

HOMOGENEOUS CONTINUA IN EUCLIDEAN $(n + 1)$ -SPACE WHICH CONTAIN AN n -CUBE ARE LOCALLY CONNECTED

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ABSTRACT. We prove that each homogeneous continuum which topologically contains an n -dimensional unit cube and lies in $(n + 1)$ -dimensional Euclidean space is locally connected.

Introduction. Although we still do not have any complete classification of homogeneous plane continua today, there are many significant results concerning homogeneous continua in the planar case. It is interesting to consider which of these results may be naturally extended to higher dimensional cases. In this paper we give an example of such an extension. In 1924 S. Mazurkiewicz [7] proved that a simple closed curve is the only homogeneous plane continuum which is nondegenerate and locally connected. Using this result, R. H. Bing showed in 1959 [1] that a simple closed curve is the only homogeneous plane continuum that contains an arc. All that he had to do to show this (and that he did) was to prove that each homogeneous plane continuum which contains an arc is locally connected. Here we extend the last statement to all finite dimensional cases.

MAIN THEOREM. *If a homogeneous continuum contains a set homeomorphic to the n -dimensional unit cube and lies in $(n + 1)$ -dimensional Euclidean space, then it is locally connected.*

It is very natural to ask whether the assumptions of this theorem imply that the considered continuum is an n -manifold. Recently the author has answered this question in the affirmative. This result, which makes heavy use of the Main Theorem, will be presented in a future paper.

The general method of the proof of the Main Theorem is similar to that of Bing's theorem, i.e. we suppose the existence of a continuum not satisfying this theorem and find a sequence of properties of it. Finally we get two properties such that one contradicts the other. However, only Property 1 here is similar to Property 4 in [1, p. 212], and the particular method of proof here differs much from Bing's. There is also another important difference from the Bing method—we use many times the so-called ε -push property of homogeneous metric continua (see Lemma 4 in [2, p. 37]) which is a corollary to the well-known Effros theorem, and which was not known to Bing in 1959.

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Throughout the paper the symbol I denotes the unit segment $[0, 1]$, E^n is n -dimensional Euclidean space, $\rho(x, y)$ is the usual Euclidean distance between points x and y of a Euclidean space. We write $\text{Lim } A_n = A$ when $\{A_n\}$ is a sequence of closed sets converging to a set A (with respect to the Hausdorff distance). Further, $B(x, \varepsilon) = \{y \in E^n : \rho(x, y) < \varepsilon\}$ and $\rho(A, B) = \inf\{\rho(a, b) : a \in A \text{ and } b \in B\}$, where $\varepsilon > 0$ and $A, B \subset E^n$. Also,

$$S_n = \{x \in E^{n+1} : \rho(x, \mathbf{0}) = 1\} \subset B_{n+1} = \{x \in E^{n+1} : \rho(x, \mathbf{0}) \leq 1\},$$

where $\mathbf{0} = (0, \dots, 0) \in E^{n+1}$. If a set A satisfies $A \stackrel{\text{top}}{=} I^n$, its combinatorial boundary is denoted by ∂A . The symbol xy denotes an arc with end points x and y . For two points $p, q \in E^n$, the symbol \overline{pq} denotes the segment between these points. A set T is called an n -dimensional simple triod if it is the union $T = A \cup xy$ with $A \stackrel{\text{top}}{=} I^n$ and $A \cap xy = \{x\} \subset A \setminus \partial A$. A point x is said to be accessible from a set Y provided there exists an arc xy such that $xy \setminus \{x\} \subset Y$. The notation $f: X \rightarrow Y$ always means that f is a continuous mapping from a space X into a space Y . A map $f: A \rightarrow X$, for $A \subset X$, is called an ε -mapping if $\rho(a, f(a)) < \varepsilon$ for each $a \in A$. A space is said to be a manifold if its topology has a basis composed of open sets homeomorphic to the same Euclidean space.

Preliminaries. We will give or remind the reader of a few facts which we need to prove the Main Theorem. We begin with a well-known theorem of Borsuk.

1. THEOREM [4, THEOREM 7, p. 495]. *If two compact sets $A, B \subset E^n$ are homeomorphic, then the numbers of components of the sets $E^n \setminus A$ and $E^n \setminus B$ are equal.*

A set $A \subset X$ separates a space X provided $X \setminus A$ is not connected. It separates X between points $p, q \in X$ provided p and q lie in distinct components of $X \setminus A$. The next theorem is an immediate consequence of the two following results: [4, Theorem 2, p. 473 and 4, Theorem 8, p. 258].

2. THEOREM. *If a compact set $F \subset S_n \setminus \{p, q\}$ separates S_n between p and q , with $p, q \in S_n$, then each set which is obtained from F by a deformation in $S_n \setminus \{p, q\}$ separates S_n between p and q .*

3. PROPOSITION. *If a set $A \subset E^{n+1}$ is homeomorphic to I^n , $p \in E^{n+1}$ and $\rho(p, \partial A) \geq \varepsilon > \rho(p, A)$, then for every component B of $B(p, \varepsilon) \cap A$ the set $B(p, \varepsilon) \setminus B$ has exactly two components.*

PROOF. Let a mapping $f: B \rightarrow S_n$ be an embedding. Define two functions $t: S_n \rightarrow E^1$ and $h: S_n \rightarrow h(S_n) \subset E^{n+1}$ by

$$t(x) = \begin{cases} \rho(x, S_n \setminus f(B)) & \text{if } x \in f(B), \\ 0 & \text{otherwise,} \end{cases}$$

and $h(x) = (1+t(x)) \cdot x$. Then h is a homeomorphism. Consider the quotient space E^{n+1}/B_{n+1} with the quotient map $\pi: E^{n+1} \rightarrow E^{n+1}/B_{n+1}$. The proofs of the two following facts are left to the reader.

- (1) The set $E^{n+1} \setminus (S_n \cup h(S_n))$ has exactly three components.
- (2) The set $E^{n+1}/B_{n+1} \setminus \pi(S_n \cup h(S_n))$ has exactly two components.

Let $q \in \text{bd } B(p, \varepsilon) \setminus \text{cl } B$ and $\rho(q, \text{cl } B) > \delta > 0$. Then

$$Y = \text{bd } B(p, \varepsilon) \setminus B(q, \delta) \underset{\text{top}}{=} I^n.$$

Consider the quotient space E^{n+1}/Y with the quotient map $\pi_1 : E^{n+1} \rightarrow E^{n+1}/Y$. Then we have

$$E^{n+1} \underset{\text{top}}{=} E^{n+1}/B_{n+1} \underset{\text{top}}{=} E^{n+1}/Y \quad \text{and} \quad \pi(S_n \cup h(S_n)) \underset{\text{top}}{=} \pi_1(\text{cl } B)$$

(the second of these equalities is a consequence of the fact that both considered sets are one-point compactifications of sets homeomorphic to B); thus it follows from Theorem 1 that the set $E^{n+1}/Y \setminus \pi_1(\text{cl } B)$ has exactly two components. But $\pi_1|_{B(p, \varepsilon)}$ is a homeomorphism and clearly the set $B(p, \varepsilon) \setminus B \underset{\text{top}}{=} \pi_1(B(p, \varepsilon) \setminus B)$ has exactly two components too.

4. PROPOSITION. *If a locally connected closed subset A of E^n separates E^n between two points $p, q \in E^n$, then there is a component of A that also does.*

PROOF. Suppose no component of A separates E^n between p and q . Consider a one-point compactification $E^n \cup \{z\} \underset{\text{top}}{=} S_n$ of E^n with the remainder z . By the local connectedness of A only a finite number of components of A intersect the segment \overline{pq} and the union of all components not intersecting this segment is closed in E^n . Let B_0 be the closure (in $E^n \cup \{z\}$) of this union and B_1, \dots, B_m be the closures (in $E^n \cup \{z\}$) of the components intersecting \overline{pq} . Of course, no set B_i for $i \in \{0, 1, \dots, m\}$ separates $E^n \cup \{z\}$ between p and q . Obviously, the claim that the union $B_0 \cup B_1 \cup \dots \cup B_m$ does not separate $E^n \cup \{z\}$ between p and q completes the proof. Indeed, if $n > 2$, we have $\dim(B_i \cap B_j) \leq \dim\{z\} = 0 \leq n - 3$ for $i, j \in \{0, 1, \dots, m\}$ and $i \neq j$. Applying m times Theorem 11 of [4, p. 471], the claim is obtained. To prove the case $n = 2$ use the same theorem and footnote (1) of [4, p. 471]. The case $n = 1$ is trivial.

REMARK. Note that the word ‘‘component’’ may be replaced by ‘‘arc component’’ in Propositions 3 and 4. Observe that in Proposition 4 the open balls in E^n may be considered instead of E^n .

5. PROPOSITION. *Let a closed subset C of a ball $B(c, \varepsilon) \subset E^n$ separate this ball between two points $p, q \in B(c, \varepsilon) \setminus C$. Then for every $\delta > 0$ such that $\delta < \rho(\{p, q\}, C \cup (E^n \setminus B(c, \varepsilon)))$ and for every δ -mapping $f : C \rightarrow E^n$ the set $f(C)$ separates the ball $B(c, \varepsilon - \delta)$ between p and q .*

PROOF. Suppose $f(C)$ does not separate $B(c, \varepsilon - \delta)$ between p and q . Let a sequence $\{x_n\}$ of points of $f(C) \cap B(c, \varepsilon - \delta)$ converge to a point $x \in B(c, \varepsilon - \delta)$. Then $\rho(x_n, \text{bd } B(c, \varepsilon)) > \delta + \delta_1$ for some $\delta_1 > 0$ and every x_n . Thus $f^{-1}(\{x_1, x_2, \dots\}) \subset C \cap \text{cl } B(c, \varepsilon - \delta_1)$. The last set is compact, so $x \in f(C)$. Hence the set $f(C) \cap B(c, \varepsilon - \delta)$ is closed in $B(c, \varepsilon - \delta)$. Thus there is an arc pq in $B(c, \varepsilon - \delta - \delta_2) \setminus f(C)$ for some positive number $\delta_2 < \rho(\{p, q\}, E^n \setminus B(c, \varepsilon)) - \delta$. So $f(C)$ does not separate $B(c, \varepsilon - \delta - \delta_2)$ between p and q . Consider the mapping $g : C \rightarrow B(\mathbf{0}, \delta)$, where $\mathbf{0} = (0, \dots, 0) \in E^n$, defined by $g(x) = f(x) - x$. By the Tietze theorem this mapping may be extended to a mapping $g^* : C_1 \rightarrow B(\mathbf{0}, \delta)$, where $C_1 = C \cup \{x \in E^n : \rho(c, x) = \varepsilon - \delta_2\}$. Thus we get the extension $f^* : C_1 \rightarrow E^n$ of the mapping g , defined by $f^*(x) = x + g^*(x)$. Observe that f^* is a δ -mapping. Consider the

deformation $G(x, t) = x + t \cdot g^*(x)$. We have $G: C_1 \times I \rightarrow E^n \setminus \{p, q\}$. Note that the set $C_1 = G(C_1 \times \{0\})$ separates E^n between p and q . By a simple application of Theorem 2 we see that the set $f^*(C_1) = G(C_1 \times \{1\})$ separates E^n between p and q . But no point of $f^*(C_1 \setminus C)$ lies in $B(c, \varepsilon - \delta - \delta_2)$. Therefore $f^*(C) = f(C)$ separates $B(c, \varepsilon - \delta - \delta_2)$ between p and q , a contradiction.

6. PROPOSITION. *If A_0, A_1, A_2, \dots are closed subsets of E^n such that there are ε_k -mappings $f_k: A_0 \rightarrow A_k$ with $\lim \varepsilon_k = 0$, then for each ball $B(c, \varepsilon) \subset E^n$ with two points $p, q \in B(c, \varepsilon) \setminus A_0$ the set A_0 separates $B(c, \varepsilon)$ between p and q if and only if for every δ satisfying $\max\{\rho(p, c), \rho(q, c)\} < \delta < \varepsilon$ almost all A_k separate the ball $B(c, \delta)$ between p and q .*

PROOF. Let A_0 separate $B(c, \varepsilon)$ between p and q . Then the fact that almost all A_k separate $B(c, \delta)$ between p and q easily follows by Proposition 5. Assume A_0 does not separate $B(c, \varepsilon)$ between p and q . Then there are an arc $pq \subset B(c, \varepsilon) \setminus A_0$ and an integer m_0 such that for each $m > m_0$ we have $\rho(pq, A_0) > \varepsilon_m$. Put $\delta = \varepsilon - \rho(pq, \text{bd } B(c, \varepsilon))/2$. It is clear that A_m does not separate $B(c, \delta)$ between p and q for $m > m_0$.

7. PROPOSITION. *Let sets $A_0, A \subset E^{n+1}$ be homeomorphic to I^n and a ball $B(c, \varepsilon) \subset E^{n+1}$ be such that $A_0 \subset A \cap B(c, \varepsilon) \subset A \setminus \partial A$. Fix a point $p \in A_0 \setminus \partial A_0$. Let B be the arc component of $A \cap B(c, \varepsilon)$ containing p , and U be a component of $B(c, \varepsilon) \setminus B$. If there is a sequence of ε_k -mappings $f_k: A_0 \rightarrow f_k(A_0) \subset U$ such that $\lim \varepsilon_k = 0$, then p is not accessible from $U \setminus (f_1(A_0) \cup f_2(A_0) \cup \dots)$.*

PROOF. Let pa be an arc with $pa \setminus \{p\} \subset U$. Denote by V the other component of $B(c, \varepsilon) \setminus B$ (see Proposition 3). Let $\{a_k\}, \{q_k\}, \{z_k\}$ be sequences such that $a_k \in pa \setminus \{p\}$ and $\lim a_k = p, q_k \in V$ and $\lim q_k = p, z_k$ is the first point of the segment $\overline{q_k a_k}$ which lies in B (in the ordering from q_k to a_k). Put $\delta = \rho(p, A \setminus A_0)/2 > 0$ and let B_0 be the component of $A \cap B(p, \delta) = A_0 \cap B(p, \delta)$ containing p . There is an integer m such that $z_m \in B_0, q_m \in B(p, \delta)$ and $a_m p \subset B(p, \delta)$, where $a_m p \subset ap$. Since B_0 is also an arc component of $A_0 \cap B(p, \delta)$, there is an arc $p z_m \subset B_0$. Put $a_m q_m = a_m p \cup p z_m \cup \overline{z_m q_m}$. A_0 separates $B(p, \delta)$ between a_m and q_m ; thus by Proposition 6 there exists an integer $i > 0$ such that $a_m q_m \cap f_i(A_0) \neq \emptyset$. But $f_i(A_0) \cap (p z_m \cup \overline{z_m q_m}) = \emptyset$; hence $\emptyset \neq a_m p \cap f_i(A_0) \subset ap \cap f_i(A_0)$. Therefore the point p is not accessible from $U \setminus (f_1(A_0) \cup f_2(A_0) \cup \dots)$.

We end this part of the paper reminding the reader of two well-known results which are essential to the remaining part of the paper.

8. THEOREM [5, COROLLARY 2, p. 276]. *Each Euclidean space E^{n+1} does not contain uncountably many mutually exclusive n -dimensional simple triods.*

9. THEOREM (ε -PUSH PROPERTY [3, LEMMA 4, p. 37]). *If X is a homogeneous continuum and $\varepsilon > 0$, then there exists $\delta > 0$ such that for all $x, y \in X$ with $\rho(x, y) < \delta$ there is a homeomorphism $h: X \rightarrow X$ moving no point more than ε , such that $h(x) = y$.*

Proof of the Main Theorem. Let M be a homogeneous continuum lying in E^{n+1} and containing a set homeomorphic to the n -dimensional unit cube I^n . Suppose M is not locally connected. We will prove some properties of M and find two of them such that one contradicts the other. Therefore we will infer that M is locally connected.

PROPERTY 1. *M does not contain an n-dimensional simple triod.*

PROOF. It is known that a homogeneous continuum which is not locally connected contains an open set with uncountably many components (it can easily be proved by the reader either directly or with help of Theorem 2 in [1, p. 211]). If *M* contained an *n*-dimensional simple triod, then by its homogeneity each component of an open subset with uncountably many components would contain such a triod, an impossibility by Theorem 8.

PROPERTY 2. *For every $\epsilon > 0$ there is $\delta > 0$ such that for each $x \in M$ there exists a set *A* homeomorphic to I^n with $x \in A \subset B(x, \epsilon) \cap M$ and $\rho(x, \partial A) > \delta$.*

PROOF. Let $\epsilon > 0$. Of course, for each $x \in M$ there is a set $A = I^n$ with $x \in A \setminus \partial A \subset B(x, \epsilon) \cap M$. Put $\sigma(x) = \sup\{\sigma > 0: \text{there is } A \stackrel{\text{top}}{=} I^n \text{ with } x \in A \subset B(x, \epsilon) \cap M \text{ and } \rho(x, \partial A) > \sigma\}$. Suppose the required δ does not exist. Then there is a convergent sequence $\{x_m\}$ of points in *M* with $\lim \sigma(x_m) = 0$. Put $x_0 = \lim x_m$. Let $I^n \stackrel{\text{top}}{=} A_0 \subset B(x_0, \epsilon) \cap M$, $x_0 \in A_0 \setminus \partial A_0$ and put $\xi = (x_0, \partial A_0)/2$. By Theorem 9 there are ϵ_m -homeomorphisms $h_m: M \rightarrow M$ with $h_m(x_0) = x_m$ and $\lim \epsilon_m = 0$. Considering the sets $h_m(A_0)$, we see that $\liminf \sigma(x_m) > \xi > 0$, a contradiction.

The next property seems to be essential in our argument. As the reader will see, it will be employed very often. Bing in [1] made heavy use of the fact that his continuum *M* contained no folded sequence of arcs converging to an arc (see Theorem 6 in [1, p. 220]). It may be observed that in case $n = 1$ this fact is hidden in the next property, and this property is its virtual generalization, which is suitable for the proof of our theorem. To formulate this property, we define a number

$$d(x, y) = \inf\{\text{diam } A: A \text{ is an arc and } x, y \in A \subset M\}$$

for each pair of points *x, y* of the same arc component of *M*.

PROPERTY 3. *There exists $\epsilon > 0$ such that for every $\delta > 0$ there is $\xi > 0$ satisfying the following condition: if *x* and *y* are points of the same arc component of *M*, $d(x, y) < \epsilon$ and $\rho(x, y) < \xi$, then $d(x, y) < \delta$.*

PROOF. Let δ_0 be a number guaranteed by Property 2 for the number $\epsilon_0 = \text{diam } M$ and put $\epsilon = \delta_0/2$. Suppose there exists $\delta > 0$ such that for every positive integer *m* there are points x_m, y_m of the same arc component of *M* fulfilling $d(x_m, y_m) < \epsilon$, $\rho(x_m, y_m) < 1/m$ and $d(x_m, y_m) > \delta$. We may assume that the sequence $\{x_m\}$ converges to a point $x_0 \in M$. By Theorem 9 there are ϵ_m -homeomorphisms $h_m: M \rightarrow M$ such that $h_m(x_m) = x_0$ and $\lim \epsilon_m = 0$. Let $A_0 \subset M$ be a set homeomorphic to I^n such that $x_0 \in A_0 \subset B(x_0, \epsilon_0) \cap M$ and $(x_0, \partial A_0) > \delta_0$ (see Property 2), and $x_m y_m$ be arcs in *M* with $\text{diam } x_m y_m < \epsilon = \delta_0/2$. Then almost all arcs $h_m(x_m y_m)$ have diameter less than δ_0 . We claim that these arcs are contained in A_0 for, if not, there is an arc $h_m(x_m y_m)$ of diameter less than δ_0 with a point $z \in h_m(x_m y_m) \setminus A_0$. Then there is an arc $zp \subset h_m(x_m y_m)$ such that $(zp \setminus \{p\}) \cap A_0 = \emptyset$ and $p \in A_0$. Since $\rho(x_0, \partial A_0) > \delta_0$, $x_0 \in h_m(x_m y_m)$ and $\text{diam } h_m(x_m y_m) < \delta_0$, we have $p \in A_0 \setminus \partial A_0$ and the set $A_0 \cup zp \subset M$ is an *n*-dimensional simple triod, which is impossible by Property 1.

Thus almost all arcs $h_m(x_m y_m)$ are contained in A_0 . By the local connectedness of A_0 there are arcs A_m in A_0 (for sufficiently great m) such that $h_m(x_m) = x_0 \in A_m$, $h_m(y_m) \in A_m$ and $\lim \text{diam } A_m = 0$ (because $\lim h_m(y_m) = x_0$). Then we have $\lim \text{diam } h_m^{-1}(A_m) = 0$ and $x_m, y_m \in h^{-1}(A_m) \underset{\text{top}}{=} I$, hence $\lim d(x_m, y_m) = 0 < \delta$, a contradiction.

Let \mathcal{C} be the family of all arc components of M . Note that the function $d: C \times C \rightarrow E^1$ yields a metric to each arc component $C \in \mathcal{C}$. This metric is equivalent to the metric ρ if and only if for each sequence $\{x_m\}$ of points of C converging to a point $x_0 \in C$ there is a sequence of arcs $x_m x_0$ with $x_m x_0 \subset C$ such that $\lim \text{diam } x_m x_0 = 0$.

PROPERTY 4. *If the metrics d and ρ are equivalent on an arc component of M , then each arc component of M is a compact manifold.*

PROOF. Let the metrics d and ρ be equivalent on a member C of \mathcal{C} and let $x \in C$. Then, by Property 1, each set A homeomorphic to I^n such that $x \in A \setminus \partial A \subset C$ is a neighborhood of x in C . Thus C is a manifold. Suppose C is not compact. However, it is locally compact, so the set $\text{cl } C \setminus C$ is compact. By (14.2) in [6, p. 33] $\text{cl } C$ is homogeneous (compare also Property 13 in [1, p. 214]). Let C_1 be another arc component of $\text{cl } C$. By the homogeneity of $\text{cl } C$ we have $\text{cl } C = \text{cl } C_1 \subset \text{cl}(\text{cl } C \setminus C) = \text{cl } C \setminus C$, a contradiction.

PROPERTY 5. *The arc components of M are not compact manifolds.*

PROOF. Suppose an arc component of M is a compact manifold. Then by the homogeneity of M , for each $C \in \mathcal{C}$ so is C . Each $C \in \mathcal{C}$ separates E^{n+1} into exactly two components (see Theorem (27.10) in [2, p. 235]). By Theorem 9, for a given sequence $\{A_m\}$ with $A_0, A_1, A_2, \dots \in \mathcal{C}$ the following conditions are equivalent:

- (i) $\text{Lim } A_m = A_0$.
- (ii) $\lim \rho(A_m, A_0) = 0$.
- (iii) The sequence $\{A_m\}$ is homeomorphically convergent to A_0 , i.e., there are ε_m -homeomorphisms $h_m: A_0 \rightarrow A_m$ with $\lim \varepsilon_m = 0$.

Therefore by Proposition 6 we conclude that if $\text{Lim } A_m = A_0$ and $p, q \in E^{n+1} \setminus A_0$, then the two following statements are equivalent:

- (i') A_0 separates E^{n+1} between p and q .
- (ii') Almost all A_m 's separate E^{n+1} between p and q .

Consequently, if $p, q \in E^{n+1} \setminus M$, the families $S(p, q) = \{C \in \mathcal{C} : C \text{ separates } E^{n+1} \text{ between } p \text{ and } q\}$ and $\mathcal{C} \setminus S(p, q)$ are closed in \mathcal{C} with the usual topology induced by the Hausdorff distance. The equivalence (i) \Leftrightarrow (ii) implies the continuity of the natural map from M to \mathcal{C} , whence it follows that \mathcal{C} is connected. Thus for each pair of points $p, q \in E^{n+1} \setminus M$ the family $S(p, q)$ or its complement is empty.

Let C and C_1 be distinct arc components of M . Again by Theorem (27.10) in [2, p. 235] the set $E^{n+1} \setminus (C \cup C_1)$ has exactly three components. Let T_1, T_2, T_3 be n -dimensional simple triods lying in mutually distinct components of $E^{n+1} \setminus (C \cup C_1)$. Then by Property 1 there are points $t_i \in T_i \setminus M$ for $i \in \{1, 2, 3\}$. Two of them, assume t_1 and t_2 , lie in the same component of $E^{n+1} \setminus C$. Clearly C_1 separates E^{n+1} between t_1 and t_2 (compare Theorem 11 of [4, p. 471]). Thus $S(t_1, t_2) \neq \emptyset \neq \mathcal{C} \setminus S(t_1, t_2)$, a contradiction.

PROPERTY 6. *If $\varepsilon > 0$ is a number guaranteed by Property 3, then for each $x \in M$ there is a sequence $\{x_m\}$ of points lying in the arc component of M containing x , such that $\lim x_m = x$ and $d(x, x_m) \geq \varepsilon$ for each m .*

PROOF. By Properties 4 and 5 the metrics ρ and d are not equivalent on an arc component $C \in \mathcal{C}$. Hence there are a sequence $\{x_m^0\}$ with $x_m^0 \in C$ and a point $x_0 \in C$ such that $\lim \rho(x_0, x_m^0) = 0$ and $\limsup d(x_0, x_m^0) > 0$. By the homogeneity of M for each point x of M we may find such a sequence $\{x_m\}$. This sequence may be chosen such that $\lim d(x, x_m) = \delta > 0$. By Property 3 we get $\delta \geq \varepsilon$.

PROPERTY 7. *There exists a number $\varepsilon > 0$ such that if $0 < \delta < \varepsilon$, $p \in E^{n+1}$, B_0 is an arc component of the nonempty set $B(p, \delta) \cap M$ and C_0 is the arc component of M containing B_0 , then*

- (a) *the set $B(p, \delta) \setminus B_0$ has exactly two components, and*
- (b) *if for a component U of $B(p, \delta) \setminus B_0$ we have $\text{cl}(U \cap C_0) \cap B_0 \neq \emptyset$, then no point of B_0 is accessible from $U \setminus C_0$, and*
- (c) *for at least one component U of $B(p, \delta) \setminus B_0$ no point of B_0 is accessible from $U \setminus C_0$.*

PROOF. Let $\varepsilon_2 > 0$ be a number guaranteed by Property 2 for the number $\text{diam } M$, let $\varepsilon_3 > 0$ be a number guaranteed by Property 3, and suppose $\varepsilon = \min\{\varepsilon_2/2, \varepsilon_3/2\}$. Assume $0 < \delta < \varepsilon$.

(a) Let $x_0 \in B_0$. By Property 2 there is $A = I^n_{\text{top}}$ such that $x_0 \in A \subset M$ and $\rho(x_0, \partial A) > \varepsilon_2 > 2\delta$. Taking into account Property 1, observe that B_0 is also the arc component of $A \cap B(p, \delta)$ containing x_0 (if not, there would be an arc $xy \subset B_0$ with $xy \setminus \{y\} \subset B_0 \setminus A$ not intersecting ∂A by the previous inequalities). Thus by Proposition 3 we see B_0 separates $B(p, \delta)$ into exactly two components.

(b) Let $\{x_m\}$ be a sequence converging to a point $x \in B_0$, with $x_m \in U \cap C_0$, where U is a component of $B(p, \delta) \setminus B_0$. For a given point $y \in B_0$ let xy be an arc in B_0 and let $h: M \rightarrow M$ be ε_m -homeomorphisms with $\lim \varepsilon_m = 0$ such that $h_m(x) = x_m$ (see Theorem 9). For sufficiently great m the arcs $h_m(xy)$ lie in $B(p, \delta)$, and for those m , we have $h_m(xy) \subset U \cap C_0$. Now we have to prove that y is not accessible from $U \setminus C_0$. In order to do this find a set $A_0 = I^n_{\text{top}}$ such that $y \in A_0 \setminus \partial A_0 \subset A_0 \subset B_0$. Then for almost all integers m we have $h_m(A_0) \subset U$. Applying Proposition 7 for the set A defined in case (a) we get the conclusion.

(c) It is a corollary to (a) and (b) and to Property 6. In fact, for a given point $x \in B_0$ take a sequence $\{x_m\}$ guaranteed by Property 6 (note that $x_m \in C_0 \setminus B_0$), find one of two components of $B(p, \delta) \setminus B_0$ with infinite number of x_m 's (see Property 7(a)) and apply Property 7(b). This completes the proof of Property 7.

Property 7 justifies the names introduced below. Let δ, p, B_0 and C_0 have the same meanings as in Property 7 such that the conclusions (a), (b), (c) of this property hold and let U, V be distinct components of $B(p, \delta) \setminus B_0$. Then U and V are called the sides of B_0 in $B(p, \delta)$ and we say that each of them is opposite to the other. If no point of B_0 is accessible from $U \setminus C_0$, then U is called a side of nonaccessibility of B_0 in $B(p, \delta)$.

PROPERTY 8. *There exists $\varepsilon > 0$ such that for every positive $\delta < \varepsilon$ there is $\xi > 0$ such that if $x_0 \in M$ and if $f: I \rightarrow C_0, g_0, g_1: I \rightarrow E^{n+1}$ are mappings*

(where C_0 is, as previously, the arc component of M containing x_0) satisfying

(i) $f(0) = x_0,$

(ii) $\rho(f(t), g_i(t)) < \xi$ for $i \in \{0, 1\}$ and $t \in I,$

(iii) the arc component A_t^i of $g_i(I) \cap B(f(t), \delta)$ containing $g_i(t)$ for $i \in \{0, 1\}$ is disjoint with the arc component B_t of $C_0 \cap B(f(t), \delta)$ containing $f(t)$, i.e., $A_t^0 \cap B_t = A_t^1 \cap B_t = \emptyset$ for every $t \in I,$

then the two following implications hold:

(a) If $g_0(0)$ lies in a side of nonaccessibility of B_0 in $B(x_0, \delta)$, then for every $t \in I$ the point $g_0(t)$ lies in a side of nonaccessibility of B_t in $B(f(t), \delta)$.

(b) If the points $g_0(0)$ and $g_1(0)$ lie in opposite sides of B_0 in $B(x_0, \delta)$, then for every $t \in I$ the points $g_0(t)$ and $g_1(t)$ lie in opposite sides of B_t in $B(f(t), \delta)$.

PROOF. Let $\varepsilon_2, \varepsilon_3 > 0$ be numbers as in the proof of Property 7 and $\varepsilon_7 > 0$ be a number guaranteed by Property 7. Put $\varepsilon = \min\{\varepsilon_2/2, \varepsilon_3/3, \varepsilon_7\}$, and let $0 < \delta < \varepsilon$. Then there is an integer k such that if $d(x, y) < \varepsilon_3$ and $\rho(x, y) < \delta/k$, then $d(x, y) < \delta$ for each points x, y of the same arc component of M (see Property 3). Let 3ξ with $0 < 3\xi < \delta/4k$ be a number guaranteed by Property 3 for the number $\delta/4k$, i.e. if $d(x, y) < \varepsilon_3$ and $\rho(x, y) < 3\xi$, then $d(x, y) < \delta/4k$ for x, y in the same arc component of M . Assume the mappings f, g_0, g_1 satisfy the statements (i)–(iii). Let $\varepsilon_0 > 0$ be a number such that if $|t_1 - t_2| < \varepsilon_0$, then

$$\max\{\rho(f(t_1), f(t_2)), \rho(g_0(t_1), g_0(t_2)), \rho(g_1(t_1), g_1(t_2))\} < \xi \quad \text{for } t_1, t_2 \in I.$$

(a) Put $P = \{t \in I : g_0(t) \text{ lies in a side of nonaccessibility of } B_t \text{ in } B(f(t), \delta)\}$. Suppose there are $t_0, t \in I$ with $0 < t - t_0 < \varepsilon_0$ such that $t_0 \in P$ and $t \notin P$. Put $a = g_0(t)$. There is a segment \overline{ax} with $x \in B_t$ and $\rho(a, x) = \rho(a, B_t) \leq \rho(a, f(t)) < \xi$. Then we have $d(f(t_0), f(t)) \leq \text{diam } f([t_0, t]) < \xi < \varepsilon_3$ and $\rho(f(t_0), f(t)) < \xi$. Thus $d(f(t_0), f(t)) < \delta$ by Property 3. Therefore

$$d(f(t_0), x) \leq d(f(t_0), f(t)) + d(f(t), x) < \delta + 2\delta < 3\varepsilon < \varepsilon_3$$

and

$$\rho(f(t_0), x) \leq \rho(f(t_0), f(t)) + \rho(f(t), g_0(t)) + \rho(a, x) < 3\xi.$$

Hence, again by Property 3 we have $d(f(t_0), x) < \delta/4k < \delta$ and $x \in B_{t_0}$. If $t' \in [t_0, t] \subset I$, then

$$\rho(f(t_0), g_0(t')) \leq \rho(f(t_0), g_0(t_0)) + \rho(g_0(t_0), g_0(t')) < 2\xi < \delta,$$

Thus $g_0([t_0, t]) \subset A_{t_0}^0 \subset B(f(t_0), \delta)$. Since $A_{t_0}^0 \cap B_{t_0} = \emptyset$, the points $g_0(t_0), g_0(t)$ lie in the same side of nonaccessibility of B_{t_0} in $B(f(t_0), \delta)$.

If $(\overline{ax} \setminus \{x\}) \cap B_{t_0} = \emptyset$, then since x is not accessible from the side of B_{t_0} containing $a = g_0(t)$, for every $y \in \overline{ax} \setminus \{x\}$ the set $\overline{yx} \setminus \{x\}$ intersects C_0 . Hence by Property 7(b) $g_0(t)$ lies in the side of nonaccessibility of B_t in $B(f(t), \delta)$, a contradiction.

If there is $y \in (\overline{ax} \setminus \{x\}) \cap B_{t_0}$, then

$$d(y, f(t)) \leq d(y, f(t_0)) + d(f(t_0), f(t)) < 2\delta + \delta < 3\varepsilon \leq \varepsilon_3$$

and

$$\rho(y, f(t)) \leq \rho(y, g_0(t)) + \rho(g_0(t), f(t)) < 2\xi < 3\xi,$$

hence $d(f(t), y) < \delta/4k < \delta$ and $y \in B_t$, which is impossible by the construction of the segment \overline{ax} .

Therefore we infer that if $t_0 \in P$, then $t \in P$ for each $t \in [t_0, t_0 + \varepsilon_0) \cap I$ and, consequently, that $P = I$ if $0 \in P$.

(b) Put $Z = \{t \in I: g_0(t), g_1(t) \text{ lie in opposite sides of } B_t \text{ in } B(f(t), \delta)\}$. Suppose there are $t_0, t \in I$ with $0 < t - t_0 < \varepsilon_0$ such that $t_0 \in Z$ and $t \notin Z$. Let $0 \in Z$. By Property 7(c) at least one of the points $g_0(0), g_1(0)$ lies in a side of nonaccessibility of B_0 in $B(x_0, \delta)$. Applying conclusion (a) we see that $g_0(t)$ and $g_1(t)$ both lie in a side of nonaccessibility of B_t in $B(f(t), \delta)$. Put $a = g_0(t), b = f(t)$ and $c = g_1(t)$. Consider the first point x of the segment \overline{ab} belonging to B_t (in the ordering from a to b) and the first point y of the segment \overline{cb} belonging to B_t (in the ordering from c to b). Then there is a sequence $\{x_m\}$ converging to x with $x_m \in (\overline{ax} \setminus \{x\}) \cap C_0$. Let $h_m: M \rightarrow M$ be ε_m -homeomorphisms with $\lim \varepsilon_m = 0$ and $h_m(x) = x_m$. We claim that there are an integer j and a point $y_0 \in h_j(B_t) \cap (\overline{cy} \setminus \{y\})$ such that x_j and y_0 are points of the same arc component of $C_0 \cap B(f(t), \delta)$. In fact, let $xy \subset B_t$ be an arc and $A_0 \subset B_t$ be a set homeomorphic to I^n with $y \in A_0 \setminus \partial A_0$. Since $x_m \in h_m(A_0 \cup xy)$, the sets $h_m(A_0 \cup xy)$ and the point c lie in the same side U_t of B_t in $B(f(t), \delta)$ for sufficiently great numbers m . By Proposition 7 it follows that y is not accessible from $U_t \setminus (h_1(A_0) \cup h_2(A_0) \cup \dots)$, hence there is $y_0 \in h_j(A_0 \cup xy) \cap (\overline{cy} \setminus \{y\}) \subset B(f(t), \delta) \cap h_j(B_t) \cap (\overline{cy} \setminus \{y\})$ for some integer j , so the claim is obtained. Therefore $d(x_j, y_0) < 2\delta < 2\varepsilon \leq \varepsilon_3$. We also have

$$\rho(x_j, y_0) \leq \rho(x_j, b) + \rho(b, y_0) \leq \rho(a, b) + \rho(b, c) < 2\xi < 3\xi,$$

hence $d(x_j, y_0) < \delta/4k$. Let $A \subset C_0$ be an arc with $x_j, y_0 \in A$ and $\text{diam } A < \delta/4k$. Consider the union $U = \overline{ax_j} \cup A \cup \overline{y_0c}$. Note that $U \subset \bigcap \{B(f(t'), \delta): t_0 \leq t' \leq t\}$. Indeed, if $u \in U$, then

$$\rho(f(t), u) \leq \rho(f(t), a) + \rho(a, x_j) + \text{diam } A + \rho(y_0, c) \leq 3\xi + \delta/4k,$$

and for $t' \in [t_0, t]$

$$\rho(f(t'), u) \leq \rho(f(t'), f(t)) + \rho(f(t), u) \leq \xi + 3\xi + \delta/4k < 2\delta/4k + \delta/4k < \delta$$

(recall that $3\xi < \delta/4k$).

By the construction of U we conclude that $U \cap B_t = \emptyset$. Suppose there is a point $x \in U \cap B_{t'}$ for some $t' \in [t_0, t]$. Then

$$d(z, f(t)) \leq d(z, f(t')) + d(f(t'), f(t)) < 2\delta + \xi < 3\delta < 3\varepsilon < \varepsilon_3$$

and

$$\rho(z, f(t)) \leq 3\xi + \delta/4k \leq 2\delta/4k < \delta/k,$$

hence $d(z, f(t)) < \delta$ and $z \in U \cap B_t \neq \emptyset$ which contradicts the previous conclusion. Therefore we have $U \cap B_{t'} = \emptyset$ for each $t' \in [t_0, t]$. If $t' \in [t_0, t]$, for $i \in \{0, 1\}$ we also get

$$\rho(g_i(t'), f(t_0)) \leq \rho(g_i(t'), f(t')) + \rho(f(t'), f(t_0)) < 2\xi < \delta,$$

and so $g_i([t_0, t]) \subset B(f(t_0), \delta)$. Note that the set $V = g_0([t_0, t]) \cup U \cup g_1([t_0, t])$ is connected, $g_0(t_0), g_1(t_0) \in V \subset B(f(t_0), \delta)$ and $V \cap B_{t_0} = \emptyset$ (see assumption (iii)). Thus $g_0(t_0)$ and $g_1(t_0)$ lie in the same side of B_{t_0} in $B(f(t_0), \delta)$, a contradiction. Similarly as in the proof of conclusion (a) for the set P , here we infer that $Z = I$. The proof is complete.

Fix a number $\varepsilon_7 > 0$ guaranteed by Property 7. For every positive number $\delta < \varepsilon_7$ and every $x \in M$ we define a number $p_\delta(x)$ as follows. Let C be the arc component

of M containing x and $C_\delta(x)$ be the arc component of $C \cap B(x, \delta)$ containing x . Then by Property 7 the set $B(x, \delta) \setminus C_\delta(x)$ has exactly two components: $V_\delta^1(x)$ and $V_\delta^2(x)$. Define two numbers

$$p_\delta^i(x) = \begin{cases} \rho(x, V_\delta^i(x) \cap C), & \text{if } V_\delta^i(x) \cap C \neq \emptyset, \\ \delta & \text{otherwise,} \end{cases}$$

for $i \in \{1, 2\}$, and put $p_\delta(x) = \max\{p_\delta^1(x), p_\delta^2(x)\}$.

Using Property 7(b) the reader may easily verify that

- (*) the condition $p_\delta(x) = 0$ is equivalent to the property that no point of $C_\delta(x)$ is accessible from $E^{n+1} \setminus C$.

This conclusion gives us an implication

- (**) if $p_\delta(x) = 0$ and $0 < \xi < \delta$, then $p_\xi(x) = 0$.

PROPERTY 9. *There exists $\varepsilon > 0$ such that for each arc component C of M if $0 < \delta < \varepsilon$, then $\inf\{p_\delta(x) : x \in C\} = 0$.*

PROOF. Let ε_3 be a number guaranteed by Property 3 and δ_2 be a number guaranteed by Property 2 for the number $\varepsilon_3/4$. Put $\varepsilon = \delta_2$ and let $0 < \delta < \varepsilon$. Fix any arc component $C \in \mathcal{C}$, any point $x \in C$ and some numbers δ_0, δ_1 with $\delta < \delta_0 < \delta_1 < \varepsilon$. Then there is a sequence of points $x_m \in C$ converging to x with $d(x, x_m) \geq \varepsilon_3$ (see Property 6). Thus $x_m \in C \setminus C_\delta(x)$.

If $p_{\delta_1}(x) = 0$ then by (**) we have $p_\delta(x) = 0$ and the conclusion holds.

Let $p_{\delta_1}(x) = \psi > 0$ and let $\varphi > 0$. We will find a point $x_j \in C$ such that $p_\delta(x_j) < \varphi$. There is $k \in \{1, 2\}$ such that $C \cap V_{\delta_1}^k(x) \cap B(x, \psi) = \emptyset$. Assume $k = 1$. Then we may find a point $y \in V_{\delta_1}^1(x) \setminus M$ and an arc

$$yx \subset (V_{\delta_1}^1(x) \cup C_{\delta_1}(x)) \cap B(x, \psi) \cap B(x, \delta/2).$$

Denote by ξ a number guaranteed by Property 3 for the number δ (i.e. if $a, b \in C' \in \mathcal{C}$, $d(a, b) < \varepsilon_3$ and $\rho(a, b) < \xi$, then $d(a, b) < \delta$). Then for some $x_i \in V_{\delta_1}^2(x)$ we have

$$\rho(x, x_i) < \min\{\xi/3, \delta/2, \varphi/2, \delta_0 - \delta\}.$$

The set $C_{\delta_1}(x)$ is contained in a set $A \subset B(x, \varepsilon_3/4) \cap M$ homeomorphic to I^n such that $\rho(x, \partial A) > \varepsilon$ (see Properties 2 and 1). Thus $\text{diam } A < \varepsilon_3/2$ and since the set $C_{\delta_1}(x)$ separates $B(x, \delta_1)$ between y and x_i , the set A does the same. Consider ε_m -homeomorphisms $h_m : M \rightarrow M$ such that $h_m(x) = x_m$ and $\lim \varepsilon_m = 0$ (see Theorem 9) and take the images $h_m(A)$. By Proposition 6 there is an integer j such that $h_j(A)$ separates $B(x, \delta_0)$ between y and x_i , $\text{diam } h_j(A) < \varepsilon_3/2$, $\rho(x_j, \partial h_j(A)) > \varepsilon$, $x_j \in V_{\delta_1}^2(x)$ and $\rho(x, x_j) < \rho(x, x_i)$. Observe that $A \cap h_j(A) = \emptyset$. Indeed, it is a consequence of inequalities $d(x, x_j) \geq \varepsilon_3 > \text{diam } A + \text{diam } h_j(A)$. For each $z \in yx$ we have

$$\rho(z, x_j) \leq \rho(z, x) + \rho(x, x_j) \leq \rho(z, x) + \rho(x, x_i) < 2\delta/2 = \delta,$$

hence $yx \subset B(x_j, \delta)$. We can prove in a similar manner that $\rho(x_i, x_j) < \delta$. Since $((V_{\delta_1}^1(x) \cap B(x, \psi)) \cup C_{\delta_1}(x)) \cap (C \setminus C_{\delta_1}(x)) = \emptyset$, $yx \subset (V_{\delta_1}^1(x) \cap B(x, \psi)) \cup C_{\delta_1}(x)$ and $h_j(A) \subset C \setminus C_{\delta_1}(x)$, we have $yx \cap h_j(A) = \emptyset$ and $h_j(A)$ separates $B(x, \delta_0)$ between x and x_i . The set $h_j(A)$ also separates the ball $B(x_j, \delta) \subset B(x, \delta_0)$ between x and

x_i . By Proposition 4 at least one arc component B of $h_j(A) \cap B(x_j, \delta)$ does the same. Let $y_0 \in \overline{x_i} \cap B$. Then $d(y_0, x_j) \leq \text{diam } h_j(A) < \varepsilon_3$ and

$$\rho(y_0, x_j) \leq \rho(y_0, x_i) + \rho(x_i, x_j) < \rho(x, x_i) + \rho(x_i, x) + \rho(x, x_j) < 3 \cdot \xi/3 = \xi;$$

thus $d(y_0, x_j) < \delta$. Since $\rho(x_j, \partial h_j(A)) > \varepsilon > \delta$, we have $C_\delta(x_j) \subset h_j(A)$ (otherwise we would get an n -dimensional simple triod in M). Consequently, $C_\delta(x_j) \subset B$ because $y_0 \in B \cap C_\delta(x_j) \neq \emptyset$. The inverse inclusion is obvious. Thus $B = C_\delta(x_j)$. Hence the points x and x_i lie in opposite sides of $C_\delta(x_j)$ in $B(x_j, \delta)$. This implies

$$p_\delta(x_j) \leq \max\{\rho(x_j, x), \rho(x_j, x_i)\} \leq \max\{\rho(x_j, x), 2 \cdot \rho(x_i, x)\} = 2 \cdot \rho(x_i, x) < \varphi,$$

which completes the proof.

PROPERTY 10. *No point of M is accessible from $E^{n+1} \setminus M$.*

PROOF. Let $x \in M$ and C be the arc component of M containing x . We will show that there is $\delta > 0$ such that $p_\delta(x) = 0$, which will imply the conclusion by (*).

Let ε_i be a number guaranteed by Property i for $i \in \{3, 7, 8, 9\}$. Put $\delta = \min\{\varepsilon_3, \varepsilon_7, \varepsilon_8, \varepsilon_9\}/5$. For a given positive number $\sigma < \delta$ let $\xi < \sigma$ be a number guaranteed by Property 8 for the number δ , $\tau < \xi$ be a number guaranteed by Property 3 for the number ξ and $\varphi < \tau$ be a number guaranteed by Theorem 9 for the number τ . By Property 9 there is $y \in C$ with $p_\delta(y) < \varphi$, i.e. there are $v_1, v_2 \in C$ lying in opposite sides of $C_\delta(y)$ in $B(y, \delta)$ with $\max\{\rho(y, v_1), \rho(y, v_2)\} < \varphi$. Let $f: I \rightarrow C$ be an embedding such that $f(0) = y$ and $f(1) = x$. By Theorem 9 there are τ -homeomorphisms $h_1, h_2: M \rightarrow M$ such that $h_1(y) = v_1$ and $h_2(y) = v_2$. Put $g_i(t) = h_i(f(t))$ for $t \in I$ and $i \in \{1, 2\}$. Thus we have $\rho(f(t), g_i(t)) < \tau$ for all $t \in I$ and $i \in \{1, 2\}$. Put $B_t = C_\delta(f(t))$ and let A_t^i be the arc component of $B(f(t), \delta) \cap g_i(I)$ containing $g_i(t)$.

Suppose $z \in B_{t_0} \cap A_{t_0}^i$. Then

$$\begin{aligned} d(f(t_0), g_i(t_0)) &\leq d(f(t_0), z) + d(z, g_i(t_0)) \leq \text{diam } B_{t_0} + \text{diam } A_{t_0}^i \\ &\leq 2\delta + 2\delta \leq 4\varepsilon_3/5 < \varepsilon_3 \end{aligned}$$

and $\rho(f(t_0), g_i(t_0)) < \tau$. Thus $d(f(t_0), g_i(t_0)) < \xi < \delta$. Let $\psi > 0$ be a number such that if $|t_1 - t_2| < \psi$, then

$$\max\{\rho(f(t_1), f(t_2)), \rho(g_i(t_1), g_i(t_2))\} < \delta \quad \text{for } t_1, t_2 \in I \text{ and } i \in \{1, 2\}.$$

Then the conditions $d(f(t_1), g_i(t_1)) < \delta$ and $|t_1 - t_2| < \psi$ imply that

$$d(f(t_2), g_i(t_2)) \leq d(f(t_2), f(t_1)) + d(f(t_1), g_i(t_1)) + d(g_i(t_1), g_i(t_2)) < 3\delta < \varepsilon_3$$

and since $\rho(f(t_2), g_i(t_2)) < \tau$, thus by Property 3 we have $d(f(t_2), g_i(t_2)) < \xi < \delta$. Applying these implications finitely many times (if t runs from t_0 to 0) we get $d(f(0), g_i(0)) = d(y, v_i) < \delta$, a contradiction.

Therefore we have $B_t \cap A_t^i = B_t \cap A_t^j = \emptyset$ for each $t \in I$. Recall that for each $t \in I$ and each $i \in \{1, 2\}$ we have $\rho(f(t), g_i(t)) < \tau < \xi$. Applying Property 8 we see that the points $g_1(1), g_2(1) \in C$ lie in opposite sides of $C_\delta(x)$ in $B(x, \delta)$ and thus $p_\delta(x) < \xi < \sigma$. Hence $p_\delta(x) = 0$.

To make the proof of the Main Theorem complete, note at least an obvious property of M (being true for each nonempty compact subset of E^{n+1}) which contradicts the previous one.

PROPERTY 11. *There exists a point in M that is accessible from $E^{n+1} \setminus M$.*

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