THE CENTRAL LIMIT THEOREM
FOR EMPIRICAL PROCESSES UNDER LOCAL CONDITIONS:
THE CASE OF RADON INFINITELY DIVISIBLE LIMITS
WITHOUT GAUSSIAN COMPONENT

NIELS T. ANDERSEN, EVARIST GINE AND JOEL ZINN

ABSTRACT. Weak convergence results are obtained for empirical processes
indexed by classes $\mathcal{F}$ of functions in the case of infinitely divisible purely
Poisson (in particular, stable) Radon limits, under conditions on the local
modulus of the processes $\{f(X) : f \in \mathcal{F}\}$ ("bracketing" conditions). They
extend (and slightly improve upon) a central limit theorem of Marcus and
Pisier (1984) for Lipschitzian processes. The law of the iterated logarithm
is also considered. The examples include Marcinkiewicz type laws of large
numbers for weighted empirical processes and for the dual-bounded-Lipschitz
distance between a probability in $\mathbb{R}$ and its associated empirical measures.

1. Introduction. Although there has been considerable activity regarding the
CLT with Gaussian limits and $n^{1/2}$ normings for empirical processes indexed by
classes of functions $\mathcal{F}$, the case of other stable limits and normings has not been
considered, at least for large classes of functions $\mathcal{F}$. The object of this paper is
to show that methods similar to those in Andersen et al. (1988) are appropriate to
derive the CLT with $n^{1/p}$ (or slightly more general) normings and $p$-stable Radon
limits (or even, in the non-i.i.d. case, infinitely divisible Radon limits), under tail
probability conditions for the local modulus of $\delta_X$ on $(\mathcal{F}, \rho)$, for appropriate
distances $\rho$, that is, under bracketing. It is not surprising that such results can be
obtained if one sees the bracketing CLT as an extension and improvement of the
CLT for Lipschitz continuous processes (i.e., the Jain-Marcus CLT). Our previous
results on bracketing may thus be viewed as the analogues of the Jain-Marcus (1974)
CLT and the Ledoux (1981) LIL for processes with Lipschitz sample paths. Mar-
cus and Pisier (1984) obtained CLT's with $n^{1/p}$ norming and stable limits for such
processes, and Ledoux (private communication) obtained recently another LIL; the
results in this paper extend and slightly improve them. Actually, our framework
for the CLT is more general and includes the non-i.i.d. case with convergence to
other (necessarily infinitely divisible) Radon limits, as well as other normings in
the stable case.

Received by the editors January 20, 1987.

1980 Mathematics Subject Classification (1985 Revision). Primary 60F17, 60F15, 60B12; Sec-
ondary 60F05, 62E20.

Key words and phrases. Central limit theorems, law of the iterated logarithm, empirical pro-
cesses, Marcinkiewicz laws of large numbers, bracketing conditions, majorizing measures.

Part of the research of the first author was carried out while visiting Texas A&M University,
supported by the Danish Natural Science Research Council.

The research of Giné and Zinn was partially supported by grants from the National Science
Foundation, U.S.A.
In §2 we give the asymptotic equicontinuity condition required to prove convergence of empirical processes to limits that are infinitely divisible measures without Gaussian component and that are Radon in $l^\infty(\mathcal{F})$. As it was to be expected, one needs the following two conditions for the CLT to hold: eventual tightness of the sums of the row laws outside neighborhoods of zero and convergence in probability to zero of the sums of the truncated variables, uniformly in $n$, as the truncation levels tend to zero (recall, in connection with this, Theorem 2.10 in Mandrekar and Zinn (1980)). Here, Hoffmann-Jørgensen's (1984) theory of weak convergence of sums of random elements taking values in nonseparable space is quite useful.

§3 contains a generalization of an inequality for sums of positive random variables that allowed replacing entropy by majorizing measure conditions in Andersen et al. (1988), and is used here for the same purpose.

In §4 we prove the main theorem, which is a CLT for the randomized empirical processes $\{\sum_{j=1}^n \varepsilon_{nj} f(X_{nj})/b_n : f \in \mathcal{F}\}$ where $X_{nj}$, $j = 1, \ldots, n$, are independent $(S, \mathcal{F})$-valued random variables and $\mathcal{F}$ is a class of measurable functions on $S$. The method of proof is similar to that of Theorem 3.1 in Andersen et al. (1988). Of course the majorizing measure and the local modulus conditions are quite different in the present situation.

In §5 we specialize the result from §4 to i.i.d. $X_i$ and $p$-stable limits, $p \in [1, 2)$, with randomization replaced by centering at expectations for $p > 1$. The norming constants $B(n)$ are a subset of all the possible ones in $\mathbb{R}$. Our methods do not seem to allow for all the normings $B(n) = n^{1/p}L(n)$, $L$ slowly varying, and the subset we can handle is related to the regularly varying functions used in Marcus (1987) for the study of infinitely divisible sample continuous processes whose Lévy measures split. For reasons of expediency, in the case $p = 1$ only the norming $n$ and the randomized sums $\sum_{i=1}^n \varepsilon_i f(X_i)$ are considered. We also obtain results about stochastic boundedness and convergence to zero (in $pr^*$ and a.s.) of the normalized sums $(n/B(n))(P_n - P)(f)$. In addition this section contains two instances of application of the general results, one to a slight extension and improvement of the CLT of Marcus and Pisier (1984) for Lipschitzian processes, and one to the CLT in $c_0$ showing that these results are sharp.

In §6 we apply a recent result of Ledoux and Talagrand that essentially reduces the law of the iterated logarithm in separable Banach spaces to a law of large numbers, to obtain a LIL under bracketing that complements the one obtained in Andersen et al. (1988) and Ledoux and Talagrand (1986). So, the unifying theme in this article is the use of majorizing measures combined with local conditions to prove convergence in probability to zero of randomized and suitably normalized empirical processes.

Finally, §7 contains Marcinkiewicz type laws of large numbers for weighted empirical processes and for the dual bounded Lipschitz norm between probability measures and their associated empiricals in the real line. Although our general theorems apply in these cases, they do not give the best results, which we obtain by other methods.

Notation and definitions are as in Andersen et al. (1988). When we refer to triangular arrays $X_{nj}$ of $S$-valued independent random variables and to independent Rademacher arrays $\{\varepsilon_{nj}\}$, we mean that these variables are defined on a product probability space, the $X_{nj}$'s being coordinate functions and the $\{\varepsilon_{nj}\}$ being defined
in another factor of the product, that is,
\[(\Omega, \Sigma, \Pr) = (\Pi_{n,j} S, \Pi_{n,j} \mathcal{F}, \Pi_{n,j} P_{n,j}) \times ([0, 1], \mathcal{B}, \lambda),\]
the \(X_{nj}\) are the coordinate functions \(\Pi_{n,j} S \rightarrow S\) and the \(\{\varepsilon_{nj}\}\) are independent Bernoulli symmetric random variables defined on \([0, 1]\).

In a few instances we use the symbols \(\geq, \leq, \approx\) to denote inequalities that hold up to constant multipliers \(c \in (10^{-6}, 10^6)\).

We recall the following two exponential inequalities, to be used in what follows: (a) Bernstein’s inequality (e.g., Bennett (1962)): if \(\xi_i\) are independent centered real random variables with ess sup \(|\xi_i| \leq c < \infty, i \leq n, then\)
\[
\left(1.1\right) P\left\{\sum_{i=1}^{n} \xi_i > t\right\} \leq \exp\left\{-\frac{t^2}{2} \left(\sum_{i=1}^{n} E \xi_i^2 + \frac{2ct}{3}\right)\right\}, \quad t > 0.
\]

(b) Upper bound for binomial probabilities (Gine and Zinn (1984)): if \(A_j\) are independent events with \(p_j = P A_j\), then
\[
\left(1.2\right) P\left\{\sum_{i=1}^{n} I_{A_j} \geq l\right\} \leq \left(\frac{e \sum_{j=1}^{n} p_j}{l}\right)^l, \quad 1 \leq l \leq n, \quad n \in \mathbb{N}.
\]

2. The CLT with non-Gaussian limits in \(l^{\infty}(\mathcal{F})\). In this section we show how the CLT in the non-Gaussian case reduces to conditions on the individual summands together with probability inequalities. Stochastic boundedness is also considered.

Let \(\mathcal{F}\) be an index set and let \(l^{\infty}(\mathcal{F})\) be the Banach space of bounded real functions on \(\mathcal{F}\) with the sup norm \(\| \cdot \|_{\mathcal{F}}\). A random element \(X\) in \(l^{\infty}(\mathcal{F})\) is a mapping \(X: (\Omega, \Sigma, \Pr) \rightarrow l^{\infty}(\mathcal{F})\) such that \(\lim_{M \rightarrow \infty} \Pr\{\|X\|_{\mathcal{F}} > M\} = 0\). The following definition is taken from Hoffmann-Jørgensen (1984).

2.1. DEFINITION. (1) \(\{X_n\}_{n=1}^{\infty}\) is an eventually tight sequence if for every \(\varepsilon > 0\) there is a compact set \(K \in l^{\infty}(\mathcal{F})\) such that
\[
\limsup_{n \rightarrow \infty} \Pr\{X_n \in G^c\} < \varepsilon
\]
for all open sets \(G \supset K\). Since \(l^{\infty}(\mathcal{F})\) is a Banach space \(K\) and \(G\) may be taken to be convex and symmetric in this definition.

(2) \(\{X_n\}_{n=1}^{\infty}\) converges in law to a Radon limit if there exists a Radon probability measure \(\gamma\) on \(l^{\infty}(\mathcal{F})\) such that for all \(H: l^{\infty}(\mathcal{F}) \rightarrow \mathbb{R}\) bounded and continuous,
\[
\lim_{n \rightarrow \infty} E^* H(X_n) = \int H \, d\gamma.
\]

(3) \(\{X_n\}_{n=1}^{\infty}\) is stochastically bounded if
\[
\lim_{M \rightarrow \infty} \limsup_n \Pr\{\|X_n\|_{\mathcal{F}} > M\} = 0.
\]

Given a random element \(X\) in \(l^{\infty}(\mathcal{F})\) define a set function \(\mathcal{L}^*(X)\) on the subsets of \(l^{\infty}(\mathcal{F})\) by
\[
(2.1) \quad \mathcal{L}^*(X)(A) = \Pr^*\{X \in A\}, \quad A \in l^{\infty}(\mathcal{F}),
\]
and for any \(B \subset l^{\infty}(\mathcal{F})\),
\[
(2.2) \quad \mathcal{L}^*(X)|_B(A) = \Pr^*\{X \in A \cap B\}, \quad A \subset l^{\infty}(\mathcal{F}).
\]
2.1'. Definition. Given \( \{X_{nj}: j \leq n, n \in \mathbb{N}\} \), we say that the sequence of set functions \( \{\sum_{j=1}^{n} \mathcal{L}^*(X_{nj})|_{\|x\|_\mathcal{F} > \delta}\}_{n=1}^{\infty} \) is eventually tight if both

\[
\sup_{n} \sum_{j} \Pr^*\{\|X_{nj}\|_\mathcal{F} > \delta\} < \infty,
\]

and for every \( \varepsilon > 0 \) there exists \( K \) compact (convex, symmetric) such that for all open (convex, symmetric) sets \( G \supseteq K \),

\[
\limsup_{n \to \infty} \sum_{j} \Pr^*\{X_{nj} \in G^c, \|X_{nj}\|_\mathcal{F} > \delta\} < \varepsilon.
\]

The following statement is Mandrekar and Zinn’s (1980, Theorem 2.10) restated in the nonseparable, nonmeasurable case. The proof is somewhat different.

2.2. Proposition. Let \( \{X_{nj}: j = 1, \ldots, n; n \in \mathbb{N}\} \) be a triangular array of \( l^\infty(\mathcal{F}) \)-valued random elements. Assume:

(i) The sequence of set functions

\[
\left\{\sum_{j=1}^{n} \mathcal{L}^*(X_{nj})|_{\|x\|_\mathcal{F} > \delta}\right\}_{n=1}^{\infty}
\]

is eventually tight for every \( \delta > 0 \), and

(ii) for all \( \varepsilon > 0 \),

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \Pr^*\left\{\left\|\sum_{j=1}^{n} X_{nj} I[\|X_{nj}\|_\mathcal{F} \leq \delta]\right\|_\mathcal{F} > \varepsilon\right\} = 0.
\]

Then the sequence of random elements \( \{\sum_{j=1}^{n} X_{nj}\}_{n=1}^{\infty} \) is eventually tight.

Proof. We first show that for all \( \delta > 0 \) \( \{\sum_{j=1}^{n} X_{nj} I[\|X_{nj}\|_\mathcal{F} \geq \delta]\}_{n=1}^{\infty} \) is an eventually tight sequence. For this we follow the proof of Lemma 2.4 in de Acosta, Araujo and Gine (1978). We let \( X_{nj}^\delta = X_{nj} I[\|X_{nj}\|_\mathcal{F} \geq \delta] \). For \( G \) convex, symmetric, open, \( q_G \) denotes its Minkowski functional. Let \( \tilde{K} \) be a compact, convex symmetric set such that

\[
\limsup_{n \to \infty} \sum_{j=1}^{n} \Pr^*\{q_G(X_{nj}^\delta) > 1\} \leq \varepsilon/2
\]

for all \( \tilde{G} \supseteq \tilde{K} \) open, convex, symmetric. Let \( m \) be such that

\[
\sum_{j=1}^{n} \Pr^*\{\|X_{nj}\|_\mathcal{F} \geq \delta\} \leq \varepsilon m/2.
\]

\( m \) and \( \tilde{K} \) exist by hypothesis. Define \( K = m\tilde{K} \) and \( G = m\tilde{G} \); then, as \( \tilde{G} \) runs over all convex symmetric open neighborhoods of \( \tilde{K} \), \( G \) runs over all such neighborhoods of \( K \). Since

\[
q_G \left(\sum_{n=1}^{m} X_{nj}^\delta\right) \leq \sum_{j=1}^{n} q_G(X_{nj}^\delta) \leq \left(\sum_{j=1}^{n} I[\|X_{nj}\|_\mathcal{F} \geq \delta]\right) \max_{j \leq n} q_G(X_{nj}^\delta),
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
we have

\[
\begin{align*}
\mathbb{P}^* \left\{ q_G \left( \sum_{j=1}^{n} X_{n,j}^\delta \right) > 1 \right\} & \leq m^{-1} \mathbb{E}^* \sum_{j=1}^{n} I_{\left\| X_{n,j} \right\| > \delta} \\
& \quad + \mathbb{P}^* \left\{ q_G \left( \sum_{j=1}^{n} X_{n,j}^\delta \right) > 1, \sum_{j=1}^{n} I_{\left\| X_{n,j} \right\| > \delta} \leq m \right\} \\
& \leq m^{-1} \sum_{j=1}^{n} \mathbb{P}^* \left\{ \left\| X_{n,j} \right\| > \delta \right\} + \mathbb{P}^* \left\{ \max_{j \leq n} q_G(X_{n,j}^\delta) > 1/m \right\} \\
& \leq m^{-1} \sum_{j=1}^{n} \mathbb{P}^* \left\{ \left\| X_{n,j} \right\| > \delta \right\} + \sum_{j=1}^{n} \mathbb{P}^* \left\{ q_G(X_{n,j}^\delta) > 1 \right\}.
\end{align*}
\]

Hence by (2.5) and (2.6), \( \limsup_{n \to \infty} \mathbb{P}^* \{ q_G(\sum_{j=1}^{n} X_{n,j}^\delta) > 1 \} < \varepsilon \). This proves that \( \{ \sum_{j=1}^{n} X_{n,j}^\delta \}_{n=1}^{\infty} \) is eventually tight.

Now we apply this fact together with hypothesis (2.4) to prove the proposition. We use Example 7.28 in Hoffmann-Jørgensen (1984) (see also Andersen and Dobric (1987, Theorem 2.12)). Set \( X_{n,j}I_{\left\| X_{n,j} \right\| > \delta} = X_{n,j}^\delta \). Given a pseudo-metric \( \rho \) on \( \mathcal{F} \), let \( w_\rho(x, \varepsilon) = \sup_{\rho(f,g) \leq \varepsilon} |x(f) - x(g)|, x \in L^\infty(\mathcal{F}), \varepsilon > 0 \), be its oscillation function. We have, for \( \delta > 0, \varepsilon > 0 \),

\[
\begin{align*}
\mathbb{P}^* \left\{ w_\rho \left( \sum_{j=1}^{n} X_{n,j}, \varepsilon \right) > \varepsilon \right\} & \leq \mathbb{P}^* \left\{ \left\| \sum_{j=1}^{n} X_{n,j} \right\| > \varepsilon/4 \right\} + \mathbb{P}^* \left\{ w_\rho \left( \sum_{j=1}^{n} X_{n,j}^\delta, \varepsilon \right) > \varepsilon/2 \right\}.
\end{align*}
\]

Given \( \varepsilon > 0 \), choose \( \delta > 0 \) such that

\[
\limsup_n \mathbb{P}^* \left\{ \left\| \sum_{j=1}^{n} X_{n,j}^\delta \right\| > \varepsilon/4 \right\} < \varepsilon/2,
\]

which is possible by (2.4). Also, by Example 7.28 in Hoffmann-Jørgensen (1984) and eventual tightness of \( \{ \sum_{j=1}^{n} X_{n,j}^\delta \}_{n=1}^{\infty} \) there exists a totally bounded pseudo-metric \( \rho \) on \( \mathcal{F} \) such that

\[
\limsup_n \mathbb{P}^* \left\{ w_\rho \left( \sum_{j=1}^{n} X_{n,j}^\delta, \varepsilon/2 \right) > \varepsilon/2 \right\} \leq \varepsilon/2.
\]

For such \( \rho \) we thus have

\[
\text{(2.7)} \quad \limsup_n \mathbb{P}^* \left\{ w_\rho \left( \sum_{j=1}^{n} X_{n,j}, \varepsilon \right) > \varepsilon \right\} \leq \varepsilon.
\]
A standard argument using the same facts about $X_{nj\delta}$ and $X_{nj}^\delta$ shows that 
$\left\{\sum_{j=1}^n X_{nj}\right\}_{n=1}^\infty$ is stochastically bounded; that is,

$$\lim_{M \to \infty} \limsup_{n \to \infty} \Pr^\ast\left(\left\|\sum_{j=1}^n X_{nj}\right\|_\mathcal{F} > M\right) = 0.$$ 

But this and (2.7) imply eventual tightness of this sequence, again by Example 7.28 in Hoffmann-Jørgensen, loc. cit. □

We say that the finite dimensional distributions (f.d.d.) of $\left\{X_n\right\}_{n=1}^\infty$ converge, if for each $k \in \mathbb{R}$ and $f_1, \ldots, f_k \in \mathcal{F}$, the sequence $(X_n(f_1), \ldots, X_n(f_k)) \in \mathbb{R}^k$ converges in law. The previous proposition together with Theorem 7.11 (3°) of Hoffmann-Jørgensen (1984) (applied with

$$\psi = \left\{\exp \left(i \sum_{\text{finite}} \alpha_k f_k\right) : \alpha_k \in \mathbb{R}, f_k \in \mathcal{F}\right\} \subset C(l^\infty(\mathcal{F})),$$

where $\sum \alpha_k f_k(x) := \sum \alpha_k x(f_k)$ for $x \in l^\infty(\mathcal{F})$ gives

2.3. COROLLARY. If conditions (i) and (ii) of Proposition 2.2 hold, and if moreover the f.d.d. of $\left\{\sum_{j=1}^n X_{nj}\right\}_{n=1}^\infty$ converge (in law) then this sequence converges in law to a Radon limit in $l^\infty(\mathcal{F})$.

2.4. REMARK. A proof similar (but easier) to that of Proposition 2.2 shows the following: Let $\left\{X_{nj} : j = 1, \ldots, n; n \in \mathbb{N}\right\}$ be a triangular array of $l^\infty(\mathcal{F})$-valued random elements. Assume

(i) $$\lim_{M \to \infty} \limsup_{n \to \infty} \sum_{j=1}^n \Pr^\ast\left(\|X_{nj}\|_\mathcal{F} > M\right) = 0$$

with $\sup_n \sum_{j=1}^n \Pr^\ast\left(\|X_{nj}\|_\mathcal{F} > M\right) < \infty$ for all $M > \delta_0$, for some $\delta_0 > 0$, and

(ii) $$\lim_{M \to \infty} \limsup_{n \to \infty} \Pr^\ast\left(\left\|\sum_{j=1}^n X_{nj}I_{\|X_{nj}\|_\mathcal{F} \leq \delta}\right\|_\mathcal{F} > M\right) = 0$$

for some $\delta > \delta_0$. Then the sequence $\left\{\sum_{j=1}^n X_{nj}\right\}_{n=1}^\infty$ is stochastically bounded.

For applications to empirical processes, the index set $\mathcal{F}$ is a subset of $\mathcal{L}_0(S, \mathcal{F})$ with envelope $F(s) := \sup_{f \in \mathcal{F}} |f(s)|$ finite for all $s \in S$, and $X_{nj} = \delta X_{nj}/b_n$ or $X_{nj} = \varepsilon_{nj} X_{nj}/b_n$, where $\left\{X_{nj}\right\}$ is a triangular array of $(S, \mathcal{F})$-valued random variables, the $\delta_n$’s are positive real constants, and $\left\{\varepsilon_{nj}\right\}$ is a Rademacher array independent of $\left\{X_{nj}\right\}$ (see the last paragraph of §1 for the precise set-up). Moreover the $X_{nj}$’s are row-wise independent (i.e., for each $n$, the random variables $X_{n1}, \ldots, X_{nn}$ are independent; in fact, they are coordinates of a large product probability space). Actually our main interest is in the case $X_{nj} = \delta X_j/b_n$ with $X_j$ i.i.d. and $b_n \uparrow \infty$, that is, the $p$-stable limit case, in fact only for $p \in [1, 2)$: the case $p = 2$ is considered in Andersen et al. (1988) and the case $p < 1$ is trivial in the sense that condition (ii) of Proposition 2.2 becomes superfluous (as $b_n/n \to \infty$) and condition (i) is also necessary for the CLT, at least in the measurable case. Another reason for not considering the case $p < 1$ is that it has no relevance regarding the law of large numbers.
In the p-stable case with \( p \neq 1 \) we are interested in the convergence in law of \( \{\sum_{j=1}^{n} (f(X_j) - Pf)/bn : f \in \mathcal{F}\}_{n=1}^{\infty} \). The following is a convenient complement to Proposition 2.2 in this case.

2.5. PROPOSITION. Let \( \{X_i\}_{i=1}^{\infty} \) be i.i.d. \((S,\mathcal{S})\)-valued random variables with law \( P \) and let \( \{\varepsilon_i\}_{i=1}^{\infty} \) be a Rademacher sequence independent of \( \{X_i\}_{i=1}^{\infty} \). Let \( \mathcal{F} \subset \mathcal{L}_1(S,\mathcal{S},P) \) have everywhere finite envelope \( F \). Assume there exist \( b_n \uparrow \infty \) such that:

(i) the f.d.d. of the sequence of random elements of \( l^\infty(\mathcal{F}) \)

\[
\left\{ \sum_{j=1}^{n} (f(X_j) - Pf)/bn : f \in \mathcal{F} \right\}_{n=1}^{\infty}
\]

converge in law to p-stable probability measures in Euclidean space, for a fixed \( p \in (1,2) \).

(ii) The sequence of set functions of \( l^\infty(\mathcal{F}) \),

\[
\{n\mathcal{L}^*(\delta_{X_i}/bn)_{||x||_{\mathcal{F}}>\delta}\}_{n=1}^{\infty}
\]

is eventually tight for all \( \delta > 0 \).

(iii) For all \( \varepsilon > 0 \),

\[
\lim_{\delta \to 0} \limsup_{n} \Pr^{*} \left\{ \left\| \sum_{j=1}^{n} \varepsilon_j f\mathbb{1}_{|f|\leq \delta bn}(X_j)/bn \right\|_{\mathcal{F}} > \varepsilon \right\} = 0.
\]

Then the sequence (2.8) converges in law to a p-stable Radon probability measure in \( l^\infty(\mathcal{F}) \).

2.6. NOTATION. If the conclusion of Proposition 2.5 holds for \( \mathcal{F},P \), and \( \{b_n\} \), we write \( \mathcal{F} \in \text{CLT}(P; b_n) \).

PROOF OF PROPOSITION 2.5 (SKETCH). Convergence of the f.d.d. of \( \{\sum_{j=1}^{n} (f(X_j) - Pf)/bn \} \) implies the same for \( \{\sum_{j=1}^{n} \varepsilon_j f(X_j)/bn \} \). Condition (ii) also holds for \( \varepsilon \delta X_i \). So, Corollary 2.3 gives convergence in law to a Radon measure of the randomized sequence \( \{\sum_{j=1}^{n} \varepsilon_j f(X_j)/bn : f \in \mathcal{F} \} \). The limit is p-stable and, in analogy to the Gaussian case, p-stable Radon measures satisfy \( \gamma_*(C_u(\mathcal{F},\mathcal{P})) = 1 \) where \( \rho(f,g) = \Lambda_{\delta,\infty}(||f-g||) = \int |f-g| d\gamma \), and \( (\mathcal{F},\mathcal{P}) \) is totally bounded (see Andersen and Dobric (1987) for the first statement and Marcus and Pisier (1985) for the second). So, the randomized sequence is eventually uniformly \( \rho \)-equicontinuous (as e.g. in the proof of Theorem 1.1.3 in Giné and Zinn (1986)). This implies, as e.g. in Giné and Zinn (1984, proof of Corollary 2.13), that

\[
\lim_{\delta \to 0} \limsup_{n} \Pr^{*} \left\{ \sup_{f,g \in \mathcal{F}, \rho(f,g) \leq \delta} |f-g|(X_i)/bn > \varepsilon \right\} = 0.
\]

Then using regular variation of \( \{b_n\} \) and the theorem in Feller (1971, p. 281) as in the proof of the stable CLT in \( R \), it follows that

\[
\lim_{\delta \to 0} \limsup_{n} \sup_{f,g \in \mathcal{F}, \rho(f,g) \leq \delta} \Pr \left\{ \left| \sum_{j=1}^{n} (f-g)(X_i) - P(f-g) \right|/bn > \varepsilon \right\} = 0.
\]
This allows use of the randomization lemmas in §2 of Giné and Zinn (1984) to obtain the eventual uniform $\rho$-equicontinuity of the nonrandomized sequence (2.8), hence its eventual tightness and, by (i), its convergence in law to a Radon limit.

2.7. REMARK. Regarding stochastic boundedness in the i.i.d. case Remark 2.4 gives the following statement: Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. $(S, \mathscr{F})$-valued random variables with law $P$ and let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a Rademacher sequence independent of $\{X_i\}_{i=1}^{\infty}$. Let $\mathcal{F} \subset L_1(S, \mathscr{F}, P)$ with envelope $F$ everywhere finite. Assume there exists a strictly increasing sequence $\{b_n\}$ regularly varying of exponent $1/p$ for some $p \in (1, 2)$ such that

(i) $\sup_n nP^*\{F > b_n\} < \infty$ and
(ii) for some $\delta > 0$,

$$\lim_{M \to \infty} \limsup_{n \to \infty} \frac{\sup \{\varepsilon_j f I_{[F \leq \delta b_n]}(X_j) \mid F \in \mathcal{F}, b_n > M\}}{b_n} = 0.$$

Then the sequence $\left\{\left\| \sum_{j=1}^{n} (f(X_j) - Pf)/b_n \right\|_\mathcal{F}\right\}_{n=1}^{\infty}$ is stochastically bounded. (In this case desymmetrization again follows from the lemmas in Giné and Zinn (1984, §2), since condition (i) implies

$$\lim_{M \to \infty} \limsup_{n \to \infty} \sup_{f \in \mathcal{F}} \frac{\sup \{\varepsilon_j f I_{[F \leq \delta b_n]}(X_j) \mid F \in \mathcal{F}, b_n > M\}}{b_n} = 0$$

by the usual computations with regularly varying functions, as mentioned in the proof of Proposition 2.5.)

3. An exponential bound for sums of positive, bounded random variables. One of the basic technical tools in proofs of limit theorems combining bracketing and majorizing measures is the inequality in this section. The present inequality generalizes the one in Andersen et al. (1988).

3.1. THEOREM. Let $\{\xi_i : i \leq n\}$ be independent, nonnegative real random variables and let $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing function. Assume there exist $h_i \mid 0$ and $r \geq 1$ such that

(i) $\sum_{i=1}^{n} P\{\xi_i > h_i\} \leq l/3$, $l \geq r$,
(ii) $h_i \leq \psi(1/\sum_{i=1}^{l} h_i)$, $l \geq r$.

Then, for $d \geq 1$, $c > 4d$ and $b > 0$ satisfying

(iii) $\psi(b) \leq h_r$,
(iv) $4rb\psi(b) \leq c$

the following inequality holds:

$$P \left\{ \sup_{a > 0} a \sum_{i=1}^{n} \xi_i I_{[\psi(a) < \xi_i \leq d\psi(b)]} > c \right\} \leq 12 \exp\{-c \ln(3/e)/2(d \vee 2) b \psi(b)\}.$$
PROOF. We have

\[
P \left\{ \sup_{0 < a < b} \frac{a}{n} \sum_{i=1}^{n} \xi_i I[\psi(a) < \xi_i \leq \psi(b)] > c \right\}
\]

(3.2)

\[
\leq P \left\{ \sup_{0 < a < b} \frac{a}{n} \sum_{i=1}^{n} \xi_i I[\psi(a) < \xi_i \leq \psi(b)] > c/2 \right\}
\]

\[
+ P \left\{ \sum_{i=1}^{n} I[\psi(b) < \xi_i \leq \psi(b)] > c/2d\psi(b) \right\} := (I) + (II).
\]

Since \( h_1 \downarrow 0 \) we can find \( l_0 \geq r \) such that \( h_{l_0 + 1} \leq \psi(b) \leq h_{l_0} \) by (iii). Then,

using (ii) and the fact that \( \psi \) is increasing, we have \( b \leq 1/\sum_{i=1}^{l_0} h_i \leq 1/l_0h_{l_0}, \) i.e. \( l_0 \leq 1/bh_{l_0} \leq 1/b\psi(b). \) So, by inequalities (1.2) and (i) we obtain the following estimate for (II):

\[
(II) \leq \left( 2c^{-1} \frac{\text{deb} \psi(b)}{\text{de} \psi(b)} \right)^{\frac{c}{2} \frac{\text{de} \psi(b)}{b}}
\]

(3.3)

\[
\leq \left( \frac{c}{4b} \frac{\text{de} \psi(b)}{\text{de} \psi(b)} \right)^{\frac{c}{2} \frac{\text{de} \psi(b)}{b}} \leq \exp \left\{ \frac{-\text{cln}(3/e)}{2 \text{de} \psi(b)} \right\}
\]

since \( c > 4d. \)

Now we estimate (I). Let \( \xi_i = \xi_i I[\xi_i \leq \psi(b)] \) and let \( \xi^*_i, 1 \leq i \leq n, \) be the nonincreasing order statistics of \( \xi_i, 1 \leq i \leq n, \) \( \xi^*_{n+1} := 0. \) Put \( \lambda = c/4b\psi(b). \) Then

\[
\psi^{-1}(\xi^*_i) \sum_{1 \leq i \leq \lambda} \xi^*_i \leq c/4, \quad 1 \leq l \leq n,
\]

and so we have

\[
(I) \leq P \left\{ \sup_{a > 0} \psi^{-1}(a) \sum_{i=1}^{n} \xi^*_i I[\xi^*_i > a] > c/2 \right\}
\]

\[
\leq P \left\{ \max_{1 \leq i \leq n} \psi^{-1}(a) \sum_{i=1}^{n} \xi^*_i I[\xi^*_i > a] > c/2 \right\}
\]

\[
\leq P \left\{ \max_{1 \leq i \leq n} \psi^{-1}(\xi^*_i) \sum_{\lambda < i \leq l} \xi^*_i > c/4 \right\}
\]

\[
\leq P \left\{ \max_{\lambda < l \leq n} \left( \sum_{\lambda < i \leq l} h_i \right) \psi^{-1}(\xi^*_i) \max_{\lambda < i \leq l} h_i^{-1} \xi^*_i > c/4 \right\}
\]

\[
\leq P \left\{ \max_{\lambda < l \leq n} \psi^{-1}(h_l) \frac{\psi^{-1}(h_l)}{h_l} > 1 \right\} + I_{\left\{ \max_{\lambda < l \leq n} \psi^{-1}(h_l) \sum_{\lambda < i \leq l} h_i > c/4 \right\}.
\]

Now, \( \lambda \geq r \) by (iv), so that (ii) and (iii) imply

\[
\max_{\lambda < l \leq n} \psi^{-1}(h_l) \sum_{\lambda < i \leq l} h_i \leq \max_{r \leq l \leq n} \psi^{-1}(h_l) \sum_{i=r}^{l} h_i \leq 1.
\]
Hence, the above indicator is zero \((c > 4d > 4)\). We apply inequality (1.2) to obtain

\[
(I) \leq P \left\{ \max_{\lambda < l \leq n} \frac{\zeta_j}{h_l} > 1 \right\}
\]

\[
\leq \sum_{\lambda < l \leq n} P \left\{ \sum_{j=1}^{n} I_{\{\zeta_j > h_l\}} > l \right\}
\]

\[
\leq \sum_{\lambda < l \leq n} \left( e \sum_{j=1}^{n} P\{\zeta_j \geq h_l\}/l \right) \leq \sum_{\lambda < l \leq n} (e/3)^l
\]

\[
\leq \frac{3}{3-e} \exp \left\{ -\frac{c \ln(3/e)}{4b\psi(b)} \right\} < 11 \exp \left\{ -\frac{c \ln(3/e)}{4b\psi(b)} \right\}.
\]

Now (3.1) follows from the estimates (3.2), (3.3) and (3.4). □

Examples of sequences \(\{h_l\}\) and functions \(\psi\) are

1. \(h_l = l^{-1/p}, \psi(x) = (px/(p - 1))^{1/(p-1)}, l \in \mathbb{N}, p \in (1, \infty), x > 0.\)
2. \(h_l = l^{-1}, \psi(x) = x/e^{x^{1/x}}, l \in \mathbb{N}, 0 < x < 1.\)

See 4.1 and 5.1 for additional examples.

4. The central limit theorem, non-i.i.d. case. In the next theorem, the function \(\varphi: \mathbb{R}^+ \to \mathbb{R}^+\) and the constants \(h_k, k \in \mathbb{N},\) will satisfy the following properties:

\[
\varphi(x) \uparrow \infty, \quad x/\varphi(x) \uparrow \infty \quad \text{as} \quad x \uparrow \infty, \quad \text{and} \quad \varphi(x) \downarrow 0, \quad x/\varphi(x) \downarrow 0 \quad \text{as} \quad x \downarrow 0,
\]

\[
\int_0^\varphi(\ln x^{-1}) dx < \infty,
\]

\[
h_k \downarrow 0 \quad \text{as} \quad k \uparrow \infty,
\]

and

\[
\text{there exist constants} \quad K > 0 \text{ and } L > 1 \text{ such that}
\]

\[
(i) \quad h_l \varphi^{-1} \left( \sum_{i=1}^l h_i \right) \leq \sum_{i=1}^l h_i, \quad l \geq L;
\]

\[
(ii) \quad h_l \leq Kh_{l+1}, \quad l \geq L;
\]

\[
(iii) \quad \sum_{i=l}^\infty h_i^2 \leq Kh_l \sum_{i=1}^l h_i, \quad l \geq L;
\]

and we can also assume, by changing \(K\) if necessary, that

\[
(iv) \quad lh_l \leq K \sum_{i=1}^l h_i, \quad l \geq L.
\]

4.1. Examples. (1) Let \(H(x), x > 0,\) be regularly varying at infinity with exponent \(1/p, p \in (1,2)\) and such that

\[
H(x) \downarrow 0 \quad \text{and} \quad x/H(x) \downarrow 0 \quad \text{as} \quad x \downarrow 0,
\]

\[
H(x) \uparrow \infty \quad \text{and} \quad x/H(x) \uparrow \infty \quad \text{as} \quad x \uparrow \infty, \quad \text{and} \quad \int_0^1 dx/H(x) < \infty.
\]
Then \( h_k = 2p'/H(k) \) and \( \varphi(x) = x/H(x) \) satisfy (4.1)-(4.4) for suitable \( L \) and \( K \) because of the well-known limits (Feller (1971)):

\[
\lim_{x \to \infty} x^{-1} \left[ H(x) \int_0^x \frac{dt}{H(t)} \right] = p', \quad \text{where } 1/p + 1/p' = 1,
\]

and

\[
\lim_{x \to \infty} x^{-1} \left[ (H(x))^2 \int_x^\infty \frac{dt}{(H(t))^2} \right] = \frac{p}{(2-p)}.
\]

(2) \( \varphi(x) = \ln x \) for \( x \geq e \), \( \varphi(x) = x^2/e^2 \) for \( 0 < x \leq e \); \( h_l = 1/l, l \in \mathbb{N} \).

These examples include those at the end of §3.

The following is the main result of this paper:

4.2. THEOREM. Let \( \varphi(x), x \in \mathbb{R}^+ \) and \( h_k, k \in \mathbb{N} \), satisfy the properties (4.1) to (4.4), and let \( b_n, n \in \mathbb{N} \), be positive finite constants. Let \( \mathcal{F} \subset L_0(S,\mathcal{F}) \) with envelope \( F(s) \) finite for all \( s \in S \). Let \( \{X_{nj}: j = 1, \ldots, n, n \in \mathbb{N}\} \) be a triangular array of row-wise independent \( S \)-valued random variables and for each \( n \) let \( \{\varepsilon_{nj}: j = 1, \ldots, n\} \) be a Rademacher sequence independent of the \( X_{nj}, j = 1, \ldots, n \). Assume:

(i) The sequence of set functions

\[
\left\{ \sum_{j=1}^n \mathcal{L}^\ast((f(X_{nj})/b_n: f \in \mathcal{F}))\right\}_{n=1}^{\infty}
\]

is eventually tight in \( l^\infty(\mathcal{F}) \).

(ii) There is a pseudo distance \( \rho \) on \( \mathcal{F} \) and a Borel probability measure \( \mu \) on \( (\mathcal{F}, \rho) \) such that

(iia) \( \lim_{\delta \to 0} \sup_{f \in \mathcal{F}} \int_0^\delta \varphi(\ln 1/\mu(B_\rho(f,\varepsilon)))d\varepsilon = 0 \) with \( \sup_{f \in \mathcal{F}} \) finite for \( \delta = \infty \), and

(iib) there are constants \( \sigma > 0, n_0 > 0 \) and \( L_1 \geq 1 \) such that for all \( f \in \mathcal{F}, l \geq L_1, n \geq n_0 \) and \( \varepsilon > 0 \),

\[
\sum_{j=1}^n P^\ast_{nj} \left\{ \left| f - g \right|/b_n > \sigma h_l \right\} \leq l/3,
\]

where \( P_{nj} = \mathcal{L}(X_{nj}) \).

(iii) For each \( k \in \mathbb{N} \) and \( f_1, \ldots, f_k \in \mathcal{F} \), the triangular array of \( \mathbb{R}^k \)-valued random variables \( \{(f_1(X_{nj}), \ldots, f_k(X_{nj}))/b_n: j = 1, \ldots, n, n \in \mathbb{N}\} \) is infinitesimal and the sequence

\[
\left\{ \sum_{j=1}^n \varepsilon_{nj}(f_1(X_{nj}), \ldots, f_k(X_{nj}))/b_n \right\}_{n=1}^{\infty}
\]

converges in law to an infinitely divisible law with a degenerate Gaussian component.

Then, the sequence of random elements \( \{\sum_{j=1}^n \varepsilon_{nj} f(X_{nj})/b_n: f \in \mathcal{F}\}_{n=1}^{\infty} \) converges in law to a Radon infinitely divisible measure \( \gamma \) on \( l^\infty(\mathcal{F}) \) with a degenerate Gaussian component.
PROOF. By Corollary 2.2, convergence in law to a Radon probability measure of \( L^\infty(\mathcal{F}) \) will follow if we show

\[
\lim_{\varepsilon \to 0} \limsup_{n} \sup_{\mathcal{F}} \Pr^* \left\{ \left\| \sum_{j=1}^{n} \varepsilon_{nj} f \mathbb{1}_{[F \leq \delta b_n]}(X_{nj})/b_n \right\|_{\mathcal{F}} > \varepsilon \right\} = 0
\]

for all \( \varepsilon > 0 \). That the limit is infinitely divisible without Gaussian component follows then by condition (iii): since the limit \( \gamma \) is Radon, it is supported by a separable Banach subspace of \( L^\infty(\mathcal{F}) \), and then \( \gamma \) is infinitely divisible with only a degenerate Gaussian component by condition (iii) above, Theorem 7.9 of Hoffmann-Jørgensen (1984) and Remark 5 there, with \( \psi = \{ \exp(i \sum \alpha_k f_k) : \alpha_k \in \mathbb{R}, f_k \in \mathcal{F} \} \) (where \( \sum \alpha_k f_k(x) := \sum \alpha_k x(f_k), x \in L^\infty(\mathcal{F}) \)).

We now prove (4.6). As shown in Andersen et al. (1988, Lemma 2.1 and Remark 2.2(3)), by (4.1) and (4.2) there is no loss of generality in assuming that \( \mu \) is discrete, with support \( \{ \pi_q f : q \in \mathbb{N}, f \in \mathcal{F} \} \subset \mathcal{F} \), \#\{\pi_q f : f \in \mathcal{F} \} < \infty for all \( q \), satisfying

\[
\rho(\pi_q f, f) \leq 2^{-q}, \quad q \in \mathbb{N}, f \in \mathcal{F}, \quad \text{and} \quad \lim_{q \to \infty} \beta_{q_0} := \lim_{q \to \infty} \sup_{f \in \mathcal{F}} \sum_{q \geq q_0} 2^{-q} \gamma_q(f) = 0
\]

with \( \sup_f \) finite for \( q_0 = 1 \), and

\[
\gamma_q(f) := \varphi \left( \frac{2^q}{\mu(\pi_1 f) \cdots \mu(\pi_q f)} \right).
\]

Letting, for each \( q \in \mathbb{N} \), \( T_q := \{ (\pi_1 f, \ldots, \pi_q f) : f \in \mathcal{F} \} \) it is obvious that \( \gamma_q \) is actually defined on \( T_q \), so that we write \( \gamma_q(t) = \gamma_q(f) \) if \( t = (\pi_1 f, \ldots, \pi_q f) \). Then, by Andersen et al. (1988, Lemma 2.4), we have

\[
\lim_{r \to \infty} \sum_{q=1}^{\infty} \sum_{t \in T_q} \exp\{-r \varphi^{-1}(s \gamma_q(t))\} = 0
\]

for every \( s > 0 \), with the sum finite for every \( r > 1 \). Now let \( \psi(x) := 1/x \varphi^{-1}(1/x) \) (\( \uparrow \infty \) as \( x \uparrow \infty \), \( \downarrow 0 \) as \( x \downarrow 0 \) by (4.1)). For \( n, q, q_0, f \), define

\[
c_{n,q}(f) = 2^{-q} b_n \psi(1/\gamma_q f),
\]

\[
\Delta_q^j(f) \leq \min_{1 \leq \tau \leq q} \left\{ \sup_{g \in B_{s}(\pi_{\tau} f, 3/2-r)} |g - \pi_{\tau} f| \right\}^*,
\]

where the * denotes the outer measurable envelope with respect to \( P_{nj} := L^\infty(X_{nj}) \), and

\[
\tau_j(f) = \min\{ q \geq q_0 : \Delta_q^j(f) > c_{n,q+1}(f) \}, \quad j = 1 \ldots n.
\]

(Sometimes we write \( \gamma_q f, c_{n,q} f, \Delta_q^j f, \tau_j f \) instead of \( \gamma_q(f), c_{n,q}(f), \Delta_q^j(f) \) and \( \tau_j(f) \) respectively.) Note that

\[
\varphi(q \ln 2) \leq \gamma_q(f) = o(2^q) \quad \text{and} \quad \gamma_q(f) \uparrow \infty \quad \text{as} \quad q \to \infty,
\]

\[
\inf_{f \in \mathcal{F}} \psi(1/\gamma_q f) > 0 \quad \text{for all} \quad q > 0,
\]

\[
\Delta_q^j(f) \downarrow \quad \text{as} \quad q \uparrow \quad \text{for all} \quad f \quad \text{and} \quad j.
\]
By summation by parts, as in Andersen et al. (1988), we have for all \( f \in \mathcal{F} \) and \( q_0 < q_1 \),

\[
\begin{align*}
f &= \pi_{q_0} f + (f - \pi_{q_0} f)I_{[\tau_f = q_0]} + (f - \pi_{q_1} f)I_{[\tau_f \geq q_1]} \\
&\quad + \sum_{q = q_0 + 1}^{q_1-1} (f - \pi_q f)I_{[\tau_f = q]} + \sum_{q = q_0 + 1}^{q_1} (\pi_q f - \pi_{q-1} f)I_{[\tau_f \geq q]}.
\end{align*}
\]

Therefore,

\[
(4.14) \quad \Pr^* \left\{ \sum_{j=1}^{n} \varepsilon_{nj} f I_{[F \leq \delta b_n]}(X_{nj}) \right\}_{\mathcal{F}} > 6\varepsilon b_n \leq \Pr \left\{ \sum_{j=1}^{n} \varepsilon_{nj} (\pi_{q_0} f)I_{[F \leq \delta b_n]}(X_{nj}) \right\}_{\mathcal{F}} > \varepsilon b_n \\
+ \Pr^* \left\{ \left| \sum_{j=1}^{n} \varepsilon_{nj} (f - \pi_q f)I_{[\tau_f = q_0]}I_{[F \leq \delta b_n]}(X_{nj}) \right|_{\mathcal{F}} > \varepsilon b_n \right\} \\
+ \Pr^* \left\{ \left| \sum_{j=1}^{n} \varepsilon_{nj} (f - \pi_q f)I_{[\tau_f \geq q_1]}I_{[F \leq \delta b_n]}(X_{nj}) \right|_{\mathcal{F}} > \varepsilon b_n \right\} \\
+ \Pr \left\{ \left| \sum_{q=q_0+1}^{q_1-1} \sum_{j=1}^{n} \Delta_q^2 f I_{[c_{n,q} + 1 < \Delta_q^2 f \leq c_{n,q} f]}(X_{nj}) \right|_{\mathcal{F}} > \varepsilon b_n \right\} \\
+ \Pr \left\{ \left| \sum_{q=q_0+1}^{q_1} \sum_{j=1}^{n} \varepsilon_{nj} (\pi_q f - \pi_{q-1} f)I_{[\tau_f \geq q]}I_{[F \leq \delta b_n]}(X_{nj}) \right|_{\mathcal{F}} > \varepsilon b_n \right\} \\
:= (I) + (II) + (III) + (IV) + (V).
\]

Now,

\[
(4.15) \quad \lim_{\delta \to 0} \limsup_{n} \Pr^* (I) = 0 \quad \text{for all } q_0 < \infty
\]

by the CLT in finite dimensions, on account of condition (iii). To handle (II) we note that for \( \delta \) small enough and \( q_0 \geq 1 \) fixed, we have by (4.11), \( \inf f c_{n,q_0+1} f / b_n > 2\delta \). Furthermore, by definition, \( |A_{\varepsilon,0}| < 2\varepsilon \). Hence,

\[
\{\tau_f = q_0\} \cap \{F \leq \delta b_n\} = \emptyset.
\]

Therefore, for such \( \delta \)'s, (II) = 0, that is,

\[
(4.16) \quad \lim_{\delta \to 0} \limsup_{n} \Pr^* (II) = 0.
\]

By (4.1) and (4.10) \( \sup_{\psi \in \mathcal{F}} \psi(1/\gamma_{q,f}) \to 0 \) as \( q \to \infty \). Hence \( n(\sup_{\psi \in \mathcal{F}} c_{n,q,f}) = nb_n^{-q_0}(1) \) and we can choose \( q_1(n) \) such that \( n(\sup_{\psi \in \mathcal{F}} c_{n,q_1(n)}(f)) < \varepsilon b_n \) (e.g., \( q_1(n) \sim \ln n \)). For such \( q_1 \) we obviously have

\[
(4.17) \quad (III) = 0.
\]
To estimate (IV) we make use of Theorem 3.1. With $t_q f = (\pi_1 f, \ldots, \pi_q f)$ and taking into account that $\Delta^j_q f$, $\gamma_q f$ and $c_{nq} f$ depend on $f$ only through $t_q f$ we have

\[(IV) \leq \sum_{q=q_0+1}^{q_1-1} \sum_{t \in T_q} \Pr^* \left\{ \sup_{(f : t_q f = t)} (\gamma_q + 1)^{-1} \sum_{j=1}^{n} b_n^{-1} 2^{q+1} \Delta^j_q f \right. \]
\[\times \left. I_{c_{nq} + 1 f < \Delta^j_q f < c_{nq} f} (X_{nq}) > \varepsilon \beta_{q_0+1}^{-1} \right\} \]
\[\leq \sum_{q=q_0+1}^{q_1-1} \sum_{t \in T_q} \Pr^* \left\{ \sup_{0 < a < 1/\gamma_q t} \frac{a}{n} \sum_{j=1}^{n} 2^{q+1} b_n^{-1} \Delta^j_q f \right. \]
\[\times \left. I_{\psi(a) < 2^{q+1} b_n^{-1} \Delta^j_q f \leq 2 \psi(1/\gamma_q t)} (X_{nq}) > \varepsilon \beta_{q_0+2}^{-1} \right\}. \]

We want to use Theorem 3.1 with $\xi_t = 2^{q+1} b_n^{-1} \Delta^j_q f$, $b = 1/\gamma_q t$, $d = 2$, $c = \varepsilon \beta_{q_0+2}$ and $r = \mathcal{L} \vee \mathcal{L}_1$ (see (4.4) and (iiib)). Since in (4.5) we may assume $\sigma = 1/6$ just by multiplying either $\rho$ or $b_n$ by a suitable constant, then (4.5) implies (i) in Theorem 3.1. We can choose $q_0$ large enough such that $c > 8$, $h_r \geq \psi(1/\gamma_q t)$ and $4r(\gamma_q t)^{-1} \psi(1/\gamma_q t) \leq c$ for $q \geq q_0$ and $t \in T_q$, so that (iii) and (iv) in 3.1 hold too. Finally, (4.4)(i) is just (ii) in Theorem 3.1. We thus obtain

\[(IV) \leq \sum_{q=q_0+1}^{q_1-1} \sum_{t \in T_q} 12 \cdot \exp \left\{ -\frac{\varepsilon \ln(3/e) \varphi^{-1}(\gamma_q t)}{4 \cdot \beta_{q_0+1}} \right\} \rightarrow 0 \quad \text{uniformly in } n, \delta \text{ and } q_1. \]

We finally estimate (V) using Bernstein’s inequality as in Andersen et al. (loc. cit.):

\[(V) \leq 2 \sum_{q=q_0+1}^{q_1} \sum_{t \in T_q} \exp \left\{ \frac{-(\varepsilon / \beta_{q_0+1})^2 2^{-2 \gamma_q t} b_n^2}{8 \cdot \sum_{j=1}^{n} P_{nq}(\Delta^j_q t)^2 I_{(\Delta^j_q t \leq c_{nq} t)} + 2^{-q+1}(\gamma_q t) \varepsilon \beta_{q_0+1}} \right\}. \]

Letting $\eta_j = 2^{q} b_n^{-1} \Delta^j_q t I_{(\Delta^j_q t \leq c_{nq} t)} (X_{nq})$, we have $\sum_{j=1}^{n} P(\eta_j > h_l) \leq l/3$. Let $l_1 = l_1(q)$ be defined by the inequality

\[(4.19) \quad h_{l_1+1} < \psi(1/\gamma_q t) \leq h_{l_1}. \]
for \( q \geq q_0 \) (\( q_0 \) is chosen large enough so that \( l_1(q) \geq L_1 \vee L \), for all \( q \geq q_0 \)). Then, using the properties in (4.4), we have

\[
\sum_{j=1}^{n} P_{n_j} (\Delta_{n_j}^{t})^2 I_{[\Delta_{n_j} t \leq c_{n,q_0} t]} = 2^{-2q} b_n^2 \sum_{j=1}^{n} E \eta_j^2 I_{[\eta_j \leq \psi(1/q_0 t)]} \\
\leq 2 \cdot 2^{-2q} b_n^2 \sum_{j=1}^{n} \int_{0}^{1/q_0 t} u P(\eta_j > u) \, du \\
\leq 2^{-2q+1} b_n^2 \sum_{j=1}^{n} \sum_{l=l_1(q)}^{\infty} \int_{h_l}^{h_{l+1}} u P(\eta_j > u) \, du \\
\leq 2^{-2q} b_n^2 \sum_{l=l_1(q)}^{\infty} (l+1)(h_l^2 - h_{l+1}^2) \\
= 2^{-2q} b_n^2 \left[ l_1(q) h_{l_1(q)}^2 + \sum_{l=l_1(q)}^{\infty} h_l^2 \right] \\
\leq 2^{-2q+1} b_n^2 K h_{l_1(q)} \sum_{l=1}^{l_1(q)} h_l \\
\leq 2^{-2q+1} b_n^2 K^2 h_{l_1(q)} + \sum_{l=1}^{l_1(q)} h_l \\
\leq 2^{-2q+1} b_n^2 K^2 \psi(1/q_0 t) q_0 t = 2^{-2q+1} b_n^2 K^2 (q_0 t)^2 / \varphi^{-1}(q_0 t)
\]

(4.20)

where in the last inequality we have used that

\[
\psi(1/q_0 t) \leq h_{l_1(q)} \leq \psi \left( \frac{1}{\sum_{i=1}^{l_1(q)} h_i} \right) \quad \text{(by (4.4)(i))}
\]
i.e. since \( \psi(x) \) is nondecreasing, \( \gamma_q t \geq \sum_{i=1}^{l_1(q)} h_i \). Using (4.20) we get

\[
(V) \leq 2 \sum_{q=q_0+1}^{q_1} \sum_{t \in T_q} \exp \left\{ -\frac{\varepsilon^2 \varphi^{-1}(q_0 t)}{(K' + 2\varepsilon/\beta_{q_0+1})\beta_{q_0+1}^2} \right\}
\]

where \( K' \) is a suitable constant (independent of \( q, n, q_1, q_0, \) etc.). Since \( \beta_{q_0+1} \to 0 \) we obtain by (4.9)

\[
\lim_{q_0 \to -\infty} (V) = 0 \quad \text{uniformly in } n, q, \text{ and } \delta.
\]

(4.21)

Now (4.15), (4.16), (4.17), (4.18) and (4.21) give

\[
\lim_{q_0 \to -\infty} \limsup_{\delta \to 0} \limsup_{n \to \infty} (II + \cdots + (V)) = 0,
\]

which by (4.14) is nothing but (4.6). \( \square \)

Regarding stochastic boundedness, we have

4.3. Proposition. Let \( \varphi, \{h_k\} \) and \( \{b_n\} \) be as in Theorem 4.2. Let \( \mathcal{F} \subset \mathcal{L}_0(S, \mathcal{F}) \) have everywhere finite envelope \( F \). Let \( \{X_{nj}: j = 1, \ldots, n \in \mathbb{N} \} \) be a
triangular array of row-wise independent $S$-valued random variables and, for each $n$,
\[ \{\varepsilon_{nj}: j = 1, \ldots, n\} \] a Rademacher sequence independent of the $X_{nj}$, $j = 1, \ldots, n$.
Assume:

(i) $\lim_{M \to \infty} \sup_n \sum_{j=1}^n \Pr[*\{F(X_{nj}) > Mb_n\} = 0$.

(ii) For all $f \in \mathcal{F}$, $l > l_0$, $n > n_0$ and $\varepsilon > 0$, and for some $\sigma > 0$, $n_0 \geq 1$ and $l_0 \geq 1$,

\[ \sum_{j=1}^n P_{nj}^* \left\{ \varepsilon^{-1} \sup_{g \in B_\rho(f, \varepsilon)} |f - g|/b_n > \sigma \mathbb{1}_j \right\} \leq l/3, \tag{4.5} \]

where $\rho$ is a pseudodistance on $\mathcal{F}$ for which there exists a probability measure $\mu$ such that

\[ \sup_{f \in \mathcal{F}} \int_0^\infty \varphi \left( \frac{1}{\mu(B_\rho(f, \varepsilon))} \right) d\varepsilon < \infty. \tag{4.22} \]

Then the sequence $\{\sum_{j=1}^n \varepsilon_{nj} f(X_{nj})/b_n: f \in \mathcal{F}\}_{n=1}^\infty$ is stochastically bounded.

We omit the proof, which is similar to that of Theorem 4.2 (and uses Remark 2.4 instead of Corollary 2.2; the main difference is that here $c_{n,q}f = M^{1/2}q^{1-b}n^{-b}q(f))$.

In the next section we apply these two theorems to $p$-stable convergence for empirical processes, but they may be useful in other situations as well. Here is an easy example.

4.4. Example. Let $U_i$ be independent uniform on $[0,1]$. Define $X_{nj} = nU_j$, $j = 1, \ldots, n$, $n \in \mathbb{N}$. Let

\[ \mathcal{F} = \{f_u(x) := (u - x)1_{[x<u]}: u \in [0,1]\}. \]

Then the sequence $\{\sum_{j=1}^n f_u(X_{nj}): u \in [0,1]\}$ converges weakly to a Radon measure of $t^\infty[0,1]$. Actually, the convergence takes place in $C[0,1]$. (Note that $f_u(nU_j) = \int_0^n I_{[nU_j \leq t]} dt$ and it is clear that, since $\sum_{j=1}^n I_{[nU_j \leq |t|}$ converges to the Poisson process—weakly in $D[0,1]$—then $\sum_{j=1}^n f_u^n I_{[nU_j \leq t]} dt$ must converge in law in $D$, hence in $C$, to the integrated Poisson processes.) We will check that this result follows easily from Theorem 4.2. Since $E \sum_{j=1}^n f_u(X_{nj}) = u^2/2$, the same arguments given in $\S$2 show that it is enough to consider the randomized version $\sum_{j=1}^n \varepsilon_j f_u(X_{nj})$. To apply Theorem 4.2, let us check first condition (i): for all $\tau > 0$

\[ \lim \inf \sup_{n \to 0} n \lambda \left\{ x: \sup_{|u-v|<\varepsilon,0 \leq u,v \leq 1} |f_u(nx) - f_v(nx)| > \tau, \sup_{0 \leq u \leq 1} f_u(nx) > \delta \right\} \leq \lim \inf \sup_{n \to 0} n I_{|\varepsilon|} = 0 \quad (\text{since } |f_u - f_v| \leq |u - v|), \]

and

\[ \lim \sup_{n \to 0} n \lambda \left\{ x: \sup_{0 \leq u \leq 1} f_u(nx) > \delta \right\} = 1 - \delta, \]
hence the sequence in (i) is eventually tight. As for (ii), we have many choices for \( \varphi, \rho \) and \( h_k \). Take \( \rho(u, v) = |u - v| \). Then \( (4.5) \) is, for \( \varepsilon \leq 1 \) and \( l \geq 6, \)

\[
\sup_{u \in [0,1]} n \lambda \{ \varepsilon^{-1} |f_{(u-\varepsilon)\vee 0}(nx) - f_{(u+\varepsilon)\wedge 1}(nx)| > \sigma h_l \} \\
\leq \sup_{u \in [0,1]} n \lambda \{ |f_{(u-\varepsilon)\vee 0}(nx) - f_{(u+\varepsilon)\wedge 1}(nx)| > 0 \} \\
\leq \sup_{u \in [0,1]} (u + \varepsilon) < 2 \leq l/3.
\]

Now, take e.g. \( h_l = l^{-1/p}, \varphi(x) = x^{1/p'} \) for any \( p \in (1,2) \) and \( \mu = \text{Lebesgue measure on } [0,1] \).

4.5. Example. Theorem 4.2 can be used to obtain weak convergence of the processes considered in Csörgő et al. (1986, §3). We only indicate how to check conditions (i) and (ii) of Theorem 4.2 for the processes \( \{ \sum_{i=1}^n \varepsilon_i f_i(nU_i) : t \geq 0 \} \) where \( f_0(x) = 0, f_t(x) := p^{-1} \int_0^t I_{[x \leq s]} ds/s^{1+1/p}, x \in (0, \infty) \), and \( \{ U_i \}_{i=1}^\infty \) are i.i.d. uniform on \((0,1)\). (Here \( S = (0,1), \mathcal{S} = \mathcal{B}, X_{nj} = nU_j \) and \( P_{nj} = \mathcal{L}(X_{nj}) \).

Just take \( \delta > 0 \) such that \( p < 2(1 - p\delta), h_l = l^{\delta-1/p}, \varphi(x) = x^{\delta+1/p'}, \) and \( \rho(s,t) = \int_s^t dx/(1 + x)^{1+\delta}, s,t \geq 0 \).

5. The CLT with stable limits. We start by applying Theorem 4.2 to the CLT for i.i.d. random variables \( X_i \) and \( \rho \)-stable limits. We note that we cannot obtain a result for all normings \( B(n) = n^{1/p} M(n) \) with \( M \) slowly varying at infinity, but only under a submultiplicativity property for \( B \) which is essentially the same that Marcus (1986) used in a different, but related, context. The reason for this constraint appears explicitly in the proof of Theorem 5.2 that follows (and also in the proof of Theorem 3.1).

Let now \( B(x) \) and \( H(x), x \in [0, \infty), \) be real valued functions satisfying:

\[
(5.1) \quad B(u)H(v) \leq KB(uv) \quad \text{for some } K > 0, u \geq 0, v \geq 1;
\]

\[
(5.2) \quad B \text{ and } H \text{ are regularly varying functions with exponent } 1/p
\]

\text{for some } p \in (1,2), \text{ continuous, strictly increasing,}

\[
B(0) = H(0) = 0 \text{ and such that } \int_0^1 dt/H(t) < \infty;
\]

\[
(5.3) \quad \text{the function } \varphi(x) := x/H(x), x > 0, \varphi(0) := 0 \text{ is strictly increasing and continuous.}
\]

5.1. Examples. The following subsets of the set of examples in 4.1 satisfy (5.1)–(5.3). Let \( Lx := \ln(e \vee x) \).

(1) \( B(x) = H(x) = x^{1/p}/(Lx)^{\alpha}, \alpha \geq 0, 1 < p < 2, \) and then \( \varphi(x) \simeq x^{1/p'}(Lx)^{\alpha} \) where \( 1/p + 1/p' = 1 \).

(2) \( B(x) = x^{1/p}(Lx)^{\alpha}, \alpha > 0, 1 < p < 2, \) and \( H(x) = x^{1/p}, \varphi(x) = x^{1/p'} \).

Theorem 4.2 gives the following corollary for stable limits:

5.2. Theorem. Let \( p \in (1,2) \) and let \( B, H, \varphi \) be functions satisfying (5.1) to (5.3) for this value of \( p \). Let \( \mathcal{F} \subset \mathcal{L}(S, \mathcal{P}, P) \) with \( F(s) < \infty \) for all \( s \in S \). Let
Let \( \{X_i\}_{i=1}^{\infty} \) be i.i.d. \( S \)-valued random variables with law \( P \). Assume:

(i) The finite dimensional distributions of the sequence

\[
\left\{ \sum_{i=1}^{n} \frac{(f(X_i) - Pf)}{B(n)} : f \in \mathcal{F} \right\}_{n=1}^{\infty}
\]

converge in law (necessarily to \( p \)-stable measures in Euclidean space);

(ii) The sequence of set functions on \( l^\infty(\mathcal{F}) \),

\[
\left\{ n \mathcal{L}^* \left( \delta_{X_1}/B(n) \right) | ||x||_\mathcal{F} > \delta \right\}_{n=1}^{\infty}
\]

is eventually tight for all \( \delta > 0 \), and

(iii) There are a pseudodistance \( \rho \) on \( \mathcal{F} \) and a Borel probability measure \( \mu \) on \( (\mathcal{F}, \rho) \) such that

(iiiia) \( \lim_{\delta \to 0} \sup_{f \in \mathcal{F}} \int_0^\delta \varphi(\ln 1/\mu(B_\rho(f, \epsilon))) \, d\epsilon = 0 \) with \( \sup_{f \in \mathcal{F}} \) finite for \( \delta = \infty \), and

(iiiib) There exists \( c < \infty \) such that for all \( f \in \mathcal{F} \) and \( \epsilon > 0 \)

\[
\sup_{t > 0} t \mathbb{P} \left\{ \epsilon^{-1} \left[ \sup_{g \in B_\rho(f, \epsilon)} |f - g| \right] > B(t) \right\} \leq c
\]

Then, the sequence (5.4) converges in law in \( l^\infty(\mathcal{F}) \) to a \( p \)-stable Radon measure, i.e. \( FECLT(P, B(n)) \).

**Proof.** By taking \( B(3ct) \) instead of \( B(t) \) if necessary, we may assume \( c \) in (5.6) to be 1/3. Now, since \( B(t)/H(l) \geq B(t/l)/K \) for all \( l \geq 1 \), (5.6) implies (4.5) with \( b_n = B(n) \), \( h_l = 1/H(l) \), \( \sigma = K \) and \( P_{nj} = P \). Also, (4.1) to (4.3) follow from the well-known properties of regularly varying functions (see Example 4.1); to obtain (4.4) it may be necessary to replace \( H \) by a suitable multiple. So, the proof of Theorem 4.2 applies to give

\[
\lim_{n \to \infty} \sup_{\epsilon > 0} \mathbb{P} \left\{ \left\| \sum_{j=1}^{n} \epsilon_j f I_{[F \leq \delta B(n)]}(X_j)/B(n) \right\|_{\mathcal{F}} > \epsilon \right\} = 0
\]

for all \( \epsilon > 0 \), and this is enough to prove Theorem 5.2 in view of Proposition 2.5. \( \Box \)

5.3. **Remarks.** (1) Talagrand’s (1986) recent result on necessary conditions for sample continuity of stable measures allows for the replacement of condition (iiia) on \( \rho \) and \( \mu \) by

(iiiia)' \( \rho \) is the \( L_1 \) (or equivalently \( \Lambda_{p, \infty} \)) pseudodistance associated to a Radon \( p \)-stable probability measure on \( l^\infty(\mathcal{F}) \).

But (iiiia)' is a stronger condition than (iiia), in contrast with the Gaussian case (Theorems 4.1 and 4.4 in Andersen et al. (1988)).

(2) Conditions (ii) and (iiib) in the previous theorem can be replaced by the stronger condition: there exists \( c < \infty \) such that for all \( \epsilon > 0 \),

\[
\sup_{t > 0} t \mathbb{P} \left\{ \epsilon^{-1} \left[ \sup_{\{f, g \in \mathcal{F} : \rho(f, g) \leq \epsilon\}} |f - g| \right] > B(t) \right\} \leq c.
\]

Since \( (\mathcal{F}, \rho) \) is totally bounded, (5.7) implies that the set functions (5.5) are uniformly bounded; moreover (5.7) expresses the fact that the sequence of set functions (5.5) is asymptotically \( \rho \)-equicontinuous, hence the sequence (5.5) is eventually tight.
We state next a result for the case $p = 1$, $B(n) = n$ and randomized empiricals (these limitations are imposed only in order to avoid technicalities such as complicated expressions for $\varphi$ and/or truncated expectations). It is an immediate consequence of Theorem 4.2 (see Example 4.1(2)).

5.4. **THEOREM.** Let $\mathcal{F} \subset L_0(S, \mathcal{S}, P)$, with $F(s) < \infty$ for all $s \in S$. Assume, for $\{X_i\}_{i=1}^\infty$ i.i.d. $S$-valued random variables with law $P$ and $\{\varepsilon_i\}_{i=1}^\infty$ a Rademacher sequence independent of $\{X_i\}_{i=1}^\infty$:

(i) The finite dimensional distributions of the sequence

$$\left\{ \sum_{j=1}^n \varepsilon_j f(X_j)/n : f \in \mathcal{F} \right\}$$

converge in law.

(ii) The sequence of set functions on $l^\infty(\mathcal{F})$,

$$\{n\mathcal{L}^*(\delta X_1/n) | \|\varphi\|_{\mathcal{F}} > \delta\}$$

is eventually tight for all $\delta > 0$.

(iii) There are a pseudodistance $\rho$ on $\mathcal{F}$ and a Borel probability measure $\mu$ on $(\mathcal{F}, \rho)$ such that

(iiiia) $\lim_{\delta \to 0} \sup_{f \in \mathcal{F}} \int_0^\delta \ln + (\ln 1/\mu(B_p(f, \varepsilon))) d\varepsilon = 0$, with $\sup_{f \in \mathcal{F}} \mu$ finite for $\delta = \infty$, and

(iiiib) There exists $c > 0$ such that for all $f \in \mathcal{F}$ and $\varepsilon > 0$,

$$\sup_{t>0} tP^* \left\{ \sup_{g \in B_p(f, \varepsilon)} |f - g| > t \right\} \leq c\varepsilon.$$

Then, the sequence (5.8) converges in law in $l^\infty(\mathcal{F})$ to a 1-stable Radon measure.

5.5. **REMARK.** Remark 5.3 with $B(n) = n$ obviously applies also in the case of Theorem 5.4.

5.6. **EXAMPLE.** A CLT in $C(T)$. Proceeding as in the proof of Theorem 4.5 of Andersen et al. (1988) we obtain the following slight generalization of the CLT in Marcus and Pisier (1984, Corollary 1.4) (see also Juknevičienė (1986)) as a corollary of the preceding Theorems 5.2 and 5.4: Let $(T, d)$ be a compact metric or pseudometric space, and let $\{X(t), t \in T\}$ be a sample continuous process on $(T, d)$, centered for $1 < p < 2$ and symmetric for $p = 1$. Let $B_p(x)$ and $\varphi_p(x)$ satisfy conditions (5.1)-(5.3) for $1 < p < 2$, and let $B_1(x) = x$, $\varphi_1(x) = \ln + x$ for $p = 1$.

Assume, for $X_i$ i.i.d. with $\mathcal{L}(X_i) = \mathcal{L}(X)$,

(i) the finite dimensional distributions of

$$\left\{ \sum_{i=1}^n X_i/B_p(n) \right\}_{n=1}^\infty$$

converge in law.

(ii) For all $\varepsilon > 0$, $\tau > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} nP \left\{ \sup_{d(s,t) < \delta} |X(t) - X(s)| > \varepsilon B_p(u), \|X\|_{\infty} > \tau B_p(n) \right\} = 0.$$
(iii) There exist a continuous pseudodistance $\rho$ on $T$ and a Borel probability measure $\mu$ on $(T, d)$ such that:

(iiiia) $\lim_{\delta \to 0} \sup_{t \in T} \int_0^\delta \varphi_\rho(\ln 1/\mu(B_\rho(t, \varepsilon))) \, d\varepsilon = 0$ with $\sup_{t \in T}$ finite for $\delta = \infty$, and

(iiiib) there exists $c < \infty$ such that for all $\varepsilon > 0,$

$$\sup_{u > 0} \left\{ \sup_{\rho(s, t) \leq \varepsilon} |X(t) - X(s)|/\varepsilon > B(u) \right\} \leq c.$$ 

Then the sequence (5.9) converges in law in $C(T)$ to the law of a $p$-stable sample continuous process.

5.6. REMARK. As a consequence of Theorem 4.3, we can drop conditions (i) and (ii) and weaken condition (iii) in Theorems 5.2 and 5.3, and still obtain stochastic boundedness, that is: Let $p \in (1, 2)$ (resp. $p = 1$) and let $B, H, \varphi$ satisfy (5.1) to (5.3) (resp. $B(n) = H(n) = n$ and $\varphi(x) = \ln^+ x$), let $\mathcal{F} \subset L_1(S, \mathcal{S}, P)$ (resp. $L_0(S, \mathcal{S}, P)$) with $F$ everywhere finite, and let $\{X_i\}_{i=1}^\infty$ be i.i.d. $S$-valued random variables with law $P$ (and $\{\varepsilon_i\}_{i=1}^\infty$ a Rademacher sequence independent of $\{X_i\}_{i=1}^\infty$ in case $p = 1$). Then, if there exist a pseudodistance $\rho$ on $\mathcal{F}$ and a probability measure $\mu$ on $(\mathcal{F}, \rho)$ such that condition (iii) of Theorem 5.2 (resp. Theorem 5.4) holds with (iiiia) weakened to only

$$\sup_{f \in \mathcal{F}} \int_0^\infty \varphi \left( \ln \frac{1}{\mu(B_\rho(f, \varepsilon))} \right) \, d\varepsilon < \infty,$$

the sequence (5.4) (resp. (5.8)) is stochastically bounded.

Theorems 5.2 and 5.4, together with standard arguments in de Acosta (1981) or in Kuelbs and Zinn (1979) about the equivalence of the weak and the strong law of large numbers in Banach spaces, give the following laws of large numbers:

5.7. PROPOSITION. Consider the situation of Theorem 5.2, without condition (i) and with condition (ii) replaced by

(ii)' $tP^*\{F > B(t)\} \to 0$ as $t \to \infty.$

Then

$$\left\| \sum_{j=1}^n (f(X_j) - Pf)/B(n) \right\|_{\mathcal{F}} \to 0 \quad \text{in pr}^*.$$

If (ii)' is replaced by

(ii)" $P(B^{-1}(F)) < \infty$

(or equivalently, by $F(X_j)/B(j) \to 0$ a.s.), then

$$\left\| \sum_{j=1}^n (f(X_j) - Pf)/B(n) \right\|_{\mathcal{F}} \to 0 \quad \text{a.s.}$$

5.8. PROPOSITION. Let $\mathcal{F} \subset L_1(S, \mathcal{S}, P)$ with $F$ everywhere finite and $\sup_{f \in \mathcal{F}} |Pf|_{\|f\| > n}| \to 0$ as $n \to \infty.$ Assume

(5.10) $tP^*(F > t) \to 0$ as $t \to \infty$
and condition (iii) from Theorem 5.4. Then
\[ \left\| \sum_{i=1}^{n}(f(X_i) - Pf)/n \right\|_{\mathcal{F}} \to 0 \quad \text{in pr}^*, \]

If instead of (5.10) we have \( P^*F < \infty \), then
\[ \sum_{i=1}^{n}(f(X_i) - Pf)/n \to 0 \quad \text{a.s.} \]

(It is easy to check that the condition \( \sup_{f \in \mathcal{F}} |PfI_{|f|>n}| \to 0 \) allows for centering at expectations.)

5.9. EXAMPLE. A stable CLT in \( c_0 \). The positive results in the following proposition are obtained as corollaries to the theorems in this section. Recall that \( c_0 \) denotes the Banach space of all real sequences that converge to zero, equipped with the sup norm. If \( X \) is a \( c_0 \)-valued random variable, we write \( X \in \text{CLT}(n^{1/p}) \), \( p \in (1, 2] \), if there is a \( p \)-stable measure \( \gamma \) in \( c_0 \) such that
\[ \mathcal{L} \left( \sum_{i=1}^{n}(X_i - EXi)/n^{1/p} \right) \to \gamma \]

where the \( X_i \) are i.i.d. copies of \( X \). If \( \gamma = \delta_0 \) we write \( X \in \text{LLN}(n^{1/p}) \) and if the sequence \( \{\|\sum_{i=1}^{n}(X_i - EXi)/n^{1/p}\|\}_{n=1}^{\infty} \) is stochastically bounded, \( X \in \text{BCLT}(n^{1/p}) \).

With this notation we have

5.10. PROPOSITION. Let \( X = (x_1, \ldots, x_k, \ldots) \) be a random vector with infinitely many real coordinates. Let \( p \in (1, 2) \) and let \( p' \) be the conjugate of \( p \). Set \( \varepsilon_n := (Ln)^{1/p'} \Lambda_{p,\infty}(\sup_{j \geq n} |x_j|) \). We have:

(i) If the sequence \( \{\varepsilon_n\} \) is bounded, then \( X \in c_0 \) and \( X \in \text{BCLT}(n^{1/p}) \) (in \( c_0 \)).

(ii) If \( (x_1, \ldots, x_k) \in \text{CLT}(n^{1/p}) \) for all \( k \in \mathbb{N} \) and if \( \lim_{n \to \infty} \varepsilon_n = 0 \), then \( X \in \text{CLT}(n^{1/p}) \).

(iii) If \( \lim_{t \to \infty} t^p P\{x_1 > t\} = 0 \) and \( \lim_{n \to \infty} \varepsilon_n = 0 \), then \( X \in \text{LLN}(n^{1/p}) \).

Moreover, the \( c_0 \)-valued random variable \( Y = \{ (Lk)^{-1/p'} \varepsilon_k \}_{k=1}^{\infty} \), where \( \{\varepsilon_k\} \) is a Rademacher sequence, does not belong to \( \text{LLN}(n^{1/p}) \), and \( Z = \{ c_k (Lk)^{-1/p'} \varepsilon_k \}_{k=1}^{\infty} \), where \( c_k \to \infty \), does not belong to \( \text{BCLT}(n^{1/p}) \).

The statements about \( Y \) and \( Z \) follow directly from Kolmogorov’s lower exponential bound (e.g. Stout (1974, p. 262, Theorem 5.2.2(iii))) as in the proof of Proposition 5.4 in Andersen et al. (1988). Details are omitted. Statement (iii) is a particular case of (ii) (with \( \delta_0 \) as limiting measure), and the proof of (i) is similar to the proof of (ii), so we will only prove (ii). The proof is based on Theorem 5.2 and the following lemma, which explicitly construct majorizing measures for some simple ultrametric spaces (this lemma is known; it is only presented here for the readers’ convenience). See Heinkel (1983) for a similar construction. See also Paulauskas, and Rackauskas (1984) for results similar to (in fact contained in) Proposition 5.10.

5.11. LEMMA. Let \( 1 < p \leq 2 \) and let \( p' \) denote the conjugate of \( p \). Let \( \mathbb{N} = \mathbb{N} \cup \{\infty\} \), let \( \{a_n : n \in \mathbb{N}\} \) be a sequence of positive numbers and set \( a_\infty = 0 \).
Define on \( \mathbb{N} \) a distance \( \rho \) by
\[
\rho(i, j) = a_i \vee a_j, \quad i, j \in \mathbb{N}, \ i \neq j; \quad \rho(i, i) = 0, \ i \in \mathbb{N}.
\]
Then, if \( a_n = o((Ln)^{-1/p'}) \) there are projections \( \pi_q : \mathbb{N} \to \mathbb{N} \), \( q \in \mathbb{N} \), and a probability measure \( \mu \) on \( \bigcup_{q=1}^{\infty} \pi_q \mathbb{N} \) satisfying:
\[
(a) \ \pi_q \text{ has finite range and } \rho(j, \pi_q j) \leq 2^{-q}, \ j \in \mathbb{N}, \ q \in \mathbb{N}, \text{ and}

(b) \ \lim_{q_0 \to \infty} \sup_{j \in \mathbb{N}} 2^{-q} (\ln 1/\mu_{\{\pi_q j\}})^{1/p'} = 0 \text{ with the sup_} j \text{ finite for } q_0 = 1. \text{ If } a_n = O((Ln)^{-1/p'}) \text{ then there are } \pi_q, \ q \in \mathbb{N}, \text{ and } \mu \text{ satisfying (a) and (b') \sup_j 2^{-q} (\ln 1/\mu_{\{\pi_q j\}})^{1/p'} < \infty.}
\]

**Proof.** Only the “little o” statement will be proved (the “big O” part is analogous). Obviously if \( (\mathbb{N}, \rho) \) satisfies (a) and (b), so does \( (\mathbb{N}, \rho) \) for any \( \rho \leq c \rho, \ c < \infty \). Hence we may assume \( a_1 = 1, \ a_n = e_n (Ln)^{-1/p'} \) with \( a_n \downarrow 0, \ e_n \downarrow 0 \) and \( e_n \geq (Ln)^{-1} \). Let
\[
n_0 := 0 \quad \text{and} \quad n_q := \inf\{j : a_j \leq 2^{-q}\} \quad \text{for } q \geq 1,
\]
and define
\[
\pi_q j = j \quad \text{if } j < n_q, \quad \pi_q j = \infty \quad \text{otherwise}.
\]
Then obviously \( \pi_q \) satisfies (a). Define a probability measure \( \mu \) on \( \mathbb{N} \) as follows:
\[
\mu\{\infty\} = \frac{1}{2}, \ \mu\{j\} = c/n_q^2 \quad \text{for } n_{q-1} < j \leq n_q, \ q \in \mathbb{N},
\]
where \( c = [2 \sum_{q=1}^{\infty} (n_q - n_{q-1})/n_q^2]^{-1} \). (Note that since \( e_n > (Ln)^{-1}, \sum n_q^{-1} < \infty \).) Now we check that \( \mu \) satisfies condition (b). The sequence \( n_q \) is monotonically increasing to \( \infty \) and, since \( e_n \downarrow 0 \), there exist \( \delta_q \to 0 \) such that
\[
n_q = \exp\{2^{2p'} \delta_q\}, \quad q \in \mathbb{N}.
\]
If \( j < n_{q_0} \) then \( \pi_q j = j \) for all \( q \geq q_0 \), and \( \mu\{j\} \geq \mu\{n_{q_0}\} \). Hence
\[
\sum_{q=q_0}^{\infty} 2^{-q} \left( \frac{1}{\mu_{\{\pi_q j\}}} \right)^{1/p'} \leq \left( \sum_{q=q_0}^{\infty} 2^{-q} \right) \left( \ln c + 2^{1+q_0 p'} \delta_{q_0} \right)^{1/p'} = (2^{1-q_0 p'} \ln c + 2^{p'+1} \delta_{q_0})^{1/p'}.
\]
If \( j \geq n_{q_0} \) and if \( q_j, j > 0 \), is defined by the inequalities \( n_{q_j-1} \leq j < n_{q_j} \), we have (since \( \pi_q j = \infty \) for \( q < q_j \), and \( \mu\{\pi_q j\} = \mu\{j\} \geq \mu\{n_{q_j}\} \) for \( q \geq q_j \):
\[
\sum_{q=q_0}^{\infty} 2^{-q} \left( \frac{1}{\mu_{\{\pi_q j\}}} \right)^{1/p'} \leq 2^{-q_0+1} (\ln 2)^{1/p'} + (2^{1-q_j p'} \ln c + 2^{p'+1} \delta_{q_j})^{1/p'} \leq 2^{-q_0+1} (\ln 2)^{1/p'} + (2^{1-q_j p'} \ln c + 2^{p'+1} \delta_{q_j})^{1/p'}.
\]
(5.15) and (5.16) prove (b). \( \square \)

**Proof of (ii) from Proposition 5.10.** Let \((S, \mathcal{F}) = (c_0, \mathcal{B})\), \( P = \mathcal{L}(\xi_1, \ldots, \xi_n, \ldots) = \mathcal{L}(X) \) and \( \mathcal{F} = \{f_n, n \in \mathbb{N} : f_n(x) = x_n \text{ for } n < \infty, f_\infty(x) = 0\} \). It is enough to prove that \( \mathcal{F} \in \text{CLT}(P, n^{1/p}) \). We apply Theo-
rem 5.2. The finite dimensional distributions satisfy the \( n^{1/p} \)-CLT by hypothesis. Let \( a_n = \varepsilon_n/(\ln n)^{1/p} \), \( n \in \mathbb{N} \), and \( a_\infty = 0 \), and define

\[
\rho(f_n, f_m) = a_n \wedge m \quad \text{if} \ n \neq m, \ \rho(f_n, f_m) = 0, \ n, m \in \mathbb{N}.
\]

Then, by Lemma 5.11, \((\mathcal{F}, \rho)\) satisfies condition (iiiia) from Theorem 5.2. To verify conditions (ii) and (iiib) from Theorem 5.2 with \( B(n) = n^{1/p} \) note that, given \( \varepsilon > 0 \), if \( a_n < \varepsilon < a_{n-1} \), then \( B_\rho(f_k, \varepsilon) = \{f_k\} \) if \( k < n \) and \( B_\rho(f_k, \varepsilon) \subseteq \{f_j\}_{j \geq n} \) for \( k \geq n \); therefore,

\[
\sup_{t>0} t \Pr \left\{ \sup_{\rho(f,g) \leq \varepsilon} |f(X) - g(X)| > \varepsilon t^{1/p} \right\} \\
\leq 2^{1+p}\varepsilon^{-p} \sup_{t>0} t^p \Pr \left\{ \sup_{j \geq n} |\xi_j| > t \right\} \leq 2^{1+p}
\]

and this gives (ii) and (iiib) by Remark 5.3(2). \( \square \)

6. A law of the iterated logarithm. In Andersen et al. (1988) and in Ledoux and Talagrand (1986) the law of the iterated logarithm of Ledoux (1981) is extended to a bracketing LIL. Here we extend the following recent result of Ledoux (private communication): Let \((T, d)\) be a compact metric space, \( \rho \) a \( d \)-continuous pseudo-distance on \( T \), \( M \) a real random variable and \( X(t), t \in T \), a sample continuous process such that:

(i) \( \mathbb{E}X^2(t) < \infty \) for all \( t \in T \) (respectively, the random variables \( \{X^2(t) : t \in T\} \) are uniformly integrable),

(ii) \( |X(t, \omega) - X(s, \omega)| \leq M(\omega)\rho(s, t) \) for all \( \omega \in \Omega, s, t \in T \),

(iii) \( \rho \) is dominated by the \( L_2 \)-distance of a sample bounded Gaussian process (respectively, sample continuous Gaussian process),

(iv) \( \mathbb{E}(M^2/L_2^2M) < \infty \).

Then \( X \) satisfies the bounded law of the iterated logarithm (BLIL) (respectively, the compact law of the iterated logarithm—the CLIL).

A consequence of our result is that the condition on \( M \) can be weakened to

\[
\sup_{t>0} (t^2/L_2 t)P(M > t) < \infty.
\]

(As usual, \( L_x = 1 \vee \ln x \) and \( L_2 = L \circ L \). We refer to Andersen et al. (1988) for notation on the LIL for empirical processes, in particular for the meaning of expressions like \( \mathcal{F} \in \text{CLIL}(P) \) and \( \mathcal{F} \in \text{BLIL}(P) \).)

The additional ingredient for the proof of the following LIL will be the Ledoux-Talagrand (1986) LIL for \( B \)-valued random variables \( X \), \( B \) a separable Banach space.

6.1. THEOREM (LEDoux AND TAлагаrand (1986)). Let \( X \) be a \( B \)-valued random variable, \( B \) a separable Banach space. Then \( X \in \text{CLIL} \) (resp. \( X \in \text{BLIL} \)) if and only if the following three conditions hold:

(i) \( \mathbb{E}f(X) = 0 \) and the random variables \( \{f^2(X) : f \in B', \|f\| < 1\} \) are uniformly integrable (resp. \( \mathbb{E}f(X) = 0 \) and \( \mathbb{E}f^2(X) < \infty \) for all \( f \in B' \)).

(ii) \( \mathbb{E}(\|X\|^2/L_2^2\|X\|) < \infty \).

(iii) If \( \{X_i\} \) are i.i.d. copies of \( X \) and \( \{\varepsilon_i\} \) a Rademacher sequence independent of \( \{X_i\} \), then \( \sum_{i=1}^{n} \varepsilon_i X_i/a_n \rightharpoonup 0 \) in pr. (resp. the sequence \( \{\|\sum_{i=1}^{n} \varepsilon_i X_i/a_n\|\}_{n=1}^{\infty} \) is stochastically bounded) where \( a_n = (nL^2n)^{1/2} \).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Actually, the proof of this theorem goes through in the nonseparable case if enough measurability is assumed. So, Theorem 5.1 reduces the LIL to a weak law of large numbers, which is essentially the type of result we ended up proving in the previous sections.

6.2. Theorem. Let $\mathcal{F} \subset \mathcal{L}_2(S, \mathcal{S}, P)$ with envelope $F$ everywhere finite, and such that for all $n \in \mathbb{N}$, the processes

$$f \rightarrow \sum_{i=1}^{n} f(X_i), \quad f \in \mathcal{F},$$

are separable (for $\mathcal{L}_1(P)$). Assume:

(i) $P(F^2 / L^2 F) < \infty$,

and that there exists a Gaussian process $G$ on $\mathcal{F}$ with bounded $d_G$-uniformly continuous sample paths ($d^2_G(f, g) = E(G(f) - G(g))^2$) (respectively, $G$ has bounded sample paths) such that

(ii) $\sup_{f, g \in \mathcal{F}} P\left\{ \frac{(f - g)^2 / d^2_G(f, g)}{L^2 |f - g| / d_G(f, g)} \right\} < \infty$

and

(iii) there exists $K < \infty$ such that for all $f \in \mathcal{F}$, $\varepsilon > 0$,

$$\sup_{t > 0} L^2 t \Pr\left\{ \varepsilon^{-1} \sup_{g \in B_{d_G}(f, \varepsilon)} |f - g| > t \right\} \leq K.$$

Then $\mathcal{F} \in \text{CLIL}(P)$ (respectively, $\mathcal{F} \in \text{BLIL}(P)$).

Proof. We will only prove the compact LIL. Since $\{G(f) : f \in \mathcal{F}\}$ is relatively compact in $\mathcal{L}_2(S, \mathcal{S}, P)$, the family of functions $\{(\sum \alpha_i G(f_i))^2 : f_i \in \mathcal{F}, \sum |\alpha_i| \leq 1\}$ is uniformly integrable by Dini's lemma. So, by Theorem 6.1, we only need to show that for all $\varepsilon > 0$,

$$\lim \Pr\left\{ \left\| \sum_{i=1}^{n} \varepsilon_i f(X_i) / a_n \right\|_{\mathcal{F}} > \varepsilon \right\} = 0.$$

Now, by Talagrand's (1987) theorem on Gaussian processes, if $\varphi(x) = x^{1/2}$ then there is a discrete probability measure $\mu$ on $\mathcal{F}$ supported by $\{\pi_q f : q \in \mathbb{N}, f \in \mathcal{F}\}$ such that (4.7) holds with $\rho = d_G$, and therefore (4.9) holds with $\varphi^{-1}(x) = x^2$. Define

$$c_{n, q}(f) = 2^{-q} (nL^2 n)^{1/2} / \gamma_q f,$$

and then $\tau f$ and $\Delta_q f$ as in the proof of Theorem 4.2 using this new $c_{n, q} f$. Since for every $q_0$, $\# \{ \pi_{q_0} f : f \in \mathcal{F} \} < \infty$ and $\pi_{q_0} f \in \mathcal{L}_2(P)$,

$$\left\| \sum_{i=1}^{n} \varepsilon_i (\pi_{q_0} f)(X_i) / a_n \right\|_{\mathcal{F}} \rightarrow 0 \quad \text{in pr.,}$$
hence (6.1) is equivalent to

\[
(6.2) \quad \lim_{q_0 \to \infty} \limsup_n \Pr \left\{ \left\| \sum_{i=1}^n \varepsilon_i (f - \pi_{q_0} f)(X_i)/a_n \right\|_\mathcal{F} > \varepsilon \right\} = 0
\]

for all \( \varepsilon > 0 \). To prove (6.2) we use a decomposition of \( f - \pi_{q_0} f \) just as in (4.14), that is

\[
\Pr \left\{ \left\| \sum_{i=1}^n \varepsilon_i (f - \pi_{q_0} f)(X_i)/a_n \right\|_\mathcal{F} > 4\varepsilon \right\} \\
\leq \Pr \left\{ \left\| \sum_{i=1}^n \varepsilon_i (f - \pi_{q_0} f)I_{[rf=q_0]}(X_i)/a_n \right\|_\mathcal{F} > \varepsilon \right\} + \Pr \left\{ \left\| \sum_{i=1}^n \varepsilon_i (f - \pi_{q_1} f)I_{[rf=q_1]}(X_i)/a_n \right\|_\mathcal{F} > \varepsilon \right\} \\
+ \Pr \left\{ \left\| \sum_{q=q_0+1}^{q_1} \sum_{i=1}^n \Delta_q f I_{[\gamma_q,\gamma_q+1]}(X_i)/a_n \right\|_\mathcal{F} > \varepsilon \right\} \\
+ \Pr \left\{ \left\| \sum_{q=q_0+1}^{q_1} \sum_{i=1}^n \varepsilon_i (\pi_q f - \pi_{q-1} f)I_{[rf=q]}(X_i)/a_n \right\|_\mathcal{F} \geq \varepsilon \right\} \\
:= (II) + (III) + (IV) + (V).
\]

The integrability condition (i) implies that

\[nP\{F > \lambda a_n\} \to 0 \quad \text{as} \quad n \to \infty,
\]

for all \( \lambda \to 0 \). So, assuming as we may that \( \gamma_{q_0+1} f \leq 2^{q_0} \), we have

\[
(6.3) \quad \limsup_n (II) = \limsup_n \Pr \left\{ \left\| \sum_{i=1}^n \Delta_q f I_{[\Delta_q f > \gamma_n,\gamma_q+1]}(X_i)/a_n \right\|_\mathcal{F} > \varepsilon a_n \right\} \\
\leq \limsup_n nP\{2F > 2^{-2^{q_0}} a_n\} = 0.
\]

As in Theorem 4.2,

\[
(6.4) \quad (III) = 0
\]

if \( n\gamma_q,\gamma_{q+1} f \leq \varepsilon a_n \) or, if \( n^{2^{-q_1}} \leq \varepsilon \gamma_{q_1+1} f \), and since \( \gamma_{q_1+1} f \geq (q_1 + 1)^{1/2} \), this holds for \( \ln n > \varepsilon^{-2} \) as long as \( q_1(n) \geq L \). For (IV), we use the proof of Theorem
3.1 with $h_t = t^{-1/2}$, $\psi(x) \simeq x$ and $d = 1$. We get, as in the estimation of (I) in Theorem 3.1,

$$
(V) \leq \sum_{q=0}^{q_1-1} \sum_{t \in T_q} \operatorname{Pr} \left\{ \sup_{f : \pi_q f = t} \sum_{i=1}^{n} \Delta_q f I_{\{r = q\}}(X_i) > \epsilon \beta_{q_0 + 2} 2^{-q} 2^{-q} \gamma_q + 1 \right\}
$$

$$
= \sum_{q=0}^{q_1-1} \sum_{t \in T_q} \operatorname{Pr} \left\{ \sup_{f : \pi_q f = t} 2^{-q} \gamma_q + 1 f \right\}
$$

$$
\times \sum_{i=1}^{n} 2^{q} \Delta_q f/a_n I_{\{2^{-1}/\gamma_q + 1 f < 2^{q} \Delta_q f/a_n \leq 1/h_q t\}}(X_i) > \epsilon/4 \beta_{q_0 + 2}
$$

$$
\leq \sum_{q=0}^{q_1-1} \sum_{t \in T_q} \sum_{c_b t^2 < i \leq n} \left[ c_n P\{2^{q} \Delta_q f/a_n > c/l\}^{1/2} \right]
$$

(6.5) 

where we let $b_t := 1/h_q t$ and $c := \epsilon/4 \beta_{q_0 + 2}$.

Now, for $L_2 c > 2$ and $L_2 n > 2$ (which we can assume), we have

$$
L_2 \left( \frac{(c_n L_2 n)^{1/2}}{l} \right) \leq L_2(c n) = L(Lc + Ln) \leq L_2 c + L_2 n < (L_2 c)(L_2 n).
$$

So, by (4.9) with $\varphi^{-1}(x) = x^2$,

$$
(V) \leq \sum_{q=0}^{q_1-1} \sum_{t \in T_q} \sum_{c_b t^2 < i \leq n} \left[ \frac{K e L_2 c}{c} \right]
$$

$$
\leq \sum_{q=0}^{q_1-1} \sum_{t \in T_q} \left( 1 - \frac{K e L_2 c}{c} \right)^{-1} \exp\{-c b_t^{-2} \ln(c/K e L_2 c)\}
$$

$$
\to 0 \quad \text{as } q_0 \to \infty.
$$

Finally, to estimate (V) we must find a bound for $P(\pi_q f - \pi_{q-1} f)^2 I_{\{r_j \geq q\}}$. Noting that $d_G^2(\pi_q f, \pi_{q-1} f) \leq 9 \cdot 2^{-2q}$, that $c_{n,q} f/3 \cdot 2^{-q} \leq n$, and using hypothesis (ii) we get

$$
P(\pi_q f - \pi_{q-1} f)^2 I_{\{r_j \geq q\}} \leq K d_G^2(\pi_q f, \pi_{q-1} f) L_2 \left( \frac{c_{n,q} f}{d_G(\pi_q f, \pi_{q-1} f)} \right)
$$

$$
\leq 9 K 2^{-2q} L_2 n.
$$
Hence, using Bernstein’s inequality as in the evaluation of (V) in Theorem 4.2, we get

(6.6)

\[
\sum_{q=q_0+1}^{q_1} \sum_{i \in T_q} \Pr \left\{ \frac{\sum_{i=1}^{n} \varepsilon_i (\pi_q t - \pi_{q-1} t) I_{[\tau \leq \eta]} (X_i)}{\varepsilon a_n 2^{-q} \gamma_q t / \beta_{q_0+1}} > \varepsilon a_n 2^{-q} \gamma_q t / \beta_{q_0+1} \right\}
\]

\[
\leq 2 \sum_{q=q_0+1}^{q_1} \sum_{i \in T_q} \exp \left\{ -\varepsilon^2 a_n^2 2^{-2q} (\gamma_q t)^2 / \beta_{q_0+1}^2 \right\}
\]

\[
\leq 2 \sum_{q=q_0+1}^{q_1} \sum_{i \in T_q} \exp \left\{ -\frac{(\gamma_q t)^2}{\beta_{q_0+1} (18K + 3\varepsilon / 2\beta_{q_0+1})} \right\} \to 0
\]

as \( q_0 \to \infty \) uniformly in \( n \).

(6.3) to (6.5) give (6.2) and the result follows. \( \square \)

For another approach to the LIL (and the CLT(\( n^{1/2} \))) for Lipschitz processes see Heinkel (1987).

7. Some additional examples: BL* norms, weighted empiricals.

(a) BL* norms. For \( f : \mathbb{R} \to \mathbb{R} \), define

\[
\|f\|_{BL} := \|f\|_\infty + \sup_{x \neq y} |f(x) - f(y)| / |x - y|,
\]

\[
BL_1(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R}, \|f\|_{BL} \leq 1 \},
\]

and for Borel measures \( Q \) of bounded variation on \( \mathbb{R} \),

\[
\|Q\|_{BL*} = \sup \left\{ \left| \int f \, dQ \right| : f \in BL_1(\mathbb{R}) \right\}.
\]

It is well known that \( \|P_n - P\|_{BL*} \to 0 \) a.s. for all probability laws \( P \) on \( (\mathbb{R}, \mathcal{B}) \), and Giné and Zinn (1986) prove that \( BL_1(\mathbb{R}) \in CLT(P, n^{1/2}) \) if and only if \( \sum_{j=-\infty}^{\infty} |P(I_j)|^{1/2} < \infty \), where \( I_j = [j-1, j) \). A natural question to ask is whether \( BL_1(\mathbb{R}) \in CLT(P, n^{1/p}) \), \( p \in (1, 2) \), under weaker conditions on \( P \). A first observation is that \( BL_1(\mathbb{R}) \in CLT(P, n^{1/p}) \) if and only if

\[
n^{1-1/p} \|P_n - P\|_{BL*} \to 0 \quad \text{a.s.}
\]

because of boundedness of the envelope function \( F \) of \( BL_1(\mathbb{R}) \). We obtain below necessary and sufficient conditions for (7.1) to hold. We do not seem to be able to obtain this result in full from Proposition 5.7, but it gets close to it. As an illustration, here is how one applies this proposition. To apply Proposition 5.7 for a given \( P \), it is enough to find a distance \( \rho \) satisfying conditions (iii) from Theorem 5.2 with \( B(t) = t^{1/p} \) and \( \mathcal{F} = BL_1(\mathbb{R}) \). For clarity of exposition, we will assume \( p_j := P(I_j) \downarrow \) and \( p_j = 0 \) for \( j \leq 0 \), and will only be interested in \( p_j \) of power type. Consider, for \( f, g \in BL_1(\mathbb{R}^+) \),

\[
\rho(f, g) = \sup_{j \in \mathbb{N}} \|f - g\|_{j}, \quad \text{with} \quad \|h\|_j = \sup \{|h(x)| : x \in I_j\},
\]
where $\tau$ is a positive number to be determined. Standard computations show that the metric entropy of $\text{BL}_1(\mathbb{R}^+)$ for $\rho$ is

$$\ln N(\text{BL}_1(\mathbb{R}^+), \rho, \epsilon) \simeq \epsilon^{-1} \sum_{j \leq \epsilon^{-1/\tau}} j^{-\tau} \begin{cases} 
\epsilon^{-1/\tau} & \text{if } \tau < 1, \\
\epsilon^{-1} \ln \epsilon^{-1} & \text{if } \tau = 1, \\
\epsilon^{-1} & \text{if } \tau > 1,
\end{cases}$$

and we want $\rho$ to satisfy (iii) in Theorem 5.2, which for metric entropy reduces to

$$\int_0^\infty (\ln N(\text{BL}_1(\mathbb{R}^+), \rho, \epsilon))^{1/p'} \, d\epsilon < \infty$$

(with $1/p + 1/p' = 1$). This integral is finite if and only if $\tau > 1/p'$.

Condition (iii) in Theorem 5.2 for $P$ becomes

$$\sup_{t>0} t^p P\{x: \sup \{|h(x)|: h \in \text{BL}_1(\mathbb{R}^+), \|h\|_j \leq \epsilon j^\tau \text{ for } j < \epsilon^{-1/\tau} \} > t\} < c\epsilon^p$$

for all $\epsilon$ near to 0. The left-hand side is bounded by

$$\sup_{t>0} t^p \left[ \sum_{j < \epsilon^{-1/\tau}} p_j I_{[\epsilon j^\tau > t]} + \sum_{j \geq \epsilon^{-1/\tau}} p_j I_{[1 > t]} \right],$$

and this is bounded by $\epsilon^p c$ for all $\epsilon$ near to 0 if and only if $\sup_{t>0} t^{p'} \sum_{j \geq t} p_j < \infty$. Thus, Proposition 5.7 shows that if $p_j \leq 1/j^{r}$ for some $r > p$, then (7.1) holds.

Now we will obtain general necessary and sufficient conditions for (7.1) to hold; in particular we will see that if $p_j = 1/j^p$ then $P$ does not satisfy (7.1). Note that with extra care in the above arguments we would have obtained (7.1) for $p_j = 1/j^p (\ln j)^r$ for certain $r$'s, but not much more, and this is not good enough as we see below.

### 7.1. THEOREM

Let $p_j = P(I_{j})$, $j \in \mathbb{Z}$. Then

$$n^{1-1/p} \|P_n - P\|_{\text{BL}^*} \to 0 \quad \text{a.s.}$$

if and only if both

$$n^{1-1/p} \sum_{\{j: p_j \leq 1/n\}} p_j \to 0$$

and

$$n^{1/2-1/p} \sum_{\{j: p_j > 1/n\}} p_j^{1/2} \to 0.$$

**PROOF.** We decompose $f(x)$, $f \in \text{BL}_1(\mathbb{R})$, as follows,

$$f(x) = \sum_{j=-\infty}^{\infty} I_{[x \in [j-1,j))} \left[ \int_{j-1}^x (f'(s) - f(j) + f(j - 1)) \, ds \right.$$ 

$$+ (f(j) - f(j - 1))(x - j + 1) + f(j - 1) \right]$$

$$= \sum_{j=-\infty}^{\infty} \int_{j-1}^j I_{[x \in [j-1,j))} I_{[j-1,x]}(s) \varphi_j(s) \, ds$$

$$+ \sum_{j=-\infty}^{\infty} a_j [(x - j + 1)I_{[x \in [j-1,j))} + (j + 1 - x)I_{[x \in [j,j+1))}].$$
where \( f' \) is the derivative of \( f \), \( \varphi_j(s) = f'(s) - (f(j) - f(j-1)) \) and \( a_j = f(j) \). Let
\[
T: \text{BL}(\mathbb{R}) \rightarrow L^\infty := \left( \bigoplus_{j=-\infty}^{\infty} L^\infty_0(j-1,j) \right) \oplus_{l=0}^{\infty} l^\infty(\mathbb{Z})
\]
where
\[
L^\infty_0(j-1,j) := \left\{ \varphi \in L^\infty(j-1,j) : \int_{j-1}^{j} \varphi(s) \, ds = 0 \right\},
\]
be defined by \( T = \{ (\varphi_j)_{j=1}^{\infty}, (a_j)_{j=1}^{\infty} \} \) with \( \varphi_j, a_j \) as above. Then \( T \) is obviously an isomorphism between these two Banach spaces.

By randomization, as usual, we can replace in (7.1) \((P_n - P)(f)\) by \( \sum_{i=1}^{n} \varepsilon_i f(X_i) \), \( X_i \) i.i.d. with law \( P \). The decomposition (7.4) gives, by duality (note that \((L^\infty)' = L^1 \oplus \oplus_{i=1}^{\infty} L^1(\mathbb{Z}))\),
\[
\sum_{i=1}^{n} \varepsilon_i f(X_i) \approx \sum_{i=1}^{n} \varepsilon_i I_{[X_i \in [j-1,j]]}
\]
where
\[
E_{\varepsilon} \left\| \sum_{i=1}^{n} \varepsilon_i f(X_i) \right\|_{\text{BL}^1(\mathbb{R})} \approx \sum_{j=-\infty}^{\infty} E_{\varepsilon} \left| \sum_{i=1}^{n} \varepsilon_i I_{[X_i \in [j-1,j]]} \right|,
\]
and by the triangle inequality (applied to the last two summands in (7.5) with a shift on \( j \) for the third),
\[
\sum_{i=1}^{n} \varepsilon_i I_{[X_i \in [j-1,j]]}
\]
Now, \( E|\sum_{i=1}^{n} \varepsilon_i I_{[X_i \in [j-1,j]]} | \) is easy to estimate. For instance, we can use Lemma 3.2 and Corollary 3.4 in Giné and Zinn (1983) to show that this expectation is bounded from above and from below by fixed linear combinations of the following three quantities:

(i) \( \delta_0 := \inf\{t > 0 : nP\{I_{[X_i \in I_j]} > t\} \leq 1/72\} \),

(ii) \( n \int_{\delta_0}^{\infty} P\{I_{[X_i \in I_j]} > t\} \, dt \),

(iii) \( \left( E\left| \sum_{i=1}^{n} \varepsilon_i I_{[X_i \in I_j]} I_{[X_i \in I_j] \leq \delta_0} \right|^2 \right)^{1/2} \).
which are respectively equal to
\[(7.8) \quad I_{[p_j > 1/2n]}, \quad np_jI_{[p_j \leq 1/2n]} \quad \text{and} \quad (np_j)^{1/2}I_{[p_j > 1/2n]} \].
Hence, from (7.6), (7.7) and (7.8) we conclude that conditions (7.2) and (7.3) are
necessary and sufficient for
\[n^{-1/p}E \left\| \sum_{i=1}^{n} \varepsilon_i f(X_i) \right\|_{\text{BL}^*} \rightarrow 0.\]
By boundedness, $L_1$ convergence is equivalent to convergence in probability for this
sequence, and (as mentioned above), equivalent to convergence in probability of the
sequence \( \{n^{1-1/p}\|P_n - P\|_{\text{BL}^*}\}_{n=1}^\infty \). By de Acosta (1981, Theorem 3.1) (applied
in the Banach space $L_1$) this sequence converges a.s. if and only if it converges in
probability. \[\square\]

7.2. REMARK. (1) The same proof shows that uniform boundedness of the
quantities in (7.1) and (7.2) is equivalent to stochastic boundedness of the sequence
(7.1). In particular this holds if $p_j = 1/j^p$.

(2) If \( \sum p_j^{1/p} < \infty \) then (7.2) and (7.3) hold; they also hold if $p_j = 1/j^p M(j)$
with $M \uparrow \infty$ and is slowly varying; but $\{p_j = 1/j^p\}$ does not satisfy (7.2) and
(7.3).

(3) Consider
\[\text{Lip}_1 \mathbb{R} = \left\{ f: \mathbb{R} \rightarrow \mathbb{R}, \sup_{x \neq y} |f(x) - f(y)|/|x - y| \leq 1 \right\} .\]
For $\text{Lip}_1 \mathbb{R}$ we also have $\text{Lip}_1 \mathbb{R} \subset \text{CLT}(P, n^{1/p})$ if and only if $n^{1-1/p}\|P_n - P\|_{\text{Lip}_1 \mathbb{R}}$
\( \rightarrow 0 \) in probability, and it is easy to show that this holds if and only if
\( n^p P\{|x| > t\} \rightarrow 0 \) as $t \rightarrow \infty$. The proof is omitted.

(b) Weighted empiricals. Let $P$ be Lebesgue measure on $[0,1]$ and let $w(t)$, $t \in
[0,1]$, be a $U$-shaped (weight) function on $(0,1)$, that is, a nonnegative measurable
function that is nonincreasing on $(0,\gamma)$ and nondecreasing on $(\gamma,1)$ for some
$0 < \gamma \leq 1/2$. Let $\mathcal{F} = \{w(t)I_{[0,t]}: t \in (0,1/2)\}$—by symmetry, it is enough to consider
just half the $U$-shape. Assume moreover $\gamma = 1/2$. The Cibisov-O'Reilly theorem
states that $\mathcal{F} \subset \text{CLT}(P, n^{1/2})$ if and only if $\mathcal{F}$ is $P$-pre-Gaussian, which amounts
to a certain integral condition for $w$. In particular if $w(t) = o((t \ln \ln 1/t)^{-1/2})$ at
zero then, $\mathcal{F} \subset \text{CLT}(P, n^{1/2})$, but if $w(t) = O((t \ln \ln 1/t)^{-1/2})$ then $\mathcal{F}$ does not
satisfy the CLT. The question is whether for these steeper weights convergence can
be achieved if one divides by larger normings, say between $n^{1/2}$ and $n$. $w(t)$ is
bounded for every $t$, so for normings $B(n)$ such that $B(n)/n^{1/2} \rightarrow \infty$ as $n \rightarrow \infty$,
the finite dimensional distributions of \( \{\sum_{i=1}^{n} \varepsilon_i w(t)I_{[0,t]}(U_i)/B(n)\} \), where $U_i$ are
i.i.d. uniform random variables, tends to zero in probability. So, as for $\text{BL}^*$, if
$\mathcal{F} \subset \text{CLT}(P, B(n))$ the limit is necessarily 0. We will only consider $B(n) = n^{1/p}$,
p $\in [1,2)$. As in the case of $\text{BL}^*$, it is possible to obtain results from Proposition
5.7 about what weights $w$ give that the sequence
\[(7.9) \quad \sup_{t \leq 1/2} \left| \sum_{i=1}^{n} w(t)(I_{[0,t]}(U_i) - t)/n^{1/p} \right| \]
converges to zero a.s. But Proposition 5.7 does not imply the complete answer, which is easy to obtain e.g. either by direct computation using order statistics and the law of iterated logarithm, or invoking a theorem of Pisier (1984) on Vapnik-Červonenkis classes. Pisier’s result is as follows: let $C$ be a class of measurable subsets of $(S, \mathcal{F})$, let $(\mathcal{M}, \| \cdot \|_{TV})$ be the set of measures on $(S, \mathcal{F})$ of bounded total variation, with the total variation norm, and let $\|Q\|_C = \sup_{C \in C} |Q(C)|$, for $Q \in \mathcal{M}$. Then the identity map

\begin{equation}
(\mathcal{M}, \| \cdot \|_{TV}) \to (\mathcal{M}, \| \cdot \|_C)
\end{equation}

is of type 2 if and only if $C$ is Vapnik-Červonenkis.

7.3. **Theorem.** Let $w(t), t \in (0, 1/2)$, be a nonnegative nondecreasing function and let $\{U_i\}_{i=1}^\infty$ be independent random variables uniformly distributed on $[0, 1]$. Then, for $p \in [1, 2]$,

1. The sequence (7.9) is stochastically bounded if and only if $\sup_{t \in (0, 1/2]} t^{1/p} w(t) < \infty$;

2. The sequence (7.9) converges to zero in probability if and only if $\lim_{t \to 0} t^{1/p} w(t) = 0$.

3. The sequence (7.9) converges to zero a.s. if and only if $\int_0^{1/2} (w(t))^p dt < \infty$.

**Proof.** As usual, we can randomize and consider

\begin{equation}
\left\{ \sum_{i=1}^n \varepsilon_i w(t) I_{[0,t]}(U_i) \right\}_{i=1}^\infty
\end{equation}

instead of $\{\sum_{i=1}^n w(t)(I_{[0,t]}(U_i) - t)\}_{i=1}^\infty$ since the necessary tail condition for $w(t)(I_{[0,t]}(U_i) - t)$ already implies boundedness of $\sup_{t>0} t^{1/p} w(t)$. Consider the processes $X_i(t) = \varepsilon_i w(t) I_{[0,t]}(U_i), 0 < t \leq \frac{1}{2}$, $X_i(t) = 0$ for $t = 0$ and for $t \in (\frac{1}{2}, 1]$, $i = 1, \ldots, X_i(t)$ has all of its trajectories of bounded variation, and the total variation norm of $X_i(t)$ is $2w(U_i)$. So, these random processes define i.i.d. random measures $\nu_i(\cdot)$ of bounded variation on $[0, 1]$ that have total variation norm $2w(U_i)$. The map (7.10) with $(S, \mathcal{F}) = ([0, 1], \mathcal{B})$ and $\mathcal{C} = \{[0, t] : t \in [0, 1]\}$ is of type 2. It is routine to check then that the type 2 inequality allows for the use of the classical computations for the CLT in the line, the stable case, about dominating truncated second moments by tails, to obtain:

$\sup_{t>0} t^p P\{\|X_1\|_{TV} > t\} < \infty \Rightarrow$ the sequence (7.11) is stochastically bounded and

$t^p P\{\|X_1\|_{TV} > t\} \to 0 \Rightarrow$ the sequence (7.11) converge to zero in pr.

But $t^p P\{\|X_1\|_{TV} > t\} = t^p P\{2w(U_1) > t\} = t^p w^{-1}(t/2)$ where $w^{-1}(s) = \sup\{t : w(t) \geq s\}$. Thus the direct implications in (1) and (2) are proved. For the converse in (1) and (2), note that, using Lévy’s inequality,

\[
P\left\{ \left\| \sum_{i=1}^n X_i(t) \right\|_\infty > cn^{1/p} \right\} \geq P\left\{ \max_{1 \leq n} \left\| X_i(t) \right\|_\infty > cn^{1/p} \right\}
\]

\[
= P\left\{ \max_{1 \leq n} w(U_i) \geq cn^{1/p} \right\} \geq nw^{-1}(cn^{1/p})/(1 + nw^{-1}(cn^{1/p}))
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Finally, (3) follows from (2) by de Acosta (1981, Theorem 3.1).

REFERENCES

13. _____, (1987), Some exponential inequalities with applications to the central limit theorem in C[0,1] (to appear).

**Aarhus Universitet, Matematisk Institut, DK-8000 Aarhus C, Denmark**

**Texas A&M University, Department of Mathematics, College Station, Texas 77843**