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HARMONICALLY IMMERSED SURFACES OF $\mathbf{R}^n$

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ABSTRACT. Some generalizations of classical results in the theory of minimal surfaces $f: M \to \mathbf{R}^n$ are shown to hold in the more general case of harmonically immersed surfaces.

Introduction. Let (M, g) be a connected Riemann surface with a prescribed metric g in its conformal class and let $f: M \to \mathbf{R}^n$ be an immersion. It is well known that f realizes M, with the induced metric from $\mathbf{R}^n$, as a minimal surface if and only if f is a conformal (with respect to g) harmonic map (cf., for example, [**3** or **8**]). That is, the theory of minimal surfaces is substantially the theory of conformally immersed harmonic surfaces. Our purpose is to analyze the case when f is simply a harmonic immersion, to introduce an appropriate Gauss map and, as the main achievement, to establish in this new setting the analogue of three fundamental results (cf. Theorems 1.1 and 2.1, and §3) in the theory of minimal surfaces: the harmonicity of the Gauss map (Ruh and Vilms [**13**]), equidistribution properties of the Gauss map in $\mathbf{C}P^n$ (Chern and Osserman [**1** and **11**]), and the Enneper-Weierstrass representation formulas recently due, for arbitrary codimension, to Hoffman and Osserman [**7**]. The last two topics (the third for $n = 3$) have already been treated by T. K. Milnor (see for instance her survey article [**9**]), but as we remark in §2, our results complement hers in an interesting way.

The method of the moving frame as well as the Einstein summation convention are used throughout this paper.

1. The Gauss map and first properties. Let (M, g) be as in the Introduction. We fix the index ranges $1 \leq A, B, \ldots \leq n$, $1 \leq i, j, \ldots \leq 2$, $3 \leq \alpha, \beta, \ldots \leq n$. With K, dV, and Δ we will indicate the Gaussian curvature, the volume element, the Laplace-Beltrami operator relative to the metric g. The geometry of $\mathbf{R}^n$ is described by its homogeneous realization $E(n)/SO(n)$ via the components of the Maurer-Cartan form of $E(n)$, the group of proper rigid motions of $\mathbf{R}^n$. Briefly, under a local section of the bundle $\pi: E(n) \to \mathbf{R}^n \cong E(n)/SO(n)$, the pulled back forms θ^A and ω^A_B with $\omega^A_B + \omega^B_A = 0$, reflecting the structure of the Lie algebra of $E(n)$, describe respectively, a local orthonormal coframe and the corresponding Levi-Civita connection forms, while the structure equations reduce to the pull-back of the Maurer-Cartan equations of $E(n)$. Let $f: M \to \mathbf{R}^n$ be an immersion. A Darboux frame along f is a map $E: U \subset M \to E(n)$, U open in M, such that

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$\pi \circ E = f$ and

$$\theta^\alpha = 0 \tag{1.1}$$

on U, where we have dropped, and usually will, the pull-back notation E^*. Thus if $\{\varphi^i\}$ is a local orthonormal coframe for the metric g and if we set

$$f^*\theta^A = R_i^A \varphi^i, \tag{1.2}$$

then condition (1.1) becomes

$$R_i^\alpha = 0. \tag{1.3}$$

Furthemore, we suppose that E preserves the orientation of M, that is, $\theta^1 \wedge \theta^2$ on M gives the correct orientation. Thus if ds^2 is the metric induced by f, $d\hat{V}$ its volume element, and $\hat{K}$ its Gaussian curvature, we have

$$ds^2 = R_i^k R_j^k \varphi^i \varphi^j, \tag{1.4}$$

$$d\hat{V} = u dV, \tag{1.5}$$

where to simplify notation we have set

$$u = \det(R_i^k) > 0. \tag{1.6}$$

On the intersection of their domains of definition, if not empty, any other Darboux frame $\tilde{E}$ is related to E by

$$\tilde{E} = EK. \tag{1.7}$$

Here K is an $E(n)$-valued function of the form

$$K = \begin{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} & 0 \\ 0 & 1 \end{pmatrix},$$

$A \in SO(2)$, and $D \in SO(n-2)$, where we are realizing $E(n)$ as the subgroup of $GL(n+1, \mathbf{R})$ given by the elements $\left(\begin{smallmatrix} e & b \\ 0 & 1 \end{smallmatrix}\right)$, where $e = (e_A) \in SO(n)$ and $e_A, b \in \mathbf{R}^n$. Therefore, under the general change of Darboux frame along f given by (1.7), and of orthonormal (oriented) coframe on M under the orthogonal transformation $C \in SO(2)$, the matrix $R = (R_j^k)$ changes according to the law

$$\tilde{R} = {}^tARC, \tag{1.8}$$

where $\tilde{}$ refers to quantities computed in the new frame $\tilde{E}$. From now on, quantities and computations will be with respect to a (local) Darboux frame $E = \left(\begin{smallmatrix} e & f \\ 0 & 1 \end{smallmatrix}\right)$, along f.

It is easy to verify that the map $\gamma_f \colon M \to \mathbf{C}P^{n-1}$ defined by

$$\gamma_f(p) = [(R_1^k(p) + iR_2^k(p))e_k(p)], \tag{1.9}$$

where square brackets denote the equivalence class, is independent of the frame. We call γ_f the Gauss map of the harmonic immersion $f \colon (M, g) \to \mathbf{R}^n$. Letting Q_{n-2} denote the quadric in $\mathbf{C}P^{n-1}$,

$$Q_{n-2} = \{[z] \in \mathbf{C}P^{n-1} \colon {}^tzz = 0\},$$

observe that $\gamma_f(p) \in Q_{n-2}$ if and only if $R(p) = (R_i^k(p))$ is a positive scalar multiple of an element of $SO(2)$. In other words $\gamma_f \colon M \to Q_{n-2}$ if and only if

f is conformal. In case M is compact, it is proved in [**12**] that f is conformal if and only if $\langle f, \tau(f) - 2u\hat{H}\rangle \geq 0$, where $\langle\ ,\ \rangle$ is the scalar product in $\mathbf{R}^n$, $\hat{H}$ is the mean curvature vector of the isometric immersion $f\colon (M, ds^2) \to \mathbf{R}^n$, and $\tau(f)$ is the tension field of the map f (cf. [**4**]). That is, $\tau(f)$ is the trace with respect to g of the generalized second fundamental tensor ∇df, which we determine as follows. Take the exterior differential of (1.2) and use the structure equations on $\mathbf{R}^n$ and (M, g) to obtain

$$(dR_k^A - R_i^A\varphi_k^i + R_k^B\omega_B^A) \wedge \varphi^k = 0,$$

where φ_k^i are the connection forms of g relative to the orthonormal coframe $\{\varphi^i\}$. Cartan's Lemma then gives

$$dR_k^A - R_i^A\varphi_k^i + R_k^B\omega_B^A = R_{ki}^A\varphi^i, \qquad R_{ki}^A = R_{ik}^A, \tag{1.10}$$

for some (local) functions R_{ki}^A which are exactly the coefficients of ∇df, that is,

$$\nabla df = R_{ki}^A\varphi^k\varphi^i \otimes e_A.$$

As an immediate consequence we deduce that

$$\tau(f) = \Delta f,$$

proving the well-known fact that f is harmonic if and only if each one of its components is a harmonic function in the usual sense.

As f is an immersion, $f^{-1}T\mathbf{R}^n$ splits into a tangential and a normal part, which decomposes ∇df into tangential, T, and normal, N, components given by

$$T = R_{ij}^k\varphi^i\varphi^j \otimes e_k, \qquad N = R_{ij}^\alpha\varphi^i\varphi^j \otimes e_\alpha. \tag{1.11}$$

Observe that T is the image under df of the difference of the Levi-Civita covariant derivatives associated, respectively, to ds^2 and g, so that $T \equiv 0$ if and only if the two connections have the same parametrized geodesics. An immersion $f\colon (M, g) \to \mathbf{R}^n$ such that $T \equiv 0$ will be called affine.

Using (1.3) in (1.10), we obtain

$$R_k^i\omega_i^\alpha = R_{ki}^\alpha\varphi^i. \tag{1.12}$$

But $\{\theta^A\}$ is a Darboux coframe along the isometric immersion $f\colon (M, ds^2) \to \mathbf{R}^n$. Hence, if h_{ij}^α are the coefficients of the corresponding second fundamental tensor II, we have

$$\omega_i^\alpha = h_{ij}^\alpha\theta^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha, \tag{1.13}$$

and using (1.2), (1.3), (1.13) in (1.12) we deduce, in matrix notation, that

$$R^\alpha = {}^tRh^\alpha R, \tag{1.14}$$

where $R^\alpha = (R_{ij}^\alpha)$, $R = (R_i^k)$, and $h^\alpha = (h_{ij}^\alpha)$. Such an equation clearly relates N to II. For instance, since by (1.6) R is invertible, we have $N \equiv 0$ if and only if $f(M)$ is contained in a two dimensional plane of $\mathbf{R}^n$.

For later use we take the exterior derivative of (1.10) and use the structure equations to obtain

$$(dR_{ij}^A - R_{kj}^A\varphi_i^\alpha - R_{ik}^A\varphi_j^k + R_{ij}^B\omega_B^A) \wedge \varphi^j = \tfrac{1}{2}R_k^A S_{kijl}\varphi^l \wedge \varphi^j \tag{1.15}$$

where S_{kijl} are the components of the Riemann curvature tensor of the metric g on M. Setting

$$dR^A_{ij} - R^A_{kj}\varphi^k_i - R^A_{ik}\varphi^k_j + R^B_{ij}\omega^A_B = R^A_{ijk}\varphi^k, \tag{1.16}$$

then from the symmetry $R^A_{ij} = R^A_{ji}$ of (1.10) we deduce that

$$R^A_{ijk} = R^A_{jik}, \tag{1.17}$$

while from (1.15) we obtain the commutation relations

$$R^A_{ijk} = R^A_{ikj} + R^A_l S_{lijk}. \tag{1.18}$$

Let $p\colon \mathbf{C}^n \to \mathbf{C}P^{n-1}$ be the natural projection. Then from the definition (1.9) of γ_f, the structure equations, and (1.10) we have

$$d\gamma_f = (R^A_{ik} + iR^A_{2k})\varphi^k \otimes p_* e_A. \tag{1.19}$$

Thus if dF^2 is the Fubini-Study metric of $\mathbf{C}P^{n-1}$ (normalized to have holomorphic sectional curvature 4), then

$$\gamma_f^* dF^2 = -\mathcal{A} g + \langle \nabla df, \tau(f)\rangle, \tag{1.20}$$

where $\mathcal{A}$ is the invariant given by

$$\mathcal{A} = \sum_A \det R^A, \qquad R^A = (R^A_{ij}) \tag{1.21}$$

In particular, we recognize that if $f\colon (M,g) \to \mathbf{R}^n$ is an isometry, then $\mathcal{A}$ is the Gaussian curvature of g.

Taking the exterior differential of (1.19) (i.e., its homogeneous variables version) and using (1.16), we compute the tension field

$$\tau(\gamma_f) = (R^A_{1kk} + iR^A_{2kk})p_* e_A \tag{1.22}$$

of γ_f which, using the commutation relations (1.17), (1.18), can be rewritten as

$$\tau(\gamma_f) = (R^A_{kk1} + iR^A_{kk2})p_* e_A. \tag{1.23}$$

DEFINITION. Let $f\colon (M,g) \to \mathbf{R}^n$ be an immersion with tension field $\tau(f)$. Then we say that f has parallel tension field if $\nabla\tau(f) \equiv 0$, where ∇ is the covariant derivative operator in $f^{-1}T\mathbf{R}^n$.

In terms of coefficients, from (1.16) we see that $\nabla\tau(f) \equiv 0$ is equivalent to the equations

$$R^A_{kki} = 0. \tag{1.24}$$

Together, (1.23) and (1.24) prove the following.

THEOREM 1.1. *Let $f\colon (M,g) \to \mathbf{R}^n$ be an immersion. Then its Gauss map $\gamma_f\colon (M,g) \to \mathbf{C}P^{n-1}$ is harmonic if and only if f has parallel tension field.*

REMARK. When f is an isometry, we easily recover the result of Ruh and Vilms **[13]**.

THEOREM 1.2. *Let $f\colon (M,g) \to \mathbf{R}^n$ be an immersion. Then the following are equivalent.*

(i) $\gamma_f\colon (M,g) \to \mathbf{C}P^{n-1}$ *is antiholomorphic*;

(ii) $\gamma_f^* dF^2 = -\mathcal{A} g$;

(iii) f *is harmonic.*

PROOF. (i) is equivalent to (iii): in fact, from (1.19), γ_f is antiholomorphic if and only if

$$0 = (R^A_{1k} + iR^A_{2k})\varphi^k \wedge \bar{\varphi} = -i(R^A_{11} + R^A_{22})\varphi^1 \wedge \varphi^2$$

for each A, where $\varphi = \varphi^1 + i\varphi^2$ is a local basis for the $(1,0)$ forms on the Riemann surface M. But this happens if and only if $\tau(f) \equiv 0$.

By (1.20), (iii) implies (ii). On the other hand by (1.20), (ii) implies

$$0 = \langle \nabla df, \tau(f) \rangle = R^A_{ij} R^A_{kk} \varphi^i \varphi^j,$$

giving $R^A_{ij}R^A_{kk} = 0$ for each i and j, so that $\sum_A (R^A_{kk})^2 = 0$. Thus (iii) follows from (ii). □

REMARKS 1. By (ii), in case f is a harmonic immersion we have that $\mathcal{A}$ is the negative of the area magnification under the Gauss map γ_f.

2. In case f is harmonic, $\mathcal{A}$ is also given by $\mathcal{A} = -\frac{1}{2}\|\nabla df\|^2$. Furthermore, by the definition (1.21) of $\mathcal{A}$, together with equations (1.14), (1.11), and (1.6) we have

$$\begin{aligned} \mathcal{A} &= \sum_\alpha \det(R^\alpha_{ij}) + \sum_k \det(R^k_{ij}) \\ &= \{\det(R^k_i)\}^2 \sum_\alpha \det(h^\alpha_{ij}) + \sum_k \det(R^k_{ij}) \\ &= u^2 \hat{K} - \tfrac{1}{2}\|T\|^2, \end{aligned} \tag{1.25}$$

where the the last equality holds because f is harmonic.

As observed above, if f is an isometry, then $\mathcal{A} = K = \hat{K}$; and if f is harmonic, then either $K \equiv 0$ or the zeros of K are isolated and of finite order, that is, K is of analytic type [**2**]. We do not know if $\mathcal{A}$ is always of analytic type, but we have the following partial result.

PROPOSITION 1.1. *If $f\colon (M,g) \to \mathbf{R}^n$ is an affine harmonic immersion, then $\mathcal{A}$ is of analytic type.*

PROOF. Since f is harmonic and affine, $\mathcal{A}$ is the negative of the square of the length of the $\mathbf{C}^{n-2}$ vector with components $R^\alpha_{11} + iR^\alpha_{12}$. Observe that because of (1.17) and (1.18) we have that $R^\alpha_{122} = R^\alpha_{221}$ and $R^\alpha_{121} = R^\alpha_{112}$. Since f is harmonic, (1.24) holds, and an easy computation using (1.16), (1.3) and $T \equiv 0$ gives

$$d(R^\alpha_{11} + iR^\alpha_{12}) \equiv (R^\beta_{11} + iR^\beta_{12})(2i\delta^\alpha_\beta \varphi^1_2 - \omega^\alpha_\beta) \mod \bar{\varphi}.$$

A result of Chern [**2**] completes the proof of the proposition. □

COROLLARY 1.1. *Let $f\colon (M,g) \to \mathbf{R}^n$ be an affine harmonic immersion. Then its Gauss map $\gamma_f\colon (M,g) \to \mathbf{C}P^{n-1}$ is weakly conformal* (*in the sense of Gulliver, Osserman and Royden* [**5**]).

PROOF. Immediate from Theorem 1.2 and Proposition 1.1. □

2. Generalization of a theorem of Chern. The result we wish to establish is the following.

THEOREM 2.1. *Let $f\colon (M,g) \to \mathbf{R}^n$ be a harmonic immersion of a complete, oriented, simply connected surface M into $\mathbf{R}^n$ such that $f(M)$ does not lie in a plane. Let $\gamma_f\colon (M,g) \to \mathbf{C}P^{n-1}$ be its Gauss map and let $(\mathbf{C}P^{n-1})^*$ be the space of hyperplanes of $\mathbf{C}P^{n-1}$. If*

$$K - \mathcal{A} \geq 0 \tag{2.1}$$

on M, then the subset of hyperplanes meeting $\gamma_f(M)$ is dense in $(\mathbf{C}P^{n-1})^*$.

PROOF. We follow Chern [**1**] with some modifications. Suppose the conclusion of the theorem be false. If $\{z^A\}$ are homogeneous coordinates in $\mathbf{C}P^{n-1}$, then we can suppose that there is a neighborhood of the hyperplane $z^1 = 0$ in $(\mathbf{C}P^{n-1})^*$ whose hyperplanes do not meet $\gamma_f(M)$. Thus, the function

$$v = \left\{ |z^1|^2 / \sum_A |z^A|^2 \right\}^{1/2} \circ \gamma_f \tag{2.2}$$

satisfies

$$v \geq \varepsilon \tag{2.3}$$

on M for some $\varepsilon > 0$. Since by Theorem 1.2 γ_f is antiholomorphic, we have

$$\partial\bar{\partial}\gamma_f^* = \gamma_f^*\bar{\partial}\partial, \tag{2.4}$$

with the usual meaning of the operators ∂ and $\bar{\partial}$. It is well known that if κ is the Kaehler form of $\mathbf{C}P^{n-1}$ then in homogeneous coordinates,

$$\kappa = \frac{i}{2}\partial\bar{\partial}\log\left(\sum_A z^A \bar{z}^A\right),$$

so that in the open set $z^1 \neq 0$ of $\mathbf{C}P^{n-1}$

$$\kappa = \frac{i}{2}\partial\bar{\partial}\log\left(1 + \frac{|z^2|^2 + \cdots + |z^n|^2}{|z^1|^2}\right). \tag{2.5}$$

Therefore, on M, (2.2), (2.4) and (2.5) give

$$\gamma_f^*\kappa = i\partial\bar{\partial}\log v. \tag{2.6}$$

On the other hand, since γ_f is antiholomorphic, (ii) from Theorem 1.2 yields

$$\gamma_f^*\kappa = \mathcal{A}\varphi^1 \wedge \varphi^2. \tag{2.7}$$

It is well known, and easy to prove by using isothermal coordinates for the metric g on M, that

$$2i\partial\bar{\partial}\log t = (\Delta \log t)\varphi^1 \wedge \varphi^2.$$

Hence comparing (2.7) with (2.6) gives

$$\Delta \log v = 2\mathcal{A}. \tag{2.8}$$

Define a new metric $\tilde{g}$ on M by setting $\tilde{g} = vg$. On M, $\tilde{g} \geq \varepsilon g$, so completeness of g implies completeness of $\tilde{g}$. If $\tilde{K}$ is the Gaussian curvature of $\tilde{g}$, then by a well-known formula we have

$$\tilde{K} = (K - \tfrac{1}{2}\Delta \log v)/v;$$

that is, by (2.8),

$$\tilde{K} = (K - \mathcal{A})/v.$$

Thus under assumption (2.1) $\tilde{K} \geq 0$, and therefore (M, g) is parabolic by a result of Huber [**6**]. But $\mathcal{A} \leq 0$ and by (2.8) and (2.3), $\log v$ is a superharmonic function bounded below. The maximum principle implies $\log v$ constant, and therefore $\mathcal{A} \equiv 0$. This in particular implies that N defined in (1.11) is identically 0 so that $f(M)$ lies in a plane, a contradiction. □

REMARKS. 1. Theorem 2.1 generalizes a result of Chern [**1**] proved under the additional hypothesis that f is an isometry. In this case $\mathcal{A} = K$ and assumption (2.1) of the theorem is automatically satisfied.

2. Theorem 2.1 provides an interesting complement to some results of T. K. Milnor, namely to Theorems 2 and 3 in [**9**]. In her theorems she assumes the completeness of the induced metric ds^2, while we assume the completeness of g together with the inequality (2.1). Although completeness of ds^2 implies completeness of her energy one metric, which is conformally equivalent to g, nevertheless there are many cases for which the induced metric is incomplete while our hypotheses hold. For example, if g is the usual flat metric on $M = \mathbf{C}$, then our hypotheses hold for any harmonic map $f\colon \mathbf{C} \to \mathbf{R}^n$. In general, the induced metric of such maps is not complete, as illustrated, for example, by taking $n = 3$, $f = (f_1, f_2, f_3)$, where $f_1 + if_2 = e^z$ and $f_3 = \operatorname{Re}(e^{z/2})$. (Thanks to Albert Baernstein for this example.)

3. The generalized Enneper representation. Let $f\colon (M, g) \to \mathbf{R}^n$ be a harmonic immersion. Then from Theorem 1.2 we have that the map $\bar{\gamma}_f\colon M \to \mathbf{C}P^{n-1}$ is holomorphic and we can construct a $\mathbf{C}^n$-valued Abelian differential α on M by setting

$$\alpha = (R_1^k e_k - iR_2^k e_k)\varphi. \tag{3.1}$$

In fact, each component α^A is well defined on all of M; that is, the above definition is independent of the choice of φ and of the Darboux frame E along f. This follows immediately from (1.7) and (1.8). One also easily verifies that each α^A has purely imaginary periods; that is, for any closed curve γ in M

$$\operatorname{Re} \int_\gamma \alpha^A = 0. \tag{3.2}$$

A simple computation shows that if $\langle\ ,\ \rangle$ is the symmetric product in $\mathbf{C}^n$, then

$$\langle \alpha, \alpha \rangle = \{|R_1|^2 - |R_2|^2 - 2i\langle R_1, R_2\rangle\}(\varphi)^2, \tag{3.3}$$

$$\langle \alpha, \bar{\alpha} \rangle = 2e(f)g, \tag{3.4}$$

where R_1 and R_2 are the column vectors of the matrix $R = (R_i^k)$ and $e(f)$ is the energy density of f. In particular, observe that $\langle \alpha, \alpha \rangle = 0$ if and only if f is conformal.

Since M is connected, we can recover f up to an additive constant vector from the Abelian differentials α^A. Fixing a point $p \in M$ and letting γ be any curve connecting p to the generic point $q \in M$, set

$$u^A + iv^A = \int_\gamma \alpha^A.$$

Then u^A is single-valued because of (3.2) and $u^A + iv^A$ is locally holomorphic. Let f^A be the Ath component of f, and U a neighborhood of p on which the Darboux frame E along f is defined as well as the coframe φ^i of (M, g) with corresponding dual frame E_i. Furthermore, let $z = x + iy$ be a local complex coordinate so that $\varphi = \lambda^2\, dz$, for some nowhere zero function λ on U. Then

$$E_1 = \lambda^{-2}\frac{\partial}{\partial x} \quad \text{and} \quad E_2 = \lambda^{-2}\frac{\partial}{\partial y}.$$

But from the definition of α,

$$\alpha^A = 2\frac{\partial f^A}{\partial z}\, dz. \tag{3.5}$$

If $z(p) = 0$ and $z(q) = z$, then

$$\begin{aligned}\alpha^A &= \left\{\frac{\partial}{\partial z}\int_0^z \alpha^A\right\} dz = \frac{\partial}{\partial z}(u^A + iv^A)dz = \frac{\partial}{\partial x}(u^A + iv^A)\, dz \\ &= \left\{\frac{\partial u^A}{\partial x} - i\frac{\partial u^A}{\partial y}\right\} dz = 2\frac{\partial u^A}{\partial z} dz\end{aligned}$$

because of the Cauchy-Riemann equations. From (3.5) we get

$$\frac{\partial f^A}{\partial z} = \frac{\partial u^A}{\partial z}$$

proving, for U connected, that f^A and u^A differ by an additive constant c^A. By standard arguments on prolongation along a curve we obtain that

$$u = \operatorname{Re}\int_\gamma \alpha \tag{3.6}$$

recovers the original immersion f up to a translation.

On the other hand, suppose we are given on (M, g) a $\mathbf{C}^n$-valued Abelian differential α with purely imaginary periods, so that (3.2) holds and such that $\langle \alpha, \bar{\alpha}\rangle \neq 0$ on M. Then f defined as in (3.6) is an immersion and is harmonic because α Abelian implies that $\Delta f = 0$. Furthermore, f will be conformal if and only if $\langle \alpha, \alpha\rangle \equiv 0$, and the energy density of f can be obtained via (3.4).

The point is to find a simple procedure to give a $\mathbf{C}^n$-valued Abelian differential on (M, g) with the above properties. Again, start with a harmonic immersion $f\colon (M, g) \to \mathbf{R}^n$ and consider the Abelian differential α on M associated to f. Suppose α^1 is not identically equal to $i\alpha^2$. Then define a nonzero Abelian differential η and meromorphic functions g^ρ, $1 \leq \rho \leq n-2$, by setting

$$\eta = \alpha^1 - i\alpha^2, \tag{3.7}$$

$$g^\rho = \alpha^{2+\rho}/(\alpha^1 - i\alpha^2). \tag{3.8}$$

Observe that the g^ρ can have poles only where η vanishes. Furthermore, introduce the quadratic Abelian differential

$$\mu = \langle \alpha, \alpha \rangle. \tag{3.9}$$

An easy computation shows that we can express the α^A in terms of η, g^ρ, and μ by

$$\begin{aligned} \alpha^1 &= \frac{1}{2}\left(1 - \sum_\rho (g^\rho)^2 + \frac{\mu}{\eta^2}\right)\eta, \\ \alpha^2 &= \frac{i}{2}\left(1 + \sum_\rho (g^\rho)^2 - \frac{\mu}{\eta^2}\right)\eta, \\ \alpha^{2+\rho} &= g^\rho \eta. \end{aligned} \tag{3.10}$$

Observe that the right-hand sides of equations (3.10) are holomorphic. Therefore, given an Abelian differential η (not identically zero), a quadratic Abelian differential μ, and $n-2$ meromorphic functions g^ρ on M, such that (3.10) defines Abelian differentials α^A, with $\langle \alpha, \bar\alpha \rangle \neq 0$ on M, and with purely imaginary periods, then equation (3.6) gives a harmonic immersion $f\colon (M,g) \to \mathbf{R}^n$ whose energy density is related to $\langle \alpha, \bar\alpha\rangle$ by (3.4). Such an immersion is conformal if and only if $\mu \equiv 0$. A simple condition to guarantee that the differentials given by (3.10) are Abelian is as follows.

At p, let μ_p be the order of the zero of η, and let ν_p be the maximum order of the poles of the g^ρ and of $\sum_\rho (g^\rho)^2 + \mu/\eta^2$. Then the α^A given by (3.10) are holomorphic if and only if $\mu_p \geq \nu_p$ for every $p \in M$. Furthermore, if M is simply connected, then the α^A have no periods.

We now analyze the case $\alpha^1 \equiv i\alpha^2$ on M. Consider the function

$$h = f^1 + if^2, \tag{3.11}$$

where $f^i = \operatorname{Re}\int_\gamma \alpha^i$, $i = 1, 2$. We have, with respect to a local coordinate z,

$$2\frac{\partial h}{\partial \bar z} d\bar z = \bar\alpha^1 + i\bar\alpha^2 \equiv 0,$$

so that the function $h\colon M \to \mathbf{C}$ is holomorphic, and $(\alpha^1)^2 + (\alpha^2)^2 = 0$ on M. Hence either h defines a minimal surface (possibly branched) or else $\alpha^1 = \alpha^2 \equiv 0$ on M, and $f(M)$ is contained in $\mathbf{R}^{n-2}$. Moreover, the map $Y\colon (M,g) \to \mathbf{R}^{n-2}$ given by $Y = (f^\beta)$, $2 \leq \beta \leq n$, is harmonic and, since $f\colon (M,g) \to \mathbf{R}^n$ is an immersion, the map $X = (f^i)$ is regular if and only if Y is not. On the other hand, suppose we are given a holomorphic map $h\colon (M,g) \to \mathbf{C}$ and a harmonic map $Y\colon (M,g) \to \mathbf{R}^{n-2}$. Define $f = {}^t(f^i, f^\alpha)$, where $h = f^1 + if^2$ and $Y = {}^t(f^\alpha)$. Then of course $f\colon (M,g) \to \mathbf{R}^n$ is a harmonic map and it is an immersion provided that Y is regular whenever the derivative of h is 0.

REMARK. The results in this section generalize results of Hoffman and Osserman [**7**].

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