\left\{f \in L_a^p\colon f(\lambda) = \frac{\partial f}{\partial z_1}(\lambda) = \frac{\partial f}{\partial z_2}(\lambda) = \frac{\partial^2 f}{\partial z_1^2}(\lambda) = 0\right\}$$

is a finite codimensional subspace of $L_a^p(\Omega)$ that is invariant under multiplication by the coordinate functions. A more complicated example is the finite codimensional invariant subspace

$$\left\{f \in L_a^p(\Omega)\colon f(\lambda) = \frac{\partial f}{\partial z_1}(\lambda) + \frac{\partial f}{\partial z_2}(\lambda) = 2\frac{\partial^2 f}{\partial z_1\partial z_2}(\lambda) + \frac{\partial^2 f}{\partial z_1^2}(\lambda) + \frac{\partial^2 f}{\partial z_2^2}(\lambda) = 0\right\}.$$

A linear partial differential operator with constant coefficients is a map L that takes each analytic function f on Ω to a linear combination of partial derivatives of f. The order of L is defined to be the highest order of any of the partial derivatives that occur in the expression defining L. We can now state the main result of this section.

THEOREM 8. *If E is a closed subspace of $L_a^p(\Omega)$ with finite codimension m and E is invariant under multiplication by the coordinate functions, then there are points $\lambda^{(1)},\ldots,\lambda^{(m)}$ in Ω and linear partial differential operators with constant coefficients $L_1,\ldots,L_m$ such that $E = \{f \in L_a^p(\Omega)\colon (L_j f)(\lambda^{(j)}) = 0$ for $j = 1,\ldots,m\}$.*

The points $\lambda^{(1)},\ldots,\lambda^{(m)}$ that occur in the statement of Theorem 8 are not necessarily distinct.

The techniques used below to prove Theorem 8 work in many contexts. For example, the proof carries over to give a classification of the closed finite codimensional subspaces of the Hardy space of the ball (or polydisk) that are invariant under multiplication by the coordinate functions. The finite codimensional invariant subspaces of the Hardy space on the polydisk are described in [**2**, Theorem 3]. Further information about finite codimensional invariant subspaces of the Hardy space on the polydisk can be found in [**1**].

To prove Theorem 8 we will use the dual of $L_a^p(\Omega)$, which is denoted $L_a^p(\Omega)^*$. Thus $L_a^p(\Omega)^*$ is the vector space of all continuous linear maps from $L_a^p(\Omega)$ to $\mathbf{C}$. For E a closed subspace of $L_a^p(\Omega)$, the annihilator of E, denoted $E^\perp$, is defined by

$$E^\perp = \{\varphi \in L_a^p(\Omega)^*\colon \varphi \mid E = 0\}.$$

The dual of $L_a^p(\Omega)/E$ can be canonically identified with $E^\perp$. So if E has finite codimension in $L_a^p(\Omega)$, then $E^\perp$ is a finite dimensional space and $\dim L_a^p(\Omega)/E$

equals $\dim E^{\perp}$. The Hahn-Banach Theorem implies that

$$E = \{f \in L^p_a(\Omega) \colon \varphi(f) = 0 \text{ for all } \varphi \in E^{\perp}\}.$$

Let E be a closed subspace of $L^p_a(\Omega)$ that is invariant under multiplication by the coordinate functions. For $j = 1, \dots, N$, define maps $M_j \colon E^{\perp} \to E^{\perp}$ by

$$(M_j\varphi)(f) = \varphi(z_j f).$$

The invariance of E under multiplication by z_j ensures that M_j maps $E^{\perp}$ to $E^{\perp}$. It is easy to check that $\{M_1, \dots, M_N\}$ is a commuting family of operators on $E^{\perp}$.

Let φ be a nonzero element of $E^{\perp}$ and let $\lambda = (\lambda_1, \dots, \lambda_N)$ be a point in $\mathbf{C}^N$. We say that φ is an eigenvector for $(M_1, \dots, M_N)$ with eigenvalue λ if

$$M_j\varphi = \lambda_j\varphi \quad \text{for each } j = 1, \dots, N.$$

In other words, φ must be a simultaneous eigenvector for $M_1, \dots, M_N$.

The following lemma will be used in the proof of Theorem 8.

LEMMA 9. *Suppose that φ is an eigenvector for $(M_1, \dots, M_N)$ with eigenvalue λ. Then λ is in Ω and there is a nonzero complex constant c such that*

$$\varphi(f) = cf(\lambda) \quad \textit{for all } f \textit{ in } L^p_a(\Omega).$$

PROOF. The hypothesis implies that

$$\varphi((z_j - \lambda_j)f) = 0 \quad \text{for every } f \text{ in } L^p_a(\Omega).$$

Now suppose that f is a polynomial. We can write

$$f = f(\lambda) + \sum_{j=1}^{N}(z_j - \lambda_j)f_j,$$

where each f_j is also a polynomial. Apply φ to both sides of the above equation, and using the equation of the previous paragraph, we obtain $\varphi(f) = f(\lambda)\varphi(1)$.

Suppose that λ were not in Ω. By our assumption on Ω, there is a polynomial g such that $g(\lambda) = 1$ and $|g(z)| < 1$ for all z in Ω. Thus $g^m \to 0$ in $L^p_a(\Omega)$ as $m \to \infty$. Thus $\varphi(g^m) \to 0$ as $m \to \infty$. But

$$\varphi(g^m) = g^m(\lambda)\varphi(1) = \varphi(1),$$

so we must have that $\varphi(1) = 0$. For every polynomial f we would then have $\varphi(f) = f(\lambda)\varphi(1) = 0$, so $\varphi = 0$. However, the definition of eigenvector excluded 0, so we have reached a contradiction. We conclude that λ is in Ω.

Let the constant c equal $\varphi(1)$, so that $\varphi(f) = cf(\lambda)$ for every polynomial f. Since the polynomials are dense in $L^p_a(\Omega)$, and since point evaluation at a point in Ω is a continuous linear functional on $L^p_a(\Omega)$, we conclude that $\varphi(f) = cf(\lambda)$ for every f in $L^p_a(\Omega)$, completing the proof. □

PROOF OF THEOREM 8. The proof is by induction on m, the codimension of E in $L^p_a(\Omega)$. As is often the case with proofs by induction, it is easier to prove a slightly stronger statement than what we need. Thus what we actually prove is Theorem 8 along with the added statement that each of the points $\lambda^{(1)}, \dots, \lambda^{(m)}$ is an eigenvalue for $(M_1, \dots, M_N)$.

Begin by considering the case where $m = 1$. In this case, $E^{\perp}$ has dimension one, and so any nonzero element φ of $E^{\perp}$ is an eigenvector for $(M_1, \dots, M_N)$. Let

$\lambda \in \mathbb{C}^N$ be the eigenvalue corresponding to the eigenvector φ. Since $\{\varphi\}$ spans $E^\perp$, we know that

$$E = \{f \in L^p_a(\Omega) \colon \varphi(f) = 0\}.$$

Lemma 9 now implies that $\lambda \in \Omega$ and

$$E = \{f \in L^p_a(\Omega) \colon f(\lambda) = 0\},$$

completing the proof for the case $m = 1$.

Now let m be an integer larger than 1, and assume that Theorem 8 has been proved for subspaces of $L^p_a(\Omega)$ whose codimension in $L^p_a(\Omega)$ is less than m. The proof will be broken into two cases.

For the first case, assume that there exists $j \in \{1, \dots, N\}$ such that M_j has at least two distinct eigenvalues. Let α be an eigenvalue of M_j, and define subspaces F_1 and F_2 of $E^\perp$ by

$$F_1 = \text{kernel}(M_j - \alpha)^m, \quad F_2 = \text{span} \bigcup_{\beta \neq \alpha} \text{kernel}(M_j - \beta)^m.$$

By writing M_j in its Jordan canonical form, it is easy to see that $F_1 + F_2 = E^\perp$ and $F_1 \cap F_2 = \{0\}$. Our assumption that M_j has at least two distinct eigenvalues implies that neither F_1 nor F_2 equals $\{0\}$.

Define closed subspaces E_1 and E_2 of $L^p_a(\Omega)$ by letting

$$E_i = \{f \in L^p_a(\Omega) \colon \varphi(f) = 0 \text{ for all } \varphi \in F_i\}.$$

The properties of F_1 and F_2 imply that $E = E_1 \cap E_2$, that E_1 and E_2 both have codimension in $L^p_a(\Omega)$ less than m, and that the sum of the codimensions of E_1 and E_2 equals m.

It is easy to see that if $\mathcal{S}$ is any commuting family of operators on a Banach space, then the kernel of any operator in $\mathcal{S}$ is invariant under all operators in $\mathcal{S}$. Since $\{M_1, \dots, M_N, (M_j - \alpha)^m\}$ is a commuting family of operators, it follows that F_1 is invariant under M_k for $k = 1, \dots, N$. Similarly, F_2 is invariant under each of the M_k.

Suppose that i equals 1 or 2, that $f \in E_i$, that $k \in \{1, \dots, N\}$, and that $\varphi \in F_i$. Then

$$\varphi(z_k f) = (M_k \varphi)(f) = 0,$$

where the last equality holds because $M_k \varphi \in F_i$. We conclude that the spaces E_1 and E_2 are invariant under multiplication by the coordinate functions.

By the induction hypothesis, E_1 and E_2 are each determined by a finite number of points in Ω ($\dim L^p_a(\Omega)/E_j$ points for E_j) at which certain linear partial differential operators with constant coefficients vanish. Since $E = E_1 \cap E_2$ and since the sum of the codimensions of E_1 and E_2 equals m, we get the conclusion of Theorem 8. For i equal to 1 or 2, it is easy to verify that if $\varphi \in E_i^\perp$ is an eigenvector with eigenvalue $\lambda \in \mathbb{C}^N$ for the M_j's associated with E_i, then φ is an eigenvector with eigenvalue λ for the M_j's associated with E. Thus our proof is finished in this case.

For the remaining case, we can assume that for each $j \in \{1, \dots, N\}$, the finite dimensional operator M_j has only one eigenvalue λ_j. Every commuting family of operators on a finite dimensional vector space can be simultaneously put in upper triangular form. Thus there is a basis $\{\varphi_1, \dots, \varphi_m\}$ of $E^\perp$ such that for each j, the matrix for M_j is upper triangular with respect to the basis $\{\varphi_1, \dots, \varphi_m\}$ and the main diagonal contains only λ_j's.

Let

$$E_1 = \{f \in L^p_a(\Omega) \colon \varphi_k(f) = 0 \text{ for } k = 1, \dots, m-1\}.$$

Suppose that $f \in E_1$, that $j \in \{1, \dots, N\}$, and that $k \in \{1, \dots, m-1\}$. Then

$$\varphi_k(z_j f) = (M_j \varphi_k)(f) = 0,$$

where the last equality holds because $M_j \varphi_k$ is a linear combination of $\{\varphi_1, \dots, \varphi_{m-1}\}$ (by the upper triangular form of M_j). Thus E_1 is invariant under multiplication by the coordinate functions.

By the induction hypothesis (along with the observation that every eigenvalue for the M_j's associated with E_1 is an eigenvalue for the M_j's associated with E), we can conclude that each φ_j (for $j = 1, \dots, m-1$) is a linear combination of partial derivatives evaluated at $\lambda = (\lambda_1, \dots, \lambda_N)$. We still must prove that φ_m is of the same form.

If $j \in \{1, \dots, N\}$, then by the upper triangular form of M_j with respect to the basis $\{\varphi_1, \dots, \varphi_m\}$ we have

$$M_j \varphi_m = \lambda_j \varphi_m + \sum_{k=1}^{m-1} a_{jk} \varphi_k,$$

where the a_{jk} are suitable complex constants. Applying an arbitrary $f \in L^p_a(\Omega)$ to both sides of the above equation, we get

$$\varphi_m((z_j - \lambda_j) f) = \sum_{k=1}^{m-1} a_{jk} \varphi_k(f).$$

Since each φ_k (for $k = 1, \dots, m-1$) is a linear combination of partial derivatives evaluated at λ, there is an integer d such that

$$\varphi_k((z_1 - \lambda_1)^{d_1} \cdots (z_N - \lambda_N)^{d_N}) = 0 \quad \text{for } k = 1, \dots, m-1$$

whenever some $d_j \geq d$. Combining the last two equations, we see that

$$\varphi_m((z_1 - \lambda_1)^{d_1} \cdots (z_N - \lambda_N)^{d_N}) = 0$$

whenever some $d_j \geq d+1$. Thus, by expanding a polynomial f into its Taylor series about the point λ, it follows that $\varphi_m(f)$ depends only upon the Taylor coefficients of the terms of degree less than $N(d+1)$. Note that each of these Taylor coefficients is equal to a partial derivative of f evaluated at λ. Since φ_m is linear and since the polynomials are dense in $L^p_a(\Omega)$, we conclude that φ_m equals a linear combination of partial derivatives evaluated at λ, completing the proof. □

A careful analysis of the proof can provide more information than we have explicitly stated. For example, we can take L_j to have order at most $j-1$.

Questions. We conclude by raising a few questions suggested by the results in the paper.

Theorem 7 shows that there are bounded domains $\Omega \subset \mathbb{C}$ and closed finite codimensional invariant subspaces E of $L^2_a(\Omega)$ such that E cannot be written in the form $qL^2_a(\Omega)$ for any $q \in H^\infty(\Omega)$. However, is every closed finite codimensional invariant subspace E singly generated in the sense that there exists a function

$q \in H^\infty(\Omega)$ such that E equals the closure of $qL_a^2(\Omega)$? Is the specific invariant subspace E constructed in Theorem 7 singly generated in this sense? Is the specific invariant subspace E constructed in Theorem 7 equal to the closure of $zL_a^2(\Omega)$? Note that by Corollary 6 of [**4**], $zL_a^2(\Omega)$ is not closed in $L_a^2(\Omega)$ for the domain Ω constructed in Theorem 8.

Theorems 11 and 13 of [**6**] show that it is unlikely that there is a nice description of all the invariant subspaces of any Bergman space. Is it possible to describe those invariant subspaces E of $L_a^p(\Omega)$ such that zE has codimension one in E?

What is the correct converse to Theorem 8? More precisely, if Ω is a bounded domain in $\mathbf{C}^N$ and $\lambda^{(1)}, \ldots, \lambda^{(m)}$ are points in Ω and $L_1, \ldots, L_m$ are linear partial differential operators with constant coefficients and

$$E = \{f \in L_a^p(\Omega) \colon (L_j f)(\lambda^{(j)}) = 0 \text{ for } j = 1, \ldots, m\},$$

what conditions on $L_1, \ldots, L_m$ and $\lambda^{(1)}, \ldots, \lambda^{(m)}$ imply that E is invariant under multiplication by the coordinate functions? This question, which was asked in the preprint version of this paper, has now been answered by Hari Bercovici [**5**].

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