

## HARDY SPACES OF VECTOR-VALUED FUNCTIONS: DUALITY

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*Dedicated to the memory of J. L. Rubio de Francia*

**ABSTRACT.** We prove here that the Hardy space of  $B$ -valued functions  $H^1(B)$  defined by using the conjugate function and the one defined in terms of  $B$ -valued atoms do not coincide for a general Banach space. The condition for them to coincide is the UMD property on  $B$ . We also characterize the dual space of both spaces, the first one by using  $B^*$ -valued distributions and the second one in terms of a new space of vector-valued measures, denoted  $\mathcal{BMO}(B^*)$ , which coincides with the classical  $\text{BMO}(B^*)$  of functions when  $B^*$  has the RNP.

**Introduction.** When the theory of Hardy spaces began to be studied by using the so-called real techniques, several characterizations for these spaces were obtained. We shall consider Hardy spaces on the circle  $\mathbf{T}$ . Let us write three equivalent formulations of  $H^1$ .

- (1)  $H_{\text{con}}^1 = \{f \in L^1(\mathbf{T}) : \tilde{f} \in L^1(\mathbf{T})\}$ ,
- (2)  $H_{\text{max}}^1 = \{f \in L^1(\mathbf{T}) : P^*f(t) = \sup_{0 < r < 1} |P_r * f(t)| \in L^1(\mathbf{T})\}$ ,
- (3)  $H_{\text{at}}^1 = \{f \in L^1(\mathbf{T}) : f = \sum \lambda_k a_k, \sum |\lambda_k| < \infty, a_k \text{ atom}\}$ ,

where  $\tilde{f}$  stands for the conjugate function of  $f$  and  $P_r$  for the Poisson kernel on the circle  $\mathbf{T}$ .

In 1971, D. L. Burkholder, R. F. Gundy and M. L. Silverstein [7] connected the spaces defined by the conjugate function (1) and by the radial maximal function (2) showing that  $H_{\text{con}}^1 = H_{\text{max}}^1$  with equivalent norms. Later, R. R. Coifman [10] gave a constructive proof of the so-called atomic decomposition of functions in  $H_{\text{max}}^1$ , stating that  $H_{\text{max}}^1 = H_{\text{at}}^1$  with equivalent norms. In this paper we shall prove that in the vector-valued case Burkholder-Gundy-Silverstein's result holds only for special kinds of Banach spaces. One of the most famous results in Hardy spaces theory was obtained by C. Fefferman (see [15, 16]) by proving that the dual space of  $H_{\text{con}}^1$  could be identified with the BMO space (functions of bounded mean oscillation) defined by F. John and L. Nirenberg [19]. This duality result leads immediately to the atomic decomposition for functions in  $H_{\text{con}}^1$ . A direct proof of the duality  $(H_{\text{at}}^1)^* = \text{BMO}$  can be found in [11 or 20].

The aim of this paper is to consider the above spaces when the functions are allowed to take values in a general Banach space  $B$ , to study the relationship between them and to characterize their dual spaces. The paper is divided into three

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sections. The first one shows that  $H_{\text{con}}^1(B)$  and  $H_{\text{max}}^1(B)$ , defined in the obvious way, need not be the same. The space  $H_{\text{con}}^1(B)$  is always included in  $H_{\text{max}}^1(B)$ , being the necessary and sufficient condition for them to coincide with the UMD property. The second part is devoted to giving a representation of the dual space of  $H_{\text{con}}^1(B)$  in terms of  $B^*$ -valued distributions. Finally in the last section we define a new space of vector-valued measures, denoted by  $\mathcal{BMO}(B)$ , so that we can write the general duality result  $(H_{\text{at}}^1(B))^* = \mathcal{BMO}(B^*)$ . Besides we find the Radon-Nikodym property on  $B$  as the right one to make the space of measures  $\mathcal{BMO}(B)$  coincide with the classical space of functions  $\text{BMO}(B)$ .

Throughout this paper  $(B, \| \cdot \|)$  will denote a real Banach space,  $L^p(B)$  will stand for the  $B$ -valued measurable functions on  $\mathbf{T}$  such that

$$\|f\|_p = \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(t)\|^p dt \right)^{1/p}.$$

**1. Two different Hardy spaces.** We can replace the absolute value in Definitions (2) and (3) by the norm of  $B$  to get the corresponding  $B$ -valued Hardy spaces

$$H_{\text{max}}^1(B) = \left\{ f \in L^1(B) : P^* f(t) = \sup_{0 < r < 1} \|P_r^* f(t)\| \in L^1 \right\},$$

$$H_{\text{at}}^1(B) = \left\{ f \in L^1(B) : f = \sum \lambda_k a_k, \sum |\lambda_k| < \infty, a_k \text{ } B\text{-atom} \right\}.$$

We define in these the following norms

$$(1.1) \quad \|f\|_{\text{max}} = \|P^* f\|_1,$$

$$(1.2) \quad \|f\|_{\text{at}} = \inf \left\{ \sum |\lambda_k| : f = \sum \lambda_k a_k \right\}.$$

The reader is referred to [10, 14, 17] for terminology and concepts used in the definitions.

The first result we would like to mention here is that Coifman's proof [10] can be merely reproduced in the  $B$ -valued setting and consequently we can write

$$(1.3) \quad H_{\text{max}}^1(B) = H_{\text{at}}^1(B) \quad \text{with equivalent norms.}$$

The definition of  $H_{\text{con}}^1(B)$  needs a slight remark. Since the existence of the conjugate function cannot be guaranteed (take  $B = l^1$  to see where the problem arises), we have to assume the existence of the conjugate function and also that this belongs to  $L^1(B)$ , that is

$$H_{\text{con}}^1(B) = \{f \in L^1(B) : \tilde{f} \in L^1(B)\}.$$

An easy way of looking at  $\tilde{f}$  is as the following limit:

$$\tilde{f}(t) = \lim_{r \uparrow 1} Q_r * f(t) \quad t\text{-a.e.}$$

where  $Q_r$  stands for the conjugate Poisson kernel.

The norm on this last space is given by

$$(1.4) \quad \|f\|_{\text{con}} = \|f\|_1 + \|\tilde{f}\|_1.$$

Our next objective is to show that  $H_{\text{con}}^1(B) \subset H_{\text{max}}^1(B)$  for a general Banach space. To do this we shall use the following lemma whose proof follows easily from the one of the real-valued case (see [4, 18]).

LEMMA 1.1. *Let  $B_0$  be a complex Banach space,  $D$  the unit disc, and  $F$  a holomorphic function on the disc with values in  $B_0$  belonging to  $H^1(D, B_0)$ , that is*

$$\|F\|_{H^1} = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|F(re^{it})\|_{B_0} dt < \infty.$$

Then

$$(1.5) \quad \frac{1}{2\pi} \int_0^{2\pi} \sup_{0 < r < 1} \|F(re^{it})\|_{B_0} dt \leq C \|F\|_{H^1}.$$

THEOREM 1.1.  $H^1_{\text{con}}(B)$  is continuously embedded in  $H^1_{\text{max}}(B)$ .

PROOF. Take  $f$  in  $H^1_{\text{con}}(B)$  and consider  $B_0 = B + iB$  with the norm  $\|a + ib\| = \|a\| + \|b\|$ . Let us define the following function

$$(1.6) \quad F(re^{it}) = P_r * f(t) + iP_r * f(t).$$

It is very easy to verify that  $F$  belongs to  $H^1(D, B_0)$  and therefore applying Lemma 1.1, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \sup_{0 < r < 1} \|P_r * f(t)\| dt \leq \frac{1}{2\pi} \int_0^{2\pi} \sup_{0 < r < 1} \|F(re^{it})\|_{B_0} dt \leq \|f\|_{\text{con}},$$

which finishes the proof.

Let us observe that this result, according to (1.3), is a different approach to the fact, proved by Bourgain [4] using Brownian motion, that every function  $f$  in  $L^1(B)$  with  $\hat{f}(n) = 0$  for  $n < 0$  can be decomposed into atoms.

Our next goal is to characterize the class of spaces  $B$  where  $H^1_{\text{con}}(B)$  coincides with  $H^1_{\text{max}}(B)$ , i.e. Burkholder-Gundy-Silverstein's result remains valid. These spaces will be those where the martingale differences are unconditional, called UMD spaces. We shall introduce this property by using a very well-known result due to Bourgain [3], MacConnell and Burkholder [6] which characterized the UMD property in terms of the boundedness of the conjugate function for functions in  $L^p(B)$ .

DEFINITION 1.1. A Banach space  $B$  is said to be a UMD space, or to have the UMD property, if there exist a value of  $p$ ,  $1 < p < \infty$ , and a constant  $C_p$  such that

$$(1.7) \quad \|\tilde{f}\|_p \leq C_p \|f\|_p \quad \text{for all } f \in L^p(B).$$

Standard techniques show that instead of taking  $L^p$  norm we can work with  $L \log L^+(B)$ , and then we can state the following theorem.

THEOREM A [21]. *Let  $1 < p < \infty$ . The following statements are equivalent.*

- (a)  $B$  is a UMD space.
- (b) There are constants  $C$  and  $C'$  such that

$$\|f\|_1 \leq C \|f\|_{L \log L^+} + C'.$$

The direct implication of the next theorem can be obtained from results in [23, 6, and 4]. Here we present a proof for the sake of completeness.

**THEOREM 1.2.** *The following statements are equivalent.*

- (a)  $H_{\text{con}}^1(B) = H_{\text{max}}^1(B)$  with equivalent norms.
- (b)  $B$  is an UMD space.

**PROOF.** Assume that every function in  $H_{\text{max}}^1(B)$  belongs to  $H_{\text{con}}^1(B)$ , and take a function  $f$  in  $L \log^+ L(B)$ . Due to the simple fact that

$$P^* f(t) = \sup_{0 < r < 1} \|P_r * f(t)\| \leq \sup_{0 < r < 1} P_r * \|f\|(t) \leq M(\|f\|)(t)$$

where  $Mf$  stands for the Hardy-Littlewood maximal function, observe that since  $\|f\| \in L \log^+ L(\mathbf{T})$  then  $M(\|f\|) \in L^1(\mathbf{T})$ , which implies that  $f$  belongs to  $H_{\text{max}}^1(B)$ , and therefore, by assumption, to  $H_{\text{con}}^1(B)$ . Moreover we have

$$\|f\|_1 \leq C \|f\|_{L \log L^+} + C.$$

Conversely, according to (1.3), we shall prove that there is a constant  $C$  such that for all  $B$ -atoms  $a$  it verifies

$$(1.8) \quad \|\tilde{a}\|_1 \leq C.$$

The first observation is that the conjugate function of an atom exists since we are assuming  $B$  is UMD, and  $a \in L^2(B)$ . Assume  $a$  is supported in  $(-\delta, \delta)$  for some  $\delta > 0$  (here we are identifying  $\mathbf{T}$  with  $[-\pi, \pi)$ ).

$$\int_{-2\delta}^{2\delta} \|a(t)\| dt \leq \left( \int_{-2\delta}^{2\delta} \|a(t)\|^2 dt \right)^{1/2} (2\delta)^{1/2} \leq C \|a\|_2 \delta^{1/2} \leq C.$$

To check the integral over  $\{t: 2\delta < |t| < \pi\}$  we use the fact that

$$\int_{2\delta < |t| < \pi} |Q_r(t-s) - Q_r(t)| dt \leq C \quad \text{for all } r, 0 < r < 1$$

and a standard computation with atoms shows that

$$\begin{aligned} \int_{2\delta < |t| < \pi} \|Q_r * a(t)\| dt &= \int_{2\delta < |t| < \pi} \left\| \int_{-\delta}^{\delta} (Q_r(t-s)a(s) - Q_r(t)a(s)) ds \right\| dt \\ &\leq \int_{-\delta}^{\delta} \left( \int_{2\delta < |t| < \pi} |Q_r(t-s) - Q_r(t)| dt \right) \|a(s)\| ds \leq C \|a\|_1 \leq C. \end{aligned}$$

From this last inequality we get

$$\int_{2\delta < |t| < \pi} \|\tilde{a}(t)\| dt \leq C.$$

Once we have proved (1.8) we can extend this to any function  $f$  in  $H_{\text{an}}^1(B)$  and easily show that

$$\|\tilde{f}\|_1 \leq C \|f\|_1 \quad \text{for } f \text{ in } H_{\text{at}}^1(B),$$

which implies, using (1.3), that  $H_{\text{max}}^1(B) \subset H_{\text{con}}^1(B)$ , and since the other inclusion was proved in Theorem 1.1, then the proof is completed.

**REMARK 1.1.** There is another very interesting space to consider in the setting of vector-valued Hardy spaces:  $H_{\text{con}}^1 \hat{\otimes} B$ , that is the tensor product with the projective norm. Several results about it can be seen in [18]. Here we only want to mention that in general  $H_{\text{con}}^1 \hat{\otimes} B \subset H_{\text{con}}^1(B)$ . The following example gives equality  $B = l^1$ . To see this it is enough to realize that  $H_{\text{con}}^1(l^1)$  can be interpreted as  $l^1(H^1)$  and then use the very well-known fact that  $l^1 \hat{\otimes} B = l^1(B)$  for any Banach space  $B$ .

**2. Duality for  $H^1_{\text{con}}(B)$ .** To characterize the dual space of  $H^1_{\text{con}}(B)$  we shall introduce the concepts of  $B$ -valued distribution and conjugate distribution.

Let us denote  $C^\infty(\mathbf{T})$  the space of  $2\pi$ -periodic functions in  $C^\infty(\mathbf{R})$  endowed with the topology given by the seminorms  $(p_m : m \in \mathbf{N})$  where

$$p_m(\phi) = \sup_{0 \leq k \leq m} \sup_{t \in \mathbf{T}} |\phi^k(t)|.$$

**DEFINITION 2.1.** The continuous linear maps from  $C^\infty(\mathbf{T})$  into  $B$  will be called  $B$ -valued distributions and the space will be denoted by  $\mathcal{D}'(B)$ . The main point to consider in this space is given by the following fact:

(2.1)  $\quad$  If  $\varphi \in C^\infty(\mathbf{T})$  then  $\tilde{\varphi} \in C^\infty(\mathbf{T})$ .

This enables us to define the conjugate of a distribution  $T$  in  $\mathcal{D}'(B)$  as the  $B$ -valued distribution given by

(2.2)  $\quad \tilde{T}(\varphi) = -T(\tilde{\varphi}) \quad \text{for all } \varphi \text{ in } C^\infty(\mathbf{T}).$

Given an element  $\Phi$  in  $(H^1_{\text{con}}(B))^*$  we can define the following  $B^*$ -valued distribution

(2.3)  $\quad \langle T_\Phi(\psi), b \rangle = \Phi(\psi \cdot b) \quad \text{for all } \psi \in C^\infty(\mathbf{T}), b \in B,$

where  $\psi \cdot b$  denotes the  $B$ -valued function  $(\psi \cdot b)(t) = \psi(t) \cdot b$ .

Under this identification we are able to look at the dual space of  $H^1_{\text{con}}(B)$  as certain class distributions in  $\mathcal{D}'(B^*)$ . We shall denote by  $\Lambda^\infty(B)$  the class of distributions which can be extended to elements in  $\mathcal{L}(L^1(\mathbf{T}), B)$ , that is, verifying that there exists a constant  $C$  such that

(2.4)  $\quad \|T(\psi)\| \leq C\|\psi\|_1 \quad \text{for all } \psi \text{ in } C^\infty(\mathbf{T}),$

where the norm in it is given by the infimum of the constant verifying (2.4).

Observe that

$$\Lambda^\infty(B^*) = \mathcal{L}(L^1(\mathbf{T}), B^*) = (L^1 \hat{\otimes} B)^* = (L^1(B))^*.$$

Finally let us denote by  $\tilde{\Lambda}^\infty(B) = \{T \in \mathcal{D}'(B) : \tilde{T} \in \Lambda^\infty(B)\}$  where  $\|T\|_{\tilde{\Lambda}^\infty} = \|\tilde{T}\|$ .

**THEOREM 2.1.**  $(H^1_{\text{con}}(B))^* = \Lambda^\infty(B^*) + \tilde{\Lambda}^\infty(B^*)$ .

**PROOF.** Let  $\Phi$  be an element in  $(H^1_{\text{con}}(B))^*$ . Since the application  $f \rightarrow (f, \tilde{f})$  maps  $H^1_{\text{con}}(B)$  into  $L^1(B) \oplus L^1(B)$  then the Hahn-Banach extension theorem allows us to find  $\Phi_1, \Phi_2$  belonging to  $(L^1(B))^*$  such that

(2.5)  $\quad \Phi(f) = \Phi_1(f) + \Phi_2(\tilde{f}) \quad \text{for all } f \text{ in } H^1_{\text{con}}(B).$

By the last remark we can consider  $T_1, T_2$ , in  $\Lambda^\infty(B^*)$  as the corresponding elements for  $\Phi_1, \Phi_2$ . We shall prove that  $T = T_1 - \tilde{T}_2$ .

$$\begin{aligned} \langle T_\Phi(\psi), b \rangle &= \Phi(\psi \cdot b) = \Phi_1(\psi \cdot b) + \Phi_2(\widetilde{\psi \cdot b}) \\ &= \langle T_1(\psi), b \rangle + \langle T_2(\tilde{\psi}), b \rangle = \langle (T_1 - \tilde{T}_2)(\psi), b \rangle. \end{aligned}$$

Conversely suppose we take  $T = T_1 + T_2$  with  $T_1, \tilde{T}_2$  belonging to  $\Lambda^\infty(B^*)$  whose corresponding elements in  $(L^1(B))^*$  are  $\Phi_1, \Phi_2$ .

Define  $\Phi(f) = \Phi_1(f) - \Phi_2(\tilde{f})$  for each  $f$  in  $H_{\text{con}}^1(B)$ . This is obviously linear and the continuity follows from the following easy inequality

$$|\Phi(f)| \leq |\Phi_1(f)| + |\Phi_2(\tilde{f})| \leq \max(\|\Phi_1\|, \|\Phi_2\|)\|f\|_{\text{con}}.$$

Since it can be shown  $T_\Phi = T_1 + T_2$  we have finished the proof.

REMARK 1.2. Duality results for Hardy spaces of vector-valued functions have been considered by several authors (see [4, 5, 25]). The following result can be stated for special kinds of Banach spaces (see [5, 25]):

$$(H_{\text{con}}^p(B))^* = H_{\text{con}}^p(B^*) \quad \text{if and only if } B \text{ is a UMD space, } (1 < p < \infty).$$

Here we shall formulate a duality result for a general Banach space. Let us define the class of distributions in  $\mathcal{D}'(B)$  such that there exists a positive function  $g$  in  $L^p(\mathbf{T})$  verifying

$$(2.6) \quad \|T(\psi)\| \leq \langle g, |\psi| \rangle \quad \text{for all } \psi \in C^\infty(\mathbf{T}).$$

This space can be identified with the space of cone absolutely summing operators from  $L^p$  into  $B$  (see [26, p. 244]). Denoting this space by  $\Lambda^p(B)$ , we can identify the dual of  $L^p(B)$  with  $\Lambda^p(B^*)$  (see [26, p. 277]). Due to this identification a proof such as for Theorem 2.1 would allow us to show the following.

$$\text{COROLLARY 2.1. } \textit{For } 1 < p < \infty, (H_{\text{con}}^p(B))^* \Lambda^{p'}(B^*) + \tilde{\Lambda}^{p'}(B^*).$$

REMARK 2.2. Theorem 2.1 can be also used to give a sufficient condition on  $B$  to get  $H_{\text{con}}^1(B) = H_{\text{con}}^1 \hat{\otimes} B$ . By using the density of  $H_{\text{con}}^1 \hat{\otimes} B$  in  $H_{\text{con}}^1(B)$  (a fact which can be seen by showing that  $\sigma_n * f$  converges to  $f$  in  $H_{\text{con}}^1(B)$ ,  $\sigma_n$  being the Féjer kernel [21]) we have only to verify that  $(H_{\text{con}}^1 \hat{\otimes} B)^*$  coincides with  $(H_{\text{con}}^1(B))^*$ . In Theorem 2.1 we have identified the dual of the second one, but the dual of a tensor product is known to be identified with  $\mathcal{L}(H_{\text{con}}^1, B^*)$  (see [12, p. 230]), and from these two facts is very easy to find a condition to get the equality between both spaces.

Let us recall that a Banach space  $X$  is said to have the Hahn-Banach extension property if for every  $Y_1$  and  $Y_2$  Banach spaces,  $\Omega: Y_1 \rightarrow Y_2$  isometric inclusion and  $T$  in  $\mathcal{L}(Y_1, X)$  then  $T$  can be extended to  $\bar{T}$  in  $\mathcal{L}(Y_2, X)$  with  $\bar{T} \cdot \Omega = T$  and  $\|\bar{T}\| \leq \|T\|$ .

Now considering the embedding from  $H_{\text{con}}^1$  into  $L^1 \oplus L^1$  given by  $\psi \rightarrow (\psi, \tilde{\psi})$  we can easily prove the next result.

$$\text{COROLLARY 2.2. } \textit{If } B^* \text{ has the Hahn-Banach extension property then } H_{\text{con}}^1(B) = H_{\text{con}}^1 \hat{\otimes} B.$$

This result has also been pointed out in [18], and proves that the example in Remark 1.1 is not a coincidence since  $(l^1)^* = l^\infty$  has this property.

**3. Duality for  $H_{\text{at}}^1(B)$ .** In this section we shall give a representation of the dual space of  $H_{\text{at}}^1(B)$ . To do this the first thing to realize is the following inclusions:

$$(3.1) \quad L^p(B) \subset H_{\text{at}}^1(B) \subset L^1(B) \quad (\text{with continuity}), \quad 1 < p < \infty.$$

Since each space is dense in the next one, we can also write

$$(3.2) \quad \Lambda^\infty(B^*) = (L^1(B))^* \subset (H_{\text{at}}^1(B))^* \subset (L^p(B))^* = \Lambda^{p'}(B^*).$$

Instead of looking at  $\Lambda^\infty(B^*)$  and  $\Lambda^{p'}(B^*)$  as spaces of distributions, or equivalently of operators in  $\mathcal{L}(L^1, B^*)$  and  $\mathcal{L}(L^p, B^*)$ , we shall regard them as spaces of  $B^*$ -valued measures. The identification is quite obvious:

Given  $T$  in  $\mathcal{L}(L^p, B)$  we define the measure

$$(3.3) \quad G(E) = T(\chi_E) \quad \text{for all measurable sets } E.$$

The spaces  $\Lambda^p(B)$  will correspond to certain classes of  $B$ -valued measures defined as follows

DEFINITION 3.1. Let  $1 < p < \infty$ . Let  $G$  be a  $B$ -valued infinitely additive measure on  $(\mathbf{T}, \mathcal{B})$ .  $G$  is said to have bounded  $p$ -variation if

$$|G|_p = \sup \left( \sum_{E \in \Pi} \frac{\|G(E)\|^p}{m(E)^{p-1}} \right)^{1/p} < \infty$$

(where the supremum is taken over all finite partitions of  $\mathbf{T}$  and where  $\lambda/0$  is interpreted by  $0$  if  $\lambda = 0$  or by  $+\infty$  if  $\lambda > 0$ ).  $G$  is said to have  $\infty$ -bounded variation if

$$|G|_\infty = \inf \{ C : \|G(E)\| \leq Cm(E) \text{ for all } E \} < +\infty.$$

We shall denote by  $V^p(B)$  the space of measures of bounded  $p$ -variation ( $1 < p \leq \infty$ ). These spaces have been connected with spaces of operators in several ways (see [13, 2]).

Before we define the spaces of vector-valued measures we shall work with, let us introduce some other notation. Given a measurable set  $E$  and a measure  $G$ , we shall denote by  $G_E$  the  $B$ -valued measure restricted to  $(E, \mathcal{B}_E, m_E)$  where  $\mathcal{B}_E = \{A \cap E : A \in \mathcal{B}\}$  and  $m_E(A) = m(A \cap E)$ . According to the definition of 2-variation we can write

$$(3.4) \quad |G_E|_2 = \sup \left( \sum_{A \in \Pi_E} \frac{\|G(A)\|^2}{m(A)} \right)^{1/2}$$

where the sup is taken over all finite partitions  $\Pi_E$  of  $E$ .

DEFINITION 3.2. Let  $G$  be a finitely additive measure with values in  $B$ . Let  $I$  be an interval, and consider the following measure on  $(I, \mathcal{B}_I, m_I)$ :

$$(3.5) \quad G_I^* = G_I - (G(I)/m(I))m_I.$$

$G$  is said to belong to  $\mathcal{BMO}(B)$  if

$$(3.6) \quad |G|_* = \sup(M(I)^{-1/2}|G_I^*|_2 : I \text{ interval}) < \infty.$$

Observe that  $|G|_* = 0$  does not imply  $G = 0$  (since  $G(E) = bm(E)$  for a constant vector  $b$  satisfies  $|G|_* = 0$ ). Therefore we shall introduce the following norm in  $\mathcal{BMO}(B)$ :

$$(3.7) \quad \|G\|_{\text{BMO}} = |G|_* + \|G(\mathbf{T})\|.$$

An equivalent useful norm can be given by replacing  $|G|_*$  by

$$(3.8) \quad |G|'_* = \sup \left\{ \inf_{b \in B} m(I)^{-1/2} |G_I - bm_I|_2 : I \text{ interval} \right\}.$$

PROPOSITION 3.1. *Let  $G$  be a finitely additive measure, then  $|G|'_* \leq |G|_* \leq 2|G|_*$ .*

PROOF. The first inequality is completely obvious. To see the converse let us take a vector  $b$  in  $B$  and an interval  $I$ . Then we can write

$$G_I^* = G_I - bm_I - (G(I)/m(I) - b)m_I.$$

Since  $V^2(B)$  is a normed space we have

$$|G_I^*|_2 \leq |G_I - bm_I|_2 + |(G(I)/m(I) - b)m_I|_2.$$

Taking a look at (3.4) and considering the particular partition given only by  $I$  we get

$$\|G(I) - bm(I)\| \leq |G_I - bm_I|_2 m(I)^{1/2}.$$

Therefore the result follows from this and the fact  $|m_I|_2 = m(I)^{1/2}$ .

PROPOSITION 3.2.  $V^\infty(B) \subset \mathcal{BMO}(B) \subset V^2(B)$  (with continuity).

PROOF. Let  $G$  be a measure in  $V^\infty(B)$ . Then  $\|G_I^*(A)\| \leq 2|G|_\infty m(A)$ . This easily implies that  $|G|_* \leq 2|G|_\infty$ . Assume now that  $G$  belongs to  $\mathcal{BMO}(B)$ , and take  $I = \mathbf{T}$ . Then we have  $G_I^* = G - G(\mathbf{T})m$  and  $|G|_2 \leq |G^*|_2 + \|G(\mathbf{T})\| \leq \|G\|_{\text{BMO}}$ .

Because of this last proposition and the good properties of measures in  $V^2(B)$  (see [13]) we can write

COROLLARY 3.1. *If  $G$  belongs to  $\mathcal{BMO}(B)$  then  $G$  is countable additive,  $m$ -continuous and with bounded variation.*

The following result connects the space of functions in  $\text{BMO}(B)$ , whose definition is like the classical one replacing the absolute value by the norm (see [17, 19 and 11] for the definitions and previous properties) with the space of measures.

PROPOSITION 3.3. *Let  $f$  belong to  $L^1(B)$  and consider  $G(E) = \int_E f(t) dm(t)$ .  $G \in \mathcal{BMO}(B)$  if and only if  $f \in \text{BMO}(B)$ .*

PROOF. This is simply based on the following result for measures in  $V^2(B)$ :  $G(E) = \int_E g(t) dm(t) \in V^2(B)$  then  $|G|_2 = (\int \|g(t)\|^2 dm(t))^{1/2}$  (see [2, 13]). Since

$$G_I^*(E) = \int_E (f(t) - f_I) dm(t) \quad \text{for all } E \text{ in } \mathcal{B}_I$$

where  $f_I = m(I)^{-1} \int_I f(t) dm(t)$ , then obviously  $G$  belonging to  $\mathcal{BMO}(B)$ , together with John-Nirenberg's lemma, is equivalent to  $f$  belonging to  $\text{BMO}(B)$ .

The last proposition implies that  $\text{BMO}(B) \subset \mathcal{BMO}(B)$  being this an isometric inclusion. The next one characterizes when both spaces coincide.

PROPOSITION 3.4.  $\text{BMO}(B) = \mathcal{BMO}(B)$  if and only if  $B$  has the Radon-Nikodym property.

PROOF. A formulation of RNP is that  $V^\infty(B) = L^\infty(B)$  (see [12, p. 63]), so assuming  $\text{BMO}(B) = \mathcal{BMO}(B)$  and taking  $G$  in  $V^\infty(B)$  we deduce that we can represent  $G$  by a function  $f$  in  $L^1(B)$ . Now a standard argument (see [12, p. 62]) shows that in fact  $f$  belongs to  $L^\infty(B)$ . Conversely when we assume  $B$  has RNP and we take  $G$  in  $\mathcal{BMO}(B)$ , then Corollary 3.1 and Proposition 3.3 finish the proof.

LEMMA 3.1. *Let  $G$  be a measure in  $\mathcal{BMO}(B)$ . Then for each positive interger  $n$  there is a simple function  $f_n$  such that*

$$(3.9) \quad \|f_n\|_{\text{BMO}} \leq 4\|G\|_{\text{BMO}}.$$

(3.10) *Denoting by  $G_n(E) = \int_E f_n(t) dm(t)$ , then  $G_n \in V^\infty(B)$  and for all measurable sets  $E$ ,  $G_n(E)$  converges to  $G(E)$  as  $n$  goes to  $\infty$ .*

PROOF. Denote by  $I_{n,k}$  the dyadic interval  $[k2^{-n}, (k + 1)2^{-n})$  and write  $x_k = G(I_{n,k})/m(I_{n,k})$ . Let us define the following simple function

$$(3.11) \quad f_n = \sum_{k=0}^{2^n-1} x_k \chi_{I_{n,k}}.$$

We shall check

$$|f_n|'_* = \sup \left\{ \inf_{b \in B} \left( m(I)^{-1} \int_I \|f_n(t) - b\|^2 dt \right)^{1/2} : I \text{ interval} \right\}$$

by considering two kinds of intervals. Let us start by taking an interval  $I$  with  $m(I) \leq 2^{-n}$ . Here we have two cases to take into account: The interval  $I$  is either contained in some  $I_{n,k}$  or it intersects two consecutive intervals  $I_{n,k}$  and  $I_{n,k+1}$  which we denote by  $I'$  and  $I''$ . By taking  $b_I = (f_n)_I$ , that is the average of  $f_n$  over the interval  $I$ , we shall have

$$(3.12) \quad m(I)^{-1} \int_I \|f_n(t) - b_I\|^2 dt = 0$$

in the first case, and therefore

$$\inf_{b \in B} (m(I)^{-1} \int_I \|f_n(t) - b\|^2 dt)^{1/2} = 0.$$

In the second case we shall have

$$(f_n)_I = x_{I'} \frac{m(I \cap I')}{m(I)} + x_{I''} \frac{m(I \cap I'')}{m(I)}$$

and then

$$\begin{aligned} & m(I)^{-1} \int_I \|f_n(t) - b_I\|^2 dt \\ &= m(I)^{-1} (\|x_{I'} - b_I\|^2 m(I \cap I') + \|x_{I''} - b_I\|^2 m(I \cap I'')). \end{aligned}$$

A very easy computation shows that

$$\begin{aligned} \|x_{I'} - b_I\| &= \|x_{I'} - x_{I''}\| m(I \cap I'') m(I)^{-1}, \\ \|x_{I''} - b_I\| &= \|x_{I''} - x_{I'}\| m(I \cap I') m(I)^{-1}. \end{aligned}$$

To compute  $\|x_{I'} - x_{I''}\|$  we shall use the triangle inequality and we compute  $\|x_{I'} - x_{I' \cup I''}\| + \|x_{I''} - x_{I' \cup I''}\|$ . Since  $I'$  and  $I''$  are consecutive intervals and  $I$  is contained in  $I' \cup I''$ , being  $2m(I') = 2m(I'') = m(I' \cup I'')$ , we can easily show

$$\|x_{I'} - x_{I' \cup I''}\| \leq 2|G|_* \quad \text{and} \quad \|x_{I''} - x_{I' \cup I''}\| \leq 2|G|_*.$$

Putting it all together we have

$$(3.13) \quad \left( m(I)^{-1} \int_I \|f_n(t) - b_I\|^2 dt \right)^{1/2} \leq 4|G|_*,$$

which shows that for an interval with  $m(I) \leq 2^{-n}$  we are done.

Let us consider now an interval  $I$  with  $m(I) > 2^{-n}$ , denoting by  $\bar{I}$  the union of dyadic intervals such that  $I \subset \bar{I} = \{I_{n,k}, k \in F(I)\}$  and  $I \leq m(\bar{I})/m(I) \leq 3$ . In this case we shall choose  $b_I = G(\bar{I})/m(\bar{I})$ .

$$\begin{aligned} m(I)^{-1} \int_I \|f_n(t) - b_I\|^2 dt &\leq 3m(\bar{I})^{-1} \int_{\bar{I}} \|f_n(t) - b_I\|^2 dt \\ &= 3m(\bar{I})^{-1} \sum_{k \in F(I)} \|x_k - b_I\|^2 m(I_{n,k}) \\ &\leq 3m(\bar{I})^{-1} |G_{\bar{I}} - b_I m_{\bar{I}}|_2^2 \leq 3|G|_*^2. \end{aligned}$$

Now joining this last inequality with (3.13) we can get easily (3.9). To obtain (3.10) it suffices to notice that the Borel  $\sigma$ -algebra  $\mathcal{B}$  is generated by  $\bigcup_n \mathcal{B}_n$  where  $\mathcal{B}_n$  is the algebra given by the dyadic intervals of length  $2^{-n}$  and to observe that when we restrict to sets in  $\mathcal{B}_n$  we have  $G_n(E) = G(E)$ . Finally we have that the fact that  $G_n$  belongs to  $V^\infty(B)$  follows from  $f_n$  being a function in  $L^\infty(B)$ .

Before we state the duality theorem, let us look at the space  $H_{\text{at}}^1(B)$  in a more convenient way for our purposes.

DEFINITION 3.3. A function  $a$  in  $L^2(B)$  is said to be a  $(B, 2)$ -atom if there exists an interval  $I$  such that

- (1)  $\text{supp } a \subset I$ ,
- (2)  $\int_I a(t) dm(t) = 0$ ,
- (3)  $\int_I \|a(t)\|^2 dm(t) \leq m(I)^{-1}$ .

We also consider  $a(t) = b$  for some  $b$  in  $B$  with  $\|b\| = 1$  as a  $(B, 2)$ -atom. A proof such as that given in [11] allows us to consider the space  $H_{\text{at}}^1(B)$  defined in terms of these atoms. So we consider

$$H_{\text{at}}^1(B) = \left\{ f \in L^1(B) : f = \sum \lambda_k a_k, a_k \text{ are } (B, 2)\text{-atoms and } \sum |\lambda_k| < \infty \right\}.$$

The norm on it is given by

$$(3.14) \quad \|f\|_{\text{at}} = \inf \left\{ \sum |\lambda_k| : f = \sum \lambda_k a_k \right\}.$$

THEOREM 3.1.  $(H_{\text{at}}^1(B))^* = \mathcal{BMO}(B^*)$ .

PROOF. First let us take  $G$  belonging to  $\mathcal{BMO}(B^*)$ . We shall define an operator acting on  $B$ -valued simple functions and we shall prove that is bounded as an operator acting from the space of simple functions with norm given by  $H_{\text{at}}^1(B)$  into  $\mathbf{R}$ , and then we shall extend it to an element in  $(H_{\text{at}}^1(B))^*$ .

Define the following operator

$$(3.15) \quad T_G \left( \sum_{k=1}^n a_k \chi_{E_k} \right) = \sum_{k=1}^n \langle G(E_k), a_k \rangle.$$

We shall show that

$$(3.16) \quad |T_G(s)| \leq |G|_{\text{BMO}} \|s\|_{\text{at}} \quad \text{for all simple functions } s.$$

First we use Lemma 3.1 to get a sequence of measures  $G_n$  in  $V^\infty(B^*)$ . Denote by  $T_n$  the operator given by (3.15) replacing  $G$  by  $G_n$ . The advantage of this one with respect to the first one is that  $T_n$  defines an element in  $(L^1(B))^*$  since  $G_n$  belongs to  $V^\infty(B^*)$ . Our aim now is to show

$$(3.17) \quad |T_n(f)| \leq C \|G_n\|_{\text{BMO}} \|f\|_{\text{at}} \quad \text{for all } f \text{ in } H_{\text{at}}^1(B).$$

Let us start with a simple atom, that is

$$s = \sum_{k=1}^N a_k \chi_{E_k}, \quad E_k \subset I, \quad \sum_{k=1}^N a_k m(E_k) = 0, \quad \sum_{k=1}^N \|a_k\|^2 m(E_k) \leq 2m(I)^{-1}.$$

Denoting  $b_I = G_n(I)/m(I)$  we can write

$$T_n(s) = \sum_{k=1}^N \langle G_n(E_k), a_k \rangle = \sum_{k=1}^N \langle G_n(E_k) - b_I m(E_k), a_k \rangle.$$

Therefore

$$\begin{aligned} |T_n(s)| &\leq \sum_{k=1}^N \|G_n(E_k) - b_I m(E_k)\|_{B^*} m(E_k)^{-1/2} m(E_k)^{1/2} \|a_k\| \\ &\leq \left( \sum_{k=1}^N \frac{\|G_n(E_k) - b_I m(E_k)\|^2}{m(E_k)} \right)^{1/2} \cdot \left( \sum_{k=1}^N \|a_k\|^2 m(E_k) \right)^{1/2} \\ &\leq 2m(I)^{-1/2} |(G_n)_I - b_I m_I|_2 \leq 2|G_n|_* . \end{aligned}$$

Given now a general  $(B, 2)$ -atom  $a$  in  $L^2(B)$  supported in  $I$  we can find a sequence of simple functions  $d_k$  converging to  $a$  in  $L^2(I, B)$ . Taking  $s_k = d_k - (\int_I d_k(t) dm(t)) \chi_I$  we have a sequence of simple atoms which also converges to  $a$  in  $L^2(I, B)$  and therefore in  $L^1(I, B)$ . Using the continuity of  $T_n$  as an element in  $(L^1(B))^*$  we can say

$$|T_n(a)| \leq 2|G_n|_* \quad \text{for nonconstant atoms } a.$$

For  $a(t) = b$  with  $\|b\| = 1$  we have  $|T_n(a)| = \|G(\mathbf{T})\|$ , and therefore

$$|T_n(a)| \leq 2\|G_n\|_{\text{BMO}} \quad \text{for all atoms } a.$$

This last inequality, together with the fact that  $T_n \in (L^1(B))^*$ , and the convergence in each representation  $f = \sum \lambda_k a_k$  is also in  $L^1(B)$ , imply (3.17). To finish the proof we invoke Lemma 3.1 which says that  $T_n(s)$  converges to  $T(s)$  for all simple functions  $s$  and  $\|G_n\|_{\text{BMO}} \leq 4\|G\|_{\text{BMO}}$ .

To prove the converse inclusion, let us take  $T$  in  $(H_{\text{at}}^1(B))^*$ , and define the following  $B^*$ -valued measure

$$(3.18) \quad \langle G(E), b \rangle = T(b \chi_E) \quad \text{for all } b \in B \text{ and } E \in \mathcal{B}.$$

Given  $n$  intervals  $I$  and a partition  $\Pi_I = \{E_1, E_2, \dots, E_n\}$  of  $I$ , we can write, according to the duality  $(l^2(B))^* = l^2(B^*)$ ,

$$\begin{aligned} &\sum \|G(E_i) - G(I)m(I)^{-1}m(E_i)\|_{B^*}^2 m(E_i)^{-1} \\ &= \sum \|G(E_i)m(E_i)^{-1/2} - G(I)m(I)^{-1}m(E_i)^{1/2}\|_{B^*}^2 \\ &= \left| \sum \langle G(E_i)m(E_i)^{-1/2} - G(I)m(I)^{-1}m(E_i)^{1/2}, b_i \rangle \right|^2 \\ &\hspace{15em} \text{for some } \sum \|b_i\|^2 = 1. \end{aligned}$$

Therefore using the definition of  $G$  we have

$$\begin{aligned} & \left| \sum (T(b_i m(E_i)^{-1/2} \chi_E) - T(b_i m(E_i)^{1/2} m(I) \chi_I)) \right| \\ &= \left| T \left( \sum b_i m(E_i)^{-1/2} \chi_E - b m(I)^{-1} \chi_I \right) \right|, \quad \text{where } b = \sum m(E_i)^{1/2} b_i. \end{aligned}$$

The reader can easily verify that

$$a = \frac{1}{2} m(I)^{-1/2} \left( \sum m(E_i)^{-1/2} b_i \chi_E - b m(I)^{-1} \chi_I \right)$$

is a  $(B, 2)$ -atom and therefore we get

$$|G_I^*|_2 \leq T(2m(I)^{1/2} a) \leq 2m(I)^{1/2} \|T\|.$$

Since also  $\|G(\mathbf{T})\|_{B^*} \leq \|T\|$ , then we have  $\|G\|_{\text{BMO}} \leq 2\|T\|$  and the proof is finished.

The following result was proved in [1] with a direct proof but now we can get it as a corollary from Theorem 3.1 and Proposition 3.4.

**COROLLARY 3.2.**  $(H_{\text{at}}^1(B))^* = \text{BMO}(B^*)$  if and only if  $B^*$  has the RNP.

**COROLLARY 3.3.** Let  $\tilde{L}^\infty(B) = \{T \in \mathcal{D}'(B) : \tilde{T} \in L^\infty(B)\}$ .

(a) If  $B^*$  has RNP then  $\text{BMO}(B^*) \subset L^\infty(B^*) + \tilde{L}^\infty(B^*)$ .

(b)  $B$  is a UMD space if and only if  $\text{BMO}(B^*) = L^\infty(B^*) + \tilde{L}^\infty(B^*)$ .

**PROOF.** (a) follows from Corollary 3.2 and Theorems 2.1 and 3.1. To see (b) it is a standard fact that if  $L^\infty(B^*) \subset \text{BMO}(B^*)$  then also  $L^2(B^*) \subset \text{BMO}(B^*)$  which implies that  $B^*$  and therefore  $B$  are UMD spaces. On the other hand, Theorem 1.2 and the duality results give us the converse.

**REMARK 3.1.** S. Y. Chang and R. Fefferman [9] considered the space  $H^1(\mathbf{T}^2)$  as the space of functions  $f$  in  $L^1(\mathbf{T}^2)$  such that

$$\sup_{0 < r < 1, 0 < s < 1} \iint P_r(t_1 - s_1) P_s(t_2 - s_2) f(s_1, s_2) ds_1 ds_2 \in L^1(\mathbf{T}^2).$$

We can relate  $H_{\text{max}}^1(H_{\text{max}}^1)$  to this space by noticing that a function in  $H_{\text{max}}^1(H_{\text{max}}^1)$  can be identified with a function  $f$  in  $L^1(\mathbf{T}^2)$  satisfying

$$P^* f(t_1) = \sup_{0 < r < 1} \left\| \int P_r(t_1 - s_1) f(s_1, s_2) ds_1 \right\|_{H_{\text{max}}^1} \in L^1(\mathbf{T}).$$

The proof of this identification uses a very nice observation due to W. Hensgen who showed that the unit ball of  $H_{\text{max}}^1$  is closed in  $L^1(\mathbf{T})$  (personal communication).

With these two identifications it is very easy to verify that

$$H_{\text{max}}^1(\mathbf{T}^2) \subset H_{\text{max}}^1(H_{\text{max}}^1).$$

Now from duality, and since BMO fails RNP, we can get that

$$\text{BMO}(\text{BMO}) \subsetneq \text{BMO}(\mathbf{T}^2),$$

where  $\text{BMO}(\mathbf{T}^2)$  is the space considered in [9] and which represents the dual of  $H_{\text{max}}^1(\mathbf{T}^2)$ .

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