CHARACTERISTIC MULTIPLIERS AND STABILITY 
OF SYMMETRIC PERIODIC SOLUTIONS OF \( \dot{x}(t) = g(x(t-1)) \)

SHUI-NEE CHOW AND HANS-OTTO WALThER

ABSTRACT. We study the scalar delay differential equation \( \dot{x}(t) = g(x(t-1)) \) with negative feedback. We assume that the nonlinear function \( g \) is odd and monotone. We prove that periodic solutions \( x(t) \) of slowly oscillating type satisfying the symmetry condition \( x(t) = -x(t-2), \ t \in \mathbb{R}, \) are nondegenerate and have all nontrivial Floquet multipliers strictly inside the unit circle. This says that the periodic orbit \( \{x_t: t \in \mathbb{R}\} \) in the phase space \( C[-1,0] \) is orbitally exponentially asymptotically stable.

1. Introduction. Let a continuous function \( g: \mathbb{R} \to \mathbb{R} \) be given with
\[ \xi g(\xi) < 0 \quad \text{for} \ \xi \neq 0. \]

In the dynamics of the equation
\[
(g) \quad \dot{x}(t) = g(x(t-1))
\]
with delayed negative feedback, periodic solutions of slowly oscillating type, i.e. solutions with zeros spaced at distances larger than the delay time \( t = 1 \), play an important role. It is very likely that any other periodic solution is necessarily unstable, see for example [7, 9].

Existence and properties of periodic solutions of slowly oscillating type depend on the graph of \( g \). One may have uniqueness and stability, or nonuniqueness [8]. In parametrized problems, bifurcation within one set of such periodic solutions exists [10].

The semiflow of equation \( (g) \) close to a periodic solution is determined by the characteristic (Floquet) multipliers [4, Chapter 10]. These multipliers are not always out of reach. They were computed in [10] for equation \( (g) \) with some additional hypotheses on \( g \), and for a nonlinear integral equation with delay in [2].

In the present paper we consider a class of odd monotone functions \( g \), and we prove that periodic solutions \( x \) of slowly oscillating type satisfying the symmetry condition:
\[
(s) \quad x(t) = -x(t-2), \quad t \in \mathbb{R},
\]
are nondegenerate, and have all nontrivial multipliers strictly inside the unit circle (Theorem 2, §6). This implies that the orbit of \( x \) is exponentially asymptotically stable with asymptotic phase (Corollary 2, §6).

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The organization of the paper is as follows. §§2–5 deal with a linear equation
(b) \( \dot{x}(t) = b(t)x(t - 1) \),
where \( b < 0 \). In applications, (b) is the linear variational equation
\( \dot{x}(t) = g'(p(t - 1))x(t - 1) \)
along a periodic solution \( p \) of equation (g) with \( g' \leq 0 \). Proceeding as in [2], we
establish relations between characteristic multipliers and slowly oscillating solutions
(§§2–4). In particular, there is a sharp restriction on multiplicities of multipliers.
§5 contains a characterization of multipliers by zeros of an analytic function \( q \). A
crucial hypothesis for this to hold true is that \( b \) has integer period \( r = 2 \). \( q \) can be
computed from a system of ordinary differential equations.

§6 starts with some facts about periodic solutions of equation (g) satisfying the
symmetry condition (s). We state Theorem 2 and reduce its proof to an investiga-
tion of real multipliers. The last section examines real zeros of the function \( q \)
associated with \( b \), and the characterization from §5 completes the proof of Theorem
2.

2. Slowly oscillating solutions of a linear nonautonomous differential
delay equation. For a given continuous function \( b: \mathbb{R} \to \mathbb{R} \) satisfying \( b(t) < 0 \)
for all \( t \), we consider solutions \( x: \mathbb{R} \to \mathbb{C} \) or \( x: [-1, \infty) \to \mathbb{C} \) of equation (b). For
every initial value \( \phi \) in the Banach space \( C_C = C([-1,0], \mathbb{C}) \), with sup norm
\[
|\phi| = \sup_{s \in [-1,0]} |\phi(s)|,
\]
(b) defines a unique solution \( x = x^\phi \) on the interval \([-1, \infty) \), i.e. a continuous
function \( x \) which is differentiable for \( t > 0 \) and satisfies (b), and \( x|[-1,0] = \phi \).
Furthermore, \( \phi([-1,0]) \subset \mathbb{R} \) implies \( x^\phi([-1, \infty)) \subset \mathbb{R} \).

DEFINITION 1. A differentiable function \( x: \mathbb{R} \to \mathbb{R} \) is called slowly oscillating
at \( t \) if either \( |x| > 0 \) on \([t - 1, t]\), or \( x \) has precisely one zero \( z \in [t - 1, t] \), and
\( \dot{x}(z) \neq 0 \). \( x \) is called slowly oscillating if \( x \) is slowly oscillating at every \( t \in \mathbb{R} \).

Note that the set \( \{ t \in \mathbb{R}: x \text{ is slowly oscillating at } t \} \) is open for every differen-
tiable function \( x: \mathbb{R} \to \mathbb{R} \).

LEMMA 1. A solution \( x: \mathbb{R} \to \mathbb{R} \) of equation (b) which is slowly oscillating at
some \( t \in \mathbb{R} \) is slowly oscillating at every \( s \geq t \).

PROOF. Suppose \( x \) is slowly oscillating at \( t \in \mathbb{R} \), and there exists \( s_1 > t \) such
that \( x \) is not slowly oscillating at \( s_1 \). The nonempty set \( A_t = \{ s \geq t: x \text{ is not }
slowly oscillating at } s \} \) is closed and contained in the open interval \((t, \infty) \). Note
that \( x \) is not slowly oscillating at \( s_0 = \inf \{ s: s \in A_t \} \). Let \( 0 < \varepsilon < s_0 - t \). We have

\( (*) \quad x \text{ is slowly oscillating at every } s \in [s_0 - \varepsilon, s_0) \).

It follows that \( x(s_0) = 0 \): Otherwise, \( |x| > 0 \) on \([s_0 - \varepsilon, s_0 + \varepsilon]\) for some \( \varepsilon > 0 \) with
\( \varepsilon < s_0 - t \), and \( x \) is slowly oscillating at \( s_0 - \varepsilon \), by \( (*) \). This implies that \( x \) is slowly
oscillating at every \( s \in [s_0 - \varepsilon, s_0 + \varepsilon] \) and is a contradiction to the definition of \( s_0 \).
We infer \( \dot{x}(s_0) = 0 \): Suppose \( \dot{x}(s_0) \neq 0 \). \( x \) is not slowly oscillating at \( x_0 \) so that
there is another zero \( z \in [s_0 - 1, s_0) \). It follows from \( (*) \) that \( \dot{x}(z) \neq 0 \) and \( |x| > 0 \)
on \((z, s_0) \). Therefore, \( \text{sign} \ \dot{x}(z) = -\text{sign} \ \dot{x}(s_0) \). By (b), \( \text{sign} x(s_0 - 1) = -\text{sign} \ \dot{x}(s_0) \).
Consequently, there must be a third zero \( z_0 \in (s_0 - 1, z) \). This contradicts (*)..

\( \dot{x}(s_0) = 0 \) gives \( x(s_0 - 1) = 0 \), by equation (b). By (*), \( \dot{x}(s_0 - 1) \neq 0 \). Choose \( \varepsilon > 0 \) so small that \( \text{sign} \, x = -\text{sign} \, \dot{x}(s_0 - 1) \) in \( (s_0 - 1 - \varepsilon, s_0 - 1) \), and \( 0 < \varepsilon < s_0 - t \). Then

\[
0 - x(s_0 - \varepsilon) = \int_{s_0 - \varepsilon}^{s_0} \dot{x}(s) \, ds = \int_{s_0 - \varepsilon}^{s_0} b(s)x(s - 1) \, ds,
\]

and

\[
\text{sign} \, x(s_0 - \varepsilon) = -\text{sign} \, \int_{s_0 - \varepsilon}^{s_0} b(s)x(s - 1) \, ds = -\text{sign} \, \dot{x}(s_0 - 1).
\]

Hence there is a third zero \( z \in (s_0 - 1, s_0 - \varepsilon) \), a final contradiction to (*), and this completes the proof.

Let \( \Sigma \) denote the set of all slowly oscillating solutions of equation (b), and \( X \) the space of continuous functions \( \mathbb{R} \to \mathbb{R} \), equipped with the topology of uniform convergence on compact sets.

**Lemma 2.**

\[
\text{cl} \, \Sigma \subset \Sigma \cup \{0\}.
\]

**Proof.** Consider \( x \in \text{cl} \, \Sigma, x \neq 0 \), and a sequence of solutions \( x^n \in \Sigma \) which converges to \( x \). It is easily seen that \( x \) is a solution of equation (b). We have \( x(t) \neq 0 \) for some \( t \in \mathbb{R} \). Equation (b) implies that there is no \( s \leq t \) with \( x = 0 \) on \( [s - 1, s] \). In view of Lemma 1, it remains to show that for every \( t_0 < t \) there exists \( t_1 \leq t_0 \) such that \( x \) is slowly oscillating at \( t_1 \). Let \( t_0 < t \) be given. If \( |x| > 0 \) on \( (-\infty, t_0] \) then \( x \) is slowly oscillating at \( t_1 = t_0 \).

If \( x \) has a zero \( z \in (-\infty, t_0] \), then consider the maximal interval \( I \) with \( \text{Sup} \, I = z \) and \( x = 0 \) on \( I \). Note that \( I \) is compact. Define \( t_1 = \min I > z - 1 \). Choose a sequence \( \tau_\nu \to t_1 \) with \( x(\tau_\nu) \neq 0 \) and \( \tau_\nu < t_1 \) for all \( \nu \). We show \( |x| > 0 \) on \( [t_1 - 1, t_1] \): Suppose \( x(z_1) = 0 \) where \( t_1 - 1 \leq z_1 < t_1 \). Hence \( z_1 < \tau_\nu < t_1 \) for \( \nu \) sufficiently large. This implies that there exist \( s_1 \) and \( s_2 \) with \( z_1 < s_1 < \tau_\nu < s_2 < t_1 \), sign \( \dot{x}(s_1) = -\text{sign} \, \dot{x}(s_2) \neq 0 \), sign \( x(s_1) = \text{sign} \, \dot{x}(s_1) \), and sign \( x(s_2) = -\text{sign} \, \dot{x}(s_2) \).

Equation (b) gives sign \( x(s_1 - 1) = -\text{sign} \, \dot{x}(s_1) \), sign \( x(s_2 - 1) = -\text{sign} \, \dot{x}(s_2) \). Together, \( 0 \neq \text{sign} \, x(s_1 - 1) = -\text{sign} \, \dot{x}(s_1) \), sign \( x(s_2 - 1) = -\text{sign} \, \dot{x}(s_2) \), sign \( x(z_1) = 0 \), \( s_1 - 1 < s_2 - 1 < z_1 < s_1 \). It follows that there are points \( s_3, s_4, s_5 \) with \( s_1 - 1 < s_3 < s_4 < z_1 < s_5 < s_1 \) and \( 0 \neq \text{sign} \, x(s_3) = -\text{sign} \, \dot{x}(s_3) = \text{sign} \, \dot{x}(s_5) \). Equation (b) gives

\[
0 \neq \text{sign} \, x(s_3 - 1) = -\text{sign} \, \dot{x}(s_4 - 1) = \text{sign} \, x(s_5 - 1).
\]

We obtain the same relations for \( x^n \) for \( n \) sufficiently large. This contradicts \( x^n \in \Sigma \). Finally, by \( x(t_1 - 1) \neq 0 \) and (b), we have \( \dot{x}(t_1) \neq 0 \), and \( x \) is slowly oscillating at \( t_1 \) (and \( I = \{t_1\} = \{z\} \)).

**Lemma 3.** For every linear space \( L \subset \Sigma \cup \{0\} \), \( \dim L \leq 2 \).

**Proof.** If there are linearly independent slowly oscillating solutions \( x^1, x^2, x^3 \) in \( L \), then choose \( a_1, a_2, a_3 \in \mathbb{R} \) with \( |a_1| + |a_2| + |a_3| > 0 \), \( a_1 x^1(0) + a_2 x^2(0) + a_3 x^3(0) = 0 \), \( a_1 x^1(-1) + a_2 x^2(-1) + a_3 x^3(-1) = 0 \). The nontrivial solution \( x = a_1 x^1 + a_2 x^2 + a_3 x^3 \in L \) is not slowly oscillating.
3. Periodic equations: solutions associated with characteristic multipliers. From now on we consider equation (b) for a periodic continuous function \( b \), with period \( r > 1 \). Characteristic multipliers (of \( b \) and \( r \)) are defined to be nonzero points \( \mu \) in the spectrum \( \sigma \) of the monodromy operator \( U = T(\tau, 0) : C_C \to C_C \), where \( T(t, 0)\phi = x^\phi_t \), \( x^\phi_t(s) = x^\phi(t + s) \) for all \( t \geq 0 \) and \( s \in [-1, 0] \). It is known that \( U \) is completely continuous \( [4, \text{Chapter 8}] \) and real, i.e. \( UC_R \subset C_R \) for \( C_R = C([-1,0], R) \). Hence, characteristic multipliers are either real or complex conjugate pairs. Each \( \mu \in \sigma \setminus \{0\} \) is an isolated point, and is an eigenvalue of \( U \) with finite algebraic multiplicity

\[
m(\mu) = \dim \bigcup_{l \in \mathbb{N}} \ker(U - \mu)^l.
\]

Let \( E_\mu \) be the geometric eigenspace \( \ker(U - \mu) \), \( d_\mu \) denote the dimension of \( E_\mu \), \( \kappa_\mu \) be the stabilizing exponent, i.e. the minimal integer \( \kappa \) with \( \ker(U - \mu)^\kappa = \ker(U - \mu)^{\kappa + 1} \), and \( G_\mu \) be the generalized eigenspace

\[
\ker(U - \mu)^{\kappa_\mu} = \bigcup_{l \in \mathbb{N}} \ker(U - \mu)^l.
\]

The index \( \mu \) will be omitted whenever possible in the following. For \( 0 \neq \mu \in \mathbb{C} \setminus \sigma \), we set \( m(\mu) = 0 = d_\mu \). We are interested in real-valued solutions \( x \) which pass through real initial values in \( E + E_\mu, G + G_\mu \), at \( t = 0 \). The properties of such solutions become rather obvious from the construction \([4, \text{Chapter 8}]\) of complex-valued solutions on \( R \) with initial value in \( E \) or \( G \). For the reader’s convenience, we briefly recall a few facts of this construction.

One starts with a basis \( \phi_1, \ldots, \phi_m \) of \( G \) such that \( \phi_1, \ldots, \phi_d \) are a basis of \( E \). Define the square matrix \( M \) by

\[
(U\phi_1, \ldots, U\phi_m) = (\phi_1, \ldots, \phi_m) \cdot M.
\]

Let \( I \) denote the unit matrix, with columns \( e^1, \ldots, e^m \in \mathbb{C}^d \), and set \( N_M = M - \mu I \). \( N_M \) is nilpotent with \( N_M^{\kappa_\mu} = 0 \neq N_M^{\kappa_\mu - 1} \). The only eigenvalue of \( M \) is \( \mu \). The first \( d \) unit vectors \( e^1, \ldots, e^d \) span the space of eigenvectors of \( M \).

Choose \( \lambda \in \mathbb{C} \) with \( e^{r\lambda} = \mu \). Note that \( \text{Re} \lambda = (\log |\mu|)/r \) is uniquely determined by \( \mu \). Set

\[
B = \text{diag}(\lambda) + N/r
\]

where

\[
N = \log \frac{1}{\mu} N_M = \sum_{l=1}^{\kappa_\mu} \frac{(-1)^{l+1}}{l!} \left( \frac{1}{\mu} N_M \right)^l.
\]

It follows that \( M = e^{rB} = e^{r\lambda}e^N, N^{\kappa_\mu} = 0 \neq N^{\kappa_\mu - 1} \). The only eigenvalue of \( B \) is \( \lambda \), and the spaces of eigenvectors of \( M \) and \( B \) coincide.

For \( t \geq 0 \), define

\[
P_t = T(t, 0)(\phi_1, \ldots, \phi_m)e^{-tB}
\]

which is a row vector with components in \( C_C \). Extend \( P \) to a \( \tau \)-periodic map on \( R \) and set \( p(t) = P_t(0) \), for all \( t \in R \). \( P \) is a continuous \( \tau \)-periodic map from \( R \) into the space of row vectors with complex components. For \( c \in \mathbb{C}^m \), \( x^c(t) = p(t)e^{tB}c \) defines a solution \( x^c : R \to C \) of equation (b) with \( x^c_0 = (\phi_1, \ldots, \phi_m) \cdot c \).
Let $Y$ denote the complex vector space of continuous functions $\mathbb{R} \to \mathbb{C}$, with the topology of uniform convergence on compact sets.

Consider the subspaces

$$
\mathcal{F}_c = \mathcal{F}_c(\mu) = \{x^c \in Y : c \in \mathbb{C}^m\}
$$
and

$$
\mathcal{E}_c = \mathcal{E}_c(\mu) = \{x^c \in Y : c_{d+1} = \cdots = c_m = 0\} = \{x^c \in Y : x_0^c \in E\}.
$$

Bases of $\mathcal{F}_C$ and $\mathcal{E}_C$ are given by $x_0^c \in \{\phi_1, \ldots, \phi_m\}$ and $c \in \{e^1, \ldots, e^d\}$, respectively; and $\dim \mathcal{F}_C = m$, $\dim \mathcal{E}_C = d$. For $\mu \neq \mu^1$, $\mathcal{F}_C(\mu) \cap \mathcal{F}_C(\mu^1) = \{0\}$. A function $x^c$ is in $\mathcal{F}_C$ if there exists $c \in \mathbb{C}^m$ with $c_{d+1} = \cdots = c_m = 0$ such that $x^c(t) = e^{\lambda t} \cdot p(t) \cdot c$ for all $t \in \mathbb{R}$. For such $x = x^c \in \mathcal{F}_C$, clearly

$$
(1) \quad x(t) = e^{\beta t} f(t) \quad \text{for all } t \in \mathbb{R}
$$
with $\beta = (\log |\mu|)/\tau$ and $f : t \to e^{i \cdot \text{Im} \lambda t} \cdot p(t) \cdot c$. Since $f$ is a finite sum of products of periodic functions, $f$ is almost periodic [1].

If $x = x^c \in \mathcal{F}_C$, then

$$
(2) \quad x(t) = e^{\beta t} \sum_{l=0}^{\kappa-1} f_l(t) t^l \quad \text{for all } t \in \mathbb{R},
$$
with $\beta$ as above and

$$
f_l : t \to p(t) \cdot e^{i \cdot m \cdot \lambda t} \frac{1}{\tau^l \cdot l!} N^l \cdot c
$$
for $l = 0, \ldots, \kappa - 1$. Each $f_l$ is almost periodic, and the coefficient function $\tilde{x} = \tilde{x}^c$, where

$$
\tilde{x}(t) = e^{\beta t} f_{\kappa-1}(t) \quad \text{for all } t \in \mathbb{R},
$$
is contained in $\mathcal{F}_C$ since $N^{\kappa-1}c$ is an eigenvector of $B$ and $M$.

We have $\tilde{x}^c \neq 0$ for $N^{\kappa-1}c \neq 0$. This implies that the set $\{x \in \mathcal{F}_C : x \neq 0\}$ is dense in $\mathcal{F}_C$.

**DEFINITION 2.** (i) Let $\mu \in \sigma \setminus \{0\}$ be given. We set

$$
\mathcal{G}(\mu) = \begin{cases} 
\mathcal{F}_C(\mu) \cap X, & \text{if } \mu \in \mathbb{R}, \\
(\mathcal{F}_C(\mu) + \mathcal{F}_C(\bar{\mu})) \cap X, & \text{if } \text{Im } \mu > 0,
\end{cases}
$$
and

$$
\mathcal{E}(\mu) = \begin{cases} 
\mathcal{F}_C(\mu) \cap X, & \text{if } \mu \in \mathbb{R}, \\
(\mathcal{F}_C(\mu) + \mathcal{F}_C(\bar{\mu}))/\mathcal{E} \cap X, & \text{if } \text{Im } \mu > 0.
\end{cases}
$$

(ii) Let $\rho > 0$ be given. We set

$$
\mathcal{G}_\rho = \begin{cases} 
\bigoplus_{|\mu| = \rho, \mu \in \sigma, \text{Im } \mu \geq 0} \mathcal{G}(\mu) & \text{if } |\mu| = \rho \text{ for some } \mu \in \sigma, \\
\{0\} & \text{if not},
\end{cases}
$$
and

$$
\mathcal{E}_\rho = \begin{cases} 
\bigoplus_{|\mu| = \rho, \mu \in \sigma, \text{Im } \mu \geq 0} \mathcal{E}(\mu) & \text{if } |\mu| = \rho \text{ for some } \mu \in \sigma, \\
\{0\} & \text{if not}.
\end{cases}
$$

(iii) $\mathcal{H}_\rho = \bigoplus_{\rho' \geq \rho} \mathcal{E}_{\rho'}$.

These subspaces of $X$ have the following properties.

$$
(3) \quad \dim \mathcal{G}(\mu) = m(\mu) \text{ for } 0 \neq \mu \in \sigma \cap \mathbb{R},
$$
and

$$
\dim \mathcal{G}(\mu) = 2m(\mu) \text{ for } \mu \in \sigma \text{ and } \text{Im } \mu > 0,
$$
\dim \mathcal{H}_\rho = \sum_{|\mu| \geq \rho} m(\mu) \text{ for } \rho > 0.
(4) For each $x \in \mathcal{G}_\rho$, (1) holds with $\beta = (\log \rho)/\tau$ and $f$ almost periodic.

(5) Let $\rho > 0$ and $\mathcal{G}_\rho \neq \{0\}$. For each $x \in \mathcal{G}_\rho$, (2) holds with

$$
\kappa = \kappa_\rho = \max\{\kappa(\mu) : \mu \in \sigma, |\mu| = \rho\} \geq 1,
\beta = (\log \rho)/\tau,
$$

$f_l \in X$ almost periodic for $l = 0, \ldots, \kappa - 1$; and the function $\tilde{x} : t \rightarrow e^{\beta t} f_{\kappa-1}(t)$ belongs to $\mathcal{G}_\rho$. The set $\{x \in \mathcal{G}_\rho : \tilde{x} \neq 0\}$ is dense in $\mathcal{G}_\rho$.

(6) Let $\rho > 0$. Then $\mathcal{H}_\rho = \mathcal{G}_\rho \oplus \mathcal{H}_\rho'$ for some $\rho' > \rho$.

4. Slowly oscillating solutions and characteristic multipliers. We shall make use of the following property of almost periodic functions $f : \mathbb{R} \rightarrow \mathbb{C}[1]$:

(AP) For every $\varepsilon > 0$ there exists $L > 0$ such that every interval of length $L$ contains $p \in \mathbb{R}$ with

$$
|f(t + p) - f(t)| < \varepsilon \quad \text{for all } t \in \mathbb{R}.
$$

LEMMA 4. Let $\rho > 0$ and $x \in \mathcal{G}_\rho$. If $x$ is slowly oscillating at some $t \in \mathbb{R}$, then $x \in \Sigma$.

PROOF. By Lemma 1, $x$ is slowly oscillating at every $s \geq t$. If $|x| > 0$ on $[t + 1, \infty)$, we set $t_0 = t + 2$. If $x(z) = 0$ for some $z \geq t + 1$, then $|x| > 0$ on $[z - 1, z)$ and $|x| > 0$ on $[t_0 - 1, t_0]$ for some $t_0 < z$ sufficiently close to $z$. In both cases, $0 < |f| < [t_0 - 1, t_0]$ for the almost periodic function $f : s \rightarrow e^{-\beta s} x(s)$ (see (4) in §3). By (AP) we can choose a sequence $(t_n)$, $t_n \rightarrow -\infty$, such that for all $s \in \mathbb{R}$ and $n \geq 0$

$$
|f(s + t_n) - f(s)| < \frac{1}{2} \min\{|f(t_0 + \theta)| : -1 \leq \theta \leq 0\}.
$$

Consider the sequence given by $s_n = t_0 + t_n$. We find $0 < |f(s_n + \theta)|$ for all $\theta \in [-1, 0]$ and $n \geq 0$. Therefore, $x$ is slowly oscillating at every $s_n$. Lemma 1 implies the assertion.

LEMMA 5. $\mathcal{G}_\rho \cap \Sigma \neq \emptyset$ implies $\mathcal{G}_\rho \subset \Sigma \cup \{0\}$.

PROOF. Let $x \in \mathcal{G}_\rho \setminus \{0\}$ be given. There exists $y \in \mathcal{G}_\rho \cap \Sigma$, by hypothesis. Consider the line segment $L$ of points

$$
y^\alpha = ax + (1 - \alpha)y, \quad \alpha \in [0, 1].
$$

Suppose $0 \in L$. Then $x = ((1 - \alpha)/\alpha)y$ for some $\alpha \in (0, 1)$, and $x$ is slowly oscillating. Suppose $0 \notin L$. Set $\alpha_0 = \sup\{\alpha \in [0, 1] : y^\alpha \text{ is slowly oscillating}\}$. If $\alpha_0 = 1$, then there is a sequence $\alpha_n \rightarrow 1$ such that each $y^{\alpha_n}$ is slowly oscillating. By Lemma 2, $x$ is slowly oscillating. If $0 \leq \alpha_0 < 1$, then the same argument shows that $y^{\alpha_0}$ is slowly oscillating. It follows that $|y^{\alpha_0}| > 0$ on some interval $[t - 1, t]$. Hence, $|y^\alpha| > 0$ on $[t - 1, t]$ for all $\alpha$ in a sufficiently small neighborhood of $\alpha_0$. Lemma 4 gives $y^\alpha \in \Sigma$ for these $\alpha$. This contradicts the fact that $\alpha_0 < 1$ is an upper bound for $y^\alpha$ to be slowly oscillating.

LEMMA 6. $\mathcal{G}_\rho \cap \Sigma \neq \emptyset$ implies $\mathcal{G}_\rho \subset \Sigma \cup \{0\}$.

PROOF. Let $y \in \mathcal{G}_\rho \setminus \{0\}$. Because of (5) there is a sequence of functions $x^n \in \mathcal{G}_\rho$ with $x^n \rightarrow y$, $\tilde{x}_n \in \mathcal{G}_\rho \setminus \{0\}$. The hypothesis and Lemma 5 yield $\tilde{x}_n \in \Sigma$ for each $n$. By Lemma 2, it remains to show that $x^n \in \Sigma$. Set $x = x^n$. By (5) and (2), we have

$$
x(t) = e^{\beta t} t^{\kappa - 1} \left\{ f_{\kappa - 1}(t) + \sum_{l=0}^{\kappa-2} t^{l-(\kappa-1)} f_l(t) \right\}
$$
for all \( t \in \mathbb{R} \setminus \{0\} \), with \( \beta \in \mathbb{R} \), \( \kappa \geq 1 \) and almost periodic functions \( f_l \in X \) for \( l = 0, \ldots, \kappa - 1 \). Since \( \tau \in \Sigma \), there exists \( t < 0 \) with \( |\tau| > 0 \) on \([t - 1, t] \). Therefore, \( |f_{\kappa - 1}| > 0 \) on \([t - 1, t]\). Property (AP) permits one to find \( \varepsilon > 0 \) and a sequence \( \{t_\nu\}, t_\nu \to -\infty \), with \( |f_{\kappa - 1}| \geq \varepsilon > 0 \) on every interval \([t_\nu - 1, t_\nu]\) (see the proof of Lemma 4).

Since all functions \( f_l \) are bounded, it follows that for \( \nu \) sufficiently large \( |x| > 0 \) on \([t_\nu - 1, t_\nu]\). Now Lemma 1 shows \( x \in \Sigma \).

**LEMMA 7.** \( \{0\} \neq \mathcal{H}_p \subset \Sigma \cup \{0\} \) implies \( \mathcal{H}_p \subset \Sigma \cup \{0\} \).

**PROOF.** By (6), \( \mathcal{A}_p = \mathcal{A}_p \cup \mathcal{A}_p' \) for some \( p' > p \). Let \( y \in \mathcal{A}_p \setminus \{0\} \). We may assume \( y = y_1 + y_2 \) with \( y_1 \in \mathcal{A}_p \) and \( 0 \neq y_2 \in \mathcal{A}_p' \). (5) shows that there is a sequence of elements \( x^n \in \mathcal{A}_p \) with \( 0 \neq \tau^n \in \mathcal{A}_p \) for all \( n \), and \( x^n \to y_1 \). Note that \( \tau^n \in \mathcal{A}_p \subset \mathcal{A}_p \subset \Sigma \cup \{0\} \) for all \( n \). We now have \( x^n + y_2 \to y \). It is enough to show that each \( x^n + y_2 \) is slowly oscillating (see Lemma 2).

Fix \( n \) and set \( x = x^n \). The definition of \( \mathcal{A}_p' \) and (5) imply that

\[
e^{-\beta t}y^2(t) \to 0 \quad \text{as} \quad t \to -\infty,
\]

for \( \beta = (\log p)/\tau \). Set \( \kappa = \kappa_p \). \( \tau \neq 0 \) and (5) yield

\[
(x + y^2)(t) = e^{\beta t}\kappa^{-1}\left\{f_{\kappa - 1}(t) + \sum_{l=0}^{\kappa-2} t^{\kappa-(\kappa-1)} f_l(t) + t^{-(\kappa-1)}e^{-\beta t}y^2(t)\right\}
\]

for all \( t < 0 \), with \( f_l \in X \) almost periodic for \( l = 0, \ldots, \kappa - 1 \) and \( f_{\kappa - 1} \neq 0 \) (if \( \kappa \geq 2 \)). Since \( \tau : t \to e^{\beta t}f_{\kappa - 1}(t) \) is slowly oscillating, we can proceed as in the proof of Lemma 6 to prove that \( x + y^2 \in \Sigma \).

For the proof of Theorem 2 we need

**COROLLARY 1.** If \( \mu \in \sigma \setminus \{0\} \) and \( \mathcal{E}(\mu) \cap \Sigma \neq \emptyset \), then \( \sum_{|\mu'| \geq |\mu|} m(\mu) \in \{1, 2\} \).

**PROOF.** By Lemmas 5, 6, and 7, \( \mathcal{E}_{|\mu|} \subset \Sigma \cup \{0\} \). We now apply Lemma 3 and (3) to obtain the result.

Another easy consequence of the preceding lemma is the following.

**THEOREM 1.** Let \( \mathcal{H}_\Sigma = \text{span}\{x \in \mathcal{H}_\rho: \rho > 0, \mathcal{H}_\rho \subset \Sigma \cup \{0\}\} \). Then \( \dim \mathcal{H}_\Sigma \leq 2 \).

**5. Characterization of \( \sigma \) for period \( \tau = 2 \).** If \( b \) in equation (b) has integer period \( \tau \), then the multipliers \( \mu \in \sigma \) are given by the zeros of an analytic function. We describe the results for \( \tau = 2 \). Proofs are analogous to those in [10, §3].

Let \( b : \mathbb{R} \to \mathbb{R}^- \) be continuous and periodic with period \( \tau = 2 \). Let \( a(t) = b(t+1) \), \( t \in \mathbb{R} \). For \( \mu \in \mathbb{C} \setminus \{0\} \), define

\[
S^\mu = \begin{pmatrix}
u^\mu_1 & \nu^\mu_2 \\
z^\mu_1 & z^\mu_2
\end{pmatrix}
\]

to be the fundamental matrix solution of the system

\[
(\mu) \quad \dot{u} = b(t)z, \quad \dot{z} = (1/\mu)a(t)u
\]

with

\[
S^\mu(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
It will be convenient to write \( \hat{u}_i^\mu, \hat{z}_i^\mu \) for the restrictions of \( u_i^\mu, z_i^\mu \) \((0 \neq \mu \in \mathbb{C}, i \in \{1, 2\})\) to the interval \([-1,0]\).

Observe that \( \det S^\mu(t) = 1 \) for all \( t \in \mathbb{R} \). Set
\[
Q(\mu) = \begin{pmatrix}
\mu z_1^\mu(-1) - 1 & \mu z_2^\mu(-1) \\
\mu z_1^\mu(-1) & \mu z_2^\mu(-1) - 1
\end{pmatrix}
\]
and
\[
q(\mu) = \det Q(\mu) = 1 - \mu - u_2^\mu(-1) - \mu z_2^\mu(-1).
\]
We note that \( q \) is analytic in \( \mathbb{C}\{0\} \) (see [3, §10.7]). \( U - \mu \) and \( S^\mu \) are related as follows.

**Lemma 8.** Let \( \mu \in \mathbb{C}\{0\} \) be given. There exists a surjective linear operator \( L(\mu): \mathbb{C}_C \rightarrow \mathbb{C}^2 \) such that \( (U - \mu)\chi = \psi \) implies
\begin{equation}
(U\chi \ x_1^\psi)
\end{equation}
for all \( t \in [-1,0] \), with \( c \in \mathbb{C}^2 \) satisfying
\begin{equation}
Q(\mu)c = L(\mu)\psi.
\end{equation}

**Proof.** (a) Set \( x: = -x^\psi \), for the solution of equation (b) with initial value \( \chi \). The functions \( x_2 = U\chi \) and \( x_1 \) satisfy, for all \( t \in [-1,0] \), the differential equations
\[
\begin{align*}
\dot{x}_2(t) &= \dot{x}(2 + t) = b(2 + t)x(2 + t - 1) = b(t)x_1(t), \\
\dot{x}_1(t) &= \dot{x}(1 + t) = b(1 + t)x(1 + t - 1) = a(t)x(t) \\
&= a(t)\left\{\frac{1}{\mu}[x_2(t) - \psi(t)]\right\}
\end{align*}
\]
since \( \psi = (U - \mu)\chi = x_2 - \mu x_1 \). Equation (7) with \( c = (x_2(0)\ x_1(0)) \) follows from the variation-of-constants formula, and from \( U\chi = x_1 \).

Define the operator \( L(\mu): \mathbb{C}_C \rightarrow \mathbb{C}^2 \) by
\begin{equation}
L(\mu)\psi = \begin{pmatrix}
-\psi(0) - I_1(\mu)\psi \\
-I_2(\mu)\psi
\end{pmatrix},
\end{equation}
where \( I_1(\mu)\psi \) is the second component of
\[
S^\mu(-1) \int_0^{-1} (S^\mu(s))^{-1} \begin{pmatrix} 0 \\ -a(s)\psi(s) \end{pmatrix} \, ds,
\]
and \( I_2(\mu)\psi \) is the first component of
\[
S^\mu(-1) \int_0^{-1} (S^\mu(s))^{-1} \begin{pmatrix} 0 \\ \frac{1}{\mu}a(s)\psi(s) \end{pmatrix} \, ds,
\]
for all \( \psi \in \mathbb{C}_C \).

The first component of equation (8) follows from \( U\chi(0) = \psi(0) + \mu x_1(-1) \) and from equation (7) with \( t = 0, t = -1 \).

Observe that equation (7) with \( t = 0 \) gives \( c_2 = x_1(0) \), and that \( x_1(0) = x(1) = x_2(-1) = U\chi(-1). \) Substituting the right-hand side of equation (7) with \( t = -1 \) for \( U\chi(-1) \) into \( c_2 = U\chi(-1) \) yields the second component of equation (8).
(b) In order to prove surjectivity of $L(\mu)$, we look for sequences $(\psi_n)$, $(\hat{\psi}_n)$ in $C_C$ such that $L(\mu)\psi_n \to (0)$, $L(\mu)\hat{\psi}_n \to (0)$ as $n \to +\infty$.

Let $n \in \mathbb{N}$ be given. Existence of an element $\psi = \psi_n \in C_C$ with $|\psi(0)| = 1$ and $|I_1(\mu)\psi| + |I_2(\mu)\psi| < 1/n$ is rather obvious.

Proof that there exists $\tilde{\psi} \in C_C$ with $0 \neq I_2(\mu)\tilde{\psi}$: We have

$$-I_2(\mu)\psi = \frac{1}{\mu} \int_0^{-1} [-u_1^\mu(-1)u_2^\mu(s) + u_2^\mu(-1)u_1^\mu(s)]a(s)\psi(s) \, ds$$

for all $\psi \in C_C$. Set $I(s) = -u_1^\mu(-1)u_2^\mu(s) + u_2^\mu(-1)u_1^\mu(s)$, for $s \in [-1,0]$. In case $u_2^\mu(-1) \neq 0$, we get

$$I(0) = u_2^\mu(-1)u_1^\mu(0) = u_2^\mu(-1) \neq 0.$$ 

Therefore $|I(s)| > 0$ on $(-\varepsilon,0)$, for some $\varepsilon \in (0,1]$, and we may take any $\tilde{\psi} \geq 0$ with $\tilde{\psi}(0) > 0$ and $\tilde{\psi}(s) = 0$ on $[-1,-\varepsilon]$.

In case $u_2^\mu(-1) = 0$, we have $I(0) = 0$. $\det S^\mu(-1) = 1$ gives $u_1^\mu(-1) \neq 0$. With $u_1^\mu(0) = b(0)u_2^\mu(0) = b(0) < 0$, we obtain $I(0) = u_1^\mu(-1)u_2^\mu(0) \neq 0$. It follows that $|I(s)| > 0$ on $(-\varepsilon,0)$ for some $\varepsilon \in (0,1]$, and we may choose $\tilde{\psi}$ as in the first case.

Multiplication by a suitable constant results in an element $\psi^* \in C_C$ with $-I_2(\mu)\psi^* = 1$. We finally change $\psi^*$ in a small interval $(-\delta,0] \subset [-1,0]$ to an element $\hat{\psi} = \hat{\psi}_n$ such that

$$| - \hat{\psi}(0) - I_1(\mu)\hat{\psi} | + | - I_2(\mu)\hat{\psi} - 1 | < 1/n.$$ 

REMARK 1. Lemma 8 is the analogue of Lemma 3.1 in [10]. In order to obtain surjectivity for the operator $L(\lambda)$, as claimed in [10], one needs that the function $[-1,0] \ni t \to g'(x(t))$, which corresponds to $[-1,0] \ni s \to a(s)$, has at most finitely many zeros. This property is satisfied for the functions $g$ in §§4-6 of [10] but was forgotten to be stated as an extra hypothesis in §3 of [10].

One might also omit surjectivity in Lemma 8 above, as well as in Lemma 3.1 [10], and derive subsequent results in a slightly different way.

For $0 \neq \mu \in C$ and $q(\mu) = 0$, let $j(\mu)$ denote the order of the zero $\mu$ of $q$. Set $j(\mu) := 0$ if $q(\mu) \neq 0$.

**LEMMA 9.** $0 \neq \mu \in C$ and $Q(\mu) \neq 0$ imply $m(\mu) = j(\mu)$.

**Proof.** (a) For $\mu \in C\setminus\{0\}$, $q(\mu) = 0$ if and only if $\mu \in \sigma$: If $0 \neq \mu \in \sigma$ then $\mu$ is an eigenvalue. $\mu \neq 0$ and $(U-\mu)X = 0$ with $X \neq 0$ given $U \neq 0$. By Lemma 8, there exists $c \neq 0$ with $Q(\mu)c = 0$. Hence $q(\mu) = \det Q(\mu) = 0$. Suppose that $0 \neq \mu \notin \sigma$. It follows that for every $\psi \in C_C$ there exists $c \in C^2$ such that $Q(\mu)c = L(\mu)\psi$. Surjectivity of $L(\mu)$ implies that the rank of $Q(\mu)$ is 2. Therefore $q(\mu) = \det Q(\mu) \neq 0$.

(b) Let $B$ denote the space of bounded linear operators $C_C \to C_C$. The analytic mapping

$$C\setminus(\sigma \cup \{0\}) \ni \mu \to q(\mu)(U-\mu)^{-1} \in B$$

admits a continuous extension $H$ to $C\setminus\{0\}$:

For $\psi \in C_C$ and $0 \neq \mu \in C\setminus\sigma$, we have

$$q(\mu)(U-\mu)^{-1} \psi = q(\mu)\mu^{-1}(U_X - \psi)$$
where $\chi = (U - \mu)^{-1}\psi$. By (a), $0 \neq q(\mu) = \det Q(\mu)$. According to Lemma 8, $U\chi$ is given by equations (7) and (8). It follows that the term $q(\mu) \cdot U\chi$ is the first component of

$$q(\mu) \left[ S^\mu(\cdot)(q(\mu))^{-1} \tilde{Q}(\mu)L(\mu)\psi 
+ S^\mu(\cdot) \int_0^1 (S^\mu(s))^{-1} \begin{pmatrix} 0 \\
-\frac{1}{\mu} a(s) \psi(s) \end{pmatrix} ds \right] \in \mathbb{C}C$$

where

$$\tilde{Q}(\mu) = \begin{pmatrix} u^\mu_2(-1) - 1 & -\mu z^\mu_2(-1) \\
- u^\mu_1(-1) & \mu z^\mu_1(-1) - 1 \end{pmatrix} = q(\mu)(Q(\mu))^{-1}.$$

Now the assertion is easily derived from continuity of the maps $0 \neq \mu \rightarrow \hat{u}_i^\mu \in \mathbb{C}C$, $0 \neq \mu \rightarrow \hat{z}_i^\mu \in \mathbb{C}C$, $i \in \{1,2\}$.

(c) $0 \neq \mu \in \sigma$ and $Q(\mu) \neq 0$ imply $H(\mu) \neq 0$:

We have $q(\mu) = 0$, by (a). Therefore the preceding part of the proof shows that for every $\psi \in \mathbb{C}C$, $H(\mu)\psi$ is the first component of $(1/\mu)S^\mu(\cdot)\tilde{Q}(\mu)L(\mu)\psi$.

$Q(\mu) \neq 0$ gives $Q(\mu) \neq 0$. By surjectivity of $L(\mu)$, there exists $\psi \in \mathbb{C}C$ such that $0 \neq Q(\mu)L(\mu)\psi = c$. In case $c_1 \neq 0$, we find $H(\mu)\psi(0) = (1/\mu)c_1 \hat{u}_1^\mu(0) + 0 = c_1/\mu \neq 0$. Hence $H(\mu)\psi \neq 0$. In case $c_1 = 0$, we get $c_2 \neq 0$, and $H(\mu)\psi = (1/\mu)c_2 \hat{u}_2^\mu$. $\hat{u}_1^\mu(0) = b(0)z_1^\mu(0) = b(0) < 0$ yields $H(\mu)\psi \neq 0$.

(d) Let $\mu \in \mathbb{C}\{\{0\}$ with $Q(\mu) \neq 0$ be given. If $\mu \notin \sigma$ then $j(\mu) = 0$ (see (a)) and $m(\mu) = 0$. In case $\mu \in \sigma$, we first observe that $Q(\mu)$ has rank 1, as follows from $Q(\mu) \neq 0$ and $0 = q(\mu) = \det Q(\mu)$. Lemma 8 implies that the geometric eigenspace $\ker(U - \mu)$ is contained in the set $\{c_1 \hat{u}_1^\mu + c_2 \hat{u}_2^\mu : Q(\mu) \cdot (c_1^2) = 0\}$. With rank $Q(\mu) = 1$, we infer $\dim \ker(U - \mu) = 1$. It follows that the algebraic multiplicity $m(\mu)$ coincides with the stabilizing exponent

$$\kappa = \kappa_\mu = \min\{k \in \mathbb{N} : \ker(U - \mu)^k = \ker(U - \mu)^{k+1}\}$$

because of

$$m(\mu) = \dim \ker(U - \mu)^k \leq \sum_{k}^{\kappa} \dim \ker(U - \mu) = \sum_{k}^{\kappa} 1 = \kappa$$

and

$$\dim \ker(U - \mu) < \dim \ker(U - \mu)^2 < \cdots < \dim \ker(U - \mu)^{\kappa}.$$

The stabilizing exponent $\kappa_\mu$ is equal to the order $K(\mu)$ of the pole of the resolvent of $U$ at $\mu$:

$$K(\mu) = \min\{k \in \mathbb{N} : \text{The map } \mathbb{C}\backslash(\sigma \cup \{0\}) \ni \lambda \rightarrow (\lambda - \mu)^k(U - \lambda)^{-1} \in \mathcal{B} $$

admits a continuous extension to $\{\mu\} \cup (\mathbb{C}\backslash(\sigma \cup \{0\}))\}$.

By definition of $j(\mu)$, we have $q(\lambda) = (\lambda - \mu)^j(\mu)h(\lambda)$, with $h$ analytic in a neighborhood of $\lambda = \mu$, and $h(\mu) \neq 0$. Using (b) and (c), one deduces $K(\mu) = j(\mu)$. Altogether, $m(\mu) = \kappa_\mu = K(\mu) = j(\mu)$.

**Lemma 10.** $\mu \in \mathbb{C}\{-1,0\}$ implies $Q(\mu) \neq 0$.

**Proof.** Suppose $Q(\mu) = 0$ and $\mu \neq 0$. We then have $u^\mu_1(-1) = 0 = z^\mu_2(-1)$ and $u^\mu_1(-1) = 1 = \mu z^\mu_1(-1)$. $1 \neq \det S^\mu(-1)$ reduces to $1 = -u^\mu_2(-1)z^\mu_1(-1) = -z_1^\mu(-1)$. Hence $\mu = -1$. 

LEMMA 11. \( \lim_{|\mu| \to \infty} q(\mu) / \mu = -1. \)

PROOF. Let

\[ S^\infty = \begin{pmatrix} u_1^\infty & u_2^\infty \\ z_1^\infty & z_2^\infty \end{pmatrix} \]

denote the matrix solution of \( \dot{u} = b(t)z, \dot{z} = 0, \) with \( S^\infty(0) = (1 \ 0) \). We have

\[ \lim_{|\mu| \to \infty} S^\mu(-1) = S^\infty(-1). \]

In particular,

\[ \lim_{|\mu| \to \infty} z_1^\mu(-1) = z_1^\infty(-1) = z_1^\infty(0) = 0 \]

and

\[ \lim_{|\mu| \to \infty} u_2^\mu(-1) = u_2^\infty(0) + \int_0^{-1} \dot{u}_2^\infty(s) \, ds = \int_0^{-1} b(s)z_2^\infty(s) \, ds = \int_0^{-1} b(s)1 \, ds. \]

It follows that \( q(\mu) / \mu = \mu^{-1} - 1 - u_2^\mu(-1) - z_1^\mu(-1) \) converges to \(-1\) as \(|\mu| \to \infty|\).

6. Symmetric periodic solutions of a nonlinear autonomous differential delay equation. Consider the nonlinear equation

\( (g) \dot{x}(t) = g(x(t - 1)). \)

Suppose that

(H1) \( g: \mathbb{R} \to \mathbb{R} \) is continuously differentiable and odd, \( \xi g(\xi) < 0 \) for all \( \xi \neq 0 \), and \( g'(0) < -\pi/2 < \lim_{\xi \to +\infty} g(\xi)/\xi. \)

It is known that \( g(\cdot) \) has symmetric periodic solutions, i.e. solutions \( x: \mathbb{R} \to \mathbb{R} \) with minimal period 4, symmetry condition

(s) \( x(t) = -x(t - 2) \) for all \( t \),

and with \( x(-1) = 0, 0 < \dot{x} \) in \([-1,0), \dot{x} < 0 \) in \((0,1]\). This can be proved by using a method from [5].

The functions \( x \) and \( y: t \to x(t - 1) \) satisfy \( \dot{x} = g(y), \dot{y} = -g(x) \), and we have

(10) \( x(t) = -y(-1 - t), \quad y(t) = -x(-1 - t), \) for all \( t \).

To prove (10), we note that the relations \( X(t) = -y(-1 - t) \) and \( Y(t) = -x(-1 - t) \) for \( t \in \mathbb{R} \) define functions with \( \dot{X} = g(Y), \dot{Y} = -g(X) \) and \( X(-1) = -y(0) = -x(-1) = 0 = x(-1), Y(-1) = -x(0) = x(-2) = y(-1). \)

The characteristic multipliers of a periodic solution \( x \) of equation \( g \) are given by the linear variational equation along \( x \),

\[ \dot{\gamma}(t) = g'(x(t - 1))\gamma(t - 1), \]

and by the minimal period of \( x \). Differentiation of equation (g) shows that \( \mu = 1 \) is a multiplier, with eigenvector \( \dot{x}_0 = \dot{x}|[-1,0] \) of the monodromy operator. In case \( m(1) = 1 \), \( x \) is called nondegenerate.

THEOREM 2. Suppose \( g \) satisfies (H1), and

(H2) \( g' \) is increasing on \([0,\infty), \) and \( g' < 0. \)

Then every symmetric periodic solution \( x \) of equation \( g \) is nondegenerate, with \( |\mu| < 1 \) for all characteristic multipliers \( \mu \) except the trivial one \( \mu = 1 \).

The orbit of a periodic solution \( x \) of equation \( g \) is the set of segments \( x_t \in C, \)

where \( x_t(s) = x(t + s) \) for \( s \in [-1,0] \) and \( t \in \mathbb{R}. \)
**Corollary 2.** Suppose in addition to the hypotheses of Theorem 1 that $g$ is of class $C^2$. Then the orbit of $x$ is exponentially asymptotically stable with asymptotic phase.

**Proof of Corollary 2.** See Corollary 3.1 in [4, Chapter 10].

**Remark 2.** The hypotheses in Theorem 2 are closely related to Nussbaum’s condition for uniqueness of periodic solutions in [8].

First, it is an easy exercise to prove that (H1) and (H2) imply the hypotheses of Theorem 1.3 in [8]. (10) shows that every symmetric periodic solution $x$ satisfies $x(-1 + t) = -x(-1 - t)$ for all $t \in \mathbb{R}$. By Theorem 1.3 in [8], there is a precisely one symmetric periodic solution, under conditions (H1) and (H2) on $g$.

Next, if we restrict $g$ to the slightly smaller class of functions with (H1), (H2) and $\xi \rightarrow g(\xi)/\xi$ strictly increasing on $(0, \infty)$, then Theorem 2.2 in [8] guarantees the uniqueness of the periodic solutions in the class of periodic solutions with $0 < x$ on $(-1, z_1)$ for a zero $z_1 > 0$, $x < 0$ on $(z_1, z_2)$ for a zero $z_2 > z_1 + 1$ and with period $z_2 + 1$.

Uniqueness within the set of all periodic solutions of slowly oscillating type (as described in the Introduction), up to translation in time, follows if $g$ is also bounded. See Remark 2.4 in [8].

In this last case, the phase plane method of Kaplan and Yorke [6] yield results on stability and attractivity, too—including information on the domain of attraction. For a proof that the domain of attraction is open and dense in $C$, see [9].

**Proof of Theorem 2.** Let $g$ and $x$ be given as in the theorem. We set $y(t) = x(t - 1)$, $b(t) = g'(y(t))$, $a(t) = b(t + 1) = g'(x(t))$, for all $t \in \mathbb{R}$. $b$ and $a$ have period $r = 2$, because of (s) and $g'(\xi) = g'(-\xi)$ on $\mathbb{R}$. We claim that

\begin{equation}
(11) \quad a(t) = b(-1 - t), \quad b(t) = a(-1 - t), \quad \text{for all } t.
\end{equation}

By (10),

\begin{equation}
a(t) = g'(x(t)) = g'(y(t + 1)) = g'(-x(-1 - (t + 1))) = g'(x(-1 - (t + 1))) = b(-1 - t)
\end{equation}

and

\begin{equation}
b(t) = g'(y(t)) = g'(-x(-1 - t)) = g'(x(-1 - t)) = a(-1 - t).
\end{equation}

This proves (11). We claim that

\begin{equation}
(12) \quad a(0) > a(-1).
\end{equation}

Since $0 = x(-1) < x(0)$, $g'(x(-1)) \leq g'(x(0))$. Suppose $g'(x(-1)) = g'(x(0))$. Then $g' = g'(x(-1)) = g'(0)$ on $[0, x(0)]$. By (s), $g' = g'(0) < -\pi/2$ on $x(\mathbb{R})$. This contradicts the well-known fact that the linear equation

\begin{equation}
\dot{w}(t) = -\alpha w(t - 1), \quad \alpha > \pi/2,
\end{equation}

has no periodic solution of slowly oscillating type (see, for example, Theorem 5 in [11] or [4, Chapter 7]). This proves (12).

The characteristic multipliers of $x$ are given by the spectrum $\sigma'$ of the monodromy operator $U' = T(4,0)$ for the linear variational equation along $x$, i.e. equation (b). Set $U = T(2,0)$. Since $b$ has period 2, $U' = U \circ U$. Let $\sigma$ denote the spectrum of $U$, as before. Then $0 \neq \xi \in \sigma'$ if and only if $\xi = \mu^2$ for some $\mu \in \sigma \setminus \{0\}$. This is most easily seen from

\begin{equation}
U' - \mu^2 = (U - \mu)(U + \mu), \quad \text{for } \mu \in \mathbb{C}.
\end{equation}
Moreover, if \( m'(\xi) \) denotes the multiplicities of the complex numbers \( \xi \neq 0 \) considered as eigenvalues of \( U'(m'(\xi)) = 0 \) if \( \xi \notin \sigma' \), then

\[
m'(\mu^2) = m(\mu) + m(-\mu), \quad \text{for all } \mu \in \mathbb{C}\{0\}.
\]

Note that \( \mu = -1 \) is an eigenvalue of \( U \) with eigenvector \( \mathbf{x}_0 = \hat{x}|_{[-1,0]} \). In the notation of §3, \( \hat{x}_0 \in E_{-1} \). So it remains to show that \( \mu = -1 \) is a simple eigenvalue of \( U \), and that there are no other eigenvalues of \( U \) with \( |\mu| \geq 1 \).

There is a solution \( y \in \mathcal{D}_C(-1) \) with \( y_0 = \mathbf{x}_0 \in E_{-1} \). Uniqueness of the initial value problem for equation (b) at \( t = 0 \) yields \( y = \hat{x} \) on \([-1,\infty) \). We have \( y(t) \in \mathbb{R} \) also for \( t < -1 \) (if \( \text{Im}y(t) \neq 0 \) for some \( t < 1 \) and \( y(t) = e^{t/2}(\text{Re}f(t) + i\text{Im}f(t)) \) with \( f \in \mathcal{Y} \) almost periodic, then \( \text{Im}y(s) \neq 0 \) for certain \( s \geq -1 \)). Therefore \( y(\mathbb{R}) \subset \mathbb{R} \), and \( y \in \mathcal{D}(-1) \subset \mathcal{D}_1 \), and \( y(t) = \mathbf{x}(t) = g(x(t-1)) < 0 \) for all \( t \in (0,2) \). Lemma 4 gives \( y \in \Sigma \). Now Corollary 1 applies, and we obtain

\[
\sum_{|\mu| \geq 1} m(\mu) \in \{1,2\}.
\]

This gives us the following possibilities: either \( m(-1) = 2 \) and there are no multipliers \( \mu \neq -1 \) with \( |\mu| \geq 1 \), or \( m(-1) = 1 \) and there are no multipliers \( \mu \in \sigma \setminus \mathbb{R} \) with \( |\mu| \geq 1 \), and

\[
\sum_{\mu \in (-\infty,1) \cup [1,\infty)} m(\mu) \leq 1.
\]

In the next section we shall employ the function \( q \) associated with \( b \) in order to show

\[
\begin{align*}
\sigma \cap [1,\infty) &= \emptyset, \\
m(-1) &= 1, \\
\sum_{-\infty < \mu < -1} m(\mu) &\in 2\mathbb{Z}.
\end{align*}
\]

This will complete the proof of Theorem 2.

### 7. Proof of (13)-(15).

I. **Computation of \( Q(-1) \).** Lemma 8 and \( U\mathbf{x}_0 + \mathbf{x}_0 = 0 \) imply \( \hat{x}_0 = -U\mathbf{x}_0 = -(c_1\mathbf{u}_1^{-1} + c_2\mathbf{u}_2^{-1}) \) where \( c = (c_1, c_2)^T \neq (0,0)^T \) and \( Q(-1)c = 0 \) and \( \dot{x}(0) = 0 \), \( u_1^{-1}(0) = 1 \) and \( u_2^{-1}(0) = 0 \) give \( c_1 = 0 \). Therefore, \( c_2 \neq 0 \) and \( Q(-1)(0,0)^T = 0 \). This shows \( z_2^{-1}(-1) = 0 \) and \( u_2^{-1}(-1) = 1 \). With \( \det S^{-1}(-1) = 1 \), we obtain \( z_1^{-1}(-1) = -1 \). Altogether

\[
S^{-1}(-1) = \begin{pmatrix}
u_1^{-1}(1) & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(1) = \begin{pmatrix} 0 & 0 \\ 1^{-1}(-1) & 0 \end{pmatrix}.
\]

\( \dot{x} = -c_2u_2^{-1} \) on \([-1,0], 0 < \dot{x} \) on \([-1,0) \), \( u_2^{-1}(0) = 0 \) and \( u_2^{-1}(0) = b(0)z_2^{-1}(0) = b(0) < 0 \) imply \( c_2 < 0 \).

II. **Computation of \( q'(\xi) \).** We have

\[
\begin{align*}
q'(\xi) &= -1 - v_2(\xi) - z_1^{-1}(\xi) - (\xi)b(1) \\
&= w_1(-1) - v_2(-1)
\end{align*}
\]
with the solutions \((v_i, w_i), i = 1, 2,\) of the initial value problems
\[
\dot{v} = b(t)w, \quad \dot{w} = -a(t)v - a(t)u_i^{-1}(t); \quad v(0) = 0 = w(0).
\]
This follows from differentiation of the initial value problems for \((u_i^\mu, z_i^\mu)\) with respect to \(\mu\) at \(\mu = -1\).

By variation of constants,
\[
q'(-1) = w_1(-1) - w_2(-1)
\]
\[
= \int_0^{-1} \left[ -a(s)u_1^{-1}(s)u_2^{-1}(s) - (u_1^{-1}(-1)a(s)u_2^{-1}(s)u_1^{-1}(s)) \right. \\
- (a(s)u_2^{-1}(s)u_1^{-1}(s)) \left. \right] ds
\]
\[
= u_1^{-1}(-1) \int_{-1}^0 a(s)[u_2^{-1}(s)]^2 ds.
\]
The last integral is negative since \(u_2^{-1}(-1) = -\dot{x}(-1)/c_2 \neq 0\).

III. Polar coordinates. Let \(\mu \in \mathbb{R}\setminus\{0\}\) be given. Then
\[
(u_i^\mu, z_i^\mu) = r_i^\mu(\cos \theta_i^\mu, \sin \theta_i^\mu)
\]
for \(i = 1, 2,\) with the solutions \((r_i^\mu, \theta_i^\mu)\) of the initial value problems
\[
\dot{r} = (b(t) + (1/\mu)a(t))r \cdot \cos \theta \cdot \sin \theta, \\
\dot{\theta} = (1/\mu)a(t)(\cos \theta)^2 - b(t)(\sin \theta)^2;
\]
\(r_i^\mu(0) = 1\) for \(i = 1, 2; \theta_i^\mu(0) = 0, \theta_i^\mu(0) = \pi/2.\) Obviously, \(r_i^\mu > 0\) on \(\mathbb{R}\) for \(i = 1, 2.\)

IV. Proof of (13). Let \(\mu \geq 1.\) Then
\[
q(\mu) = 1 - \mu - u_2^\mu(-1) - \mu z_1^\mu(-1) \leq -r_2^\mu(-1) \cos \theta_1^\mu(-1) - \mu r_1^\mu(-1) \sin \theta_1^\mu(-1).
\]
We have \(r_i^\mu(-1) > 0\) for \(i = 1, 2.\) The vectorfield for the \(\theta\)-equation in the \((r, \theta)\)-plane points to the right, and upward for \(\theta = \pi/2,\) downward for \(\theta = 0,\) with nonzero vertical components. It follows that both \(\theta_i^\mu(-1)\) are contained in the interval \((0, \pi/2).\) Hence \(q(\mu) < 0,\) or \(\sigma \cap [1, \infty) = \emptyset,\) by Lemmas 9 and 10.

V. Proof of \(u_1^{-1}(-1) > 0.\) It is enough to show \(\theta_1^{-1}(-1) \in (-\pi/2, 0].\)

(a) \(\theta_2^{-1}(-1) = 0.\)

PROOF. \(0 < -c_2^{-1} \dot{x} = u_2^{-1} = r_2^{-1} \cos \theta_2^{-1}\) on \([-1, 0), \theta_2^{-1}(0) = \pi/2\) and
\[
\dot{\theta}_2^{-1} = -a(t)(\cos \theta_2^{-1})^2 - b(t)(\sin \theta_2^{-1})^2 > 0
\]
imply \(\theta_2^{-1}(t) \in (-\pi/2, \pi/2]\) for all \(t \in [-1, 0].\) \(0 = z_2^{-1}(-1) = r_2^{-1}(-1) \sin \theta_2^{-1}(-1)\) gives \(\theta_2^{-1}(-1) = 0.\)

(b) For all \(t \in \mathbb{R}, \theta_2^{-1}(t) = \pi/2 - \theta_2^{-1}(-1 - t),\) and \(\theta_2^{-1}(-1/2) = \pi/4.\)
PROOF. The function $\theta: t \to \pi/2 - \theta_2^{-1}(-1 - t)$ satisfies $\theta(0) = \pi/2 = \theta_2^{-1}(0)$ (see (a)). Because of (11),

$$
\dot{\theta}(t) = \theta_2^{-1}(-1 - t) = -a(-1 - t)(\cos \theta_2^{-1}(-1 - t))^2 - b(-1 - t)(\sin \theta_2^{-1}(-1 - t))^2
$$

$$
= -b(t)(\sin(\pi/2 - \theta_2^{-1}(-1 - t)))^2 - a(t)(\cos(\pi/2 - \theta_2^{-1}(-1 - t)))^2,
$$

$$
= -a(t)(\cos \theta(t))^2 - b(t)(\sin \theta(t))^2, \text{ for all } t \in \mathbb{R}.
$$

It follows that $\theta = \theta_2^{-1}$.

(c) Let $\theta_*$ denote the solution of

$$
\dot{\theta} = (b(t) - a(t))(\cos \theta)^2 - b(t),
$$

$$
= -a(t)(\cos \theta)^2 - b(t)(\sin \theta)^2, \quad \theta(-1/2) = -\pi/4.
$$

Then,

$$
\dot{\theta}_2^{-1} - \dot{\theta}_* = ((\cos \theta_2^{-1})^2 - (\cos \theta_*)^2)(b(t) - a(t)).
$$

Hypothesis (H2) implies that $a = g' \circ x$ is increasing on $[-1, 0]$. (11) shows that $b$ is decreasing on $[-1, 0]$, with $b(-1/2) = a(-1/2)$. It follows that $b - a$ is nonnegative on $[-1, -1/2]$ and nonpositive on $[-1/2, 0]$. $\theta_2^{-1}$ and $\theta_*$ are both strictly increasing.

(d) $\theta_*(t) > -\pi/2$ on $[-1, -1/2]$. Proof: Suppose there exists $t \in (-1, -1/2)$ with $\theta_*(t) = -\pi/2$ and $-\pi/2 \leq \theta_* \leq -\pi/4$ in $[t, -1/2]$. For $t \leq s \leq -1/2$, we have $0 \leq \theta_2^{-1}(s) \leq \pi/4$ and

$$
\dot{\theta}_2^{-1}(s) - \dot{\theta}_*(s) = ((\cos \theta_2^{-1}(s))^2 - (\cos \theta_*(s))^2)(b(s) - a(s)) \geq 0.
$$

Therefore

$$
\pi/4 = \theta_*(-1/2) - \theta_*(t) \leq \theta_2^{-1}(-1/2) - \theta_2^{-1}(t) = \pi/4 - \theta_2^{-1}(t).
$$

On the other hand, $\theta_2^{-1}(-1) = 0$ and $\dot{\theta}_2^{-1} > 0$ imply $\theta_2^{-1}(t) > 0$, a contradiction.

(e) $\theta_*(-1) > -\pi/2$.

Proof. Suppose $\theta_*(-1) = -\pi/2$. We have $\dot{\theta}_2^{-1} \geq \dot{\theta}_*$ on $[-1, -1/2]$ (compare (c) and the proof of (d)). The assumption, (11) and (12) imply

$$
\dot{\theta}_*(-1) = -b(-1) = -a(0) < -a(-1) = \dot{\theta}_2^{-1}(-1).
$$

Hence

$$
-\pi/4 - \theta_*(-1) = \theta_*(-1/2) - \theta_*(-1) < \theta_2^{-1}(-1/2) - \theta_2^{-1}(-1) = \pi/4.
$$

This is a contradiction to $\theta_*(-1) = -\pi/2$.

(f) In the same way, one can show that $\theta_*(0) \leq 0 (= \theta_2^{-1}(0))$. It follows that $\theta_* \leq \theta_1^{-1}$. In particular, $-\pi/2 < \theta_*(1) \leq \theta_1^{-1}(-1)$. $\theta_1^{-1}(-1) \leq 0$ is obvious from $\dot{\theta}_1^{-1} > 0$, $\theta_1^{-1}(0) = 0$.

VI. $u_2^{-1}(-1) \neq 0$ yields $Q(-1) \neq 0$ (see I) and $q'(-1) \neq 0$ (see II). Therefore Lemma 9 applies, and we find

$$
m(-1) = j(-1) = 1.
$$
Moreover, $u_1^{-1}(0) > 0$ gives $q'(-1) < 0$. Lemma 11 and $q(-1) = 0 > q'(-1)$ imply that the sum of the orders of zeros of $q$ in $(-\infty,-1)$ must be even. With Lemmas 10 and 9 we obtain

$$
\sum_{\infty < \mu < -1} m(\mu) = \sum_{\infty < \mu < -1} j(\mu) \in 2\mathbb{Z}.
$$

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824

MATHEMATHISCHES INSTITUT, UNIVERSITAT MUNCHEN, THERESIENSTR. 39, D 8000 MUNCHEN 2, FEDERAL REPUBLIC OF GERMANY