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THE CONNECTION MAP FOR ATTRACTOR-REPELLER PAIRS

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ABSTRACT. In the Conley index theory, the connection map of the homology attractor-repeller sequence provides a means of detecting connecting orbits between a repeller and attractor in an isolated invariant set. In this work, the connection map is shown to be additive: under suitable decompositions of the connecting orbit set, the connection map of the invariant set equals the sum of the connection maps of the decomposition elements. This refines the information provided by the homology attractor-repeller sequence. In particular, the properties of the connection map lead to a characterization of isolated invariant sets with hyperbolic critical points as an attractor-repeller pair.

1. The homology attractor-repeller sequence. One of the methods by which the Conley index theory studies isolated invariant sets is to decompose them into subinvariant sets and connecting orbits between the sets. The simplest such decomposition of a set S is an attractor-repeller pair (A, A^*): an invariant subset A of S with $A = \omega(U)$ for some S-neighborhood U of A; and its dual repeller $A^* = \{x \in S \mid \omega(x) \cap A = \varnothing\}$. Then A^* is a repeller in S, and S decomposes into $A \cup C(A^*, A; S) \cup A^*$, where

$$C(A^*, A; S) = \{x \in S \mid \omega^*(x) \subseteq A^*, \omega(x) \subseteq A\}$$

is the connecting orbit set.

If S is an isolated invariant set with attractor-repeller pair (A, A^*), then A and A^* are also isolated, so S, A and A^* all have a Conley index in X, defined in terms of index pairs for each in X. These index pairs are related by the construction of an index triple: a compact triple $N_0 \subseteq N_1 \subseteq N_2$ such that (N_2, N_0) is an index pair for S in X; (N_1, N_0) is an index pair for A in X; (N_2, N_1) is an index pair for A^* in X. Kurland [**4**] shows that index triples exist for all $(S; A, A^*)$. Further, these index triples can always be taken to be regular [**7, 8**] (i.e. each index pair induced by the triple is an NDR pair).

From the index triple of $(S; A, A^*)$, Kurland constructs a long coexact sequence relating the (homotopy) Conley indices of S, A, and A^* in X. Franzosa [**3**] derives from this an exact sequence relating the homology Conley indices. Namely, given a regular index triple (N_2, N_1, N_0) for an attractor-repeller pair (A, A^*) in S, there exists an exact sequence

$$\cdots \xrightarrow{\partial} H_*(N_1, N_0) \xrightarrow{i_*} H_*(N_2, N_0) \xrightarrow{j_*} H_*(N_2, N_1) \xrightarrow{\partial} \cdots.$$

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This defines an exact sequence of homology Conley indices

$$\cdots \xrightarrow{\partial} CH_*(X;A) \xrightarrow{i_*} CH_*(X;S) \xrightarrow{j_*} CH_*(X;A^*) \xrightarrow{\partial} \cdots$$

called the homology index sequence of the attractor-repeller pair, or the homology attractor-repeller sequence. This sequence provides an algebraic condition for the existence of connections. Namely, if the connection map ∂ is nontrivial, then $C(A^*, A; S)$ is nonempty. In [**2**, **3**] Franzosa generalizes the connection map for an attractor-repeller pair to a connection matrix for a Morse decomposition. Connection maps and matrices are used by many authors (e.g. [**1**–**4**, **7**–**10**]) to detect and (when combined with other techniques) describe connecting orbit sets in a variety of problems.

It is not true that ∂ is nontrivial when $C(A^*, A; S)$ is nontrivial. For example, consider the gravity flow on the k-sphere S^k ($k > 0$). This gives a gradient flow with the south pole x_0 an attractor; the north pole x_1 a repeller; and $C(x_1, x_0; S^k)$ is nonempty. Further, $CH_*(S^k; S^k) \cong H_*(S^k)$; $CH_*(S^k; x_1) \cong \tilde{H}_*(S^k)$; and $CH_*(S^k; x_0) \cong \tilde{H}_*(S^0)$. That is, for all k,

$$CH_*(S^k; S^k) \cong CH_*(S^k; x_1) \oplus CH_*(S^k; x_0),$$

and ∂ is trivial. For $k > 1$, this is simply the fact that the dimensions of the attractor and repeller indices do not admit a nontrivial degree 1 map. (If dimensions admit a nontrivial degree 0 map, the transition matrix of [**10**] provides information similar to that of the connection map.) However, when $k = 1$, $C(x_1, x_0; S^k)$ consists of two orbits and the dimensions admit a nontrivial map

$$\partial \colon CH_1(S^k; x_1) \to CH_0(S^k; x_0),$$

yet ∂ is trivial.

This triviality can be understood by considering each connecting orbit separately. The closure of each orbit is isolated, with (x_1, x_0) an attractor-repeller pair and the connection map of the orbit an isomorphism. However, these isomorphisms are of opposite orientation, so their sum is trivial. That is, the sum of the connection maps of the individual orbits gives the connection map of the circle.

In this work we formalize and generalize this additivity by producing a sum theorem for the connection map which holds for any attractor-repeller pair (A, A^*) of an isolated invariant set S. This requires identifying isolated subsets of S which each have (A, A^*) as an attractor-repeller pair and whose union is S. In §2, conditions for such decompositions are found (Theorem 2.2) and the sum theorem is proved for such decompositions (Theorem 2.5). This provides new methods of detecting connecting orbits (2.6–3.3), but it also can be reapplied to the flows on spheres which motivated it, refining the information available there. Namely, it is shown (Theorem 3.4) that the k-sphere remains isolated whenever it is embedded into a C^2 flow on a manifold so that x_1 and x_0 remain hyperbolic with indices k and 0, and (with one exception when $k = 1$) it is the only isolated set which can be so embedded.

The reader is referred to [**1**, **11**] for attractor-repeller pairs and index constructions, and to [**3**] for connection maps. The notation in this paper follows that of [**6**]: IS denotes the category of pairs $(X; S)$ with X a locally compact metric flow and $S \subseteq X$ an isolated invariant set; $\mathscr{N}(X; S)$ denotes the collection of index pairs

(N, L) for S in X such that L is the "immediate exit set" of N, and such that N with the "immediate entrance set" L^- is an index pair for the reverse flow; $CH_*(X; S)$ is the homology Conley index of S in X, defined by $CH_*(X; S) \cong H_*(N, L)$ for any $(N, L) \in \mathscr{N}(X; S)$.

I would like to thank Henry Kurland for pointing out counterexamples to the converse of Lemma 2.1. The examples, and an alternate proof of Lemma 2.1, can be found in [**5**].

2. The sum theorem for attractor-repeller pairs. To develop the sum theorem, the appropriate method of decomposing an attractor-repeller decomposition will be to take separations of the connecting orbit set. Recall that if $\mathscr{A}$ is an indexing set, X a topological space, an $\mathscr{A}$-separation of X is a collection $\{X_\alpha \mid \alpha \in \mathscr{A}\}$ of disjoint open subsets of X such that $X = \bigcup_{\alpha \in \mathscr{A}} X_\alpha$. Note that $U_\alpha = \bigcap_{\beta \neq \alpha} (X \setminus U_\beta)$ is also closed in X. For a space X with a flow defined on it, an invariant $\mathscr{A}$-separation will refer to a separation with each X_α invariant under the flow. If S is a compact invariant set with $\{S_\alpha\}_{\alpha \in \mathscr{A}}$ an invariant $\mathscr{A}$-separation of S, then $\mathscr{A}$ is finite. Similarly, if (A, A^*) is an attractor-repeller pair for S with $\{C_\beta\}_{\beta \in \mathscr{B}}$ an invariant $\mathscr{B}$-separation of $C(A^*, A; S)$, then $\mathscr{B}$ is finite. Each such C_i is then a union of components of $C(A^*, A; S)$, so $C(A^*, A; S)$ will have a finest separation if and only if it has a finite number of components.

LEMMA 2.1. *Suppose S is a compact invariant set in X with attractor-repeller pair (A, A^*). If A^* and A are isolated in X and $C(A^*, A; S)$ is open in $C(A^*, A; X)$, then S is isolated in X.*

PROOF. Choose disjoint index pairs $(N_1, L_1) \in \mathscr{N}(X; A^*)$, $(N_0, L_0) \in \mathscr{N}(X; A)$. There exist $T_1 < 0 < T_0$ such that $S \cap L_1 = C(A^*, A; S) \cap L_1$ has $(S \cap L_1) \cdot (-\infty, T_1] \subseteq \operatorname{int}_X(N_1)$, $(S \cap L_1) \cdot [T_0, \infty) \subseteq \operatorname{int}_X(N_0)$. Let $T = \frac{1}{2}(T_0 - T_1)$, and replace (N_1, L_1), (N_0, L_0) by (N_1^T, L_1^T), (N_0^T, L_0^T), where $N^T = \bigcap_{-T \leq t \leq T} N \cdot t$ and $L^T = L \cdot (-T) \cap N^T$. Then $S \cap L_1^T$ is a section of $C(A^*, A; S)$ with the property that every $x \in S \cap L_1^T$ has $x \cdot (-\infty, 0] \subseteq N_1$, $x \cdot (0, \infty) \cap N_1 = \varnothing$. Since $C(A^*, A; S)$ is open in $C(A^*, A; X)$, there exists a compact neighborhood U_1 of $S \cap L_1^T$ in L_1^T so that $U_1 \cap C(A^*, A; X) = S \cap L_1^T$. U_1 and $\varepsilon \geq 0$ may be chosen so that $M_1 = N_1 \cup U_1 \cdot [0, 4T + \varepsilon] \cup N_0$ is a compact neighborhood of S in X with $U_1 \cdot (0, 4T + \varepsilon] \cap N_1 = \varnothing$. Similarly, in $(L_0^-)^T$ (the "entrance set" of N_0) there exists a compact neighborhood U_0 of $S \cap (L_0^-)^T$ such that $U_0 \cap C(A^*, A; X) = S \cap (L_0^-)^T$, $U_0 \cdot [-4T - \varepsilon, 0) \cap N_0 = \varnothing$, and $M_0 = N_1 \cup U_0 \cdot [-4T - \varepsilon, 0] \cup N_0$ is a compact neighborhood of S in X.

Let $M = M_1 \cap M_0$. M is a compact neighborhood of S. As N_1 and N_0 are disjoint compact sets, $I(N_1 \cup N_0) = I(N_1) \cup I(N_0) = A^* \cup A$. If $x \in I(M) \setminus A^* \cup A$, then $x \cdot \mathbf{R}$ intersects U_1 and U_0, so there exist times $t_1 < t_0$ such that $x \cdot (-\infty, t_1] \subseteq N_1$, $x \cdot (t_1, t_0) \subseteq X \setminus (N_1 \cup N_0)$, $x \cdot [t_0, \infty) \subseteq N_0$. Thus $\omega^*(x) \subseteq A^*$, $\omega(x) \subseteq A$, and $x \in C(A^*, A; X)$. That is, $I(M) \subseteq A^* \cup C(A^*, A; X) \cup A$. Then by the choice of U_1 and $U_0, I(M) = S$. □

COROLLARY 2.2. *Suppose S is a compact invariant set in X.*

(i) *If $\{S_i\}$ is an invariant separation of S, then S is isolated in X if and only if each S_i is.*

(ii) *If S is isolated with attractor-repeller pair (A, A^*) and $\{C_i\}$ a separation of $C(A^*, A; S)$, then each $S_i = A \cup C_i \cup A^*$ is isolated.*

PROOF. (i) If $\{S_i\}$ is a separation of S with each S_i isolated in X, choose disjoint isolating neighborhoods N_i for the S_i in X. Then $N = \bigcup N_i$ is an isolating neighborhood for S in X. If S is isolated, choose an isolating neighborhood for S in X. $S \setminus S_i$ and S_i are closed in S, so they are closed in N. Choose disjoint open neighborhoods U, V of S_i and $S \setminus S_i$ in $\operatorname{int}_X N$. Then $\overline{U}$ is a compact neighborhood of S_i in X with $I(\overline{U}) = S_i$.

(ii) If S is isolated in X, then A^* and A are isolated in X [**1**, III, 7.1], hence in S, and each C_i is open in $C(A^*, A; S)$. Thus S_i is isolated in S by 2.1, and S is isolated in X, so S_i is isolated in X. □

If S is isolated with attractor-repeller pair (A, A^*) and separation $\{C_i\}$ of $C(A^*, A; S)$, then there exist exact sequences

$$\cdots \to CH_n(X; A) \to CH_n(X; S) \to CH_n(X; A^*) \to CH_{n-1}(X; A) \to \cdots,$$

the homology attractor-repeller sequence of $(S; A, A^*)$, and

$$\cdots \to CH_n(X; A) \to CH_n(X; S_i) \to CH_n(X; A^*) \to CH_{n-1}(X; A) \to \cdots,$$

the homology attractor-repeller sequence of $(S_i; A, A^*)$. The $(S; A, A^*)$ sequence has connection map $\partial\colon CH_n(X; A^*) \to CH_{n-1}(X; A)$; the $(S_i; A, A^*)$ sequence has connection map $\partial_i\colon CH_n(X; A^*) \to CH_{n-1}(X; A)$. To relate these maps, the following index triple constructions are required:

LEMMA 2.3. *If $(X; S) \in \mathrm{IS}$ with $\{C_i\}$ a separation of $C(A^*, A; S)$ and $S_i = A \cup C_i \cup A^*$, there exists an isolating neighborhood for $(X; S)$ of the form $\tilde{N} = N \cup (\bigcup N_i) \cup N^*$, with N, N_i, N^* compact such that*

(i) *N is positively invariant, N^* negatively invariant, relative to $\tilde{N}$.*

(ii) *$N \cap N^* = \varnothing$, $N_i \cap N_j = \varnothing$ for $i \neq j$, and N_i intersects $N \cup N^*$ along their common boundaries.*

(iii) *N is an isolating neighborhood for A, N^* is an isolating neighborhood for A^*, and $\tilde{N}_i = N \cup N_i \cup N^*$ an isolating neighborhood for S_i.*

PROOF. Let M be an isolating neighborhood for S in X. Choose a regular index triple (M_2, M_1, M_0) positively invariant relative to M. Then M_1 is an isolating neighborhood for A and $\overline{M_2 \setminus M_1}$ is an isolating neighborhood for A^*. There exists a $T > 0$ so that no $x \in \partial_{M_2} M_1$ has $x \cdot [-T, T] \subseteq \partial_{M_2} M_1$. Take $N = \{x \in M_1 | x \cdot [-T, 0] \cap \overline{M_2 \setminus M_1} = \varnothing\}$, $N^* = \{x \in \overline{M_2 \setminus M_1} | x \cdot [0, T] \cap M_1 = \varnothing\}$. N and N^* are then disjoint isolating neighborhoods for A and A^*, positively (resp. negatively) invariant relative to M_2.

For $i \neq j$, let $r_{ij} = \min\{d(x, y) | x \in \overline{C_i \setminus (N \cup N^*)},\ y \in \overline{C_j \setminus (N \cup N^*)}\}$ and let $r = \frac{1}{2}\min\{r_{ij}\}$. Let $N_i = \{x \in \overline{M_2 \setminus (N \cup N^*)} |\, d(x, \overline{C_i \setminus (N \cup N^*)}) \leq r\}$. The N_i are clearly compact, and satisfy (ii). For each i, $N \cup N_i \cup N^*$ is a neighborhood for S_i which misses some element of C_j for all $j \neq i$. Thus it is an isolating neighborhood for S_i. Further, $S = \bigcup S_i \subseteq \bigcup (N \cup N_i \cup N^*) = \tilde{N} \subseteq M_2$, so $\tilde{N}$ is an isolating neighborhood for S, and N (resp. N^*) is positively (resp. negatively) invariant relative to $\tilde{N}$. □

LEMMA 2.4. *In the notation of Lemma 2.3, if $(\overline{N},\overline{L})$ is a (regular) index pair for S in X which is positively invariant in $\tilde{N}$, then $(\overline{N},\overline{L}\cup(\overline{N}\cap N),\overline{L})$ and $(\overline{N},\overline{L}\cup \mathrm{cl}_X(\overline{N}\setminus N^*),\overline{L})$ are (regular) index triples for $(S;A,A^*)$.*

PROOF. For the triples to be index triples, the pairs $(\overline{L}\cup(\overline{N}\cap N),\overline{L})$ and $(\overline{L}\cup \mathrm{cl}_X(\overline{N}\setminus N^*),\overline{L})$ must be index pairs for A in X. But

$$\mathrm{cl}_X((\overline{L}\cup(\overline{N}\cap N))\setminus\overline{L}) = \mathrm{cl}_X(\overline{N}\setminus\overline{L})\cap N$$

and

$$\mathrm{cl}_X((\overline{L}\cup \mathrm{cl}_X(\overline{N}\setminus N^*))\setminus\overline{L}) = \mathrm{cl}_X(\overline{N}\setminus(\overline{L}\cup N^*))$$

are isolating neighborhoods of A; and $\overline{L}$ is a positively invariant exit set for $\overline{N}$, hence for $\overline{L}\cup(\overline{N}\cap N)$ and $\overline{L}\cup \mathrm{cl}_X(\overline{N}\setminus N^*)$. □

The same argument shows that $(\overline{N}\cap\tilde{N}_i,(\overline{L}\cap\tilde{N}_i)\cup(\overline{N}\cap N),\overline{L}\cap\tilde{N}_i)$ and $(\overline{N}\cap\tilde{N}_i,(\overline{L}\cap\tilde{N}_i)\cup(\overline{N}\cap(N\cup N_i)),\overline{L}\cap\tilde{N}_i)$ are index triples for $(S_i;A,A^*)$.

THEOREM 2.5. *Suppose S is an isolated invariant set with attractor-repeller pair (A,A^*) and separation $\{C_i\}$ of $C(A^*,A;S)$. If ∂ is the connection map of $(S;A,A^*)$ and ∂_i is the connection map of $(S_i;A,A^*)$, then $\partial=\sum_{i=1}^n\partial_i$.*

PROOF. It suffices to prove the theorem for $n=2$. Take a regular index triple $(\overline{N}_2,\overline{N}_1,\overline{N}_0)=(\overline{N},\overline{L}\cup(\overline{N}\cap N),\overline{L})$ as in 2.4. Then $\overline{N}$ can be viewed as an identification of the spaces $N\cup N_1\cup N^*$ and $N\cup N_2\cup N^*$, formed by identifying the copies of N and N^*. Form M by identifying the $N\cup N_1\cup N^*$ and $N\cup N_2\cup N^*$ along N^* only, and let $p\colon M\to\overline{N}_2$ be the natural projection map. Then the map of triples $(M_2,M_1,M_0)=(M,p^{-1}(\overline{N}_1),p^{-1}(\overline{N}_0))\to(\overline{N}_2,\overline{N}_1,\overline{N}_0)$ induces a commutative diagram

$$\begin{array}{ccccccc}
\cdots\to H_*(M_2,M_0) & \longrightarrow & H_*(M_2,M_1) & \longrightarrow & H_*(M_1,M_0) & \longrightarrow & H_*(M_2,M_0)\to\cdots\\
\downarrow p_{20*} & & \downarrow p_{21*} & & \downarrow p_{10*} & & \downarrow p_{20*}\\
\cdots\to H_*(\overline{N}_2,\overline{N}_0) & \longrightarrow & H_*(\overline{N}_2,\overline{N}_1) & \longrightarrow & H_*(\overline{N}_1,\overline{N}_0) & \longrightarrow & H_*(\overline{N}_2,\overline{N}_0)\to\cdots
\end{array}$$

(i) $p_{21}\colon(M_2,M_1)\to(\overline{N}_2,\overline{N}_1)$ is a relative homeomorphism, so p_{21*} is an isomorphism.

(ii) By excision, there is a commutative diagram

$$\begin{array}{ccc}
H_*(p^{-1}(\overline{N}\cap N),p^{-1}(\overline{L}\cap(\overline{N}\cap N))) & \xrightarrow{\cong} & H_*(M_1,M_0)\\
\downarrow p_* & & \downarrow p_{10*}\\
H_*(\overline{N}\cap N,\overline{L}\cap(\overline{N}\cap N)) & \xrightarrow{\cong} & H_*(\overline{N}_1,\overline{N}_0)
\end{array}$$

Over $N\cap\overline{N}, p$ is a disjoint double cover, so

$$\begin{aligned}
&H_*(p^{-1}(\overline{N}\cap N),p^{-1}(\overline{L}\cap(\overline{N}\cap N)))\\
&\qquad\cong H_*(\overline{N}\cap N,\overline{L}\cap(\overline{N}\cap N))\oplus H_*(\overline{N}\cap N,\overline{L}\cap(\overline{N}\cap N)),
\end{aligned}$$

with $p_*=\mathrm{id}_*+\mathrm{id}_*$.

(iii) As $(\overline{N}_2,\overline{N}_1,\overline{N}_0)$ is an index triple for $(S;A,A^*)$, ∂_N is the connection map of $(S;A,A^*)$.

(iv) By excision, the components of the composition

$$H_*(M_2, M_1) \xrightarrow{\partial_M} H_*(p^{-1}(\overline{N} \cap N), p^{-1}(\overline{L} \cap (\overline{N} \cap N)))$$
$$\xrightarrow{\cong} H_*(\overline{N} \cap N, \overline{L} \cap (\overline{N} \cap N)) \oplus H_*(\overline{N} \cap N, \overline{L} \cap (\overline{N} \cap N))$$

are (∂_1, ∂_2). Namely, to compute one component, excise the other copy of $N \cap \overline{N}$ in $p^{-1}(N \cap \overline{N})$. The resulting triple is then an index triple for $(S_i; A, A^*)$, so the map is ∂_i.

Then $\partial = \partial \circ p_{21*} = p_{10*} \circ \partial = (\mathrm{id}_* + \mathrm{id}_*) \circ (\partial_1, \partial_2) = \partial_1 + \partial_2$. □

COROLLARY 2.6. *Suppose S is isolated with attractor-repeller pair (A, A^*), $\{C_i\}$ a separation of $\bigcup C_i \subseteq C(A^*, A; S)$ such that each $S_i = A \cup C_i \cup A^*$ is isolated. If $\partial \neq \sum_{i=1}^n \partial_i$, then $\bigcup C_i \neq C(A^*, A; S)$.*

In [**9**], Mischaikow studies homoclinic orbits in Hamiltonian systems by relating them to connecting orbits of an attractor-repeller pair in an associated gradient-like system. The connection map associated to each homoclinic is an isomorphism, but the connection map of the total set does not change when the associated heteroclinic orbit is attached. Corollary 2.6 shows that attaching the heteroclinic forces other connecting orbits to be created as well.

3. Hyperbolic critical point attractor-repeller pairs. The simplest example of a connection map is obtained by taking a C^2 flow on a manifold M. Suppose S is an isolated invariant set with attractor-repeller pair $(\{x_0\}, \{x_1\})$, x_1 and x_0 hyperbolic critical points of index k_0 and k_1 respectively. When $k_1 = k_0 + 1$, $CH_*(X; \{x_1\})$ consists of a single copy of $\mathbf{Z}$ in dimension $k_0 + 1$; $CH_*(X; \{x_0\})$ consists of a single copy of $\mathbf{Z}$ in dimension k_0. The connection map is then a homomorphism $\partial\colon \mathbf{Z} \to \mathbf{Z}$. We now compute this homomorphism in the case of transverse intersection of $W^u(x_1)$ and $W^s(x_0)$.

THEOREM 3.1. *If x_1, x_0 are hyperbolic critical points in a manifold M with indices $k+1$ and k and $W^u(x_1)$ transverse to $W^s(x_0)$, then every connecting orbit $\gamma \in C(x_1, x_0)$ has $\bar{\gamma} = \{x_1\} \cup \gamma \cup \{x_0\}$ an isolated invariant set with $I(M; \bar{\gamma}) = \overline{0}$.*

PROOF. As x_1 and x_0 are hyperbolic, they are isolated. $W^u(x_1)$ is transverse to $W^s(x_0)$, so $C(x_1, x_0)$ consists of a finite number of orbits. Thus γ is a component of $C(x_1, x_0)$ and $\bar{\gamma}$ is isolated.

To show $I(M; \bar{\gamma}) = \overline{0}$, consider first $M = \mathbf{R}^n$ with a flow such that $x_1 = (-1, 0, \ldots, 0)$ and $x_0 = (1, 0, \ldots, 0)$ are hyperbolic critical points of index $k+1$ and k respectively, $\bar{\gamma}_0 = \{(x, 0, \ldots, 0) \mid |x| \leq 1\}$, and the x_1-axis is invariant under the flow. We show $I(\mathbf{R}^n; \bar{\gamma}_0) = \overline{0}$ by a series of continuations. Write $\mathbf{x} \in \mathbf{R}^n$ as $(x, \mathbf{y})$, with $\mathbf{y} \in \{0\} \times \mathbf{R}^{n-1}$, and suppose the flow is given by vectorfield $X(\mathbf{x})$.

By the rescaling $(t, x, \mathbf{y}) \mapsto (t, x, \varepsilon^{-1}\mathbf{y})$, X can be perturbed to its first order (in $\mathbf{y}$) terms. $\bar{\gamma}_0$ is unchanged by this rescaling, and continues throughout the rescaling. That is, $\overline{\gamma}_0$ in the X flow is related by continuation to $\bar{\gamma}_0$ in the flow induced by $X_1(x, \mathbf{y}) = (X(x, \mathbf{0}) + X_1(x)\mathbf{y}, X_2(x)\mathbf{y}, \ldots, X_n(x)\mathbf{y})$. Next, continue to $X_2(x, \mathbf{y}) = (X(x, \mathbf{0}),\ X_2(x)\mathbf{y}, \ldots, X_n(x)\mathbf{y})$. For every x, $\mathbf{0}$ is an isolated rest point in the $A(x)\mathbf{y} = (X_2(x)\mathbf{y}, \ldots, X_n(x)\mathbf{y})$ flow on $\mathbf{R}^{n-1}$ with $I(\mathbf{R}^{n-1}; \{\mathbf{0}\}) = \Sigma^k$. As $\overline{\gamma}_0$ is isolated in the $X(x, \mathbf{0})$ flow on $\mathbf{R}$, $\overline{\gamma}_0 = \overline{\gamma}_0 \times \{\mathbf{0}\}$ is isolated in the X_2 flow, and related by continuation to $\overline{\gamma}_0$ in the X_1 flow. The X_2 flow continues to a product flow $X_3(x, \mathbf{y}) = (X(x, \mathbf{y}),\ A(0)\mathbf{y})$, with $\overline{\gamma}_0$ continuing throughout. Then in the X_3 flow, $I(\mathbf{R}^n; \overline{\gamma}_0) = I(\mathbf{R}; \overline{\gamma}_0) \times I(\mathbf{R}^{n-1}; \{\mathbf{0}\})$, with $I(\mathbf{R}; \overline{\gamma}_0) = \overline{0}$. That is, $\overline{\gamma}_0$ has an isolating neighborhood $\{|t| \leq 1 + \varepsilon\}$, with exit set $\{-1 - \varepsilon\}$.

For the general case, choose a map $f \colon \gamma \to \mathbf{R}^n$ taking γ diffeomorphically onto $\gamma_0 = (-1, 1) \times \mathbf{R}^{n-1}$. Choose orbits $\gamma_u \in W^u(x_1)$, $\gamma_s \in W^s(x_0)$ whose unit tangent vectors at x_1, x_0 are negatives of those of γ, and a smooth embedding

$$f \colon \Gamma = \gamma_u \cup \{x_1\} \cup \gamma \cup \{x_0\} \cup \gamma_s \to \mathbf{R} \times \{\mathbf{0}\} \subseteq \mathbf{R}^n.$$

(The flow can be perturbed away from γ if necessary to guarantee that f is an embedding.) As $\Gamma \cong \mathbf{R}$, f extends to a tubular neighborhood $f \colon E \to \mathbf{R}^n$. The flow $\dot{\mathbf{x}} = Df(f^{-1}(\mathbf{x}))X(f^{-1}(\mathbf{x}))$ on $\mathbf{R}^n$ has $I(\mathbf{R}^n; \bar{\gamma}_0) = \overline{0}$, and f is a flow map. Thus $f^{-1}(\bar{\gamma}_0) = \bar{\gamma}$ is isolated in E (hence in M) and $I(f)$ is an isomorphism, so $I(M; \bar{\gamma}) = I(E; \bar{\gamma}) = I(\mathbf{R}^n; \bar{\gamma}_0) = \overline{0}$. □

COROLLARY 3.2. *For each such γ, $(\{x_1\}, \{x_0\})$ is an attractor-repeller pair for $\overline{\gamma}$, with connection map $\partial \colon CH_{k+1}(M; x_1) \to CH_k(M; x_0)$ an isomorphism.*

COROLLARY 3.3. *If $k_1 = k_0 + 1$ and $W^u(x_1)$ is transverse to $W^s(x_0)$, then $S = \mathrm{cl}_M C(x_1, x_0)$ is an isolated invariant set. $C(x_1, x_0)$ consists of finitely many orbits, with $|C(x_1, x_0)/\mathbf{R}| \geq |\partial|$ and $|C(x_1, x_0)/\mathbf{R}| \equiv |\partial| \bmod 2$, where $|\partial| = n$ if and only if $\mathrm{coker}(\partial) \cong \mathbf{Z}/n\mathbf{Z}$.*

PROOF. By transversality, $C(x_1, x_0)/\mathbf{R}$ is finite and the collection of connecting orbits is an invariant separation of $C(x_1, x_0)$. For each orbit, the connection map is an isomorphism, so $\mathrm{coker}(\partial_i) = 0$ and $|\partial_i| = 1$. As $\partial = \sum_{i=1}^n \partial_i$,

$$|\partial| \leq \sum_{i=1}^{n} |\partial_i| = |C(x_1, x_0)/\mathbf{R}|, \quad \text{and} \quad |\partial| \equiv \partial \equiv \sum_{i=1}^{n} \partial_i \equiv \sum_{i=1}^{n} |\partial_i| \mod 2. \quad \square$$

Thus, in the case of transverse intersection, $C(x_1, x_0; S)$ has a finest invariant separation (with elements consisting of single connecting orbits) and ∂_S is trivial only when the isomorphisms ∂_i of the orbits cancel. If $W^u(x_1)$ and $W^s(x_0)$ are not transverse, there exists isolated $(S; x_0, x_1)$ such that $C(x_1, x_0; S)$ is connected and ∂_S is trivial. However, if $W^u(x_1)$ and $W^s(x_0)$ are not transverse, they can be made transverse by an arbitrarily small perturbation of the flow. For sufficiently small perturbations, $(S; x_0, S_1)$ continues as an attractor-repeller decomposition, so ∂_S remains unchanged. That is, every transverse perturbation has connecting orbit set $C(x_1, x_0; S')$ with $|C(x_1, x_0; S')/\mathbf{R}| \geq |\partial_S|$.

We now return to the flows on spheres from §1 in the following setting: M a manifold with C^2 flow, x_0, x_1 hyperbolic critical points of index 0 and k, S an isolated invariant set with $(\{x_0\}, \{x_1\})$ an attractor-repeller pair. In this setting, $W^u(x_1)$ and $W^s(x_0)$ are necessarily transverse.

THEOREM 3.4. *Suppose S is a compact invariant set with attractor-repeller pair $(\{x_0\}, \{x_1\})$. Then S is isolated in M if and only if S is a topological embedding of a 0-sphere or a k-sphere, or $k = 1$ and S is a topological embedding of the unit interval. If S is a sphere, the connection map of $(S; \{x_1\}, \{x_0\})$ is trivial; if S is an interval, the connection map of $(S; \{x_1\}, \{x_0\})$ is an isomorphism.*

PROOF. (i) Suppose S is an embedding of S^0, S^k, or an interval. For $k = 1$, it follows from 3.1 that S is isolated in M. For $k > 1$, we apply 2.1. As x_1

and x_0 are hyperbolic, they are isolated. Embeddings of spheres are compact, so S will be isolated if $C(x_1, x_0; S)$ is open in $C(x_1, x_0)$. If S is an embedding of S^0, $C(x_1, x_0; S) = \varnothing$; if S is an embedding of S^k, $C(x_1, x_0; S) = C(x_1, x_0)$.

(ii) Suppose S is isolated and $k = 1$. $W^u(x_1)$ consists of x_1 and two orbits unstable to x_1. Thus $C(x_1, x_0; S)$ consists of 0, 1 or 2 orbits. The intersection $W^u(x_1) \cap W^s(x_0)$ is necessarily transverse, so S is a topological embedding of $S^0, [0,1]$, or S^1.

The relevant segment of the homology attractor-repeller sequence is

$$0 \to CH_1(M; S) \to CH_1(M; x_1) \xrightarrow{\partial} CH_0(M; x_0) \to CH_0(M; S) \to 0.$$

$CH_0(M; S) \cong \tilde{H}_0(N/L)$ for some index pair (N, L). In particular, it is free abelian. As $CH_0(M; x_0) \cong \mathbf{Z}$ maps onto it, it is either 0 or $\mathbf{Z}$. From the exact sequence, $CH_0(M; S) = \operatorname{coker}(\partial)$, so by 3.3, $|\partial| \leq 1$.

If $S \cong S^0$, then $C(x_1, x_0) = \varnothing$ and $\partial = 0$.

If $S \cong [0,1]$, then ∂ is an isomorphism by 3.2.

If $S \cong S^1$, then by deleting the connecting orbits one at a time, isolated invariant sets S_1, S_2 are formed with $\partial = \partial_1 + \partial_2$ and ∂_1 and ∂_2 isomorphisms. Thus $|\partial| = 0, 2$. But $|\partial| \leq 1$, so $\partial = |\partial| = 0$.

(iii) If S is isolated and $k > 1$, dimensions force the connection map to be trivial. If $C(x_1, x_0; S) = \varnothing$, then S is a 0-sphere. If $C(x_1, x_0; S) \neq \varnothing$, then $W^u(x_1) = \{x_1\} \cup C(x_1, x_0; S)$.

That is, $\{x_1\} \cup C(x_1, x_0; S) \subseteq W^u(x_1)$ by definition. Suppose

$$W^u(x_1) \not\subseteq \{x_1\} \cup C(x_1, x_0; S).$$

If $C(x_1, x_0; S) \neq \varnothing$, choose a $y_1 \in C(x_1, x_0; S)$ and a $y_2 \in W^u(x_1) \setminus C(x_1, x_0; S)$. Let $M_0 = \mathrm{cl}_M(\{x_0, x_1, y_1, y_2\} \cdot \mathbf{R})$. M_0 is a closed invariant subset of M with $M_0 \cap S = \{x_0, x_1, y_1\} \cdot \mathbf{R}$. In M_0, x_1 and x_0 are hyperbolic with indices 1 and 0 and $y_1 \cdot \mathbf{R}$ is the sole connecting orbit. Then from (ii), the homology indices are

$$\begin{aligned} CH_*(M_0; x_0) &= (\mathbf{Z}, 0, 0, \dots), \\ CH_*(M_0; M_0 \cap S) &= (0, 0, 0, \dots), \\ CH_*(M_0; x_1) &= (0, \mathbf{Z}, 0, \dots), \end{aligned}$$

and the M_0 connection map $\partial_0 \colon CH_1(M_0; x_1) \to CH_0(M_0; x_0)$ is an isomorphism.

But $CH_*(M; x_0) = (\mathbf{Z}, 0, 0, \dots)$, $CH_*(M; x_1) = (0, \dots, 0, \mathbf{Z}, 0, \dots)$ (with the $\mathbf{Z}$ in dimension $k > 1$), so it must have $\partial_M = 0$. Further, the inclusion $i \colon M_0 \to M$ must have $i_* \colon CH_*(M_0; x_1) \to CH_*(M; x_1)$ trivial and

$$i_* \colon CH_*(M_0; x_0) \to CH_*(M; x_0)$$

an isomorphism. That is, $i_* \circ \partial_0$ is an isomorphism and $\partial_M \circ i_*$ is trivial, so $i_* \circ \partial_0 \neq \partial_M \circ i_*$. But in [**7**] it is shown that $i_* \circ \partial_0 = \partial_M \circ i_*$, so $W^u(x_1) \subseteq \{x_1\} \cup C(x_1, x_0; S)$.

Thus $\{x_1\} \cup C(x_1, x_0; S)$ is an injective immersion of $\mathbf{R}^k$ in M by the stable manifold theorem. As $C(x_1, x_0; S) \subseteq W^s(x_0)$, this extends to a continuous bijection of S^k onto S by sending ∞ to x_0. S^k is compact and M is Hausdorff, so S is a topological embedding of S^k in M. □

REMARKS. (i) This result differs from Reeb's theorem in that Reeb's hypothesis that S be a compact manifold is replaced by the assumptions that S be isolated in a manifold and have $k_0 = 0$ and ∂ trivial.

(ii) S need not be a submanifold of M, as the embedding of $W^u(x_1)$ need not extend differentiably to x_0.

(iii) The homology Conley index by itself is not sufficiently refined to distinguish the 0-sphere and k-sphere cases. However, by considering additional structures on the index (such as the mappings introduced in [**6**]) it may be possible to distinguish them by some homological invariant.

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