TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 307, Number 1, May 1988

ISOMETRY GROUPS OF RIEMANNIAN SOLVMANIFOLDS

CAROLYN S. GORDON AND EDWARD N. WILSON

ABSTRACT. A simply connected solvable Lie group R together with a left-invariant Riemannian metric g is called a (simply connected) Riemannian solvmanifold. Two Riemannian solvmanifolds (R, g) and (R', g') may be isometric even when R and R' are not isomorphic. This article addresses the problems of (i) finding the "nicest" realization (R, g) of a given solvmanifold, (ii) describing the embedding of R in the full isometry group $I(R, g)$, and (iii) testing whether two given solvmanifolds are isometric. The paper also classifies all connected transitive groups of isometries of symmetric spaces of noncompact type and partially describes the transitive subgroups of $I(R, g)$ for arbitrary solvmanifolds (R, g).

Introduction. A Riemannian solvmanifold is a connected Riemannian manifold $\mathcal{M}$ which admits a transitive solvable group of isometries. It is well known (see §1) that every such manifold actually admits an almost simply transitive solvable group R of isometries, simply transitive if $\mathcal{M}$ is simply connected. We will assume for simplicity in our introductory remarks that $\mathcal{M}$ is simply connected, although this assumption is dropped throughout much of the paper. The simply transitive group R is in general not unique even up to isomorphism. Given any choice of R, $\mathcal{M}$ is isometric to R equipped with a left-invariant metric $\langle\ ,\ \rangle$. We address the following problems:

(i) Given the data $(R, \langle\ ,\ \rangle)$ for $\mathcal{M}$, find the "nicest" simply transitive solvable group S of isometries of $\mathcal{M}$, i.e. find the nicest realization $(S, \langle\ ,\ \rangle')$ of $\mathcal{M}$.

(ii) Given $(R, \langle\ ,\ \rangle)$, describe the full isometry group of $\mathcal{M}$.

(iii) Develop a test to determine whether any two given Riemannian solvmanifolds are isometric.

In §1, we construct a single conjugacy class of simply transitive solvable subgroups of the full isometry group $I(\mathcal{M})$ which we call the subgroups in "standard position". They are defined entirely in terms of their embeddings in $I(\mathcal{M})$. These play the role of the "nicest" groups in (i). §2 develops the abstract theory of modifications of solvable Lie algebras needed for §3. In §3, beginning with the data $(R, \langle\ ,\ \rangle)$ and no other knowledge of $I(\mathcal{M})$, we algorithmically modify $(R, \langle\ ,\ \rangle)$ to obtain $(S, \langle\ ,\ \rangle')$ where S is a simply transitive subgroup of $I(\mathcal{M})$ in standard position. We also compute the normalizer of S in $I(\mathcal{M})$. This together with the results of §1 gives considerable information about $I(\mathcal{M})$. In many cases, such as when R is unimodular (see §4), S is normal in $I(\mathcal{M})$ and hence a complete solution to problem (ii) is obtained.

Received by the editors March 25, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C20, 53C30; Secondary 53C35.

The first author was partially supported by the National Science Foundation grant 8502084.

Problem (iii) is answered in §5. In addition we develop a test for determining when two locally homogeneous metrics on $\Gamma\backslash R$ are isometric, where R is a simply connected Lie group and Γ is a uniform (i.e. cocompact) discrete subgroup. We indicate some applications at the conclusion of the section.

Related to problems (i) and (ii) is the problem of finding all transitive subgroups of $I(\mathcal{M})$ (not necessarily simply transitive or solvable). We address this problem in §6, first in case $\mathcal{M}$ has nonpositive curvature. In particular, when $\mathcal{M}$ is a symmetric space of noncompact type, we classify all connected transitive subgroups of $I(\mathcal{M})$. We apply this special case to partially analyze the transitive subgroups of $I(\mathcal{M})$ for an arbitrary solvmanifold.

Problems analogous to (i)–(iii) have been addressed for various other types of homogeneous Riemannian manifolds in, for example, [**A-W1**, **2**, **G1**, **G2**, **O-T**, **On**, **Oz** and **W**]. In [**G-W2**], we will combine the results of this paper with the other special cases to study arbitrary homogeneous Riemannian manifolds.

1. Subgroups in standard position. Lie groups will always be denoted by capital italic letters, sometimes with subscripts, superscripts, etc. Their Lie algebras will be denoted by the corresponding lowercase German letters with the same subscripts, superscripts, etc.

(1.1) DEFINITIONS. (i) A *Riemannian solvmanifold* is a connected Riemannian manifold $\mathcal{M}$ which is acted upon transitively by a solvable Lie group of isometries. A transitive subgroup R of $I(\mathcal{M})$, the group of all isometries on $\mathcal{M}$, is said to act *almost simply transitively* on $\mathcal{M}$ if for some (then every) $p \in \mathcal{M}$, the subgroup of R fixing p is discrete. We note that $I(\mathcal{M})$ and consequently R act effectively on $\mathcal{M}$; i.e. the only element of $I(\mathcal{M})$ which acts as the identity transformation on $\mathcal{M}$ is the identity element.

(ii) $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ is *a data triple for a Riemannian solvmanifold* if $\mathfrak{r}$ is a solvable Lie algebra finite dimensional over $\mathbb{R}$, $\langle\cdot,\cdot\rangle$ is an inner product on $\mathfrak{r}$, and for $\tilde{R}$ the unique connected, simply connected Lie group with Lie algebra $\mathfrak{r}$, D is a discrete subgroup of $\{x \in \tilde{R} \colon \operatorname{Ad}(x) \text{ is orthogonal relative to } \langle\cdot,\cdot\rangle\}$.

Given the triple $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$, we associate with it the manifold $\mathcal{M} = \tilde{R}/D$ equipped with the unique Riemannian metric which agrees with $\langle\cdot,\cdot\rangle$ on the tangent space $\mathcal{M}_{eD} \approx \mathfrak{r}$ and for which the natural left action of $\tilde{R}$ on $\mathcal{M}$ is by isometries. Then $\mathcal{M}$ is a Riemannian solvmanifold and for C the intersection of D with the center of $\tilde{R}$, $R = \tilde{R}/C$ acts effectively, isometrically, and almost simply transitively on $\mathcal{M}$.

If $\mathcal{M}$ is simply connected, then $D = \{e\}$. At times when we are restricting attention to simply connected manifolds, we will replace the data triple by the *data pair* $(\mathfrak{r}, \langle\ ,\ \rangle)$.

(iii) Two data triples $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ and $(\mathfrak{r}', \langle\cdot,\cdot\rangle', D')$ will be said to be *isomorphic* if there exists a Lie algebra isomorphism $\varphi \colon \mathfrak{r} \to \mathfrak{r}'$ such that $\langle\cdot,\cdot\rangle' = (\Phi^{-1})^*\langle\cdot,\cdot\rangle$ and $D' = \Phi(D)$ where $\Phi \colon \tilde{R} \to \tilde{R}'$ is the corresponding isomorphism of simply connected Lie groups. The data pairs $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$ and $(\mathfrak{r}', \langle\cdot,\cdot\rangle)$ will be said to be isomorphic if the triples $(\mathfrak{r}, \langle\cdot,\cdot\rangle, \{e\})$ and $(\mathfrak{r}', \langle\cdot,\cdot\rangle', \{e\})$ are isomorphic. It is easily verified that isomorphic data triples (pairs) define isometric Riemannian solvmanifolds.

(1.2) LEMMA. *Every Riemannian solvmanifold is defined by a data triple in the sense above. In particular, every Riemannian solvmanifold admits almost simply transitive isometry groups.*

PROOF. Let $\mathcal{M}$ be a Riemannian solvmanifold, H a transitive solvable isometry group on $\mathcal{M}$, p a chosen base point in $\mathcal{M}$, and L_p the subgroup of $I(\mathcal{M})$ fixing p. Since L_p is compact, there is an inner product on the Lie algebra $\mathfrak{g}$ of $I(\mathcal{M})$ for which the adjoint action of L_p on $\mathfrak{g}$ is orthogonal. For $x \in N$, the connected nilradical of H, $\mathrm{Ad}(x)$ is unipotent on the Lie algebra of H. Since L_p acts effectively on $\mathcal{M} = H/(H \cap L_p)$, $L_p \cap N$ is the identity. Since any subspace of $\mathfrak{g}$ containing $\mathfrak{n}$ is an $\mathfrak{h}$-ideal, we have

$$\mathfrak{h} = (\mathfrak{l}_p \cap \mathfrak{h}) + \mathfrak{r}$$

for some ideal $\mathfrak{r}$. The corresponding connected subgroup R of H satisfies $H = (L_p \cap H)R$ with $L_p \cap R$ discrete. Under the covering map $r \to r \cdot p$ from R to $\mathcal{M}$, the Lie algebra $\mathfrak{r}$ is identified with the tangent space $\mathcal{M}_p$ and the metric on $\mathcal{M}$ thereby assigns to $\mathfrak{r}$ an inner product $\langle \cdot, \cdot \rangle_p$. $\mathcal{M}$ is defined up to isometry by the data triple $(\mathfrak{r}, \langle \cdot, \cdot \rangle_p, D_p)$ where D_p is the inverse image of $L_p \cap R$ under the covering of R by its universal covering group $\tilde{R}$.

(1.3) DEFINITION AND REMARKS. The inner product $\langle \cdot, \cdot \rangle_p$ on $\mathfrak{r}$ defined in the proof of Lemma 1.2 will be called the p-inner product. Note that if $q \in \mathcal{M}$, then there is an element of R (the connected subgroup of $I(\mathcal{M})$ with Lie algebra $\mathfrak{r}$) which carries p to q. The corresponding inner automorphism defines an isomorphism between $(\mathfrak{r}, \langle\ ,\ \rangle_p, D_p)$ and $(\mathfrak{r}, \langle\ ,\ \rangle_q, D_q)$. Thus we will usually drop the subscripts p. $\mathcal{M}$ may admit many distinct conjugacy classes of almost simply transitive isometry groups. In this section we will identify a canonical conjugacy class whose elements we call the groups *in standard position* within $I_0(\mathcal{M})$, the connected component of the identity in $I(\mathcal{M})$. As we proceed toward the definition of standard position, it may be helpful to keep the following examples in mind.

(1.4) EXAMPLES. (i) When $\mathcal{M}$ is a Riemannian nilmanifold, i.e. $\mathcal{M}$ is a Riemannian solvmanifold with a transitive nilpotent isometry group H, Wilson [**W**] showed that the connected component N of the identity in $\mathfrak{h}$ acts simply transitively on $\mathcal{M}$, is normal in $I(\mathcal{M})$, and is the unique simply transitive connected nilpotent Lie subgroup of $I(\mathcal{M})$. However, $\mathcal{M}$ may admit many other simply transitive solvable groups of isometries. For example, when $\mathcal{M}$ is n-dimensional Euclidean space, N is the group $\mathbb{R}^n$ of translations. For $n \geq 3$, it is easy to construct other simply transitive solvable subgroups of $I_0(\mathbb{R}^n) = SO(n) \cdot \mathbb{R}^n$. As we will see later, the definition of standard position given below implies that when $\mathcal{M}$ is a nilmanifold, the nilpotent group N is the unique group in standard position.

(ii) When $\mathcal{M}$ is a homogeneous Riemannian manifold with nonpositive sectional curvature, Azencott and Wilson [**A-W1, 2**] showed that $\mathcal{M}$ is a Riemannian solvmanifold and defined a notion of standard position. The definition we shall give here is an extension of theirs to the general Riemannian solvmanifold. In the special case when $\mathcal{M}$ is a symmetric space of the noncompact type, the groups in standard position are those of the form $S = AN$ for KAN an Iwasawa decomposition of $G = I_0(\mathcal{M})$. For R an arbitrary simply transitive subgroup of G, it is easy to see (e.g. cf. [**A-W2**]) that there exists an Iwasawa decomposition of G for which $N_G(R) \subset N_G(AN)$. Even when the normalizers coincide, AN is preferable to R

on the grounds of structural simplicity. Similar comments apply to the groups in standard position constructed by Azencott and Wilson in the general nonpositive curvature case.

(1.5) DEFINITIONS AND NOTATIONS. (i) We shall systematically use the following subscript conventions to denote important subgroups of a Lie group H and its Lie algebra $\mathfrak{h}$:

H_2 = connected radical of H,

H_1 = any maximal semisimple subgroup of H,

H_{nc} (resp., H_c) = maximal normal subgroup of H_1 of the noncompact (resp., compact) type, once H_1 is chosen,

$\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_c, \mathfrak{h}_{nc}$ = Lie algebras of H_1, H_2, H_c, H_{nc}.

Thus H has the Levi decomposition $H = H_1H_2$ and $H_1 = H_{nc}H_c$ with discrete intersections. Recall that H_1 is unique up to conjugacy by elements of the connected nilradical (e.g. see Jacobson [**J**]) and H_{nc} and H_c are uniquely determined by H_1. At the Lie algebra level, $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ is a vector space direct sum with $\mathfrak{h}_2$ the radical of $\mathfrak{h}$. Also $\mathfrak{h}_1 = \mathfrak{h}_{nc} + \mathfrak{h}_c$ with $\mathfrak{h}_{nc}$ (resp., $\mathfrak{h}_c$) the sum of all ideals of $\mathfrak{h}_1$ of the noncompact (resp. compact) type.

(ii) When $H = H_{nc}$ is semisimple of the noncompact type, an Iwasawa decomposition of H (respectively, $\mathfrak{h}$) will be denoted by $H = KAN$ (resp. $\mathfrak{h} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$). Thus $\mathfrak{s} \equiv \mathfrak{a} + \mathfrak{n}$ is a solvable Lie algebra with nilradical $\mathfrak{n}$, $\mathfrak{a}$ is abelian and the roots of $\mathrm{ad}_{\mathfrak{h}}\,\mathfrak{a}$ are real, and $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{h}$. At the group level, K, A and N are connected, the center Z of H is contained in K, and K/Z is compact. Recall that K is unique up to conjugacy in H and, once K is chosen, A is unique up to conjugacy by elements of K, and N is unique up to conjugacy by elements of the normalizer M of A in K. The subgroup MAN of H contains a maximal connected solvable subgroup of H. Note that our M is usually denoted M' in the literature; otherwise our terminology is standard. For a further discussion of Iwasawa decompositions, see [**H**].

(iii) When H is compact, recall that H_2 is central in H and $\mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}]$ is the unique maximal semisimple subalgebra of $\mathfrak{h}$. Hence the Levi decomposition $H = H_1H_2$ is unique.

(iv) For $\mathfrak{h}$ arbitrary, a Lie subalgebra $\mathfrak{l}$ of $\mathfrak{h}$ is said to be *compactly embedded* in $\mathfrak{h}$ if $\mathfrak{h}$ admits an inner product relative to which the operators $\mathrm{ad}_{\mathfrak{h}} X$, $X \in \mathfrak{l}$, are skew-symmetric. In particular, if $\mathfrak{h}$ is the Lie algebra of a transitive group H of isometries of a Riemannian manifold M and L is the isotropy subgroup of H at a point $p \in M$, then $\mathfrak{l}$ is compactly embedded in $\mathfrak{h}$. Note however that unless H is closed in $I(\mathcal{M})$, L need not be compact in the topology of H.

It is well known that if $\mathcal{M}$ is a Riemannian manifold of nonpositive curvature and C is any compact subgroup of $I(\mathcal{M})$, then C has a fixed point in $\mathcal{M}$. For a general Riemannian solvmanifold, we have the following weaker result:

(1.6) LEMMA. *Let $\mathcal{M}$ be a Riemannian solvmanifold and C any connected compact semisimple group of isometries on $\mathcal{M}$. Then there exists $p \in \mathcal{M}$ for which C is contained in the isotropy subgroup L of $I(\mathcal{M})$ at p.*

PROOF. Let $G = I_0(\mathcal{M})$ and let V be a maximal compact subgroup of G containing the isotropy subgroup of G at some point $q \in \mathcal{M}$. By the conjugacy of maximal compact subgroups of G, a conjugate U of V contains both C and the isotropy subgroup L at a point $p \in \mathcal{M}$. Let $\mathfrak{r}$ be the Lie algebra of a solvable transitive subgroup R. Then $\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$ whence $\mathfrak{u} = \mathfrak{l} + \mathfrak{u} \cap \mathfrak{r}$ with $\mathfrak{u} \cap \mathfrak{r}$ both compact and solvable, thus abelian. Theorem 1.1 of Oniscik [**On**] states that if $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{d}$ are compact Lie algebras satisfying $\mathfrak{a} = \mathfrak{b} + \mathfrak{d}$, then $[\mathfrak{a}, \mathfrak{a}] = [\mathfrak{b}, \mathfrak{b}] + [\mathfrak{d}, \mathfrak{d}]$. Thus $[\mathfrak{u}, \mathfrak{u}] = [\mathfrak{l}, \mathfrak{l}]$ and $\mathfrak{c} \subset \mathfrak{u}_1 = \mathfrak{l}_1 \subset \mathfrak{l}$, so $C \subset L$.

In order to define the solvable subgroups of $I_0(\mathcal{M})$ in "standard position", we need to first define a conjugacy class of subgroups F of $I_0(\mathcal{M})$. These groups will turn out to be the normalizers of the subgroups in standard position.

(1.7) PROPOSITION. *Let $\mathcal{M}$ be a Riemannian solvmanifold, $G = I_0(\mathcal{M})$ and, in the notation of* 1.5, *let G_1G_2 be a Levi decomposition of G and KAN an Iwasawa decomposition of G_{nc}. Let $F = MANG_cG_2$. Then*

(i) *F is a self-normalizing subgroup of G.*

(ii) *F has Levi decomposition F_1F_2 with $F_2 = M_2ANG_2$ and $F_1 = M_1G_c$. (Note that M_1 and M_2 are uniquely defined by* 1.5(iii).) *In particular, $F_{nc} = \{e\}$, i.e. F contains no nontrivial noncompact semisimple subgroups.*

(iii) *Among Lie subgroups of G, F is maximal with respect to the property that $F_{nc} = \{e\}$.*

(iv) *F acts transitively on M. Moreover if R is any connected, solvable, transitive subgroup of G, then F contains a conjugate of R.*

(v) *There exists $p \in \mathcal{M}$ such that for L the isotropy subgroup of G at p, we have $L \cap (\operatorname{nilrad} F) = \{e\}$ and $M_1G_c \subset L \cap F \subset MG_cG_2$.*

PROOF. (i) follows from the fact that if $x \in G_{nc}$ normalizes MAN, it also normalizes the nilradical N of MAN and then must lie in MAN.

(ii) is trivial and (iii) follows from the fact that any subgroup of G_{nc} properly containing MAN must contain a noncompact semisimple subgroup (e.g. see §6).

(iv) Let π be the canonical projection from G onto $\tilde{G} = G/ZG_cG_2$ where Z is the center of G. Then $\tilde{G} = \pi(L)\pi(R)$ is semisimple of noncompact type with $\pi(L)$ compact and $\pi(R)$ solvable. Choose a maximal compact subgroup $\tilde{K}$ of $\tilde{G}$ containing $\pi(L)$. Lemma 6.6 of [**A-W2**] states that if $\tilde{G}$ is noncompact semisimple with finite center, $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$, and Σ is a solvable subgroup such that $\tilde{G} = \tilde{K}\Sigma$, then there exists an Iwasawa decomposition $\tilde{G} = \tilde{K}\tilde{A}\tilde{N}$ such that $\Sigma \subset \tilde{M}\tilde{A}\tilde{N}$ where $\tilde{M}$ is the normalizer of $\tilde{A}\tilde{N}$ in $\tilde{K}$. We apply this lemma with $\Sigma = \pi(R)$. It follows that there exists an inner automorphism Φ of G such that $\pi(\Phi(R)) \subset \pi(MAN)$. But then $\Phi(R) \subset F$. Since $\Phi(R)$ is transitive on $\mathcal{M}$, so is F.

For (v), note first that Lemma 1.6 proves the existence of L such that $M_1G_c \subset L \cap F$. For the second containment, the homomorphic projection of $\mathfrak{f}$ onto $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ carries $\mathfrak{l} \cap \mathfrak{f}$ onto a compactly embedded subalgebra of $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ containing $\mathfrak{m}_1$. Any such subalgebra is contained in $\mathfrak{m}$. Finally the fact that $L \cap \operatorname{nilrad}(F) = \{e\}$ follows from (iv) and the argument given in the proof of Lemma 1.2.

(1.8) LEMMA. *Let $\mathcal{M}$ be a Riemannian solvmanifold and H a subgroup of $I_0(\mathcal{M})$ which contains a solvable subgroup R normal in H and transitive on $\mathcal{M}$. Let p be any point in $\mathcal{M}$, L the isotropy subgroup of H at p and $\mathfrak{s}$ the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{h}$ relative to the Killing form B of $\mathfrak{h}$. Then*

(i) *$\mathfrak{s}$ is a solvable ideal of $\mathfrak{h}$ which contains* $\operatorname{nilrad}(\mathfrak{h})$;

(ii) $\mathfrak{s}$ *is independent of the choice of* p;

(iii) *the connected Lie subgroup* S *of* H *with Lie algebra* $\mathfrak{s}$ *is normal in* H *and acts almost simply transitively on* $\mathcal{M}$.

PROOF. Since $H = LR$, we have $\mathfrak{h} = \mathfrak{l} + \mathfrak{r}$ with $\mathfrak{r}$ a solvable ideal in $\mathfrak{h}$. In particular, $\mathfrak{r} \subset \mathfrak{h}_2$ and $\mathfrak{l}_1 = [\mathfrak{l}, \mathfrak{l}]$ is a semisimple Levi factor of $\mathfrak{h}$. Since $B(\mathfrak{h}_1, \mathfrak{h}_2) = 0$ for any Levi decomposition $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ of $\mathfrak{h}$, we have $B(\mathfrak{l}_1, \mathfrak{h}_2) = 0$. Moreover, B is negative definite on the compactly embedded algebra $\mathfrak{l}$. It follows that $\mathfrak{s} \subset \mathfrak{h}_2$. Since $\mathfrak{s} \supset \ker B \supset \operatorname{nilrad}(\mathfrak{h})$, $\mathfrak{s}$ is a solvable ideal in $\mathfrak{h}$. If q is another point of $\mathcal{M}$ with $\tilde{\mathfrak{l}}$ the Lie algebra of the isotropy subgroup of H at q, then $\tilde{\mathfrak{l}} = \operatorname{Ad}(x)\mathfrak{l}$ for some $x \in H$ such that $x \cdot p = q$. Since B is $\operatorname{Ad}(x)$ invariant, $\operatorname{Ad}(x)\mathfrak{s}$ is the orthogonal complement of $\tilde{\mathfrak{l}}$ relative to B. But $\operatorname{Ad}(x)\mathfrak{s} = \mathfrak{s}$ since $\mathfrak{s}$ is an ideal in H. Hence $\mathfrak{s}$ is defined independently of p. By construction $\mathfrak{h} = \mathfrak{l} + \mathfrak{s}$ with $\mathfrak{l} \cap \mathfrak{s} = \{0\}$. Since S is normal in H, we have $H = LS$ and (iii) follows.

(1.9) THEOREM. *Let* $F = MANG_cG_2$ *be as in* 1.7 *and let* L *be the isotropy subgroup of* G *at* $p \in \mathcal{M}$. *Define* $\mathfrak{s}$ *as the orthogonal complement in* $\mathfrak{f}$ *of* $\mathfrak{l} \cap \mathfrak{f}$ *relative to the Killing form on* $\mathfrak{f}$. *Then*

(i) $\mathfrak{s}$ *is a solvable ideal of* $\mathfrak{f}$ *and is independent of the choice of* p.

(ii) *The connected Lie subgroup* S *of* F *with Lie algebra* $\mathfrak{s}$ *acts almost simply transitively on* $\mathcal{M}$..

(iii) $\mathfrak{a} + \mathfrak{n} + [\mathfrak{g}, \mathfrak{g}_2] \subset \mathfrak{s} \subset \mathfrak{f}_2 = \mathfrak{m}_2 + \mathfrak{a} + \mathfrak{n} + \mathfrak{g}_2$.

PROOF. By Proposition 1.7(iv) and Lemma 1.6, F_2 acts transitively on $\mathcal{M}$. Therefore (i) and (ii) are immediate from Lemma 1.8. By (i) we have $\mathfrak{s} \subset \mathfrak{f}_2$ and by Lemma 1.8(i), we have

$$\mathfrak{s} \supset \operatorname{nilrad}(\mathfrak{f}) = \mathfrak{n} + \operatorname{nilrad}(\mathfrak{g}_2).$$

Since $[\mathfrak{g}, \mathfrak{g}_2] \subset \operatorname{nilrad}(\mathfrak{g}_2)$, it remains only to show that $\mathfrak{a} \subset \mathfrak{s}$. By Proposition 1.7(v), $\mathfrak{l} \cap \mathfrak{f} \subset \mathfrak{m} + \mathfrak{g}_c + \mathfrak{g}_2$, so it suffices to show that $B(\mathfrak{a}, \mathfrak{m} + \mathfrak{g}_c + \mathfrak{g}_2) = \{0\}$ for B the Killing form of $\mathfrak{f}$. For B' the Killing form of $\mathfrak{g}$, we have $B'(\mathfrak{g}_1, \mathfrak{g}_2) = \{0\}$. But for $X \in \mathfrak{g}$, $Y \in \mathfrak{g}_2$,

$$B(X, Y) = \operatorname{trace}_{\mathfrak{f}}(\operatorname{ad} X)(\operatorname{ad} Y) = \operatorname{trace}_{\mathfrak{g}_2}(\operatorname{ad} X)(\operatorname{ad} Y) = B'(X, Y).$$

Since $\mathfrak{a} \subset \mathfrak{g}_1$, we obtain $B(\mathfrak{a}, \mathfrak{g}_2) = \{0\}$. A similar argument shows that $B(\mathfrak{a}, \mathfrak{g}_c) = \{0\}$. Finally by well-known properties of semisimple algebras and their representations, there exists an inner product on $\mathfrak{g}$ such that for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{m}$, $\operatorname{ad}(X)$ is symmetric while $\operatorname{ad}(Y)$ is skew-symmetric. By restricting the inner product to the $\operatorname{ad}(\mathfrak{a} + \mathfrak{m})$-invariant subspace $\mathfrak{f}$ of $\mathfrak{g}$, we see that $\operatorname{ad}_{\mathfrak{f}} X$ is symmetric and $\operatorname{ad}_{\mathfrak{f}} Y$ is skew-symmetric, so $B(X, Y) = 0$, i.e. $B(\mathfrak{a}, \mathfrak{m}) = \{0\}$.

(1.10) DEFINITIONS. Let $\mathcal{M}$ be a Riemannian solvmanifold and $\mathfrak{g}$ the Lie algebra of $G = I_0(\mathcal{M})$. A subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is said to be in *standard position* if for some choice of $F = MANG_cG_2$ as in Proposition 1.7 and for some $p \in \mathcal{M}$, $\mathfrak{s}$ is the orthogonal complement in $\mathfrak{f}$ of the isotropy subalgebra of $\mathfrak{f}$ at p relative to the Killing form on $\mathfrak{f}$. A Lie subgroup S of G is said to be in *standard position* in G if S is connected and its Lie algebra is in standard position.

(1.11) THEOREM. *Let* $\mathcal{M}$ *be a Riemannian solvmanifold and* $G = I_0(\mathcal{M})$.

(i) $\{S\colon S$ *is in standard position in* $G\}$ *is a conjugacy class of subgroups of* G.

(ii) *Let S be in standard position in G and F the subgroup of G appearing in the definition of S. Then F is the normalizer of S in G.*

(iii) *Let R be a solvable subgroup of G for which there exists $p \in \mathcal{M}$ such that $\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$ where $\mathfrak{l}$ is the Lie algebra of the isotropy subgroup L of G at p. Then there exists a subgroup S of G in standard position such that R lies within the normalizer F of S.*

PROOF. (i) The subgroups F defined in Proposition 1.7 form a conjugacy class of subgroups of G. By Theorem 1.9(i), there is associated to each F a subgroup S in standard position defined independently of the base point p. Use the temporary notation $B_{\mathfrak{f}}$ for the Killing form of $\mathfrak{f}$ and $\mathfrak{l}_p$ for the Lie algebra of the isotropy subgroup of G at p. Let

$$\mathfrak{s} = \mathfrak{f} \ominus (\mathfrak{l}_p \cap \mathfrak{f})$$

relative to $B_{\mathfrak{f}}$. If $F' = xFx^{-1}$ for $x \in G$, then

$$\mathfrak{f}' = \mathrm{Ad}(x)(\mathfrak{f}) \quad \text{and} \quad \mathrm{Ad}(x)(\mathfrak{l}_p \cap \mathfrak{f}) = \mathfrak{l}_{x \cdot p} \cap \mathfrak{f}'.$$

Also

$$B_{\mathfrak{f}'}(\mathrm{Ad}(x)X, \mathrm{Ad}(x)(Y)) = B_{\mathfrak{f}}(X, Y)$$

for all $X, Y \in \mathfrak{f}$. It follows that $\mathfrak{s}' \equiv \mathrm{Ad}(x)(\mathfrak{s})$ is the orthogonal complement in $\mathfrak{f}'$ of $\mathfrak{l}_{x \cdot p} \cap \mathfrak{f}'$ relative to $B_{f'}$ and thus $\mathfrak{s}'$ is the algebra in standard position associated with $\mathfrak{f}'$ by Definition 1.10.

(ii) By Theorem 1.9(i), $\mathfrak{s}$ is an ideal in $\mathfrak{f}$ and thus F lies in the normalizer of S. For the opposite inclusion, note that since $G = G_{nc}G_cG_2 = G_{nc}F$, it suffices to show that the normalizer of S in G_{nc} lies in F. Suppose $x \in G_{nc}$ and x normalizes S. Then x normalizes $\mathrm{nilrad}(S \cap G_{nc}) = N$. It follows that $x \in MAN \subset F$.

If R acts transitively on $\mathcal{M}$, then (iii) is immediate from Proposition 1.7(iv) and the conjugacy of the subgroups F. In the general case, let $\bar{R}$ be the closure of R in G. Since L is compact, $L\bar{R}$ is closed. But $\mathfrak{g} = \mathfrak{l} + \bar{\mathfrak{r}}$ implies $L\bar{R}$ is open and thus $G = L\bar{R}$, i.e. $\bar{R}$ acts transitively on $\mathcal{M}$. Since $\bar{R}$ is again a solvable subgroup of G, the remark above applies to $\bar{R}$ and thus yields (iii).

In §3, we shall see that, in retrospect, the argument just given is unnecessary since the present hypothesis on R will be shown to be equivalent to R being transitive on $\mathcal{M}$.

(1.12) COROLLARY. *Let $\mathcal{M}$ be a Riemannian solvmanifold and $G = I_0(\mathcal{M})$. The following are equivalent:*

(i) *G contains a solvable subgroup which is normal in G and transitive on $\mathcal{M}$.*

(ii) *G contains a subgroup in standard position which is normal in G.*

(iii) *G contains no noncompact simple subgroups.*

(iv) *There is a unique subgroup S in standard position in G.*

Moreover when (iv) *holds, S is normal in $I(\mathcal{M})$.*

PROOF. (i) implies (ii): By (i) and Lemma 1.8, G contains an almost simply transitive solvable normal subgroup S. It is easily checked, using the construction of S in Lemma 1.8 and Definition 1.10, that S is in standard position.

(ii) implies (iii): If R is a normal solvable subgroup of G which is transitive on $\mathcal{M}$, then for L the isotropy subgroup of G at some point, we have $G = LR$ with L compact and $R \subset G_2$. Hence G/G_2 is compact and (iii) follows.

(iii) implies (iv): The condition (iii) implies that there is a unique F as in Definition 1.10, namely $F = G$. This proves (iv).

(iv) implies (i) follows from Theorem 1.11.

For the last statement, let $x \in I(\mathcal{M})$ and let Φ be the automorphism of G defined by conjugation by x. By (iii) and Definition 1.10, $\mathfrak{s} = \mathfrak{g} \ominus \mathfrak{l}$ relative to the Killing form $B_{\mathfrak{g}}$, where $\mathfrak{l}$ is the isotropy algebra at some $p \in \mathcal{M}$. Then $\Phi_*(\mathfrak{l})$ is the isotropy algebra at $x \cdot p$, and $\Phi_*(\mathfrak{s}) = \mathfrak{g} \ominus \Phi_*(\mathfrak{l})$ relative to $B_{\mathfrak{g}}$. Hence $\Phi_*(\mathfrak{s})$ is in standard position and by (iv), we have $\Phi_*(\mathfrak{s}) = \mathfrak{s}$; i.e. $xSx^{-1} = S$.

(1.13) REMARKS. (i) We will show in Theorem 3.1, that if R is almost simply transitive, then $N_G(R) \subset F$ where F is as in Theorem 1.11(iii). Thus, the subgroups in standard position have maximal normalizers among all solvable transitive subgroups of G.

(ii) Starting from a data triple $(\mathfrak{s}, \langle \cdot, \cdot \rangle, D)$ for a Riemannian solvmanifold $\mathcal{M}$ where the corresponding group S happens to be in standard position in $G = I_0(\mathcal{M})$, Theorem 1.11(iii) implies that it is enough to compute $F = N_G(S)$ in order to have enough of G to construct a representative of each conjugacy class of subgroups of G which are solvable and almost simply transitive on $\mathcal{M}$. In §§3 and 4 we will reverse the process and show how to construct in a concrete manner a group S in standard position from an arbitrary data triple presentation $(\mathfrak{r}, \langle \cdot, \cdot \rangle, D)$ of $\mathcal{M}$. At each stage in the construction, the only part of G which is needed is the normalizer of the almost simply transitive subgroup at hand.

2. Modifications of solvable Lie algebras.

(2.1) REMARKS. In this section we will develop the theory of modifications in a context more general than that of isometry algebras of Riemannian solvmanifolds. The general theory is no more complicated than the special one and will be needed in a later paper.

(2.2) DEFINITION. Let $\mathfrak{r}$ be a solvable subalgebra of a Lie algebra $\mathfrak{g}$. An $\mathfrak{r}$-*modification map* is a linear map $\varphi \colon \mathfrak{r} \to \mathfrak{g}$ such that

(i) $\varphi(\mathfrak{r})$ is contained in a compactly embedded subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ with $\mathfrak{l} \cap \mathfrak{r} = \{0\}$;

(ii) $[\varphi(\mathfrak{r}), \mathfrak{r}] \subset \mathfrak{r}$; and

(iii) for Id the identity map on $\mathfrak{r}$, the space $\mathfrak{r}' \equiv (\mathrm{Id} + \varphi)(\mathfrak{r})$ is a solvable subalgebra of $\mathfrak{g}$.

φ is said to be a *normal* $\mathfrak{r}$-modification map if, in addition to (i)–(iii), φ satisfies

(iv) $[\varphi(\mathfrak{r}), \mathfrak{r}'] \subset \mathfrak{r}'$.

A subalgebra $\mathfrak{r}'$ of $\mathfrak{g}$ is said to be a (normal) *modification* of $\mathfrak{r}$ in $\mathfrak{g}$ if $\mathfrak{r}' = (\mathrm{Id} + \varphi)\mathfrak{r}$ for some (normal) $\mathfrak{r}$-modification map φ. If R and R' are connected solvable subgroups of a Lie group G, then R' is said to be a (normal) modification of R if the Lie algebra of R' is a (normal) modification of the Lie algebra of R.

(2.3) REMARKS. (i) It is possible to have distinct $\mathfrak{r}$-modification maps φ and ψ such that $(\mathrm{Id} + \varphi)(\mathfrak{r}) = \mathfrak{r}' = (\mathrm{Id} + \psi)(\mathfrak{r})$. In this case $\varphi(X) - \psi(X) \in \mathfrak{r}'$ for all $X \in \mathfrak{r}$, so φ is normal if and only if ψ is normal. Thus normality of the modification $\mathfrak{r}'$ is defined independently of the choice of φ.

(ii) Suppose $\mathcal{M}$ is a Riemannian solvmanifold, $G = I_0(\mathcal{M})$, and R and R' are solvable subgroups of G each acting almost simply transitively on $(\mathcal{M})$. Then for $\mathfrak{l}$ the isotropy algebra at any point $p \in \mathcal{M}$, we have $\mathfrak{g} = \mathfrak{l} + \mathfrak{r} = \mathfrak{l} + \mathfrak{r}'$ with $\mathfrak{l} \cap \mathfrak{r} = \{0\} = \mathfrak{l} \cap \mathfrak{r}'$. Hence there exists $\varphi \colon \mathfrak{r} \to \mathfrak{l}$ such that $\mathfrak{r}' = (\mathrm{Id} + \varphi)(\mathfrak{r})$. We conclude that $\mathfrak{r}'$ is a modification of $\mathfrak{r}$ if and only if $\mathfrak{r}'$ is contained in the normalizer of $\mathfrak{r}$ in $\mathfrak{g}$.

(iii) Alekseevskiĭ [**A12**] defined a notion of "twisting" of a simply transitive solvable isometry group. In our terminology a twisting is a normal modification.

(2.4) PROPOSITION. *Let* φ *be an* $\mathfrak{r}$*-modification map and* $\mathfrak{r}' = (\mathrm{Id}+\varphi)(\mathfrak{r})$. *Then*

(i) $\varphi(\mathfrak{r})$ *is abelian and* $[\mathfrak{r}',\mathfrak{r}'] \subset \ker\varphi$.

(ii) *The following are equivalent*:

(a) $\mathfrak{r}'$ *is a normal modification of* $\mathfrak{r}$;

(b) $[\varphi(\mathfrak{r}),\mathfrak{r}] \subset \ker\varphi$;

(c) $\mathfrak{r}$ *is a modification of* $\mathfrak{r}'$ *in* $\mathfrak{g}$ *with modification map* $\psi\colon \mathfrak{r}' \to \mathfrak{g}$ *defined by* $\psi(X+\varphi(X)) = -\varphi(X)$;

(d) φ *is a homomorphism, i.e.* $[\mathfrak{r},\mathfrak{r}] \subset \ker\varphi$.

PROOF. (i) By Definition 2.2(i)–(iii), $\varphi(\mathfrak{r})$ is a compact subalgebra of $\mathfrak{g}$ isomorphic to the image of $\mathfrak{r}'$ in $(\varphi(\mathfrak{r})+\mathfrak{r})/\mathfrak{r}$. Since $\mathfrak{r}'$ is solvable, $\varphi(\mathfrak{r})$ is solvable and compact, hence abelian. Thus $[\mathfrak{r}+\varphi(\mathfrak{r}),\mathfrak{r}+\varphi(\mathfrak{r})] \subset \mathfrak{r}$; so $[\mathfrak{r}',\mathfrak{r}'] \subset \mathfrak{r}\cap\mathfrak{r}' = \ker\varphi$.

(ii) The equivalence of (a) and (b) is immediate from Definition 2.2(ii), (iv) and Proposition 2.4(i). The equivalence of (a) and (c) follows from the fact that the map ψ of (c) satisfies $\psi(\mathfrak{r}') = \varphi(\mathfrak{r})$ and is a modification map precisely when $\varphi(\mathfrak{r})$ normalizes $\mathfrak{r}'$. Next by reversing the roles of $\mathfrak{r}$ and $\mathfrak{r}'$ in (i), we see that (c) implies (d). We shall complete the proof by showing that (d) implies (b).

In general, for $X, Y \in \mathfrak{r}$, (i) implies

$$[X+\varphi(X), Y+\varphi(Y)] = [X,Y] + [\varphi(X),Y] - [\varphi(Y),X] \in \ker(\varphi).$$

Therefore if (d) holds, then

$$(1) \qquad [\varphi(X),Y)] - [\varphi(Y),X] \in \ker(\varphi) \quad \text{for all } X,Y \in \mathfrak{r}.$$

In particular, when $Y \in \ker(\varphi)$, (1) yields

$$(2) \qquad [\varphi(\mathfrak{r}), \ker\varphi] \subset \ker\varphi.$$

Now let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{r}$ relative to which $\mathrm{ad}_\mathfrak{r} U$ is skew-symmetric for all $U \in \varphi(\mathfrak{r})$ and define $\mathfrak{a}$ as the orthogonal complement of $\ker\varphi$ relative to $\langle\cdot,\cdot\rangle$. From (2), $[\varphi(\mathfrak{r}),\mathfrak{a}] \subset \mathfrak{a}$ and thus by (1)

$$(3) \qquad [\varphi(X),Y] = [\varphi(Y),X] \quad \text{for all } X,Y \in \mathfrak{a}.$$

For $Y \in \mathfrak{a}$, we then have $[\varphi(Y),Y] \in \mathfrak{a}$ and, for all $X \in \mathfrak{a}$, skew-symmetry and (3) yield

$$0 = \langle[\varphi(X),Y],Y\rangle = \langle[\varphi(Y),X],Y\rangle = -\langle X,[\varphi(Y),Y]\rangle.$$

Whence $[\varphi(Y),Y] = 0$. Replacing Y by $X+Y$ and expanding with the aid of (3), we obtain

$$0 = [\varphi(X+Y),X+Y] = [\varphi(X),Y] + [\varphi(Y),X] = 2[\varphi(X),Y]$$

for all $X,Y \in \mathfrak{a}$, i.e. $[\varphi(\mathfrak{a}),\mathfrak{a}] = 0$. But then (2) yields

$$[\varphi(\mathfrak{r}),\mathfrak{r}] = [\varphi(\mathfrak{a}),\mathfrak{a}+\ker\varphi] \subset \ker\varphi$$

and this is (b).

(2.5) THEOREM. *Let* $\mathfrak{n}$ *be a nilpotent subalgebra of a Lie algebra* $\mathfrak{g}$. *Then every modification of* $\mathfrak{n}$ *in* $\mathfrak{g}$ *is a normal modification.*

PROOF. Let $\varphi\colon \mathfrak{n} \to \mathfrak{g}$ be a modification map and define $\mathfrak{t} = \varphi(\mathfrak{n})$. By Proposition 2.4(ii), it suffices to prove that $[\mathfrak{t}, \mathfrak{n}] \subset \ker \varphi$. Since $\mathfrak{t}$ is compactly embedded in $\mathfrak{g}$, there exists an inner product relative to which the operators $\operatorname{ad} U$, $U \in \mathfrak{t}$, are skew-symmetric.

Let $m+1$ be the step size of $\mathfrak{n}$ and let $\mathfrak{n} = \mathfrak{z}_0 \supset \mathfrak{z}_1 \supset \cdots \supset \mathfrak{z}_m \supset \mathfrak{z}_{m+1} = \{0\}$ be the central series of $\mathfrak{n}$, i.e. $\mathfrak{z}_m$ is the center of $\mathfrak{n}$ and for $0 \le j \le m-1$, $\mathfrak{z}_j = \{X \in \mathfrak{n}\colon \operatorname{ad}(X)(\mathfrak{n}) \subset \mathfrak{z}_{j+1}\}$. By the Jacobi identity, $[\mathfrak{t}, \mathfrak{z}_j] \subset \mathfrak{z}_j$ for all j. Define

$$\mathfrak{b}_j = (\mathfrak{z}_j \cap \ker\varphi) + \mathfrak{z}_{j+1} \quad \text{and} \quad \mathfrak{a}_j = \mathfrak{z}_j \ominus \mathfrak{b}_j \tag{4}$$

where $\ominus$ denotes the orthogonal difference relative to $\langle\cdot,\cdot\rangle$. Note that $\mathfrak{a}_i \perp \mathfrak{a}_j$ for $i \ne j$. Let

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \cdots \oplus \mathfrak{a}_m. \tag{5}$$

Step (i). $\mathfrak{n} = \mathfrak{a} + \ker\varphi$ (vector space direct sum):

Since $\mathfrak{z}_j = \mathfrak{a}_j + \mathfrak{b}_j$ and $\mathfrak{b}_j \subset \mathfrak{z}_{j+1} \ker\varphi$, we have

$$\begin{aligned} \mathfrak{n} = \mathfrak{z}_0 = \mathfrak{a}_0 + \mathfrak{b}_0 &\subset \mathfrak{a}_0 + \mathfrak{z}_1 + \ker\varphi \subset \mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{z}_2 + \ker\varphi \\ &\subset \cdots \subset \mathfrak{a} + \mathfrak{z}_{m+1} + \ker\varphi = \mathfrak{a} + \ker\varphi. \end{aligned}$$

If $0 \ne X \in \mathfrak{a} \cap \ker\varphi$, then there is a maximal index $j \le m$ for which $X \in \mathfrak{z}_j \cap \ker\varphi$. Since $\mathfrak{z}_j \subset \mathfrak{z}_i$ for $i < j$, X is orthogonal to $\mathfrak{a}_0 \oplus \cdots \oplus \mathfrak{a}_j$ and thus lies in $\mathfrak{a}_{j+1} \oplus \cdots \oplus \mathfrak{a}_m \subset \mathfrak{z}_{j+1}$. This contradicts the maximality of j. Thus $\mathfrak{a} \cap \ker(\varphi) = \{0\}$.

Step (ii). $\mathfrak{t}$ normalizes each of $\mathfrak{a}_j$, $\mathfrak{b}_j$, $0 \le j \le m$:

Suppose $Y \in \mathfrak{z}_j \cap \ker(\varphi)$ and $T \in \mathfrak{t}$. $T = \varphi(X)$ for some $X \in \mathfrak{n}$. By (1) and 2.4(i), we have

$$[X + \varphi(X), Y + \varphi(Y)] = [X, Y] + [\varphi(X), Y] \in \mathfrak{z}_j \cap \ker\varphi. \tag{6}$$

Since $[X, Y] \in \mathfrak{z}_{j+1}$, (6) implies that $[\mathfrak{t}, \mathfrak{z}_j \cap \ker\varphi] \subset \mathfrak{b}_j$ and consequently $[\mathfrak{t}, \mathfrak{b}_j] \subset \mathfrak{b}_j$. By skew-symmetry, $[\mathfrak{t}, \mathfrak{a}_j] \subset \mathfrak{a}_j$ for all j.

Step (iii). $[\mathfrak{t}, \mathfrak{a}] = \{0\}$:

By step (i), $\mathfrak{t} = \varphi(\mathfrak{a})$. Hence by (5) and linearity, it suffices to show that $[\varphi(\mathfrak{a}_i), \mathfrak{a}_j] = \{0\}$, $0 \le i, j \le m$. Let $X \in \mathfrak{a}_i$, $Y \in \mathfrak{a}_j$ with $i \le j$. Since $\mathfrak{t}$ is abelian,

$$[\varphi(Y), X] = [\varphi(X), Y] + [X, Y] - [X + \varphi(X), Y + \varphi(Y)]. \tag{7}$$

By 2.4(i) and step (ii), the three terms on the right side of (7) lie in $\mathfrak{a}_j$, $\mathfrak{z}_{i+j}$ and $\mathfrak{z}_i \cap \ker\varphi$, respectively. If $i < j$, all three of these spaces are contained in $\mathfrak{b}_i$, so

$$[\varphi(Y), X] \in \mathfrak{a}_i \cap \mathfrak{b}_i = \{0\}. \tag{8}$$

But then by (7),

$$[X + \varphi(X), Y + \varphi(Y)] \in (\mathfrak{a}_j + \mathfrak{z}_{i+j}) \cap \ker\varphi \subset \mathfrak{z}_j \cap \ker\varphi. \tag{9}$$

Since $\mathfrak{z}_{i+j} \subset \mathfrak{z}_{j+1}$, (7), (8) and (9) yield

$$[\varphi(X), Y] \in \mathfrak{a}_j \cap (\mathfrak{z}_{j+1} + \mathfrak{z}_j \cap \ker\varphi) = \mathfrak{a}_j \cap \mathfrak{b}_j = \{0\}. \tag{10}$$

Hence $[\varphi(\mathfrak{a}_i), \mathfrak{a}_j] = \{0\} = [\varphi(\mathfrak{a}_j), \mathfrak{a}_i]$ for $i < j$ by (8) and (10). It remains to show that $[\varphi(\mathfrak{a}_i), \mathfrak{a}_i] = \{0\}$. For $X, Y \in \mathfrak{a}_i$, (7) implies

$$[\varphi(X), Y] - [\varphi(Y), X] \in \mathfrak{a}_i \cap \mathfrak{b}_i = \{0\}. \tag{11}$$

But exactly the same argument which we applied to (3) in the proof of Proposition 2.4 applies to (11) to yield $[\varphi(\mathfrak{a}_i), \mathfrak{a}_i] = \{0\}$. This completes step (iii).

Note that the decomposition of step (i) is not necessarily orthogonal so the proof of Theorem 2.5 is not yet complete. Let $\mathfrak{n}_j = \mathfrak{z}_j \ominus \mathfrak{z}_{j+1}$. Then $\mathfrak{n} = \bigoplus_{j=0}^{m} \mathfrak{n}_j$. By skew-symmetry of $\operatorname{ad} \mathfrak{t}$, $[\mathfrak{t}, \mathfrak{n}_j] \subset \mathfrak{n}_j$. We show $[\mathfrak{t}, \mathfrak{n}_j] \subset \ker \varphi$, $0 \leq j \leq m$, to complete the proof. For $j = m$,

$$\mathfrak{n}_m = \mathfrak{z}_m = \mathfrak{a}_m \oplus \mathfrak{b}_m = \mathfrak{a}_m \oplus (\mathfrak{z}_m \cap \ker \varphi).$$

Since $[\mathfrak{t}, \mathfrak{a}_m] = 0$ by step (iii), skew-symmetry of $\operatorname{ad} \mathfrak{t}|_{\mathfrak{n}_m}$ implies $[\mathfrak{t}, \mathfrak{n}_m] \subset \ker \varphi$. We now proceed inductively and assume $[\mathfrak{t}, \mathfrak{n}_i] \subset \ker \varphi$ for all $i > j$. Fix $U \in \mathfrak{t}$. There exists a basis $\{X_1, \ldots, X_r, Y_1, \ldots, Y_r\}$ of $[U, \mathfrak{n}_j]$ and positive constants $\alpha_1, \ldots, \alpha_r$ such that $[U, X_k] = \alpha_k Y_k$ and $[U, Y_k] = -\alpha_k X_k$ for $1 \leq k \leq r$. Fix any index k and write X, Y, α for X_k, Y_k, α_k. By step (i), $U = \varphi(A)$ for some $A \in \mathfrak{a}$. Also $[A, \varphi(X)] = 0$ by step (iii), so we have

$$\begin{aligned} [A + \varphi(A), X + \varphi(X)] &= [A, X] + \alpha Y, \\ [A + \varphi(A), Y + \varphi(Y)] &= [A, Y] - \alpha X. \end{aligned} \tag{12}$$

By Proposition 2.4(i), the expressions on the left in (12) lie in $\ker \varphi$, so applying φ to both sides yields

$$\varphi[A, X] = -\alpha\varphi(Y), \qquad \varphi[A, Y] = \alpha\varphi(X). \tag{13}$$

On the other hand, $[A, X] \in \mathfrak{z}_{j+1} = \mathfrak{n}_{j+1} \oplus \cdots \oplus \mathfrak{n}_m$ so $[\varphi(A), [A, X]] \in \ker \varphi$ by the induction hypothesis. Using the Jacobi identity and the facts that $[\varphi(A), A] = 0$ (step (iii)) and $[\varphi(A), X] = \alpha Y$, we obtain $\alpha[A, Y] \in \ker \varphi$ and deduce from (13) that $X \in \ker \varphi$. Similarly $Y \in \ker \varphi$. Since the index k and $U = \varphi(A)$ were arbitrary, we have shown that $[\mathfrak{t}, \mathfrak{n}_j] \subset \ker \varphi$ and by induction $[\mathfrak{t}, \mathfrak{n}] \subset \ker \varphi$.

We will see in Example 2.8 that there exist modifications of solvable Lie algebras which are not normal. However Theorem 2.7 below shows that all modifications can be obtained by successive normal modifications.

(2.6) LEMMA. *Let $\mathfrak{s}$ be a solvable Lie algebra and let $\mathfrak{n}$ denote the nilradical of $\mathfrak{s}$. All derivations of $\mathfrak{s}$ map $\mathfrak{s}$ into $\mathfrak{n}$. In particular, when $\mathfrak{s}$ is endowed with an inner product, all skew-symmetric derivations are zero on the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{s}$.*

PROOF. See [**A-W2**, p. 36]. The additional hypothesis in [**A-W2**] that $\mathfrak{s}$ is an "NC algebra" and that $\operatorname{nilrad}(\mathfrak{s}) = [\mathfrak{s}, \mathfrak{s}]$ are not used in the proof. To correct a misprint in the proof given in [**A-W2**], the phrase "$X \in \mathfrak{n}$" should be replaced by "$X \in \mathfrak{s}$" wherever it appears.

(2.7) THEOREM. *Let $\mathfrak{s}$ be a solvable subalgebra of a Lie algebra $\mathfrak{g}$, φ an $\mathfrak{s}$ modification map, and $\mathfrak{r} = (\mathrm{Id} + \varphi)\mathfrak{s}$ the associated modification. Choose an $\operatorname{ad}(\varphi(\mathfrak{s}))$-invariant complement $\mathfrak{a}$ of $\mathfrak{n} = \operatorname{nilrad}(\mathfrak{s})$ in $\mathfrak{s}$ and let $\mathfrak{r}' = \mathfrak{n} + (\mathrm{Id} + \varphi)(\mathfrak{a})$. Then $\mathfrak{r}'$ is* *a normal modification of* $\mathfrak{s}$ *and* $\mathfrak{r}$ *is a normal modification of* $\mathfrak{r}'$. *In particular,* $\varphi(\mathfrak{n})$ *normalizes both* $\mathfrak{r}$ *and* $\mathfrak{r}'$.

PROOF. Let $\mathfrak{a}' = (\mathrm{Id}+\varphi)(\mathfrak{a})$ and $\mathfrak{n}' = (\mathrm{Id}+\varphi)(\mathfrak{n})$. We may define an inner product on $\mathfrak{s}$ relative to which $\mathfrak{a} \perp \mathfrak{n}$ and the operators $\operatorname{ad}\varphi(X)|_{\mathfrak{s}}$, $X \in \mathfrak{s}$, are skew-symmetric. By Lemma 2.6, $\mathfrak{r}' = \mathfrak{a}' + \mathfrak{n}$ is a normal modification of $\mathfrak{s}$. Also $\mathfrak{n}'$ is a modification of $\mathfrak{n}$ in $\mathfrak{g}$, so $[\varphi(\mathfrak{n}), \mathfrak{n}'] \subset \mathfrak{n} \cap \mathfrak{n}'$ by Theorem 2.5. Moreover since $\mathfrak{a}' \subset \mathfrak{a} + \varphi(\mathfrak{s})$ and $\varphi(\mathfrak{s})$ is abelian, Lemma 2.6 implies that $[\varphi(\mathfrak{n}), \mathfrak{a}'] = \{0\}$. Hence $\varphi(\mathfrak{n})$ normalizes $\mathfrak{r} = \mathfrak{a}' + \mathfrak{n}'$ and, consequently, $\mathfrak{r}$ is a normal modification of $\mathfrak{r}'$.

(2.8) EXAMPLE. Let $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}$ where $\mathfrak{n}$ is a six-dimensional abelian ideal and $\mathfrak{a} = \mathbb{R}A$ is one dimensional. Assume that the matrix of $\operatorname{ad} A|_{\mathfrak{n}}$ relative to a basis $\{X_1, \dots, X_6\}$ of $\mathfrak{n}$ is given by

$$\operatorname{ad} A|_{\mathfrak{n}} = \left(\begin{array}{cc|cc|c} 0 & 1 & 0 & -2 & \\ -1 & 0 & -2 & 0 & 0 \\ \hline & & 0 & -1 & \\ \multicolumn{2}{c|}{0} & 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

(The -2's may be replaced by 0's without affecting this example. They are included only in anticipation of Example 3.9.) Next let $\mathfrak{g} = \mathfrak{l} + \mathfrak{s}$ where $\mathfrak{l}$ is a two-dimensional abelian algebra normalizing $\mathfrak{n}$ and centralizing $\mathfrak{a}$. Let $\mathfrak{l}$ have bases V_1, V_2 with

$$\operatorname{ad} V_{1|\mathfrak{n}} = \left(\begin{array}{cc|cc|c} 0 & 1 & & & \\ -1 & 0 & \multicolumn{2}{c|}{0} & 0 \\ \hline & & 0 & 1 & \\ \multicolumn{2}{c|}{0} & -1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 \end{array}\right)$$

and

$$\operatorname{ad} V_{2|\mathfrak{n}} = \left(\begin{array}{c|c|cc} 0 & 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right).$$

Let

$$\mathfrak{r} = \operatorname{span}_{\mathbb{R}}\{A - V_1, X_1, X_2, X_3 + aV_2, X_4 + bV_2, X_5, X_6\}$$

where $a, b \in \mathbb{R}$ with at least one of a, b nonzero. $\mathfrak{r}$ is a solvable subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{l}$ and hence is a modification of $\mathfrak{s}$. It is not a normal modification since $V_1 = \varphi(A)$ does not normalize $\mathfrak{r}$. However setting

$$\mathfrak{r}' = \operatorname{span}_{\mathbb{R}}\{A - V_1, X_1, X_2, X_3, X_4, X_5, X_6\},$$

then $\mathfrak{r}'$ is a normal modification of $\mathfrak{s}$ and $\mathfrak{r}$ is a normal modification of $\mathfrak{r}'$.

(2.9) THEOREM. *Let U, R, and R' be subgroups of a connected Lie group G with R and R' connected and solvable. Suppose R' is a modification of R. Then $G = UR$ if and only if $G = UR'$.*

PROOF. By Theorem 2.7, it suffices to assume that R' is a normal modification of R. By Proposition 2.4, R is then a normal modification of R' as well, so we need only prove the "only if" statement. Since $\mathfrak{g} = \mathfrak{u} + \mathfrak{r}$, we may choose a modification map φ with values in $\mathfrak{u}$. Let $\mathfrak{k} = \varphi(\mathfrak{r})$. By Proposition 2.4, $\mathfrak{k}$ is a subalgebra of $\mathfrak{u}$ and $\mathfrak{k}$ normalizes $\mathfrak{r}'$. Hence for K the corresponding connected subgroup of G, we see that KR' is a subgroup of G with Lie algebra $\mathfrak{k}+\mathfrak{r}' = \mathfrak{k}+\mathfrak{r}$. Hence $KR' = KR$. Thus

$$G = UR = UKR = UKR' = UR'$$

since $K \subset U$.

(2.10) COROLLARY (PRESERVATION OF TRANSITIVITY). *Let $\mathcal{M}$ be a connected homogeneous solvmanifold, $G = I_0(\mathcal{M})$ and R an almost simply transitive connected solvable subgroup of G. Then every modification of R in G also acts almost simply transitively on $\mathcal{M}$.*

PROOF. The conditions for almost simple transitivity are that $G = LR$ with $\mathfrak{l} \cap \mathfrak{r} = \{0\}$. If R' is a modification of R, Theorem 2.9 gives $G = LR'$ and, by a dimension check, $\mathfrak{l} \cap \mathfrak{r}' = \{0\}$.

(2.11) EXAMPLE. While modifications preserve transitivity, they need not preserve simple transitivity. Consider the subgroup R of $GL(6, \mathbb{R})$ with Lie algebra

$$\mathfrak{r} = \left\{ \left[\begin{array}{cc|cc|cc} 0 & t & 0 & 0 & 0 & x_1 \\ -t & 0 & 0 & 0 & 0 & x_2 \\ \hline 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 & x_6 \end{array}\right] : t, x_i \in \mathbb{R} \right\}.$$

Let T, X_i be the elements of $\mathfrak{r}$ with 1's in the t and x_i entries, respectively, and 0's elsewhere. Give R a left-invariant Riemannian metric g so that $\{T, X_1, \dots, X_4\}$ is an orthonormal basis relative to the Riemannian inner product at the identity e of R. Let $\mathcal{M} = (R, g)$. Set

$$V = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$\operatorname{ad} V_{|\mathfrak{r}}$ is a skew-symmetric derivation of $\mathfrak{r}$, so V lies in the isotropy algebra $\mathfrak{l}$ (see Proposition 3.3). Let $\mathfrak{r}'$ be the modification of $\mathfrak{r}$ with modification map φ given by $\varphi(X_i) = 0$, $i = 1, \dots, 4$, and $\varphi(T) = \frac{1}{2}V$ and let R' be the corresponding connected subgroup of $\exp(\mathbb{R}V)R \subset I_0(\mathcal{M})$. Note that $\exp(2\pi T) = e$. Hence $\exp 2\pi(T + \varphi(T)) = \exp(\pi V) \in R' \cap L$, and R' does not act simply transitively.

3. The standard modification algorithm. Combining results of §§1 and 2 we obtain

(3.1) THEOREM. *Let $\mathcal{M}$ be a connected homogeneous Riemannian solvmanifold, $G = I_0(\mathcal{M})$ and L the isotropy subgroup of G at $p \in \mathcal{M}$. Let R be a connected solvable subgroup of G. The following are equivalent:*

(a) $\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$ *with* $\mathfrak{l} \cap \mathfrak{r} = \{0\}$.

(b) *R acts almost simply transitively on $\mathcal{M}$.*

(c) *R is a modification of a subgroup S of G in standard position.*

Moreover when (c) *holds, $N_G(R) \subset N_G(S)$.*

PROOF. By Corollary 2.10, (c) implies (b) and trivially (b) implies (a). Suppose (a) holds. By Theorem 1.11, part (iii), R is contained in the normalizer of some subgroup S of G in standard position. As in Remark 2.3(ii), it follows that $\mathfrak{r}$ is a modification of $\mathfrak{s}$. Thus (a) implies (c).

Now suppose (c) holds. We show $N_G(R) \subset N_G(S) \equiv F$. Note that $R \subset F$ by definition of modification. $F = (MAN)G_cG_2$ as in Proposition 1.7. If $G_{nc} = \{e\}$, then $F = G$ and we are done. Otherwise $\mathfrak{a} + \mathfrak{n} \subset \mathfrak{s}$ by Theorem 1.9 and Definition 1.10, and $\mathfrak{r}$ contains a modification of $\mathfrak{a} + \mathfrak{n}$. We will show that $\mathfrak{n} \subset \mathfrak{r}$. It will then follow that the normalizer of $\mathfrak{r}$ in $\mathfrak{g}_{nc}$ lies in $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ and hence $N_G(R) \subset F$.

Let $\mathfrak{q}$ be any modification of $\mathfrak{a}+\mathfrak{n}$ in $\mathfrak{f}$ and φ the associated modification map. Set $\mathfrak{t} = \varphi(\mathfrak{a}+\mathfrak{n})$. Then $\mathfrak{t}$ normalizes both $\mathfrak{a}+\mathfrak{n}$ and $\mathfrak{n}$ $(= [\mathfrak{a}+\mathfrak{n}, \mathfrak{a}+\mathfrak{n}])$. An examination of the structure of $\mathfrak{f}$ shows that the normalizer in $\mathfrak{f}$ of $\mathfrak{a}+\mathfrak{n}$ centralizes $\mathfrak{a}$; i.e. $[\mathfrak{t}, \mathfrak{a}] = \{0\}$. Hence for $X \in \mathfrak{a}$, $Y \in \mathfrak{n}$,

$$[X + \varphi(X), Y + \varphi(Y)] = [X, Y] + [\varphi(X), Y]. \tag{1}$$

Choose $X \in \mathfrak{a}$ such that $\mathfrak{n}$ is the direct sum of positive root spaces of $\operatorname{ad} X$. Since $\operatorname{ad}\varphi(X)_{|\mathfrak{n}}$ is a semisimple operator with purely imaginary eigenvalues (recall $\mathfrak{t}$ is compactly embedded in $\mathfrak{f}$), it follows from (1) that $[(\mathrm{Id} + \varphi)(X), (\mathrm{Id} + \varphi)(\mathfrak{n})] = \mathfrak{n}$ and consequently $\mathfrak{n} \subset \mathfrak{q}$. Since $\mathfrak{r}$ contains a modification of $\mathfrak{a} + \mathfrak{n}$, we have shown that $\mathfrak{n} \subset \mathfrak{r}$ and, as noted above, the theorem follows.

(3.2) REMARK. Let $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ be a data triple for a Riemannian solvmanifold $\mathcal{M}$ with basepoint p. As in Definition 1.1(ii), the subgroup R of $I(\mathcal{M})$ with Lie algebra $\mathfrak{r}$ is given by $R = \tilde{R}/C$ where $\tilde{R}$ is the simply connected covering of R and C is a discrete central subgroup. $\mathcal{M} = \tilde{R}/D = R/\bar{D}$ where $\bar{D}$ is the image of D under the covering projection $\tilde{R} \to R$. Let L be the isotropy subgroup of $I(\mathcal{M})$ at p. Let $x \in N_L(R)$. The isotropy action of x on $\mathcal{M} = R/\bar{D}$ is given by the automorphism $a \to xax^{-1}$ of R leaving $\bar{D}$ invariant. This automorphism lifts to an automorphism of $\tilde{R}$ leaving D invariant. Since L acts effectively on $\mathcal{M}$, this automorphism is nontrivial unless $x = e$. Thus when convenient, we will view $N_L(R)$ as a subgroup of $\mathrm{Aut}(R)$ and also of $\mathrm{Aut}(\tilde{R})$.

(3.3) PROPOSITION. *In the notation of Remark* 3.2,

$$N_L(R) = \{\sigma \in \mathrm{Aut}(\tilde{R}) \colon \sigma(D) = D \text{ and } \sigma^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle\}.$$

In case $\mathcal{M}$ is simply connected (and hence $D = \{e\}$), the Lie algebra $N_{\mathfrak{l}}(\mathfrak{r})$ of $N_L(R)$ is the algebra of all skew-symmetric derivations of $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$.

PROOF. See [**A-W2**, Corollary 2.13].

(3.4) DEFINITION. For $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ a data triple for a Riemannian solvmanifold $\mathcal{M}$, we define the *standard modification* $(\mathfrak{r}', \langle\cdot,\cdot\rangle', D')$ of $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ as follows: Let $\mathfrak{h}$ denote the normalizer of $\mathfrak{r}$ in the Lie algebra $\mathfrak{g}$ of $G = I_0(\mathcal{M})$ ($\mathfrak{h} = N_{\mathfrak{l}}(\mathfrak{r}) + \mathfrak{r}$ in the notation of Proposition 3.3). $\mathfrak{r}'$ is defined to be the orthogonal complement of $N_{\mathfrak{l}}(\mathfrak{r})$ in $\mathfrak{h}$ relative to the Killing form of $\mathfrak{h}$. By Lemma 1.8, $\mathfrak{r}'$ is a solvable ideal of $\mathfrak{h}$ and the corresponding connected subgroup R' of $H = N_G(R)$ acts almost simply transitively on $\mathcal{M}$. Set $D' = R' \cap L$, and let $\langle\cdot,\cdot\rangle'$ be the inner product on $\mathfrak{r}'$ defined by the Riemannian metric on $\mathcal{M}$ as in Lemma 1.2. Note that $\mathfrak{r}'$ is a modification of $\mathfrak{r}$ (Remark 2.3(ii)); let φ be the modification map. Then define $\langle\cdot,\cdot\rangle'$ by

$$\langle(\mathrm{Id} + \varphi)X, (\mathrm{Id} + \varphi)Y\rangle' = \langle X, Y\rangle.$$

We will frequently abuse notation and say that $\mathfrak{r}'$ (respectively R') is the standard modification of $\mathfrak{r}$ (respectively R) when the data D, $\langle\cdot,\cdot\rangle$ is understood.

(3.5) THEOREM. *Let $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ be a data triple for a Riemannian solvmanifold, let $\mathfrak{g} = I_0(\mathcal{M})$, and let $\mathfrak{s}$ be a solvable subalgebra of $\mathfrak{g}$ in standard position whose normalizer contains $\mathfrak{r}$. ($\mathfrak{s}$ exists by Theorem* 3.1.) *Let $(\mathfrak{r}', \langle\cdot,\cdot\rangle', D')$ be the standard modification of $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ and let $(\mathfrak{r}'', \langle\cdot,\cdot\rangle'', D'')$ be the standard modification of $(\mathfrak{r}', \langle\cdot,\cdot\rangle', D')$. Then $\mathfrak{r}'' = \mathfrak{s}$.*

(3.6) COROLLARY. *There exists a unique subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ in standard position such that $\mathfrak{r} \subset N_{\mathfrak{g}}(\mathfrak{s})$.*

(3.7) COROLLARY. *$\mathfrak{r}$ is in standard position if and only if $\mathfrak{r}$ equals its standard modification.*

PROOF OF THEOREM 3.5. Let $\mathfrak{f} = N_{\mathfrak{g}}(\mathfrak{s})$, $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{r})$ and $\mathfrak{h}' = N_{\mathfrak{g}}(\mathfrak{r}')$. By hypothesis $\mathfrak{r} \subset \mathfrak{f}$, so $\mathfrak{r}$ is a modification of $\mathfrak{s}$ (Remark 2.3(ii)). By Theorem 3.1, $\mathfrak{h} \subset \mathfrak{f}$. Hence the definition of standard modification implies $\mathfrak{r}' \subset \mathfrak{f}$, and thus $\mathfrak{r}'$ is also a modification of $\mathfrak{s}$ and $\mathfrak{h}' \subset \mathfrak{f}$. Let φ and φ' be the modification maps associated with the modifications $\mathfrak{r}$ and $\mathfrak{r}'$ of $\mathfrak{s}$, respectively.

Let $\mathfrak{n} = \operatorname{nilrad}(\mathfrak{s})$. By Theorem 2.7, $\varphi(\mathfrak{n}) \subset \mathfrak{h}$ and hence $\mathfrak{n} \subset \mathfrak{h}$. But $\mathfrak{n}$ is a nilpotent $\mathfrak{f}$-ideal, so $\mathfrak{n} \subset \operatorname{nilrad}\mathfrak{h} \subset \mathfrak{r}'$ by Lemma 1.8(i). In particular, $[\mathfrak{s},\mathfrak{s}] \subset \mathfrak{n} \subset \mathfrak{r}'$, so φ' is a homomorphism. By Proposition 2.4, $\mathfrak{r}'$ is a normal modification of $\mathfrak{s}$; i.e. $\varphi'(\mathfrak{s}) \subset \mathfrak{h}'$ and hence $\mathfrak{s} \subset \mathfrak{h}'$. Let B and B' denote the Killing forms of $\mathfrak{f}$ and $\mathfrak{h}'$, respectively. If $X \in N_{\mathfrak{l}}(\mathfrak{r}') \subset N_{\mathfrak{l}}(\mathfrak{s})$ and $Y \in \mathfrak{s}$, then since $\mathfrak{s}$ is an ideal in both $\mathfrak{f}$ and $\mathfrak{h}'$, we have

$$B'(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y_{|\mathfrak{s}}) = B(X, Y).$$

But $B(X, Y) = 0$ by Definition 1.10 of $\mathfrak{s}$, so $B'(N_{\mathfrak{l}}(\mathfrak{r}'), \mathfrak{s}) = 0$ and $\mathfrak{s}$ is the standard modification of $\mathfrak{r}'$.

(3.8) REMARKS. (i) In case $\mathcal{M}$ is simply connected, $D = \{e\}$ for any data triple $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$, so $\mathcal{M}$ is specified by the data pair $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$. Theorem 3.5 then allows us to compute the subgroup $F = (MAN)G_cG_2$ of G defined in Proposition 1.7 from the data $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$ as follows:

(a) Compute $N_{\mathfrak{l}}(\mathfrak{r})$, the algebra of skew-symmetric derivations of $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$ (see Proposition 3.3).

(b) Compute the standard modification $(\mathfrak{r}', \langle\cdot,\cdot\rangle')$ of $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$ as in 3.4.

(c) Repeat (i) and (ii) with $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$ replaced by $(\mathfrak{r}', \langle\cdot,\cdot\rangle')$ to obtain a pair $(\mathfrak{s}, \langle\cdot,\cdot\rangle'')$ with $\mathfrak{s}$ in standard position.

(d) $\mathfrak{f} = N_{\mathfrak{l}}(\mathfrak{s}) + \mathfrak{s}$ where $N_{\mathfrak{l}}(\mathfrak{s})$ is the algebra of skew-symmetric derivations of $(\mathfrak{s}, \langle\cdot,\cdot\rangle'')$.

(e) Let S be the simply connected Lie group with Lie algebra $\mathfrak{s}$ and U the connected subgroup of $\operatorname{Aut}(S)$ with Lie algebra $N_{\mathfrak{l}}(\mathfrak{s})$. Then F is the semidirect product US.

The algorithm is more complex in case $\mathcal{M}$ is not simply connected, since in the presence of the discrete groups D, D' and D'', the computations must be performed at the Lie group rather than Lie algebra level.

(ii) Let $\mathcal{R}$ be the set of all almost simply transitive solvable subgroups of $I_0(\mathcal{M})$. For $R, R' \in \mathcal{R}$, write $R \sim R'$ if R' can be obtained from R by a series of modifications. By Proposition 2.4 and Theorem 2.7, $\sim$ is an equivalence relation. Suppose that S is in standard position. We claim that the equivalence class of S in R is given by $[S] = \{R \in \mathcal{R} \colon R \subset N_G(S)\}$. Theorem 3.5 says that $R \in [S]$ whenever

$R \subset N_G(S)$. Conversely if R is a modification of S, then $R \subset N_G(S)$ and by Theorem 3.1, $N_G(R) \subset N_G(S)$. Thus any modification of R lies in $N_G(S)$ and, inductively, $R' \subset N_G(S)$ whenever $S \sim R'$. The claim is proved. By Corollary 3.6, each equivalence class contains exactly one element S in standard position. S can be recovered from any equivalent element R by performing two standard modifications. Note that by using the conjugacy of the subgroups of G in standard position, it is easily checked that any two equivalence classes in $\mathcal{R}$ are conjugate by an element of G.

(3.9) EXAMPLES. (i) Let $\mathcal{M}$ be the Euclidean space $\mathbb{R}^n$. Then $G = SO(n,\mathbb{R})\cdot\mathbb{R}^n$, and $\mathbb{R}^n$ is in standard position. If $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$ is any data pair, then $\mathfrak{r}$ is a modification of $\mathbb{R}^n$. The image of the modification map φ is an abelian subalgebra of $\mathfrak{l} = so(n,\mathbb{R})$. $N_{\mathfrak{l}}(\mathfrak{r})$ is the centralizer in $\mathfrak{l}$ of $\varphi(\mathfrak{r})$. The standard modification of $\mathfrak{r}$ is $\mathbb{R}^n$.

(ii) Let $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{r}'$ and $\mathfrak{l}$ be the Lie algebras of Example 2.8. Define $\langle\cdot,\cdot\rangle$ on $\mathfrak{r}$ so that $\{A - V_1, X_1, X_2, X_3 + aV_2, X_4 + bV_2, X_5, X_6\}$ is an orthonormal basis. A straightforward computation shows that $\mathbb{R}V_2$ is the space of all skew-symmetric derivations of $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$ and that $(\mathfrak{r}', \langle\cdot,\cdot\rangle')$ is the standard modification of $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ is the obvious inner product on $\mathfrak{r}'$. Similarly, we find that $\mathfrak{l}$ is the algebra of skew-symmetric derivations of $(\mathfrak{r}', \langle\cdot,\cdot\rangle')$ and that $(\mathfrak{s}, \langle\cdot,\cdot\rangle'')$ is its standard modification. Here $\{A, X_1, \ldots, X_6\}$ is an orthonormal basis of $\mathfrak{s}$ with respect to $\langle\cdot,\cdot\rangle''$. $\mathfrak{l}$ is also the algebra of all skew-symmetric derivations of $(\mathfrak{s}, \langle\cdot,\cdot\rangle'')$. We will see in Theorem 4.2 that since $\mathfrak{s}$ is unimodular, $\mathfrak{l} + \mathfrak{s}$ is actually the full isometry algebra of the associated simply connected Riemannian solvmanifold.

4. Special cases.

(4.1) REMARKS. (i) Let $\mathcal{M}$ be a Riemannian solvmanifold and $G = I_0(\mathcal{M})$. Suppose G contains no noncompact simple subgroups, i.e. $G_{nc} = \{e\}$. By Corollary 1.12, G contains a unique standard position subgroup S and S is normal in G. The algorithm described in 3.8(i) then allows us to construct G from any data triple $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$, at least in the case when $\mathcal{M}$ is simply connected. However, beginning with the data triple, it is in general difficult to determine whether $G_{nc} = \{e\}$ and hence whether the algorithm has produced the full group G. Theorem 4.2 below gives a sufficient condition on $\mathfrak{r}$ to guarantee that $G_{nc} = \{e\}$. Recall that a Lie group R is unimodular if $\operatorname{tr}(\operatorname{ad}_{\mathfrak{r}}(X)) = 0$ for all $X \in \mathfrak{r}$. Thus the hypothesis of Theorem 4.2 may be viewed as a condition on the Lie algebra $\mathfrak{r}$.

(ii) Theorem 4.3 gives an algebraic condition on $\mathfrak{r}$ which guarantees that if $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ is a data triple for a Riemannian solvmanifold, then $\mathfrak{r}$ is in standard position. Recall that $\mathfrak{r}$ is said to have only real roots if all generalized eigenvalues of $\operatorname{ad}_{\mathfrak{r}}(X)$ are real for each $X \in \mathfrak{r}$. Corollary 4.4 provides a new proof of the main result of [**W**].

(4.2) THEOREM. *Suppose the Riemannian solvmanifold $\mathcal{M}$ admits an almost simply transitive solvable unimodular group R of isometries. Then $I_0(\mathcal{M})$ contains no nontrivial noncompact simple subgroups and thus contains a unique normal subgroup S in standard position. Moreover every almost simply transitive solvable subgroup of $I_0(\mathcal{M})$ is unimodular.*

PROOF. By Remark 3.8(ii) and Theorem 2.7, the last statement of the theorem will follow if we prove that every normal modification of a unimodular Lie algebra is unimodular. Thus let $\mathfrak{r}'$ be a normal modification of $\mathfrak{r}$ with modification map φ.

Choose an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{r}$ relative to which the operators $\operatorname{ad}\varphi(X)_{|\mathfrak{r}}$ are skew-symmetric, and define $\langle\cdot,\cdot\rangle'$ on $\mathfrak{r}'$ by

$$\langle X+\varphi(X), Y+\varphi(Y)\rangle' = \langle X, Y\rangle. \tag{1}$$

We claim that for all $X, Y \in \mathfrak{r}$,

$$\langle [X+\varphi(X), Y+\varphi(Y)], Y+\varphi(Y)\rangle' = \langle [X,Y], Y\rangle. \tag{2}$$

Since $\operatorname{ad}\varphi(X)$ is skew-symmetric, $\langle[\varphi(X),Y],Y\rangle = 0$. Thus (2) will follow from (1) if we show that

$$\langle [\varphi(Y), X], Y\rangle = 0. \tag{3}$$

By Proposition 2.4, $[\varphi(Y),X] \in \ker\varphi$, so (3) holds if $Y \perp \ker\varphi$ and holds trivially if $Y \in \ker\varphi$. This proves (2).

By (2),

$$\operatorname{trace}(\operatorname{ad}_{\mathfrak{r}'}(X+\varphi(X))) = \operatorname{trace}(\operatorname{ad}_{\mathfrak{r}} X)$$

for all $X \in \mathfrak{r}$. Thus $\mathfrak{r}'$ is unimodular if and only if $\mathfrak{r}$ is unimodular, and as noted above, the last statement of the theorem is proved.

In particular, any subgroup S of $G = I_0(\mathcal{M})$ in standard position is unimodular. Let $\mathfrak{f} = N_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{m}+\mathfrak{a}+\mathfrak{n}+\mathfrak{g}_c+\mathfrak{g}_2$ as in Proposition 1.7 and Theorem 1.9. Suppose $G_{nc} \neq \{e\}$. Then $\mathfrak{a}$ and $\mathfrak{n}$ are nontrivial. By 1.9(iii),

$$\mathfrak{a}+\mathfrak{n} \subset \mathfrak{s} \subset \mathfrak{f}_2 \equiv \mathfrak{m}_2+\mathfrak{a}+\mathfrak{n}+\mathfrak{g}_2.$$

Since $\mathfrak{s}$ is an ideal in $\mathfrak{f}_2$ and $[\mathfrak{a}, \mathfrak{m}_2+\mathfrak{a}] = \{0\}$, we have

$$\operatorname{tr}(\operatorname{ad}_{\mathfrak{s}}(X)) = \operatorname{tr}(\operatorname{ad}_{\mathfrak{f}_2}(X)) = \operatorname{tr}(\operatorname{ad}(X)_{|\mathfrak{n}+\mathfrak{g}_2}) \tag{4}$$

for all $X \in \mathfrak{a}$. Since $\mathfrak{g}_{nc}$ is semisimple and $Y \to \operatorname{ad}(Y)_{|\mathfrak{g}_2}$ is a representation of $\mathfrak{g}_{nc}$, $\operatorname{tr}(\operatorname{ad}(Y)_{|\mathfrak{g}_2}) = 0$ for all $Y \in \mathfrak{g}_{nc}$, and in particular for $Y \in \mathfrak{a}$. However we may choose $X \in \mathfrak{a}$ such that all eigenvalues of $\operatorname{ad}(X)_{|\mathfrak{n}}$ are real and positive. For such X, $\operatorname{tr}(\operatorname{ad}_{\mathfrak{s}}(X)) > 0$ by (4), contradicting unimodularity of $\mathfrak{s}$. Hence $G_{nc} = \{e\}$ and the theorem follows by Remark 4.1.

(4.3) THEOREM. *Suppose that R is an almost simply transitive solvable group of isometries of the Riemannian solvmanifold $\mathcal{M}$ and that all roots of $\mathfrak{r}$ are real. Then R is in standard position in $I_0(\mathcal{M})$. If in addition R is unimodular, then $G = LR$ where L, the isotropy group at a point of $\mathcal{M}$, is contained in* $\operatorname{Aut}(R)$.

PROOF. Let $(\mathfrak{r}, \langle\cdot,\cdot\rangle, D)$ be a data triple associated with R. Write $\mathfrak{r} = \mathfrak{a}_0 \oplus \mathfrak{n}_0$, orthogonal direct sum with respect to $\langle\cdot\rangle$, with $\mathfrak{n}_0 = \operatorname{nilrad}(\mathfrak{r})$. By Proposition 3.3, $N_{\mathfrak{l}}(\mathfrak{r})$ is contained in the algebra of skew-symmetric derivations of $(\mathfrak{r}, \langle\cdot,\cdot\rangle)$, so by Lemma 2.6, $[N_{\mathfrak{l}}(\mathfrak{r}), \mathfrak{a}_0] = \{0\}$. Moreover the elements of $N_{\mathfrak{l}}(\mathfrak{r})$ act on $\mathfrak{r}$ with purely imaginary roots, while the elements of $\operatorname{ad}_{\mathfrak{r}}(\mathfrak{a}_0)$ have only real roots. Hence $\operatorname{tr}(\operatorname{ad}X)(\operatorname{ad}Y)_{|r} = 0$ for all $X \in N_{\mathfrak{l}}(\mathfrak{r})$, $Y \in \mathfrak{a}_0$. Since $\mathfrak{r}$ is an ideal in $\mathfrak{h} \equiv N_{\mathfrak{l}}(\mathfrak{r})+\mathfrak{r}$, it follows that $B_{\mathfrak{h}}(\mathfrak{a}_0, N_{\mathfrak{l}}(\mathfrak{r})) = 0$ for $B_{\mathfrak{h}}$ the Killing form of $\mathfrak{h}$. Trivially $B_{\mathfrak{h}}(\mathfrak{n}_0, N_{\mathfrak{l}}(\mathfrak{r})) = 0$, so by Definition 3.4, $\mathfrak{r}$ coincides with its standard modification. By Corollary 3.7, R is therefore in standard position. The last statement of the theorem follows from Theorem 4.2 and Remark 3.2.

As a corollary, we obtain a new proof of the following:

(4.4) COROLLARY (WILSON [**W**]). *If the Riemannian solvmanifold $\mathcal{M}$ admits an almost simply transitive nilpotent group R of isometries, then R coincides with the nilradical of $G = I_0(\mathcal{M})$. In particular, R is unique and $G = LR$ with $L \subset \mathrm{Aut}(R)$.*

PROOF. $\mathfrak{r}$ is unimodular with only real roots, so by Theorem 4.3, $G = LR$ with $L \subset \mathrm{Aut}(R)$. Hence $R \subset \mathrm{nilrad}(G)$. If $X \in \mathfrak{l} \cap \mathrm{nilrad}(\mathfrak{g})$, then $\mathrm{ad}_{\mathfrak{g}} X$ is both semisimple (since $X \in \mathfrak{l}$) and nilpotent, hence trivial. But L is assumed to act effectively on $\mathcal{M}$, so the representation $X \to \mathrm{ad}_{\mathfrak{g}} X$ of $\mathfrak{l}$ is faithful. Consequently $\mathfrak{l} \cap \mathrm{nilrad}(\mathfrak{g}) = \{0\}$ and $R = \mathrm{nilrad}(G)$.

5. Tests for isometry of solvmanifolds.

(5.1) REMARKS. We now use the results of the previous sections to determine when two given solvmanifolds are isometric. The first theorem considers the simply connected case. Given data for two simply connected Riemannian solvmanifolds $\mathcal{M}$ and $\mathcal{M}'$, the algorithm in Remark 3.8(i) allows us to obtain new data $(\mathfrak{s}, \langle\ ,\ \rangle)$ for $\mathcal{M}$ and $(\mathfrak{s}', \langle\ ,\ \rangle')$ for $\mathcal{M}'$ with $\mathfrak{s}$ and $\mathfrak{s}'$ in standard position in $I_0(\mathcal{M})$ and $I_0(\mathcal{M}')$, respectively. Hence it suffices to develop a test for isometry when $\mathcal{M}$ and $\mathcal{M}'$ are specified by data in standard position. Theorem 5.2 gives such a test.

The second theorem concerns locally homogeneous solvmanifolds. Suppose S is a simply connected solvable Lie group and Γ is a discrete subgroup. Each left-invariant Riemannian metric g on S induces a Riemannian metric $\bar{g}$ on $\Gamma\backslash S$ so that (S, g) is a Riemannian covering of $(\Gamma\backslash S, \bar{g})$. The locally defined left translations of $\Gamma\backslash S$ by elements of S are local isometries, and $(\Gamma\backslash S, \bar{g})$ is a locally homogeneous space. In Theorem 5.4, we give a test for isometry of two locally homogeneous metrics on $\Gamma\backslash S$ under certain conditions on S.

(5.2) THEOREM. *Let $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ and $(\mathfrak{s}', \langle\cdot,\cdot\rangle')$ be data in standard position for the simply connected Riemannian solvmanifolds $\mathcal{M}$ and $\mathcal{M}'$, respectively. Then $\mathcal{M}$ is isometric to $\mathcal{M}'$ if and only if $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is isomorphic to $(\mathfrak{s}', \langle\cdot,\cdot\rangle')$.*

PROOF. First note that by Remark 1.3(i) and the conjugacy of the algebras in standard position, any two data pairs $(\mathfrak{s}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{s}_2, \langle\cdot,\cdot\rangle_2)$ in standard position for the same solvmanifold $\mathcal{M}$ are isomorphic. Thus it suffices to prove that $\mathcal{M}$ and $\mathcal{M}'$ are isometric if and only if $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is isomorphic to some data pair for $\mathcal{M}'$ in standard position. The "if" statement is trivial. Conversely, any isometry $\tau\colon \mathcal{M} \to \mathcal{M}'$ defines an automorphism $\Psi\colon G \to G'$ by $\tau(x) = \tau \circ x \circ \tau^{-1}$, where $G = I_0(\mathcal{M})$ and $G' = I_0(\mathcal{M}')$. Ψ carries the isotropy group L at the basepoint $p \in \mathcal{M}$ to the isotropy group L' at $p' = \tau(p)$. Let S denote the connected subgroup of G with Lie algebra $\mathfrak{s}$ and set $F = N_G(S)$. Then F and $\Psi(F)$ satisfy the conditions of Proposition 1.7 in G and G', respectively. Since $L' \cap \Psi(F) = \Psi(L \cap F)$, it follows easily from Definition 1.10 that $\Psi(S)$ is in standard position in G'. Thus $(\Psi_*\mathfrak{s}, (\Psi^{-1})^*\langle\cdot,\cdot\rangle)$ is a data pair for $\mathcal{M}'$ isomorphic to $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$.

(5.3) COROLLARY (ALEKSEEVSKIĬ [**Al2**]). *Let S be a simply connected solvable Lie group whose Lie algebra has only real roots. Let g and g' be two left-invariant metrics on S. Then g is isometric to g' if and only if $g' = \Psi^* g$ for some $\Psi \in \mathrm{Aut}(S)$.*

PROOF. Theorems 5.2 and 4.2.

(5.4) THEOREM. *Let S be a simply connected solvable unimodular Lie group whose Lie algebra has only real roots, let Γ be a discrete subgroup of S, let $C = \{\Phi \in \mathrm{Aut}(S) \colon \Phi(\Gamma) = \Gamma\}$ and let $\mathrm{Inn}(S)$ denote the group of inner automorphisms of S. Let g_1 and g_2 be left-invariant Riemannian metrics on S and $\bar{g}_1$ and $\bar{g}_2$ the induced metrics on $\Gamma\backslash S$. Then $\bar{g}_1$ is isometric to $\bar{g}_2$ if and only if $g_2 = \Psi^* g_1$ for some $\Psi \in \mathrm{Inn}(S) \cdot C$.*

PROOF. First suppose $g_2 = \Psi^* g_1$ with $\Psi = L_x \circ R_x^{-1} \circ \gamma$ for some $x \in S$, $\gamma \in C$. (L_x and R_x^{-1} denote left and right translation.) Since g_1 is left-invariant, $\Psi^* g_1 = (R_x^{-1}\gamma)^* g_1$. But $R_{x^{-1}} \circ \gamma$ is the lift of a mapping $\Gamma\backslash S \to \Gamma\backslash S$. Hence $(\Gamma\backslash S, g_1)$ and $(\Gamma\backslash S, g_2)$ are isometric.

Conversely, suppose $\bar{\tau} \colon (\Gamma\backslash S, \bar{g}_1) \to (\Gamma\backslash S, \bar{g}_2)$ is an isometry. Then τ lifts to an isometry $\tau \colon (S, g_1) \to (S, g_2)$. By Corollary 5.3, $g_1 = \Phi^* g_2$ for some $\Phi \in \mathrm{Aut}(S)$. Hence $\tau \circ \Phi^{-1}$ is an isometry of (S, g_2). Choose $x \in S$ so that $\sigma \equiv L_x \circ \tau \circ \Phi^{-1}$ is an isometry of (S, g_2) preserving the identity element e. By Remark 3.2 and the last statement of Corollary 1.12, $\sigma \in \mathrm{Aut}(S)$. Hence

$$R_x \circ \tau = L_{x^{-1}} \circ R_x \circ \sigma \circ \Phi \in \mathrm{Aut}(S). \tag{1}$$

But $R_x \circ \tau$ is the lift of a mapping $\bar{R}_x \circ \bar{\tau} \colon \Gamma\backslash G \to \Gamma\backslash G$. Hence $R_x \circ \tau \in C$ and, by (1), $\sigma \circ \Phi \in \mathrm{Inn}(S) \cdot C$. Since σ is an isometry of (S, g_2),

$$(\sigma \circ \Phi)^* g_2 = \Phi^* \sigma^* g_2 = \Phi^* g_2 = g_1$$

and the theorem is proved.

(5.5) APPLICATIONS. (i) Two compact Riemannian manifolds are said to be isospectral if their Laplace-Beltrami operators have the same spectra. Let S be a simply connected nilpotent group and Γ a cocompact discrete subgroup. In [**G-W1**] we give sufficient conditions for two given locally homogeneous metrics on $\Gamma\backslash S$ to be isospectral. Using Theorem 5.4, we then establish the existence of continuous families of isospectral, nonisometric, locally homogeneous metrics on $\Gamma\backslash S$ for suitable S.

(ii) Deloff [**D**] constructs examples of homogeneous Einstein metrics of strictly negative curvature. It follows from Corollary 5.3 that all of his examples are distinct; i.e. no two are isometric.

6. Groups transitive on manifolds of nonpositive curvature.

(6.1) REMARKS. Throughout the previous sections, we have focused our attention on the various simply transitive solvable subgroups of $I_0(\mathcal{M})$. However $I_0(\mathcal{M})$ may contain many nonsolvable transitive subgroups as well. In fact Example 6.2 below shows that a solvmanifold can even admit a simply transitive semisimple group of isometries. The classification of all transitive subgroups of $I_0(\mathcal{M})$ would be very difficult in general. However when $\mathcal{M}$ has negative curvature, many of the complications such as that illustrated in Example 6.2 do not arise. We will use the theory developed in [**A-W1**, **2**] together with some structural results on semisimple Lie algebras to be developed in Lemmas 6.3–6.5 to give necessary and sufficient conditions for a subgroup H of $I_0(\mathcal{M})$ to act transitively on $\mathcal{M}$ in the negative curvature case. We will then specialize further to symmetric spaces $\mathcal{M}$ of noncompact type and give a complete classification of all transitive subgroups of $I_0(\mathcal{M})$. Finally, we utilize this specialized result to classify for arbitrary simply connected solvmanifolds the noncompact semisimple parts H_{nc} of all transitive subgroups H of $I_0(\mathcal{M})$.

(6.2) EXAMPLE. Let H be the universal covering of $SL(2,\mathbb{R})$ and let $H = KAN$ be an Iwasawa decomposition. Let g be a left-invariant Riemannian metric on H which is also $\mathrm{Ad}(K)$-invariant and set $\mathcal{M} = (H, g)$. Let Z denote the center of H. Z is discrete and lies in K. $I_0(\mathcal{M}) = H \times K/\Delta(Z)$ where $\Delta(Z) = \{(z,z)\colon z \in Z\} \subset H \times K$ (see [**G1**]). Note that $K \cong \mathbb{R}$. The isotropy subgroup of $I_0(\mathcal{M})$ is $K \times K/\Delta(Z)$. The subgroup $S = AN \times K$ is a simply transitive solvable subgroup of $I_0(\mathcal{M})$, so $\mathcal{M}$ is a simply connected homogeneous solvmanifold. However the simple Lie group H acts simply transitively on $\mathcal{M}$.

(6.3) LEMMA. *Let $\mathfrak{g}$ be a semisimple Lie algebra of noncompact type and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ an Iwasawa decomposition. Let Δ be the roots of $\mathfrak{a}$ in $\mathfrak{g}$ and, for $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ denote the root space of α. Order Δ so that $\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$. Let π be a basis of simple roots of Δ^+ and let π_0 be any subset of π. Then the subalgebra $\mathfrak{b}(\pi_0)$ of $\mathfrak{g}$ generated by $\sum_{\pm\alpha\in\pi_0} \mathfrak{g}_\alpha$ is semisimple of noncompact type. $\mathfrak{b}(\pi_0)$ has an Iwasawa decomposition $\mathfrak{k}^{(0)} + \mathfrak{a}^{(0)} + \mathfrak{n}^{(0)}$ compatible with $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. (I.e. $\mathfrak{k}^{(0)} \subset \mathfrak{k}$, $\mathfrak{a}^{(0)} \subset \mathfrak{a}$ and $\mathfrak{n}^{(0)} \subset \mathfrak{n}$.) Moreover $\mathfrak{n}^{(0)} = \sum_{\alpha\in(\Delta_0)^+} \mathfrak{g}_\alpha$ where Δ_0 is the largest subset of Δ consisting of linear combinations of elements of π_0.*

PROOF. A special case of this lemma is proved in [**H**, Proposition 2.1, Lemma 2.3, Chapter IX]. The general case is proved in the same way. We note that the analogous result for complex semisimple Lie algebras is well known. (See [**Hu**, pp. 87–88].)

(6.4) LEMMA. *In the notation of Lemma* 6.3, *let*

$$\mathfrak{a}^{(1)} = \bigcap_{\alpha\in\pi_0} \ker\alpha, \qquad \mathfrak{n}^{(1)} = \sum_{\substack{\alpha\in\Delta^+ \\ \alpha(\mathfrak{a}^{(1)})\neq 0}} \mathfrak{g}_\alpha$$

and

$$\mathfrak{c}(\pi_0) = \mathfrak{b}(\pi_0) + \mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}.$$

Let $\mathfrak{m}$ be the normalizer of $\mathfrak{a} + \mathfrak{n}$ in $\mathfrak{k}$. Then

(i) $\mathfrak{a} = \mathfrak{a}^{(0)} + \mathfrak{a}^{(1)}$ *and* $\mathfrak{n} = \mathfrak{n}^{(0)} + \mathfrak{n}^{(1)}$, *vector space direct sums.*

(ii) $[\mathfrak{a}^{(1)}, \mathfrak{n}^{(1)}] = \mathfrak{n}^{(1)}$ *and* $[\mathfrak{a}^{(1)}, \mathfrak{b}(\pi_0)] = \{0\}$.

(iii) $\mathfrak{c}(\pi_0)$ *is a subalgebra of $\mathfrak{g}_{nc}$ with radical $\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$ and nilradical $\mathfrak{n}^{(1)}$.*

(iv) $\mathfrak{m} = \mathfrak{m}^{(0)} \oplus \mathfrak{m}^{(1)}$, *direct sum of ideals, with $\mathfrak{m}^{(0)} \subset \mathfrak{b}(\pi_0)$ and $[\mathfrak{m}^{(1)}, \mathfrak{b}(\pi_0)] = \{0\}$.*

(v) $\mathfrak{m}$ *normalizes $\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$. Thus $\mathfrak{c}(\pi_0) + \mathfrak{m}$ is a subalgebra of $\mathfrak{g}$ with semisimple Levi factor $\mathfrak{b}(\pi_0) \oplus \mathfrak{m}_1^{(1)}$ and radical $\mathfrak{m}_2^{(1)} + \mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$. (As usual $\mathfrak{m}_1^{(1)} + \mathfrak{m}_2^{(1)}$ denotes the Levi decomposition of $\mathfrak{m}^{(1)}$.)*

PROOF. (ii) and the second statement in (i) are immediate from the definitions of $\mathfrak{a}^{(1)}$ and $\mathfrak{n}^{(1)}$. Next π_0 is a basis for the positive roots of $\mathfrak{a}^{(0)}$ in $\mathfrak{b}(\pi_0)$, so $\dim(\mathfrak{a}_0) = \sharp\pi_0$. By definition of $\mathfrak{a}^{(1)}$

$$\dim(\mathfrak{a}^{(1)}) \geq \sharp\pi - \sharp\pi_0 = \dim(\mathfrak{a}) - \dim(\mathfrak{a}^{(0)}).$$

Since $\mathfrak{a}^{(0)} \cap \mathfrak{a}^{(1)} = \{0\}$, (i) follows.

For (iii), we need only show that $[\mathfrak{b}(\pi_0), \mathfrak{n}^{(1)}] \subset \mathfrak{n}^{(1)}$. Note that

$$\Delta_0 = \{\alpha \in \Delta\colon \alpha(\mathfrak{a}^{(1)}) = 0\}.$$

Set $\Delta_1 = \Delta - \Delta_0$ and $\Delta_1^+ = \Delta_1 \cap \Delta^+$. If $\alpha \in \Delta_0$, $\beta \in \Delta_1$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta_1$. If in addition $\beta \in \Delta_1^+$ and $\pm\alpha \in \pi_0$, then $\alpha + \beta \in \Delta_1^+$. Thus $[\mathfrak{g}_\alpha, \mathfrak{n}^{(1)}] \subset \mathfrak{n}^{(1)}$ whenever $\pm\alpha \in \pi_0$ and consequently $[\mathfrak{b}(\pi_0), \mathfrak{n}^{(1)}] \subset \mathfrak{n}^{(1)}$.

Since $\mathfrak{m}$ commutes with $\mathfrak{a}$ and normalizes each eigenspace of $\mathrm{ad}(\mathfrak{a})$, $\mathfrak{m}$ normalizes both $\mathfrak{b}(\pi_0)$ and $\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$. Thus $\mathfrak{m}^{(0)} \equiv \mathfrak{m} \cap \mathfrak{b}(\pi_0)$ is an $\mathfrak{m}$-ideal. Since $\mathfrak{m}$ is compact, $\mathfrak{m}$ splits into a direct sum $\mathfrak{m}^{(0)} \oplus \mathfrak{m}^{(1)}$. To see that $[\mathfrak{m}^{(1)}, \mathfrak{b}(\pi_0)] = \{0\}$, note that the adjoint representation of $\mathfrak{b}(\pi_0)$ carries $\mathfrak{b}(\pi_0) + \mathfrak{m}^{(1)}$ to $\mathfrak{b}(\pi_0)$. Since every representation of the semisimple algebra $\mathfrak{b}(\pi_0)$ is completely reducible, $\mathfrak{b}(\pi_0) + \mathfrak{m}^{(1)} = \mathfrak{b}(\pi_0) + \mathfrak{t}$ for some subspace t commuting with $\mathfrak{b}(\pi_0)$. Since $[\mathfrak{t}, \mathfrak{a}^{(0)}] = \{0\}$, $\mathfrak{t} \subset \mathfrak{a}^{(0)} + \mathfrak{m}$. But for $X \in \mathfrak{a}^{(0)}$, $Y \in \mathfrak{m}$, $[X + Y, \mathfrak{n}^{(0)}] \neq \{0\}$ unless $X = 0$. Hence $\mathfrak{t} \subset \mathfrak{m}$ and so $\mathfrak{t} = \mathfrak{m}^{(1)}$. (iv) and (v) follow.

(6.5) LEMMA. *Let* $\mathfrak{u}$ *be a subalgebra of a semisimple Lie algebra* $\mathfrak{g}$ *of noncompact type. Suppose that* (i) *the semisimple Levi factor* $\mathfrak{u}_1$ *of* $\mathfrak{u}$ *is of noncompact type with Iwasawa decomposition* $\mathfrak{k}^{(0)} + \mathfrak{a}^{(0)} + \mathfrak{n}^{(0)}$ *compatible with the Iwasawa decomposition* $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ *of* $\mathfrak{g}$ *and* (ii) $\mathfrak{u}_2 = \mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$ *with* $[\mathfrak{a}^{(1)}, \mathfrak{n}^{(1)}] = \mathfrak{n}^{(1)}$, $\mathfrak{a} = \mathfrak{a}^{(0)} + \mathfrak{a}^{(1)}$, $\mathfrak{n} = \mathfrak{n}^{(0)} + \mathfrak{n}^{(1)}$, *and* $[\mathfrak{a}^{(1)}, \mathfrak{u}_1] = \{0\}$. *Then in the notation of Lemmas* 6.3–6.4, $\mathfrak{u} = \mathfrak{c}(\pi_0)$ *for some* $\pi_0 \subset \pi$, *where* π *is a basis of the root system* Δ *of* $\mathfrak{a}$.

PROOF. Let $\Delta_0 = \{\alpha \in \Delta\colon \alpha(\mathfrak{a}^{(1)}) = 0\}$, $\Delta_1 = \Delta - \Delta_0$, $\Delta_0^+ = \Delta_0 \cap \Delta^+$ and $\Delta_1^+ = \Delta_1 \cap \Delta^+$. Since $[\mathfrak{a}^{(1)}, \mathfrak{u}_1] = \{0\}$, it follows that

$$\mathfrak{n}^{(0)} = \sum_{\alpha \in \Delta_0^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^{(1)} = \sum_{\alpha \in \Delta_1^+} \mathfrak{g}_\alpha.$$

Let $\bar{\Delta}_0 = \{\bar{\alpha} \equiv \alpha|_{\mathfrak{a}^{(0)}} : \alpha \in \Delta_0\}$. Note that the natural map $\Delta_0 \to \bar{\Delta}_0$ is a bijection. $\bar{\Delta}_0$ is the root system of $\mathfrak{a}^{(0)}$ in $\mathfrak{u}_1$. Let $\bar{\pi}_0$ be a basis of simple roots in $\bar{\Delta}_0^+$ and π_0 the corresponding subset of Δ_0. We show that π_0 is contained in a basis of Δ. Choose $A \in \mathfrak{a}^{(1)}$ such that $\alpha(A) > 0$ for all $\alpha \in \Delta_1^+$. Suppose $\alpha \in \pi_0$ and $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Delta^+$. If either β or γ belongs to Δ_1^+, then $\alpha(A) > 0$, contradicting the definition of π_0. But if both β and γ belong to Δ_0, then $\bar{\alpha} = \bar{\beta} + \bar{\gamma}$ is a decomposable element of $\bar{\pi}_0$, again a contradiction. Hence π_0 consists of indecomposable roots in Δ^+ and extends to a basis π of Δ^+. Thus $\mathfrak{u}_1 = \mathfrak{b}(\pi_0)$ and $\mathfrak{u} = \mathfrak{c}(\pi_0)$.

(6.6) REMARKS. We are now ready to study the transitive subgroups of $I_0(\mathcal{M})$ when $\mathcal{M}$ is a solvmanifold of negative (i.e. nonpositive) curvature with trivial Euclidean DeRham factor. In Theorem 4.6 of [**A-W2**], Azencott and Wilson showed that if $(\mathfrak{r}, \langle\ ,\ \rangle)$ is a data pair for such a manifold $\mathcal{M}$, then $\mathfrak{r}$ is an "*NC* algebra without flat part". (See Definition 6.2 of [**A-W1**] and Definition 4.3 of [**A-W2**].) We summarize here and in Lemma 6.7 those properties of $(\mathfrak{r}, \langle\ ,\ \rangle)$ and of $I_0(\mathcal{M})$ which will be needed in Theorem 6.8.

(a) The orthogonal complement $\mathfrak{a}$ of the nilradical $\mathfrak{n}$ of $\mathfrak{r}$ is abelian and satisfies $[\mathfrak{a}, \mathfrak{n}] = \mathfrak{n}$.

(b) Every modification $\mathfrak{r}' = (\mathrm{Id} + \varphi)\mathfrak{r}$ of $\mathfrak{r}$ is normal and is given by $\mathfrak{r}' = \mathfrak{a}' + \mathfrak{n}$ with $\mathfrak{a}' = (\mathrm{Id} + \varphi)(\mathfrak{a})$. Moreover $\mathfrak{a}'$ is abelian and $[\mathfrak{a}', \mathfrak{n}] = \mathfrak{n}$. (See p. 45 of [**A-W2**].)

(c) There exists a closed normal subgroup $S^{(2)}$ of G_2 such that a solvable subgroup S of G is in standard position if and only if S is the semidirect product

$(AN)S^{(2)}$ of $S^{(2)}$ with an Iwasawa subgroup AN of G_{nc} for some semisimple Levi factor $G_1 = G_{nc}G_c$. Moreover $\mathfrak{s}^{(2)}$ is itself an NC algebra without flat part. We will write $\mathfrak{s}^{(2)} = \mathfrak{a}^{(2)} + \mathfrak{n}^{(2)}$ as in (a).

(6.7) LEMMA. *Let $\mathcal{M}$ be a simply connected Riemannian solvmanifold of negative curvature with no Euclidean factor and let U be a connected subgroup of $G = I_0(\mathcal{M})$ which acts transitively on $\mathcal{M}$. Fix $p \in \mathcal{M}$. Then there exists a semisimple Levi factor U_1 of U and an Iwasawa decomposition $U_{nc} = K^{(0)}A^{(0)}N^{(0)}$ such that $K^{(0)}$ fixes p and such that for any simply transitive solvable subgroup R of U of the form $R = A^{(0)}N^{(0)}Q$, where Q is a closed normal subgroup of U and $A^{(0)}N^{(0)} \cap Q = \{e\}$ (such R exist), the p-inner product on $\mathfrak{r}$ has the following properties* :

(i) $(\mathfrak{a}^{(0)} + \mathfrak{n}^{(0)}) \perp \mathfrak{q}$;

(ii) *the orthogonal complement $\tilde{\mathfrak{a}}^{(2)}$ of $\tilde{\mathfrak{n}}^{(2)} = [\mathfrak{q}, \mathfrak{q}]$ in $\mathfrak{q}$ is abelian and commutes with $\mathfrak{u}_1$*;

(iii) $\mathrm{ad}_{\mathfrak{q}}(\theta X) = -(\mathrm{ad}_{\mathfrak{q}} X)^t$ *for all $X \in \mathfrak{u}_{nc}$ where θ is the Cartan involution of $\mathfrak{u}_{nc}$ fixing $\mathfrak{k}^{(0)}$.*

Moreover $\mathfrak{q}$ is an NC algebra without flat part. In particular, $[\tilde{\mathfrak{a}}^{(2)}, \tilde{\mathfrak{n}}^{(2)}] = \tilde{\mathfrak{n}}^{(2)}$.

PROOF. Except for the final statement concerning $\mathfrak{q}$, the lemma is a reformulation of part of Theorem 5.6 of [**A-W2**]. The version of this theorem quoted here is stated in the first paragraph of the proof on p. 38 of [**A-W2**]. We note however that we have deleted the hypothesis of Theorem 5.6 that U be closed in G. A careful look at the proof of the theorem shows that this hypothesis is not needed. Alternatively, Proposition 2.5 of [**A-W1**] asserts that if $K^{(0)}A^{(0)}N^{(0)}$ is any Iwasawa decomposition of U_{nc}, then $K^{(0)}$ is compact (equivalently U_{nc} has finite center) and there exists a closed normal subgroup Q of U such that $Q \subset U_2$ and $R \equiv A^{(0)}N^{(0)}Q$ acts simply transitively on $\mathcal{M}$. Lemma 2.4 of [**A-W1**] states that any simply transitive solvable subgroup of G is closed in G. Thus for any choice of Q, we have that R and consequently $U_{nc}Q$ $(= K^{(0)}R)$ are closed in G. Theorem 5.6 can then be applied to the subgroup $U_{nc}Q$ of U. The final statement of the lemma is included in Proposition 5.3 of [**A-W2**].

(6.8) THEOREM. *Let $\mathcal{M}$ be a simply connected Riemannian solvmanifold of negative curvature with no Euclidean factor and let H be a connected subgroup of $G = I_0(\mathcal{M})$. Then H acts transitively on $\mathcal{M}$ if and only if there exists a semisimple Levi factor G_1 of G, an Iwasawa decomposition $G_{nc} = KAN$, a basis π of the positive roots of $\mathfrak{a}$ and a subset π_0 of π such that, in the notation of Lemmas* 6.3–6.4 *and* (c) *of Remark* 6.6, $\mathfrak{b}(\pi_0) + \mathfrak{r} \subset \mathfrak{h} \subset \mathfrak{c}(\pi_0) + \mathfrak{m}^{(1)} + \mathfrak{g}_c + \mathfrak{g}_2$ *for some normal modification $\mathfrak{r}$ of $\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)} + \mathfrak{s}^{(2)}$.*

PROOF. The "if" statement is immediate from Remark 6.6 since $\mathfrak{b}(\pi_0) + \mathfrak{r}$ contains a modification of $\mathfrak{a} + \mathfrak{n} + \mathfrak{s}^{(2)}$ where $\mathfrak{a} + \mathfrak{n} \equiv \mathfrak{a}^{(0)} + \mathfrak{a}^{(1)} + \mathfrak{n}^{(0)} + \mathfrak{n}^{(1)}$ is an Iwasawa subalgebra of $\mathfrak{g}_{nc}$. For the converse, we will apply Lemma 6.7 with H and G alternately playing the role of U. For $U = H$, we use the same notation as in the lemma for the various subgroups of H. For $U = G$, we drop the superscripts (0) and write $G_{nc} = KAN$, and we let the group $S^{(2)}$ of Remark 6.6(c) play the role of Q and $S \equiv (AN)S^{(2)}$ play the role of R. Thus $\mathfrak{q} = \tilde{\mathfrak{a}}^{(2)} + \tilde{\mathfrak{n}}^{(2)} \subset \mathfrak{h}$ and $\mathfrak{s}^{(2)} = \mathfrak{a}^{(2)} + \mathfrak{n}^{(2)} \subset \mathfrak{g}$.

As seen in the proof of Proposition 1.7(iv), there exists an Iwasawa decomposition $G_{nc} = KA'N'$ (K as above) such that $R \subset (M'A'N')G_cG_2$ for M' the normalizer of $A'N'$ in K. Also $S' \equiv (A'N')S^{(2)}$ acts simply transitively on $\mathcal{M}$. By the conjugacy of Iwasawa decompositions, $A'N' = x(AN)x^{-1}$ for some element $x \in G_{nc}$ which normalizes K. Necessarily $x \in K$ and since K fixes p, $\mathrm{Ad}(x)$ defines an isometry between the p-inner products on $\mathfrak{s}$ and $\mathfrak{s}'$. Hence S' can also play the role of R in Lemma 6.7 when $U = G$. I.e. we may assume that

$$R \subset (MAN)G_cG_2 \equiv F \tag{1}$$

for our original choice of AN. Thus by Remark 6.6, $\mathfrak{r}$ is a normal modification of $\mathfrak{s}$ and

$$\mathfrak{n} + \mathfrak{n}^{(2)} = \mathfrak{n}^{(0)} + \tilde{\mathfrak{n}}^{(2)}. \tag{2}$$

Since $\mathfrak{n}^{(2)}$ is an ideal in both $\mathfrak{g}$ and $\mathfrak{h}$, $\mathfrak{n}^{(2)} \subset \tilde{\mathfrak{n}}^{(2)}$.

Let $\mathfrak{l}$ denote the isotropy subalgebra of $\mathfrak{g}$ at p. The p-inner products on $\mathfrak{r}$ and $\mathfrak{s}$ extend to a positive semidefinite symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ with kernel $\mathfrak{l}$.

We claim that $\mathfrak{h}_{nc} \subset \mathfrak{g}_{nc}$ and that the decompositions $K^{(0)}A^{(0)}N^{(0)}$ and KAN are compatible. $\mathfrak{h}_{nc}$ lies in some semisimple Levi factor of $\mathfrak{g}$ and hence by the Levi-Malcev Theorem, $\mathrm{Ad}(x)\mathfrak{h}_{nc} \subset \mathfrak{g}_{nc}$ for some $x \in \mathrm{nilrad}(G)$. Since $\mathfrak{g} = \mathfrak{l} + \mathfrak{s}$ and $\mathfrak{l} \cap \mathrm{nilrad}(\mathfrak{g}) = \{0\}$, it is easily checked that $\mathrm{nilrad}(\mathfrak{g}) = \mathfrak{n}^{(2)}$. Hence

$$\mathrm{Ad}(x)\mathfrak{h}_{nc} \subset (\mathfrak{h}_{nc} + \mathfrak{n}^{(2)}) \cap \mathfrak{g}_{nc}. \tag{3}$$

But since $\mathfrak{n}^{(2)}$ lies in both $\mathfrak{s}^{(2)}$ and $\mathfrak{q}$, property (i) of Lemma 6.7 implies

$$\langle \mathfrak{h}_{nc}, \mathfrak{n}^{(2)} \rangle = 0 = \langle \mathfrak{g}_{nc}, \mathfrak{n}^{(2)} \rangle. \tag{4}$$

Since $\langle\ ,\ \rangle$ is positive-definite on $\mathfrak{n}^{(2)}$, it follows from (3) and (4) that $\mathrm{Ad}(x)\mathfrak{h}_{nc} = \mathfrak{h}_{nc}$; i.e. $\mathfrak{h}_{nc} \subset \mathfrak{g}_{nc}$. Thus $\mathfrak{h}^{(0)} \subset \mathfrak{l} \cap \mathfrak{g}_{nc} = \mathfrak{k}$ and $\mathfrak{n}^{(0)} \subset (\mathfrak{n} + \mathfrak{n}^{(2)}) \cap \mathfrak{g}_{nc} = \mathfrak{n}$. Since $\mathfrak{a}^{(0)} \perp \mathfrak{n}$, we have $\mathfrak{a}^{(0)} \subset \mathfrak{a} + \mathfrak{m}$. But by property (iii) of Lemma 6.7 applied to H, both $\mathfrak{a}$ and $\mathfrak{a}^{(0)}$ act symmetrically on $\mathfrak{n}^{(0)} + \mathfrak{n}^{(2)} \equiv \mathfrak{n} + \mathfrak{n}^{(2)}$ under the adjoint action, whereas m acts skew-symmetrically. Hence $\mathfrak{a}^{(0)} \subset \mathfrak{a}$ and the claim is proved. Let

$$\mathfrak{n}^{(1)} = \mathfrak{n} \ominus \mathfrak{n}^{(0)} \quad \text{and} \quad \mathfrak{a}^{(1)} = \mathfrak{a} \ominus \mathfrak{a}^{(0)} \tag{5}$$

relative to $\langle\ ,\ \rangle$. We will verify the hypotheses of Lemma 6.5 with $\mathfrak{u} = \mathfrak{h}_{nc} + \mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$. Since $\mathfrak{s}$ is a normal modification of $\mathfrak{r}$ (Proposition 2.4), we have that $\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)} + \mathfrak{s}^{(2)}$ is a normal modification of $\mathfrak{q}$, say with modification map ψ. By Proposition 2.4, $\mathfrak{q}$ is a modification of $\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)} + \mathfrak{s}^{(2)}$ with modification map φ given by $\varphi(X + \psi(X)) = -\psi(X)$. By (2), $\mathfrak{n}^{(1)} \subset \tilde{\mathfrak{n}}^{(2)}$. By (2), (5), and the definition of $\mathfrak{s}$, $\mathfrak{a}^{(1)} \perp \tilde{\mathfrak{n}}^{(2)}$ relative to $\langle\ ,\ \rangle$, so $\mathfrak{a}^{(1)} \subset (\mathrm{Id} + \psi)(\tilde{\mathfrak{a}}^{(2)})$. Thus $(\mathrm{Id} + \varphi)\mathfrak{a}^{(1)} \subset \tilde{\mathfrak{a}}^{(2)}$. The map φ takes values in $\mathfrak{l} \cap \mathfrak{f} \subset \mathfrak{m} + \mathfrak{g}_c + \mathfrak{g}_2$. Let $X \in \mathfrak{a}^{(1)}$ and write $\varphi(X) = A + B$ with $A \in \mathfrak{m}$ and $B \in \mathfrak{g}_c + \mathfrak{g}_2$. Since $[\tilde{\mathfrak{a}}^{(2)}, \mathfrak{h}_{nc}] = \{0\}$ (Lemma 6.7(ii)) and $[X + A, \mathfrak{h}_{nc}] \subset \mathfrak{g}_{nc}$ while $[B, \mathfrak{g}_{nc}] \subset \mathfrak{g}_2$, we have $[X + A, \mathfrak{h}_{nc}] = \{0\}$. Since $\mathrm{ad}_{\mathfrak{g}_{nc}} X$ has only real eigenvalues while $\mathrm{ad}_{\mathfrak{g}_{nc}} A$ has only imaginary ones, it follows that $[X, \mathfrak{h}_{nc}] = \{0\}$. Thus $[\mathfrak{a}^{(1)}, \mathfrak{h}_{nc}] = \{0\}$. Next

$$\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)} + \mathfrak{s}^{(2)} = (\mathrm{Id} + \psi)(\mathfrak{q}) = (\mathrm{Id} + \psi)(\tilde{\mathfrak{a}}_2) + \tilde{\mathfrak{n}}_2$$

by Remark 6.6(b). By the same remark,

$$[\mathfrak{a}^{(1)} + \mathfrak{a}^{(2)}, \mathfrak{n}^{(1)} + \mathfrak{n}^{(2)}] = \mathfrak{n}^{(1)} + \mathfrak{n}^{(2)}. \tag{6}$$

But $[\mathfrak{a}^{(2)}, \mathfrak{n}^{(1)}] = \{0\}$ by Lemma 6.7(ii), $[\mathfrak{a}^{(2)}, \mathfrak{n}^{(2)}] = \mathfrak{n}^{(2)}$ and

$$[\mathfrak{a}^{(1)}, \mathfrak{n}^{(1)}] \subset (\mathfrak{n}^{(1)} + \mathfrak{n}^{(2)}) \cap \mathfrak{g}_{nc} = \mathfrak{n}^{(1)},$$

so (6) implies $[\mathfrak{a}^{(1)}, \mathfrak{n}^{(1)}] = \mathfrak{n}^{(1)}$. We conclude from Lemma 6.5 that

$$\mathfrak{h}_{nc} + \mathfrak{a}^{(1)} + \mathfrak{n}^{(1)} = \mathfrak{c}(\pi_0)$$

for some subset π_0 of a basis π of positive roots of $\mathfrak{a}$, and the theorem follows.

If $\mathcal{M}$ is a symmetric space of noncompact type, then $\mathcal{M}$ has negative curvature and $G = I_0(\mathcal{M})$ is semisimple of noncompact type. Thus Theorem 6.8 yields

(6.9) THEOREM. *Let $\mathcal{M}$ be a symmetric space of noncompact type and let H be a connected subgroup of $G = I_0(\mathcal{M})$. Then H acts transitively on $\mathcal{M}$ if and only if there exists an Iwasawa decomposition $G = KAN$, a basis π of the positive roots of $\mathfrak{a}$ in $\mathfrak{g}$, and a subset π_0 of π such that $\mathfrak{b}(\pi_0) + \mathfrak{r} \subset \mathfrak{h} \subset \mathfrak{c}(\pi_0) + \mathfrak{m}^{(1)}$ for some normal modification $\mathfrak{r}$ of $\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$ by elements of $\mathfrak{m}^{(1)}$.*

(6.10) THEOREM. *Let $\mathcal{M}$ be a Riemannian solvmanifold and let H be a transitive subgroup of $G = I_0(M)$. Choose semisimple Levi factors H_1 of H and G_1 of G such that $H_1 \subset G_1$. Let $\pi_{nc}\colon \mathfrak{g} \to \mathfrak{g}_{nc}$ be the homomorphic projection. Then there exists an Iwasawa decomposition $G = KAN$, a basis π of simple roots of $\mathfrak{a}$ and a subset π_0 of π such that in the notation of Lemmas 6.3–6.4,*

(i) $\mathfrak{h}_{nc} = \mathfrak{b}(\pi_0)$,

(ii) $\mathfrak{h}_c \subset \mathfrak{m}^{(1)} \oplus \mathfrak{g}_c$,

(iii) $\pi_{nc}(\mathfrak{h}_2) \subset \mathfrak{m}^{(1)} + \mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$ *and* $\pi_{nc}(\mathfrak{h}_2)$ *contains a normal modification of* $\mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$.

PROOF. By Lemma 1.6, H_c fixes a point $p \in \mathcal{M}$. Let L be the isotropy subgroup of G at p. Let Z be the discrete center of G_{nc} and let $\rho\colon G \to G/(ZG_2G_2)$ be the canonical projection. Write $\tilde{G} = \rho(G)$. Note that ρ has differential $\pi_{nc}\colon \mathfrak{g} \to \mathfrak{g}_{nc}$. As in the proof of Proposition 1.7(iv), $\rho(L) \subset \tilde{K}$ for some maximal compact subgroup $\tilde{K}$ of $\rho(G)$, and $\tilde{\rho} = \tilde{K}\rho(H)$. Thus $\rho(H)$ acts transitively on $\tilde{G}/\tilde{K}$. Under a left-invariant metric, $\tilde{G}/\tilde{K}$ is a symmetric space of noncompact type. Thus we may apply Theorem 6.9 to conclude that in the notation of the theorem, $\mathfrak{b}(\pi_0) + \mathfrak{r} \subset \pi_{nc}(\mathfrak{h}) \subset \mathfrak{c}(\pi_0) + \mathfrak{m}^{(1)}$ for a suitable choice of Iwasawa decomposition $\mathfrak{g}_{nc} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and of π_0. In particular, since $\mathfrak{h}_{nc} \subset \mathfrak{g}_{nc}$, (i) follows. (iii) is also immediate. For (ii), $H_c \subset G_1$ and $\pi_{nc}(\mathfrak{h}_c)$ commutes with $\mathfrak{h}_{nc} = \mathfrak{b}(\pi_0)$ and normalizes $\pi_{nc}(\mathfrak{h}_2)$. It follows that $\pi_{nc}(\mathfrak{h}_c)$ commutes with $\mathfrak{a}$ and therefore lies in $\mathfrak{m}^{(1)} + \mathfrak{a}^{(1)}$. Then $\pi_{nc}(\mathfrak{h}_c) \subset \mathfrak{m}^{(1)}$ and $\mathfrak{h}_c \subset \mathfrak{m}^{(1)} + \mathfrak{g}_c$.

NOTE ADDED IN PROOF. Equation (2) in the proof of Theorem 4.2 is incorrect. To correct the proof, we show that $\mathfrak{r}'$ is unimodular. Let $\mathfrak{h} = \mathfrak{r} + \varphi(\mathfrak{r})$. By Proposition 2.4, $\mathfrak{r}$ and $\mathfrak{r}'$ are both $\mathfrak{h}$-ideals containing $[\mathfrak{h}, \mathfrak{h}]$. Since $\mathfrak{r}$ is unimodular and $\operatorname{tr}\operatorname{ad}\varphi(X)_{\mathfrak{l}\mathfrak{r}} = 0$ for $X \in \mathfrak{r}$, it follows that $\mathfrak{h}$ and hence $\mathfrak{r}'$ are unimodular.

REFERENCES

[Al1] D. V. Alekseevskiĭ, *Conjugacy of polar factorizations of Lie groups*, Mat. Sb. **84 (126)** (1971), 14–26.

[Al2] ——, *Homogeneous Riemannian spaces of negative curvature*, Mat. Sb. **96 (138)** (1975), 87–109.

[**A-W1**] R. Azencott and E. N. Wilson, *Homogeneous manifolds with negative curvature*. I, Trans. Amer. Math. Soc. **215** (1976), 323–362.

[**A-W2**] ——, *Homogeneous manifolds with negative curvature*. II, Mem. Amer. Math. Soc. **8** (1976), No. 178.

[**D**] E. D. Deloff, *Naturally reductive metrics and metrics with volume preserving geodesic symmetries on NC algebras*, Thesis, Rutgers Univ., New Brunswick, N. J., 1979.

[**G1**] C. S. Gordon, *Riemannian isometry groups containing transitive reductive subgroups*, Math. Ann. **248** (1980), 185–192.

[**G2**] ——, *Transitive Riemannian isometry groups with nilpotent radicals*, Ann. Inst. Fourier (Grenoble) **31** (1981), 193–204.

[**G-W1**] C. S. Gordon and E. N. Wilson, *Isospectral deformations of compact solvmanifolds*, J. Differential Geom. **19** (1984), 241–256.

[**G-W2**] ——, *The fine structure of transitive Riemannian isometry groups*. I, Trans. Amer. Math. Soc., **289** (1985), 367–380; II, in preparation.

[**H**] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.

[**Hu**] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972.

[**J**] N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

[**O-T**] T. Ochiai and T. Takahashi, *The group of isometries of a left invariant Riemannian metric on a Lie group*, Math. Ann. **223** (1976), 91–96.

[**On**] A. L. Oniscik, *Inclusion relations among transitive compact transformation groups*, Amer. Math. Soc. Transl. (2) **50** (1966), 5–58.

[**Oz**] H. Ozeki, *On a transitive transformation group of a compact group manifold*, Osaka J. Math. **14** (1977), 519–531.

[**W**] E. N. Wilson, *Isometry groups on homogeneous nilmanifolds*, Geom. Dedicata **12** (1982), 337–346.

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, 1 BROOKINGS DRIVE, ST. LOUIS, MISSOURI 63130