ON THE DUAL OF AN EXPOSNENTIAL SOLVABLE LIE GROUP

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Abstract. Let $G$ be a connected, simply connected exponential solvable Lie group with Lie algebra $g$. The Kirillov mapping $\eta: g^*/\text{Ad}^*(G) \to \hat{G}$ gives a natural parametrization of $\hat{G}$ by co-adjoint orbits and is known to be continuous. In this paper a finite partition of $g^*/\text{Ad}^*(G)$ is defined by means of an explicit construction which gives the partition a natural total ordering, such that the minimal element is open and dense. Given $\pi \in \hat{G}$, elements in the enveloping algebra of $g_c$ are constructed whose images under $\pi$ are scalar and give crucial information about the associated orbit. This information is then used to show that the restriction of $\eta$ to each element of the above-mentioned partition is a homeomorphism.

1. Introduction. Let $G$ be a real, connected, simply connected exponential solvable Lie group with Lie algebra $g$. By a representation of $G$ we shall mean a strongly continuous, unitary representation of $G$ in some Hilbert space, and we denote the dual of $G$ by $\hat{G}$, that is, the set of unitary equivalence classes of topologically irreducible representation of $G$. Denote by $\eta$ the natural mapping of the set $g^*/\text{Ad}^*(G)$ of co-adjoint orbits in the dual $g^*$ of $g$ onto $\hat{G}$. When $g^*/\text{Ad}^*(G)$ is given the quotient topology and $\hat{G}$ the hull kernel topology, $\eta$ is continuous. It was first conjectured by A. A. Kirillov in [8] and proved by I. Brown in [3] that if $G$ is nilpotent, $\eta$ is a homeomorphism. K. Joy in a later paper [7] gives a much shorter proof of Brown's Theorem using results of J. M. G. Fell pertaining to the space $S(G)$ of subgroup representation pairs $(\pi, H)$, where $H$ is a closed connected subgroup of $G$ and $\pi$ is a unitary equivalence class of representations of $H$. Two results on the bicontinuity of $\eta$ when $G$ is exponential are due to J. Boidol [2] and H. Fujiwara [6]. Boidol shows that $\eta^{-1}$ is continuous provided that $G$ is *-regular; *-regularity is seen to fail however even for a completely solvable group of dimension four. On the other hand, Fujiwara proves the existence of a dense open subset $U$ of $\hat{G}$ such that $V = \eta^{-1}(U)$ is dense and such that the restriction of $\eta$ to $V$ is a homeomorphism. However, Fujiwara's result provides no explicit characterization of $U$. Finally, it is known that $\eta$ is a homeomorphism for all $G$ of dimension less than six. Those cases which are not *-regular are handled by constructing elements $W$ in the center of the enveloping algebra $U(g_c)$, and using the fact that the mapping $\phi_W$ on $\hat{G}$ given by $\rho(W) = \phi_W(\rho)I$ is continuous. $\phi_W \circ \eta$ can be regarded as an $\text{Ad}^*(G)$-invariant polynomial function of $g^*$, and as such provides enough information to conclude convergence of the corresponding orbits. In the general case the center of $U(g_c)$ is not large enough to yield sufficient information about $\eta^{-1}$.

Now let $n$ be the nilradical of $g$, and let $\rho \in \hat{G}$ such that $\rho$ is extended from $N = \exp(n)$. A generalization of the construction mentioned above is given whereby

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elements \( w_i \in U(\mathfrak{g}_c) \) are constructed such that \( \{ \rho(W_i) \} \) are scalar operators whose values allow one to systematically obtain \( \eta^{-1}(\rho) \) from the orbit of \( \rho|_N \). The Kirillov mapping has a natural generalization in the context of the space of subgroup representation pairs \( (\rho, H) \) such that \( H \supseteq N \) and \( \rho \in \hat{H} \), and a theorem regarding this mapping is proved which has as a corollary the following. There is a finite partition \( \{ U_\alpha \} \) of \( \mathfrak{g}^* / \text{Ad}^*(G) \)—obtained by an explicit construction depending only on a choice of Jordan-Hölder sequence for \( \eta \)—on each element of which \( \eta \) is open.

2. Preliminaries. Let \( \mathfrak{g} \) be a real, solvable Lie algebra of exponential type. For any subspace \( \mathfrak{h} \) of \( \mathfrak{g} \), let \( \mathfrak{h}^* \) denote the dual space of \( \mathfrak{h} \), and if \( j \) is a subspace of \( \mathfrak{g} \) such that \( [\mathfrak{h}, \mathfrak{h}] \subseteq j \) and \( \lambda \in j^* \), denoted by \( B_\lambda \) the bilinear form defined on \( \mathfrak{h} \) by \( B_\lambda(X, Y) = \lambda([X, Y]) \), \( X, Y \in \mathfrak{h} \). For any subset \( s \) of \( \mathfrak{h} \), denote by \( s^{\lambda, \mathfrak{h}} \) the orthogonal complement of \( s \) in \( \mathfrak{h} \) with respect to \( B_\lambda \). The radical \( \mathfrak{h}^{\lambda, \mathfrak{h}} \) of \( B_\lambda \) will also be denoted by \( R(\lambda, \mathfrak{h}) \).

Let \( \{ \mathfrak{h}_n \}_{n=1}^\infty \) be a sequence of subspaces of \( \mathfrak{g} \). We shall say that \( \mathfrak{h}_n \) converges to a subspace \( \mathfrak{h} \) (or write \( \mathfrak{h}_n \to \mathfrak{h} \)) if there are positive integers \( K \) and \( d \) such that for each \( n > K \), there is a basis \( X_1^{(n)}, X_2^{(n)}, \ldots, X_d^{(n)} \) of \( \mathfrak{h}_n \) and a basis \( X_1, X_2, \ldots, X_d \) of \( \mathfrak{h} \) with \( X_j = \lim_n X_j^{(n)} \), \( 1 \leq j \leq d \). Suppose that \( \mathfrak{h}_n \to \mathfrak{h} \), and let \( W_n \in \mathfrak{h}_n \), \( n \geq 1 \), such that for some \( W \in \mathfrak{g} \), \( W = \lim_n W_n \). Then \( W \in \mathfrak{h} \), and it follows that if for some \( \mathfrak{h}', \mathfrak{h}_n \to \mathfrak{h}' \), then \( \mathfrak{h}' = \mathfrak{h} \), and if \( \mathfrak{h}_n \) is a subalgebra (ideal) for infinitely many \( n \), then \( \mathfrak{h} \) is a subalgebra (ideal). Clearly every sequence \( \{ \mathfrak{h}_n \} \) of nontrivial subspaces of \( \mathfrak{g} \) has a subsequence which converges, and it is easily seen that \( \mathfrak{h}_n \to \mathfrak{h} \) if and only if every convergent subsequence of \( \{ \mathfrak{h}_n \} \) converges to \( \mathfrak{h} \).

**Lemma 2.1.** Let \( \{ j_n \}_{n=1}^\infty \) be a sequence of subspaces of \( \mathfrak{g} \) such that for each \( n \), \( j_n \subseteq \mathfrak{h}_n \), and suppose that \( j_n \to j \) and \( \mathfrak{h}_n \to \mathfrak{h} \). Let \( \{ \lambda_n \}_{n=1}^\infty \) be a sequence in \( \mathfrak{g}^* \) such that for some \( \lambda \in \mathfrak{g}^* \), \( \lambda||_{[\mathfrak{g}, \mathfrak{g}]} = \lim_n \lambda_n||_{[\mathfrak{g}, \mathfrak{g}]} \), and \( \dim R(j^\lambda) = \lim \inf_n \dim R(j_n^\lambda, h_n) \). Then \( j_n^\lambda, h_n \to j^\lambda, h \).

**Proof.** Let \( K \) and \( d \) be positive integers such that for each \( n > K \), there is a basis \( Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_d^{(n)} \) of \( j_n \) with \( \lim_n Y_j^{(n)} = Y_j \), \( 1 \leq j \leq d \), and \( \{ Y_j \} \) a basis of \( j \). Let \( \{ j_k^{\lambda, h_k} \} \) be any convergent subsequence, \( j_k^{\lambda, h_k} \to j_0 \), and let \( W \in j_0 \). Then for each \( k \), there is \( W(k) \in j_k^{\lambda, h_k} \) such that \( W = \lim_k W(k) \), and we have \( \lambda([W, Y_j]) = \lim_k \lambda([W(k), Y_j^{(k)}]) = 0 \), \( 1 \leq j \leq d \). Thus \( j_0 \subseteq j_{\lambda, \mathfrak{h}} \). On the other hand,

\[
\dim R(j_0) \geq \lim \inf_n \dim R(j_n^\lambda, h_n)
\]

so that \( j_0 = j_{\lambda, \mathfrak{h}} \), and hence \( j_n^\lambda, h_n \to j_{\lambda, \mathfrak{h}} \). \( \Box \)

Let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{g} \). We denote by \( U(\mathfrak{h}) \) the enveloping algebra of \( \mathfrak{h} \) and regard \( U(\mathfrak{h}) \) as a subalgebra of \( U(\mathfrak{g}) \). We denote the complexification \( \mathfrak{h} \otimes_\mathbb{R} \mathbb{C} \) by \( \mathfrak{h}_c \) and regard \( U(\mathfrak{h}_c) \) as a subalgebra of \( U(\mathfrak{g}_c) \).

Let \( G \) be a connected, simply connected Lie group with Lie algebra \( \mathfrak{g} \), and let \( H \) be the closed, connected subgroup of \( G \) with Lie algebra \( \mathfrak{h} \). Denote by \( \eta_H \) Kirillov mapping \( \mathfrak{h}^* / \text{Ad}^*(H) \to \hat{H} \), let \( \pi \) be a representation of \( H \), and let \( \lambda \in \mathfrak{h}^* \). We shall say that \( \pi \) corresponds to \( \lambda \) if \( \pi \in \eta_H(\text{Ad}^*(H)\lambda) \). If \( \mathfrak{p} \in \mathfrak{h}^* \) is a polarization at \( \lambda \), we occasionally use the notation \( \text{ind}(\lambda, \mathfrak{p}) \) for the irreducible representation.
ind(\chi_\lambda, P, H) of \ H \ induced \ by \ the \ character \ \chi_\lambda \ of \ P = \exp(p) \ with \ differential 
\ \ \ i(\lambda|_p).

Now let \ \lambda \in \mathfrak{g}^*, \ and \ let \ \mathfrak{m} \ be \ a \ nilpotent \ subalgebra \ of \ \mathfrak{g}.

**DEFINITION 2.2.** A pair \ (m_1, m_0) \ of \ \mathfrak{m}-ideals \ such \ that \ m_0 \subset m_1, \ dim \mathbb{R}(m_1/m_0) \ \ \ \ = 1, \ m_0 \subset R(\lambda, \mathfrak{m}) \ and \ m_1 \not\subset \ R(\lambda, \mathfrak{m}) \ will \ be \ called \ a \ Kirillov \ pair \ in \ \mathfrak{m} \ at \ \lambda.

Let \ (m_1, m_0) \ be \ a \ Kirillov \ pair \ in \ \mathfrak{m} \ at \ \lambda, \ and \ let \ \mathfrak{l} = m_1^\lambda \cdot m. \ Then \ \mathfrak{l} \ is \ a \ codimension \ 1 \ subalgebra \ of \ \mathfrak{m}. \ Let \ \pi_1 \ be \ an \ irreducible \ representation \ of \ L = \exp(\mathfrak{l}) \ corresponding \ to \ \lambda|_\mathfrak{l}, \ and \ let \ \mathfrak{X} \in \mathfrak{m} \sim \mathfrak{l}. \ Then \ the \ representation \ \pi = \pi(\pi_1, \mathfrak{X}) \ defined \ in \ L^2(\mathbf{R}, H(\pi_1)) \ by \ the \ formula 

\begin{equation}
(\pi(y \exp sX)f)(t) = \pi_1(\exp \mathfrak{X}y \exp -tX)f(t + s) \quad (y \in L, \ s, t \in \mathbb{R})
\end{equation}
corresponds \ to \ \lambda|_\mathfrak{m}. \ The \ primary \ representation \ \tilde{\pi}_1 \ defined \ in \ L^2(\mathbf{R}, H(\pi_1)) \ by \ \tilde{\pi}_1(y)f(t) = \pi_1(y)f(t), \ y \in L, \ can \ be \ differentiated \ in \ the \ space \ \mathcal{C}^\infty(\pi) \ of \ smooth \ vectors \ for \ \pi, \ that \ is, \ \mathcal{C}^\infty(\tilde{\pi}_1) \supset \mathcal{C}^\infty(\pi). \ The \ following \ lemma \ is \ more \ or \ less \ well \ known, \ but \ crucial \ in \ this \ paper.

**LEMMA 2.3.** There is an explicit construction by which, given \ \textit{any} \ \mathfrak{W} \in U(\mathfrak{l}), \ one \ obtains \ \tilde{\mathfrak{W}} \in U(\mathfrak{m}) \ such \ that \ \pi(\tilde{\mathfrak{W}}) = \tilde{\pi}_1(\mathfrak{W}).

**PROOF.** Let \ \mathfrak{m} \ be \ a \ positive \ integer \ such \ that \ ad \mathfrak{X}^{m+1} = 0, \ and \ let \ \mathfrak{W} \in U(\mathfrak{m}). \ We \ construct \ an \ element \ \mathfrak{W}_m \in U(\mathfrak{m}) \ as \ follows. \ Let \ \mathfrak{t} \ denote \ the \ operator \ on \ \mathcal{C}^\infty(\mathfrak{m}) \ defined \ by \ \phi(t) \rightarrow t\phi(t). \ We \ have \ \pi(\mathfrak{W}) = \sum_{j=0}^{m} (\mathfrak{t}^j/\mathfrak{j}!) \tilde{\pi}_1(ad \mathfrak{X}^j) \ so \ that \ \pi(ad \mathfrak{X}^m) = \tilde{\pi}_1(ad \mathfrak{X}^m). \ Let \ \mathfrak{Y} \ be \ the \ element \ in \ \mathfrak{m} \sim \mathfrak{m}_0 \ such \ that \ \lambda(\mathfrak{Y}) = 0, \ and \ \mathfrak{B}_\lambda(\mathfrak{X}, \mathfrak{Y}) = 1, \ so \ that \ \pi(\mathfrak{Y}) = \mathfrak{t}. \ Define \ \mathfrak{W}_1 \in U(\mathfrak{m}) \ by 

\[ \mathfrak{W}_1 = \mathfrak{W} - \frac{ad \mathfrak{X}^m \cdot (-i\mathfrak{Y})^m}{m!}. \]

Then \ \pi(\mathfrak{W}_1) = \sum_{j=0}^{m-1} (\mathfrak{t}^j/\mathfrak{j}!) \tilde{\pi}_1(ad \mathfrak{X}^j) \ and \ \pi(ad \mathfrak{X}^{m-1}) \mathfrak{W}_1) = \tilde{\pi}_1(ad \mathfrak{X}^{m-1}). \ If \ m > 1, \ set 

\[ \mathfrak{W}_2 = \mathfrak{W}_1 - \frac{ad \mathfrak{X}^{m-1} \mathfrak{W}_1 \cdot (-i\mathfrak{Y})^{m-1}}{(m-1)!} \]

and we find that \ \pi(ad \mathfrak{X}^{m-2} \mathfrak{W}_2) = \tilde{\pi}_1(ad \mathfrak{X}^{m-2}). \ Continue \ in \ this \ way \ until \ \mathfrak{W}_m = \tilde{\mathfrak{W}} \ is \ obtained. \ Q.E.D.

**3. A partition of the dual of a nilpotent Lie group.** Now let us assume that \ \mathfrak{g} \ is \ nilpotent; fix \ \lambda \in \mathfrak{g}^* \. By induction on the dimension of \ \mathfrak{g} \ it is easily seen that there is a sequence of subalgebras \ \mathfrak{g} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_d \ satisfying \ the \ conditions 

(i) \ \mathfrak{m}_d \ is \ a \ polarization \ at \ \lambda.

(ii) \ If \ \mathcal{R}(\lambda, \mathfrak{g}) \neq \mathfrak{g}, \ then \ d > 0 \ and \ for \ each \ k = 1, 2, \ldots, d - 1, \ there \ is \ a \ Kirillov \ pair \ (\mathfrak{m}_{k+1}, \mathfrak{m}_k) \ in \ \mathfrak{m}_k \ at \ \lambda \ such \ that \ \mathfrak{m}_{k+1} = \mathfrak{m}_{k1}^{\lambda_{m}}. \ Thus \ if \ \mathcal{R}(\lambda, \mathfrak{g}) \neq \mathfrak{g}, \ then \ d = \frac{1}{2} \dim(\text{Ad}^*(\mathfrak{g} (\mathfrak{G}) \lambda).

**DEFINITION 3.1.** A sequence of subalgebras satisfying conditions (i) and (ii) above \ will \ be \ called \ a \ Kirillov \ sequence \ for \ \lambda \ in \ \mathfrak{g}.

Let \ \mathfrak{d} \ be \ a \ nonnegative \ integer. \ Let \ us \ say \ that \ an \ operator \ \mathfrak{d} \ on \ \mathcal{C}^\infty(\mathfrak{R}^d) \ (\mathcal{C}^\infty(\mathfrak{R}^0) \equiv \mathbb{C}) \ is \ a \ \textit{polynomial} \ differential \ operator \ if \ there \ is \ a \ polynomial \ \mathfrak{P} \ in \ \ \ \ 2d \ indeterminants \ with \ complex \ coefficients \ such \ that 

\[ \mathfrak{d} = \mathfrak{P}(t_1, \ldots, t_d, \partial/\partial t_1, \ldots, \partial/\partial t_d). \]
A theorem of Kirillov (cf. [8, Theorem 7.1]) states that \( \eta(\text{Ad}^*(G)\lambda) \) has a realization \( \pi \) in a space of functions on \( \mathbb{R}^d \) such that the image of \( U(g_c) \) under \( \pi \) is the set of polynomial differential operators. In this section we shall determine when it is possible, given a sequence \( \{\lambda_n\}_{n=1}^\infty \) in \( g^* \) such that \( \lambda_n \to \lambda = \lambda_0 \), to obtain a corresponding sequence \( \{\pi_n\}_{n=0}^\infty \) of irreducible representations such that given any \( D \) as above, there is a sequence \( \{W_n\}_{n=0}^\infty \) in \( U(g_c)^{(m)} \) for some \( m \), with \( W_n \to W_0 \) and \( \pi_n(W_n) = D \) for each \( n \).

Let \( g = g_p \supset g_{p-1} \supset \cdots \supset g_0 = (0) \) be a Jordan-Hölder sequence for \( g \). Define subsets \( e(\lambda) \), \( j(\lambda) \) and \( i(\lambda) \) of \( \{1, 2, \ldots, p\} \) as follows. Set

\[
e(\lambda) = \{t | g_t + R(\lambda, g) \supsetneq g_{t-1} + R(\lambda, g)\}
\]

and let \( p(\lambda) = \sum_t R(\lambda, g_t) \). Define \( j(\lambda) \subset e(\lambda) \) by

\[
j(\lambda) = \{t | g_t + p(\lambda) \supsetneq g_{t-1} + p(\lambda)\}
\]

and let \( i(\lambda) = e(\lambda) \setminus j(\lambda) \). Then \( \text{card}(e(\lambda)) = \dim(\text{Ad}^*(G)\lambda) \) and it is shown in [1] that \( p(\lambda) \) is a polarization at \( \lambda \), hence \( \text{card}(j(\lambda)) = \frac{1}{2} \text{card}(e(\lambda)) \). If \( e(\lambda) \neq \{\phi\} \), we shall write \( e(\lambda) = \{e_1 < e_2 < \cdots < e_{2d}\} \). We define a sequence of subalgebras \( g = g^0(\lambda) \supset g^1(\lambda) \supset \cdots \supset g^d(\lambda) \) as follows. Setting \( g^0(\lambda) = g \), assume that for some \( k \geq 0 \), \( g^k(\lambda) \) is defined and \( g^k(\lambda) \neq R(\lambda, g^k(\lambda)) \). Let \( i_{k+1} \) be the smallest index such that \( g_{i_{k+1}} \cap g^k(\lambda) \not\subset R(\lambda, g^k(\lambda)) \) and set

\[
g^{k+1}(\lambda) = \left(g_{i_{k+1}} \cap g^k(\lambda)\right)\lambda \cdot g^k(\lambda).
\]

Note that \( g^{k+1}(\lambda) \) is codimension 1 in \( g^k(\lambda) \). If \( g^k(\lambda) = R(\lambda, g^k(\lambda)) \), then let the sequence terminate at \( g^k(\lambda) \), and set \( k = d \). Thus \( g^k(\lambda) \) is isotropic with respect to \( B_\lambda \). If \( e(\lambda) \neq \{\phi\} \), then in this way we obtain a sequence of indices \( i_1, i_2, \ldots, i_d \). Note that if \( R(\lambda, g) \subset g^k(\lambda) \), then \( R(\lambda, g) \subset g^{k+1}(\lambda) \), \( 0 \leq k \leq d \); thus we have

\[
R(\lambda, g) \subset R(\lambda, g^k(\lambda)) \subset g^k(\lambda), \quad 0 \leq k \leq d.
\]

Now for each \( k = 1, 2, \ldots, d \), let \( j_k \) be the smallest index such that \( g_{j_k} \cap g^{k-1}(\lambda) \not\subset g^k(\lambda) \).

**Lemma 3.2.** For each \( k = 1, 2, 3, \ldots, d \), \( i_k \in e(\lambda) \) and \( j_k \in e(\lambda) \). If \( k < d \), \( i_k < i_{k+1} \), and for \( k \leq d \), \( i_k < j_k \). Moreover \( g^d(\lambda) = p(\lambda) \), \( i(\lambda) = \{i_k\}_{k=1}^d \), and \( j(\lambda) = \{j_k\}_{k=1}^d \).

**Proof.** If \( i_k \notin e(\lambda) \), there is \( Y \in R(\lambda, g) \) such that \( g_{i_k} = RY + g_{i_k-1} \). But since \( R(\lambda, g) \subset R(\lambda, g^{k-1}(\lambda)) \),

\[
g_{i_k} \cap g^{k-1}(\lambda) = RY + (g_{i_k-1} \cap g^{k-1}(\lambda)) \subset R(\lambda, g^{k-1}(\lambda))
\]

a contradiction. If \( j_k \notin e(\lambda) \), let \( X \in R(\lambda, g) \) such that \( g_{j_k} = RX + g_{j_k-1} \). Since \( R(\lambda, g) \subset g^k(\lambda) \),

\[
g_{j_k} \cap g^{k-1}(\lambda) = RX + (g_{j_k-1} \cap g^{k-1}(\lambda)) \subset g^k(\lambda)
\]

a contradiction. This proves the first statement of the lemma.

Now by definition of \( i_k \),

\[
g_{i_k} \cap g^{k-1}(\lambda) = RY + g_{i_k-1} \cap g^{k-1}(\lambda) \subset RY + R(\lambda, g^{k-1}(\lambda)).
\]
Thus
\[ [\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda), \mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda)] \subset [Y, R(\lambda, \mathfrak{g}^{k-1}(\lambda))] \]
\[ + [R(\lambda, \mathfrak{g}^{k-1}(\lambda)), R(\lambda, \mathfrak{g}^{k-1}(\lambda))] \subset \text{Ker}(\lambda). \]

By definition of \( \mathfrak{g}^k(\lambda) \) and \( j_k \), it follows that \( \mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset \mathfrak{g}^k(\lambda) \), and hence that \( i_k < j_k \), and that
\[ \mathfrak{g}_{i_k} \cap \mathfrak{g}^k(\lambda) = \mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset R(\lambda, \mathfrak{g}^k(\lambda)); \]
therefore \( i_k < i_{k+1} \).

Next we show that \( \mathfrak{g}^d(\lambda) = \mathfrak{p}(\lambda) \). For this, let \( X \in R(\lambda, \mathfrak{g}_t) \), and suppose that \( X \in \mathfrak{g}^k(\lambda) \) for some \( k < d \). We show that \( X \notin \mathfrak{g}^{k+1}(\lambda) \). Suppose \( t < i_k+1 \). By choice of \( i_{k+1} \), \( X \notin \mathfrak{g}_{i_k} \cap \mathfrak{g}^{k}(\lambda) \subset R(\lambda, \mathfrak{g}^{k}(\lambda)) \subset \mathfrak{g}^{k+1}(\lambda) \). Suppose \( t \geq i_{k+1} \); then \( X \in R(\lambda, \mathfrak{g}_{i_k}) \cap \mathfrak{g}^{k}(\lambda) \subset (\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k}(\lambda))^{\lambda, \mathfrak{g}^{k}(\lambda)} \subset \mathfrak{g}^{k+1}(\lambda) \). Since \( X \in \mathfrak{g}^0(\lambda) = \mathfrak{g}^0 \), it follows that \( X \notin \mathfrak{g}^d(\lambda) \).

Let us show now that \( i(\lambda) = \{i_k\}_{k=1}^d \). Note that for each \( k \), \( R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{g}^{k+1}(\lambda) \) so that \( R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{g}^{k+1}(\lambda) \) and hence \( R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{p}(\lambda) \). Thus \( \mathfrak{g}_{i_k} \subset \mathfrak{g}_{i_k-1} + \mathfrak{p}(\lambda) \) and \( i_k \notin j(\lambda) \). It follows that \( i(\lambda) = \{i_k\}_{k=1}^d \).

Finally, to see that \( \{j_k\}_{k=1}^d = j(\lambda) \), note that \( j_k \in j(\lambda) \), since if not, then \( \mathfrak{g}_{j_k} \subset \mathfrak{g}_{j_k-1} + \mathfrak{g}^k(\lambda) \), hence \( \mathfrak{g}_{j_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset (\mathfrak{g}_{j_k-1} + \mathfrak{g}^k(\lambda)) \cap \mathfrak{g}^{k-1}(\lambda) = \mathfrak{g}_{j_k-1} \cap \mathfrak{g}^{k-1}(\lambda) + \mathfrak{g}^k(\lambda) = \mathfrak{g}^k(\lambda) \) a contradiction.

Let \( j \in j(\lambda) \), and let \( k_0 \) be the smallest \( k \), \( 1 \leq k \leq d \), such that \( \mathfrak{g}_j \subset \mathfrak{g}_{j-1} + \mathfrak{g}^{k_0}(\lambda) \). We claim that \( j = j_{k_0} \). Now \( \mathfrak{g}_j \subset \mathfrak{g}_{j-1} + \mathfrak{g}^{k_0}(\lambda) \), so we may write \( \mathfrak{g}_j = \mathbb{R} X + \mathfrak{g}_{j-1} \) where \( X \in \mathfrak{g}^{k_0-1}(\lambda) \), and by choice of \( k_0 \), \( X \in \mathfrak{g}^{k_0}(\lambda) \). Hence \( \mathfrak{g}_j \cap \mathfrak{g}^{k_0-1}(\lambda) \subset \mathfrak{g}^{k_0}(\lambda) \) and \( j \geq j_{k_0} \), by choice of \( j \geq j_{k_0} \). If \( j > j_{k_0} \), then choose \( \tilde{X} \in \mathfrak{g}_{j_0} \) such that \( \mathfrak{g}^{k_0-1}(\lambda) = \mathbb{R} \tilde{X} + \mathfrak{g}^{k_0}(\lambda) \). Since \( \dim(\mathfrak{g}^{k_0-1}(\lambda)/\mathfrak{g}^{k_0}(\lambda)) = 1 \), there are elements \( a \neq 0 \), \( b \neq 0 \) in such that \( W = a \tilde{X} + b X \in \mathfrak{g}^{k_0}(\lambda) \). But \( \mathfrak{g}_j = \mathbb{R} W + \mathfrak{g}_{j-1} \subset \mathfrak{g}_{j-1} + \mathfrak{g}^{k_0}(\lambda) \), contradicting our choice of \( k_0 \); therefore \( j = j_{k_0} \), and the proof of the lemma is finished. □

Now let \( E \) denote the set of pairs
\[ E = \{(e(\lambda), j(\lambda))|\lambda \in \mathfrak{g}^*\} \]
and for \( d \) a positive integer, let
\[ E_d = \{(e, j) \in E| \text{card}(j) = d\}. \]
Let us regard elements of \( E_d \) as ordered \( 3d \)-tuples of integers
\[ (e, j) = (e_1, e_2, \ldots, e_{2d}, j_1, j_2, \ldots, j_d) \]
where \( e_1 < e_2 < \ldots \), and \( \{j_1, \ldots, j_d\} = j \) is indexed by the inductive process above. We define a total order on \( E \) in the following way. Let \( (\phi, \phi) \) be the maximal element of \( E \), and regarding \( E_d \) as above, let \( E_d \) have the natural lexicographic ordering. If \( d > d' \), let us say that for any \( \alpha \in E_d \), \( \alpha' \in E_{d'} \), \( \alpha < \alpha' \).

Now for each \( \alpha \in E \), set \( \Omega_{\alpha} = \{\lambda \in \mathfrak{g}^*|(e(\lambda), j(\lambda)) = \alpha\} \) and for each \( e_0 = e(\lambda_0) \),
\[ \Omega_{e_0} = \{\lambda \in \mathfrak{g}^*|e(\lambda) = e_0\} \]
so that \( \Omega_{e_0} = \bigcup \Omega_{(e,j)} | e = e_0 \). If \( s \in G \), and \( \lambda \in \mathfrak{g}^* \), then \( \mathfrak{g}^k(\text{Ad}^{s}(\lambda)) = \text{Ad}(s)(\mathfrak{g}^k(\lambda)) \) and it follows that each \( \alpha \in E \), \( \Omega_{\alpha} \) is \( G \)-invariant. The sets \( \Omega_{e_0} \) were first considered by Pukanszky in [10], and the sets \( \Omega_{\alpha} \) are considered by N. V. Pedersen in a paper to appear.
Let \( \{Z_1, Z_2, \ldots, Z_p\} \) be a basis compatible with the Jordan-Hölder sequence chosen at the beginning of this section. Let \( e = e(\lambda_0) \) for some \( \lambda_0 \), let \( P^{ij}_e(\lambda) = \lambda([Z_{e_i}, Z_{e_j}]) \), \( e_i, e_j \in e \), and set \( P_e(\lambda) = \det((P^{ij}_e(\lambda))) \). Letting the set \( \{e(\lambda) | \lambda \in \mathfrak{g}^*\} \) have the total ordering inherited from \( E \), it is shown in [9] that

\[
\Omega_e = \{ \lambda \in \mathfrak{g}^* | P_{e'}(\lambda) = 0, e' < e \text{ and } P_e(\lambda) \neq 0 \}.
\]

Now for each \( e = e(\lambda) \), let \( J_e = \{ j | (e, j) \in E \} \), and let \( J_e \) have the total ordering inherited from \( E \).

**Proposition 3.3.** There are polynomials \( P_{(e,j)} \), \( j \in J_e \), such that for each \( j \in J_e \),

\[
\Omega_{(e,j)} = \{ \lambda \in \Omega_e | P_{(e,j')}(\lambda) = 0, j' < j \text{ and } P_{(e,j')}(\lambda) \neq 0 \}.
\]

**Proof.** Let \( j \in J_e \), and write \( j = \{ j_1, \ldots, j_d \} \) and \( i = e - j = \{ i_1, \ldots, i_d \} \) as in the inductive process above. Let \( \lambda \in \Omega_e \) and for each \( k = 1, 2, \ldots, d \) set \( e^{(k)} = e - \{ i_1, \ldots, i_k, j_1, \ldots, j_k \} \) and define elements \( Z_k(\lambda) \in g \), \( t \in e^{(k)} \) as follows. Note that \( \{ Z_t \}_{t \in e} \) is a basis for \( g \) modulo \( R(A, \mathfrak{g}) \). Let \( Z_1^i(\lambda) = Z_t \) if \( t \in e^{(1)} \), \( t < j_1 \) and for \( t > j_1 \), set

\[
Z^i_1(\lambda) = B_1(Z_{i_1}, Z_{i_1})Z_{i_1} - B_1(Z_{j_1}, Z_{j_1})Z_{j_1}.
\]

Suppose that \( \lambda \in \Omega_{(e,j')} \) with \( j' \geq j \) and write \( j' = \{ j'_1, \ldots, j'_d \} \), \( i = e - j' = \{ i'_1, \ldots, i'_d \} \). Since \( e_1 = i_1 = i'_1 \), clearly \( j'_1 = j_1 \) if and only if \( B_1(Z_{j_1}, Z_{i_1}) \neq 0 \), and in this case, \( \{ Z_t^1(\lambda) \}_{t \in e^{(1)}} \) is a basis of \( g^1(\lambda) \) modulo \( R(\lambda, g^1(\lambda)) \). Therefore, by definition of \( j_1, j_2 \), we have \( j'_1 = j_1 \) and \( j'_2 = j_2 \) if and only if \( B_1(Z_{j_1}, Z_{i_1}) \neq 0 \) and \( B_2(Z_{j_2}(\lambda), Z_{i_2}(\lambda)) \neq 0 \). Now define \( Z_t^2(\lambda) \), \( t \in e^{(2)} \) in the same way as \( \{ Z_t^1(\lambda) \}_{t \in e^{(1)}} \), so that if \( j'_1 = j_1, j'_2 = j_2 \), then \( \{ Z_t^2(\lambda) \}_{t \in e^{(2)}} \) is a basis of \( g^2(\lambda) \) modulo \( R(\lambda, g^2(\lambda)) \). Continuing in this way, set

\[
P_{e,j}(\lambda) = B_1(Z_{j_1}, Z_{i_1})B_1(Z_{j_2}(\lambda), Z_{i_2}(\lambda)) \cdots B_1(Z_{j_d-1}(\lambda), Z_{i_d-1}(\lambda))
\]

and the proposition follows. Q.E.D.

**Corollary 3.4.** Let \( \alpha_0 \) be the minimal element of \( E \). Then \( \Omega_{\alpha_0} \) is Zariski open in \( \mathfrak{g}^* \). Moreover, for each \( \alpha \in E \), \( \Omega_\alpha \) is open in \( \bigcup_{\beta \geq \alpha} \Omega_\beta \).

**Proof.** That \( \Omega_{\alpha_0} \) is Zariski open is clear. As for the second statement, let \( \alpha \in E \) with \( \alpha = (e, j) \). \( \Omega_\alpha \) is Zariski open in \( \bigcup_{j' \geq j} \Omega_{(e, j')} \) by Proposition 3.3. But \( \Omega_e \) is open in \( \bigcup_{e' \geq e} \Omega_{e'} \), and hence \( \bigcup_{j' \geq j} \Omega_{(e, j')} = \Omega_e \cap \bigcup_{\beta \geq \alpha} \Omega_\beta \) is open in \( \bigcup_{\beta \geq \alpha} \Omega_\beta \), and it follows that \( \Omega_\alpha \) is open in \( \bigcup_{\beta \geq \alpha} \Omega_\beta \). Q.E.D.

Now for each \( \lambda \in \mathfrak{g}^* \) such that \( \{ \phi \} \neq e(\lambda) = i(\lambda) \cup j(\lambda), i(\lambda) = \{ i_1, i_2, \ldots, i_d \}, j(\lambda) = \{ j_1, j_2, \ldots, j_d \} \), define for each \( 0 \leq k < d \), \( m_{k1}(\lambda) = g^k(\lambda) \cap g_{k+1} \) and \( m_{k0}(\lambda) = g^k(\lambda) \cap g_{k+1} \). Then \( \{ m_{k1}(\lambda), m_{k0}(\lambda) \} \) is a Kirillov pair in \( g^k(\lambda) \) at \( \lambda \), and \( m_{k1}g^k(\lambda) = g^{k+1}(\lambda), 0 \leq k < d \). Thus the sequence \( g^0(\lambda) \supset g^1(\lambda) \supset \cdots \supset g^d(\lambda) \) is a Kirillov sequence for \( \lambda \) in \( g \).

**Theorem 3.5.** Let \( d \) be a positive integer and let \( \alpha \in E_d \). Let \( \{ \lambda_n \}_{n=0}^\infty \) be a sequence in \( \Omega_\alpha \) which converges to \( \lambda_0 \). Then for each \( n \geq 0 \), there is an irreducible representation \( \pi_n \) corresponding to \( \lambda_n \) such that

\[
D = P(t_1, t_2, \ldots, t_d, \partial/\partial t_1, \partial/\partial t_2, \ldots, \partial/\partial t_d)
\]

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is any polynomial differential operator, then there is an integer \( m > 0 \) and a sequence \( \{W_n\}_{n=0}^\infty \) in \( U(\mathfrak{g}_c)^{(m)} \) which converges to \( W_0 \) and such that \( \pi_n(W_n) = D, \) \( n = 0, 1, 2, \ldots. \)

PROOF. Clearly we may assume that for some \( 1 \leq k \leq d, \) either \( D = t_k \) or \( D = \partial/\partial t_k. \) For each \( n \geq 0, \) we have the data \( \{g^k(\lambda_n)\}_{k=0}^d, \{m_{k1}(\lambda_n), m_{k0}(\lambda_n)\}_{k=0}^{d-1} \) as in the remarks preceding the theorem. Note that for each \( n \geq 0, 1 \leq t \leq p, \) and \( 1 \leq k \leq d, \) \( \dim(\mathfrak{g}_t \cap g^k(\lambda_n)) = \text{card}(\{j_1 | s \leq k, j_1 < t\}). \) Now by Lemma 1.1, we have that \( g^k(\lambda_n) \rightarrow g^l(\lambda_0). \) Since \( \dim_R(\mathfrak{g}_t \cap g^l(\lambda_n)) = \dim_R(\mathfrak{g}_t \cap g^l(\lambda_0)), \) \( n = 1, 2, 3, \ldots, \) it follows that \( \mathfrak{g}_t \cap g^l(\lambda_n) \rightarrow \mathfrak{g}_t \cap g^l(\lambda_0), \) and in particular, \( m_{11}(\lambda_n) \rightarrow m_{11}(\lambda_0) \) and \( m_{10}(\lambda_n) \rightarrow m_{10}(\lambda_0). \) But then Lemma 1.1 implies that \( g^2(\lambda_n) \rightarrow g^2(\lambda_0). \) Continuing in this way, we obtain for each \( k = 0, 1, \ldots, d-1, \) \( g^{k+1}(\lambda_n) \rightarrow g^{k+1}(\lambda_0), \) \( m_{k1}(\lambda_n) \rightarrow m_{k1}(\lambda_0) \) and \( m_{k0}(\lambda_n) \rightarrow m_{k0}(\lambda_0). \) Now, for each \( 0 \leq k \leq d, n \geq 0, \) we shall define an irreducible representation \( \pi_k^{(n)} \) of \( G^k(\lambda_n) = \exp G^k(\lambda_n). \) Let \( \pi_d^{(n)} \) be the character of \( G^d(\lambda_n) \) with differential \( i(\lambda_n|g_d(\lambda_n)). \) Choose \( X_d^{(0)} \in g^{d-1}(\lambda_0) \sim g^d(\lambda_0), \) and since \( g^{d-1}(\lambda_n) \rightarrow g^{d-1}(\lambda_0) \) and \( g^d(\lambda_n) \rightarrow g^d(\lambda_0), \) we can choose \( X_d^{(n)} \in g^{d-1}(\lambda_n) \sim g^d(\lambda_n), \) \( n = 1, 2, 3, \ldots, \) such that \( X_d^{(n)} \rightarrow X_d^{(0)}. \) Now for each \( n, \) define \( \pi_{d-1} = \pi(\pi_d^{(n)}, X_d^{(n)}) \) as in formula (1) above, that is, for each \( f \in L^2(\mathcal{R}, H(\pi_d^{(n)})) = L^2(\mathcal{R}), y \in G^d(\lambda_n), \) \( s, t \in \mathcal{R}, \)

\[
(\pi_{d-1}^{(n)}(y \cdot \exp_G(sX_d^{(n)}))f)(t) - \pi_{d-1}^{(n)}(\exp_G(tX_d^{(n)})) \cdot y \cdot \exp(-tX_d^{(n)}))f(t + s).
\]

We continue in this way, choosing \( X_k^{(0)} \in g^{k-1}(\lambda_0) \sim g^k(\lambda_0) \) and \( X_k^{(n)} \in g^{k-1}(\lambda_n) \sim g^k(\lambda_n), \) \( n = 1, 2, 3, \ldots, \) such that \( X_k^{(n)} \rightarrow X_k^{(0)}, \) for each \( k, \) so that

\[
H(\pi_k^{(n)}) = L^2(\mathcal{R}, H(\pi_k^{(n)})), n \geq 0.
\]

For each \( k < d, \) denote elements of \( \mathbb{R}^{d-k} \) by \((t_{k+1}, t_{k+2}, \ldots, t_d), \) set \( U_{d-1} = \text{identity mapping on } L^2(\mathcal{R}), \) and define for \( k < d-1, U_k: H(\pi_k) \rightarrow L^2(\mathbb{R}^{d-k}) \) by \( U_k(f(t_{k+1}, t_{k+2}, \ldots, t_d)) = U_{k+1}(f(t_{k+1}))(t_{k+2}, \ldots, t_d). \) Set for each \( n \geq 0, \)

\[
\pi_n = U_0(\pi_0^{(n)})U_0^{-1}.
\]

Now suppose that \( D = t_k, \) and set \( j = k - 1. \) For each \( n \geq 0, \) let \( Y_k^{(n)} \in m_{j1}(\lambda_n) \sim m_{j0}(\lambda_n) \) such that \( \lambda(Y_k^{(n)}) = 0 \) and \( B_{\lambda_n}(X_k^{(n)}), Y_k^{(n)}) = 1. \) It is easily seen that \( Y_k^{(n)} \rightarrow Y_k^{(0)}, \) and that for each \( n, U_j\pi_j^{(n)}U_j^{-1}(-iY_k^{(n)}) = t_k. \) If \( j = 0, \) we are done. Otherwise, we apply Lemma 2.3 to obtain, for each \( n, W^{(n)} \in U(g_k(\lambda_n)c) \) such that \( \pi_{j-1}^{(n)}(W^{(n)}) = \pi_{j-1}^{(n)}(-iY_k^{(n)}). \) The construction whereby \( W^{(n)} \) is obtained involves only \( \text{ad} X_j^{(n)} = Y_j^{(n)} = Y_{k-1}^{(n)} \in m_{k1}(\lambda_n) \cap \ker(\lambda) \sim m_{k0}(\lambda_n) \) where \( B_{\lambda_n}(X_j^{(n)}, Y_j^{(n)}) = 1, n = 0, 1, 2, 3, \ldots, \) and we have \( Y_{j+1}^{(n)} \rightarrow Y_j^{(0)} \) as well as \( \text{ad} X_j^{(n)} \rightarrow \text{ad} X_j^{(0)} \). Hence it is clear that for some \( m, W^{(n)} \in U(\mathfrak{g}_c)^{(m)}, n \geq 0, \) and \( W^{(n)} \rightarrow W^{(0)} \), and from the definition of \( U_j \) it is clear that for each \( n, \)

\[
U_{j-1}\pi_{j-1}^{(n)}U_{j-1}^{-1}(W^{(n)}) = t_k. \]

If \( j = 1, \) then we are done. If \( j > 1, \) then we continue this process applying Lemma 2.3 at each step. This finishes the case \( D = t_k. \) If \( D = \partial/\partial t_k, \) the proof is similar. Q.E.D.

4. A theorem. We now drop the assumption that \( \mathfrak{g} \) is nilpotent, that is, let \( \mathfrak{g} \) be a real solvable Lie algebra of exponential type, and \( G \) a connected, simply connected Lie group with Lie algebra \( \mathfrak{g}. \) Let \( n \) be the nilradical of \( \mathfrak{g}, \) and for the
remainder of this paper, let \( n = n_p \supset n_{p-1} \supset \cdots \supset n_0 = (0) \) be a Jordan-Hölder sequence for \( n \) having the property that for each \( t = 1, 2, 3, \ldots, p-1 \), if \([g, n_t] \not\subset n_t\), then \([g, n_{t+1}] \subset n_{t+1}\). Let \( E \) be the index set and \( \{\Omega_\alpha\}_{\alpha \in E} \) the \( AD^*(N) \)-invariant partition of \( n^* \) corresponding to this Jordan-Hölder sequence as constructed in the previous section.

Let \( \lambda \in n^* \), \( p(\lambda) = \sum \lambda R(\lambda, n_t) \). Suppose that there is \( A \in R(\lambda, g) \sim n \), and set \( h = RA + n \), \( H = \exp(h) \). It is shown in [1] that \( p(\lambda) \) is invariant under \( ad A \), and we may extend the equivalence class of \( \sigma = \text{ind}(\lambda, p(\lambda)) \) in \( \tilde{N} \) to \( H \) by setting

\[
(2) \quad (\sigma(\exp sA)f)(y) = f(\exp(-sA) \cdot y \cdot \exp sA) \exp \left( \frac{1}{2} \text{tr}(\text{ad}_t p(\lambda)^-) A t) \right), \quad y \in N.
\]

The corresponding extension of \( \lambda \) to \( h \) is obtained by setting \( \lambda(A) = 0 \). Now let \( n = n^0(\lambda) \supset n^1(\lambda) \supset \cdots \supset n^d(\lambda) = p(\lambda) \) be the Kirillov sequence in \( \lambda \) as constructed in the previous section, let \( X_k \in n^{k-1}(\lambda) \sim n^k(\lambda) \), \( k = 1, 2, \ldots, d \), and let \( \pi = \pi_0 \) be the irreducible representation of \( N \) corresponding to \( \lambda \) as constructed in Theorem 3.5. Then \( \pi_0 = U\sigma U^{-1} \) where \( U : H(\pi_0) \to L^2(\mathbb{R}^d) \) is defined by

\[
Uf(t_1, t_2, \ldots, t_d) = f(\exp t_1 X_1 \cdot \exp t_2 X_2 \cdots \exp t_d X_d).
\]

We extend \( \pi_0 \) as indicated above (that is, so as to be isomorphic with the above extension of \( \sigma \)).

Now let \( \alpha \in E \) such that \( \lambda \in \Omega_\alpha \), and suppose that \( \{\lambda_n\}_{n=1}^\infty \) is a sequence in \( \Omega_\alpha \) such that \( \lambda_n \to \lambda \). By Theorem 3.5, we have a corresponding sequence \( \{\pi_n\}_{n=1}^\infty \) of irreducible representations of \( N \) such that if \( D \) is a polynomial differential operator on \( \mathbb{R}^d \), then there is \( W_n \in U(n_c)^m, n = 0, 1, 2, \ldots, \) for some \( m \), such that \( W_n \to W_0 \) and \( \pi_n(W_n) = D \) for each \( n \). Recall that for each \( n \), we have \( X_k^{(n)} \in n^{k-1}(\lambda_n) \sim n^k(\lambda_n), k = 1, 2, \ldots, d \) such that \( \text{ind}(\lambda, p(\lambda_n)) \) is equivalent to \( \pi_n \) via the isomorphism

\[
Uf(t_1, t_2, \ldots, t_d) = f(\exp t_1 X_1^{(n)} \cdot \exp t_2 X_2^{(n)} \cdots \exp t_d X_d^{(n)}).
\]

and for each \( k, X_k^{(n)} \to X_k \). Suppose that we have \( A_n \in R(\lambda_n, g) \sim n, n = 1, 2, 3, \ldots \) such that \( A_n \to A \), and set \( h_n = RA_n + n, H_n = \exp h_n \) for each \( n \). Extend \( \pi_n \) to \( H_n \), as above, so that the corresponding extension of \( \lambda_n \) is obtained by setting \( \lambda_n(A_n) = 0 \). Let the algebra \( \mathcal{D} \) of polynomial differential operators on \( \mathbb{R}^d \) have the (obvious) filtration \( \mathcal{D}^{(0)} \subset \mathcal{D}^{(1)} \subset \mathcal{D}^{(2)} \subset \cdots \) so that \( D \in \mathcal{D}^{(m)} \) if and only if there is a polynomial \( P \) of degree \( \leq m \) such that \( D = P(t_1, t_2, \ldots, t_d, \partial/\partial t_1, \partial/\partial t_2, \ldots, \partial/\partial t_d) \), and for each \( m \), let \( \mathcal{D}^{(m)} \) have the usual topology as a finite dimensional vector space over \( C \).

**Lemma 4.1.** There is an integer \( m > 0 \) such that \( \pi_0(A) \in \mathcal{D}^{(m)}, \{\pi_n(A_n)\}_{n=0}^\infty \subset \mathcal{D}^{(m)}, \) and \( \pi_n(A_n) \to \pi_0(A) \) in \( \mathcal{D}^{(m)} \).

**Proof.** Clearly we may assume that \( d > 0 \). Let us use the notation \( T = (t_1, t_2, \ldots, t_d), U = (u_1, u_2, \ldots, u_p) \) and \( Z = (z_{11}, z_{12}, \ldots, z_{ij}, \ldots, z_{pp}) \) for elements of \( \mathbb{R}^d, \mathbb{R}^p, \) and \( \mathbb{R}^{p^2} \), respectively, and denote the objects associated with \( \pi_0 \) by \( \lambda_0, A_0, \) etc. For each \( n \geq 0 \), let \( \{X_k^{(n)}\}_{k=d+1}^p \) be a basis of \( p(\lambda_n) \) such that \( X_k^{(n)} \to X_k^{(0)} \) \( d < k < p \), and for each \( i, j = 1, 2, \ldots, p \) and \( s \in \mathbb{R} \), let \( a_{ij}^{(n)}(s) \) denote
the coefficient of $X_j(n)$ in the expansion of $e^{s\text{ad}A_n}X_j(n)$ in terms of the ordered basis $X_1(n), X_2(n), \ldots, X_p(n)$ of $\mathfrak{g}$. Denote the element $(a_{11}(s), a_{12}(s), \ldots, a_{ij}(s), \ldots, a_{pp}(s))$ of $\mathbb{R}^p$ by $a(n)(s)$.

By the Campbell-Hausdorff formula, we have for each $n$, polynomials $P_1(n), P_2(n), \ldots, P_p(n)$ in $T$ such that

$$
\exp sA_n \left( \prod_{j=1}^p \exp t_j X_j(n) \right) = \exp \left( \sum_{j=1}^p P_j(n)(T) X_j(n) \right).
$$

Let $q > 0$ and such that $N$ is step $q$. Then for each $n, j$, $\deg(P_j(n)) < q$, and the coefficients of $P_j(n)$ depend only on the structure constants $(b^j_k)(n), [X_i(n), X_j(n)] = \sum b^j_k(n) X_k(n)$. Clearly for each $i, j, k$, $(b^j_k)(n) \to (b^j_k)(0)$ and hence $P_j(n) \to P_j(0)$ in the vector space $C[T](q), 1 \leq j \leq p$. Now let the polynomials $\tilde{P}_j(n)$ in $T$ and $Z$ be defined by $\tilde{P}_j(n)(T, Z) = \sum P_i(n)(T) z_{ij}$; then we have

$$
\exp -sA_n \left( \prod_{j=1}^d \exp t_j X_j(n) \right) \exp sA_n = \exp \sum_{j=1}^p \tilde{P}_j(n)(T, a(n)(s)) X_j(n),
$$

$s \in \mathbb{R}$, $n = 0, 1, 2, 3, \ldots$.

On the other hand, there are polynomials $R_j(n), 1 \leq j \leq p$, in $U$ such that

$$
\exp \sum_{j=1}^d u_j X_j(n) = \exp \sum_{j>d} R_j(n)(U) X_j(n) \cdot \prod_{j=1}^d \exp R_j(n)(U) X_j(n).
$$

As with $P_j(n)$, we see that for each $n, j$, $\deg(R_j(n)) \leq q$, and for each $j$, $R_j(n) \to R_j(0)$ in $C[U](q)$. Now let $Q_j(n) = R_j(n)(\tilde{P}_1(n), \ldots, \tilde{P}_p(n), 1 \leq j \leq p$. Then $Q_j(n) \to Q_j(0)$ in $C[T, Z](q^2)$ and from the definition of $\pi_n$ we have, for each $\phi \in C^\infty(\pi_n)$,

$$
(\pi_n(A_n)\phi)(T) = \left. \frac{d}{ds} \right|_{s=0} \exp i \sum_{j>d} Q_j(n)(T, a(n)(s)) \lambda_n(X_j(n)) \\
\cdot \phi \left( Q_1(n)(T, a(n)(s)), Q_2(n)(T, a(n)(s)), \ldots, Q_d(n)(T, a(n)(s)) \right) \\
\cdot \exp \left( -\frac{1}{2} \text{tr(\text{ad}_n/P(\lambda_n)A_n)} \right)
$$

for each $n$. Let $\tilde{Q}_j(n)$ be the polynomial in $C[T](q)$ such that

$$
\left. \frac{d}{ds} \right|_{s=0} Q_j(n)(T, a(n)(s)) = \tilde{Q}_j(n)(T), \quad 1 \leq j \leq p, \ n \geq 0.
$$

Note that $a_{ij}(n)(0) = \delta_{ij}$ for each $i, j$ and $n$, and for each $i, j$, $d/ds|_{s=0} a_{ij}(n)(s) \to d/ds|_{s=0} a_{ij}(0)(s)$, whence $\tilde{Q}_j(n) \to \tilde{Q}_j(0)$. Since for each $n$,

$$
\pi_n(A_n) = i \sum_{j>d} \tilde{Q}_j(n)(T) \lambda_n(X_j(n)) + \sum_{j=1}^d \tilde{Q}_j(n)(T) \frac{\partial}{\partial t_j} - \frac{1}{2} \text{tr(\text{ad}_n/P(\lambda_n)A_n)}
$$

the result follows. Q.E.D.

For each $n > 0$, define $A_n^* \in \mathfrak{h}_n^*$ by setting $A_n^*(A_n) = 1, A_n^*|_n \equiv 0.$
COROLLARY 4.2. There is an integer \( m > 0 \) and for each \( n = 0, 1, 2, \ldots \), there is \( W_n \in U((h_n)_c) \) such that \( \{W_n\} \subseteq U(g_c)^{(m)} \), \( W_n \rightarrow W_0 \) in \( U(g_c)^{(m)} \), and such that for any real sequence \( \{c_n\} \), \( (\chi_{c_n} \otimes \pi_n)(W_n) = c_n \) where \( \chi_{c_n} \) is the character of \( H_n \) with differential \( i c_n A_n^* \).

PROOF. By Lemma 4.1, for each \( n \) we may write \( \pi_n(A_n) = \sum_{\mu} a_{\mu}^{(n)} D_{\mu} \) where \( \{D_{\mu}\} \) is a finite collection of polynomial differential operators and for each \( \mu \), \( \{a_{\mu}^{(n)}\} \) is a sequence of complex numbers such that \( a_{\mu}^{(0)} = \lim_{n \to 0} a_{\mu}^{(n)} \). By Theorem 3.5, for each \( \mu \), there is a sequence \( \{V_{\mu}^{(n)}\} \subseteq U(n_c)^{(m_\mu)} \) such that \( V_{\mu}^{(n)} \rightarrow V_{\mu}^{(0)} \) in \( U(n_c)^{(m_\mu)} \) and such that for each \( n \), \( \pi_n(V_{\mu}^{(n)}) = D_{\mu} \). Thus \( \pi_n(A_n - \sum_{\mu} a_{\mu}^{(n)} V_{\mu}^{(n)}) = 0, n = 0, 1, 2, \ldots \), and we may take \( m = \max_{\mu} \{m_\mu\} \) and

\[
W_n = -i \left( A_n - \sum_{\mu} a_{\mu}^{(n)} V_{\mu}^{(n)} \right), \quad n = 0, 1, 2, \ldots, \quad \text{Q.E.D.}
\]

Let \( K(G) \) be the space of all closed, connected subgroups of \( G \) (with the compact-open topology), and let \( S(G) \) be the space of all pairs \( (\rho, H) \) where \( H \in K(G) \) and \( \rho \) is an unitary equivalence class of representations of \( H \) with the topology of Fell (cf. [5]). Let \( K_N(G) \) be the set of all \( H \in K(G) \) such that \( N \subset H \), and \( S_N(G) \) the set of all \( (\rho, H) \in S(G) \) such that \( H \in K_N(G) \) and \( \rho \in \tilde{H} \). For each \( H \in K_N(G) \) we have a topological embedding of \( \tilde{H} \) in \( S_N(G) \). The proof that \( \eta_G \) is continuous (cf. [10, Proposition 2]) is easily generalized to show that the mapping \( \Theta : g^* \times K_N(G) \rightarrow S_N(G) \) given by

\[
\Theta((l, H)) = (\eta_H(Ad^*(H))(l|_{\tilde{l}}), H)
\]

is continuous, where \( g^* \times K_N(G) \) has the product topology. If \( (\rho, H) \in S_N(G) \) we denote the \( Ad^*(H) \)-orbit \( \eta_H^{-1}(<\rho>) \) by \( O_{\rho} \), and if \( J \subset H, j = \log(J), \) let \( O_{\rho|J} = \{l|_{J} : l \in O_{\rho}\} \).

The following two facts are well-known consequences of the general theory (cf. [4 and 5]).

**LEMMA 4.3.** Let \( (\rho, H) \in S_N(G) \) and let \( J \in K_N(G) \) be a subgroup of \( H \). Then the set of all \( \sigma \in \tilde{J} \) such that \( O_{\sigma} \subset O_{\rho|\log(J)} \) is a dense subset of \( Sp(\rho|J) \).

**LEMMA 4.4.** Let \( (\rho, H) \in S_N(G) \), and let \( \{(\rho_n, H_n)\} \subseteq S_N(G) \) such that \( (\rho_n, H_n) \rightarrow (\rho, H) \). Let \( J \in K_N(G) \) and for each \( n, J_n \in K_N(G) \) such that \( J \subset H, J_n \subset H_n, \) and \( J_n \rightarrow J \). Let \( (\sigma, J) \in S_N(G) \) such that \( O_{\sigma} \subset O_{\rho|\log(J)} \). Then for each \( n \), there is \( \sigma_n \in \tilde{J}_n \) such that \( O_{\sigma_n} \subset O_{\rho_n|\log(J_n)} \) and such that the sequence \( \{(\sigma_n, J_n)\} \) converges to \( (\sigma, J) \).

We define a partition of \( S_N(G) \) as follows. For each \( (\rho, H) \in S_N(G) \), let \( \alpha(\rho) \) be the smallest index in \( E \) such that \( O_{\rho^n|\cap \Omega_0} \neq \{\phi\} \). For each \( \alpha \in E, \) let

\[
\hat{V}_\alpha = \{(\rho, H) \in S_N(G) : \alpha(\rho) = \alpha\}.
\]

From Brown's Theorem [3] and Lemma 4.3 above it follows that \( (\rho, H) \in \hat{V}_\alpha \) if and only if \( Sp(\rho|N) \cap \eta_N(\Omega_\alpha) \neq \{\phi\} \) and \( Sp(\rho|N) \cap \eta_N(\Omega_0) = \{\phi\} \) for all \( \beta < \alpha \). For each \( \alpha \), set \( V_\alpha = \hat{V}_\alpha \cap \tilde{G}, \) and \( U_\alpha = \eta_{G^{-1}}(V_\alpha) \). Then \( U_\alpha = \{O \in g^* / Ad^*(G) : O|_{\cap \Omega_0} \neq \{\phi\} \) and \( O|_{\cap \Omega_0} \neq \{\phi\} \) for all \( \beta < \alpha \).
LEMMA 4.5. For each α, \( \tilde{V}_\alpha \) is open in \( \bigcup_{\beta \geq \alpha} \tilde{V}_\beta \). If \( \alpha_0 \) is the smallest element of \( E \), then \( \tilde{V}_{\alpha_0} \) is dense in \( S(N(G)) \).

PROOF. Let \((\rho, H) \in \tilde{V}_\alpha\) and suppose that \( \{(\rho_n, H_n)\}_{n=1}^{\infty} \) is a sequence in \( \bigcap_{\beta \geq \alpha} \tilde{V}_\rho \) such that \( (\rho_n, H_n) \to (\rho, H) \). Let \( \sigma_0 \in \text{Sp}(\rho|_N) \) such that \( O_{\sigma_n} \subset O_{\rho_n|_N} \cap \Omega_\alpha \). By Lemma 4.4, there is \( \{\sigma_n\} \subset \tilde{N} \) such that \( \sigma_n \to \sigma \) and for each \( n \), \( O_{\sigma_n} \subset O_{\rho_n|_N} \). By Brown's Theorem, \( O_{\sigma_n} \to O_\sigma \). Since \( \{O_{\sigma_n}\} \subset \bigcup_{\beta \geq \alpha} \Omega_\beta \) and \( \Omega_\alpha \) is open in \( \bigcup_{\beta \geq \alpha} \Omega_\beta \), \( \{O_{\sigma_n}\} \) is eventually in \( \Omega_\alpha \), thus \( \{(\rho_n, H_n)\}_{n=1}^{\infty} \) is eventually in \( \tilde{V}_\alpha \).

Let \( \alpha_0 \) be the minimal element of \( E \) and let \((\rho, H) \in S(N(G)), \tilde{h} = \log(H) \). The set \( \{O \in \mathfrak{h}^* / \text{Ad}^*(H)| \ O|_n \cap \Omega_\alpha = \{0\} \} \) is dense in \( \mathfrak{h}^* / \text{Ad}^*(H) \), hence the set \( \{\rho \in \tilde{H}| \ O|_n \cap \Omega_\alpha = \{0\} \} \) is dense in \( \tilde{H} \) (by continuity of \( \eta_H \)). The embedding of this set in \( S(N(G)) \) is contained in \( \tilde{V}_{\alpha_0} \) and \((\rho, H) \) is contained in its closure. □

LEMMA 4.6. Let \( \alpha \in E \) and let \( \lambda_n \in \Omega_\alpha, n = 0, 1, 2, \ldots \), such that the sequence \( \{\lambda_n\}_{n=1}^{\infty} \) converges to \( \lambda_0 \), for each \( n \geq 0 \), let \( A_n \in R(\lambda_n|_n, g) \sim n, h_n = RA_n + n, H_n = \exp(h_n) \), extend \( \lambda_n \) to \( h_n \) by setting \( \lambda_n(A_n) = 0 \), define \( A_n^* \in \mathfrak{h}^* \) by \( A_n^*(A_i) = \delta_{ij} \) and \( A_n^*|_n \equiv 0 \). Recall then that \( \tau_N \in \hat{N} \), and that if \( \lambda \) is extended to \( j^* \) by setting \( \lambda(A_j) = 0 \), \( 1 \leq j \leq r \), then there is a unique \( t = (t_1, t_2, \ldots , t_r) \in \mathbb{R}^r \) such that \( \lambda + \sum_{j=1}^r t_j A_j^* \in \Omega_\sigma \). For any \( j \), \( 1 \leq j \leq r \), if \( \nu = \sigma|_N \) is extended to \( H_j = \exp(RA_i + N) \) by formula (2), and \( \chi_t \) is the character of \( H_j \) having differential \( it_j(A_j^*|_j) \), then \( \sigma|_{H_j} = \chi_t \otimes \nu \).

PROOF. We need only prove the “only if” part. Suppose that \( (\rho_n, H_n) \to (\rho_0, H_0) \). Let \( \pi_0 \) an irreducible representation corresponding to \( \lambda_0 \) and let \( \{\pi_n\}_{n=1}^{\infty} \) a sequence of representations corresponding to \( \{\lambda_n\}_{n=1}^{\infty} \) as obtained in Theorem 3.5, so that \( \pi_n \subset \pi_n|_N, n \geq 0 \). Extend \( \pi_n \) to \( H_n \) as in formula (2) so as to correspond to \( \lambda_n \), and let \( \chi_n \) be the character of \( H_n \) such that \( \gamma_n = \chi_n \otimes \tau_n \subset \pi_n \). Then by Corollary 4.2, there is \( m > 0 \) and \( \{W_n\}_{n=1}^{\infty} \subset \bigcup U(\mathfrak{g}_c)^{(m)} \) such that \( W_n \to W_0 \) and such that for each \( n \), \( W_n \subset U((h_n)_c) \) and \( \gamma_n(W_n) = t_n \). Now the general theory implies that \( t_n \to t_0 \). To see this, let \( \Psi_0 \in C_c^\infty(G) \) and \( v_0 \in H(\gamma_0) \) such that \( \langle \gamma_0(\Psi_0)v_0, v_0 \rangle \neq 0 \). For each \( n \), let \( \Gamma_n \) be the representation of \( C_s^*(G) \) lifted from \( \gamma_n \). Note that any \( \Psi \in C_c^\infty(G) \) defines in a natural way an element \( \tilde{\Psi} \in C_s^*(G) \) by setting \( \tilde{\Psi}(K, x) = \Psi(x), K \in K(G), x \in K \) such that for any \( v \in H(\gamma_n), \langle \Gamma_n(\tilde{\Psi})v, v \rangle = \langle \gamma_n(\Psi)v, v \rangle \). Now let \( V_1, V_2, \ldots , V_q \in U(\mathfrak{g}_c) \) such that for each \( n \), \( W_n = \sum_{j=1}^q a_j^{(n)}V_j \) with \( a_j^{(n)} \in C, 1 \leq j \leq q \), for each \( j \), \( a_j^{(0)} = \lim_{n \to \infty} a_j^{(n)} \). Set \( \tilde{\Psi} = V_j \Psi_0, 1 \leq j \leq q \). Then by Lemma 2.2 of [4], there is, for each \( n > 0 \), \( v_n \in H(\gamma_n) \) such that \( \langle \Gamma_n(\tilde{\Psi})v_n, v_n \rangle \to \langle \Gamma_0(\tilde{\Psi})v_0, v_0 \rangle \) as \( n \to \infty, 0 \leq j \leq q \). Thus \( \langle \Gamma_n(W_n(\Psi_0)v_n, v_n) \to \langle \Gamma_0(W_0(\Psi_0)v_0, v_0) \rangle = t_0 \). Q.E.D.

For each \( \alpha \in E \), set \( U_\alpha = \Theta^{-1}(\tilde{V}_\alpha) \).
THEOREM 4.7. \( \Theta|_{\bar{U}_\alpha} : \bar{U}_\alpha \to \bar{V}_\alpha \) is open, for each \( \alpha \in E \).

PROOF. Let \((\rho_0, H_0) \in \bar{V}_\alpha \), and suppose that \( \{(\rho_n, H_n)\}_{n=1}^\infty \) is a sequence in \( \bar{V}_\alpha \) which converges to \((\rho_0, H_0)\). Let \( h_n = \log(H_n) \), \( n = 0, 1, 2, \ldots \), and let \( l_0 \in g^* \) such that \( l_0 \mid h_n \in O_{\rho_n} \). It is enough to show that there is a subsequence \( \{(\rho_k, H_k)\}_{k=1}^\infty \) of \( \{(\rho_n, H_n)\}_{n=1}^\infty \) and a corresponding sequence \( \{l_k\}_{k=1}^\infty \) in \( g^* \) such that \( l_k \mid h_n \in O_{\rho_k} \) for each \( k \) and \( l_k \to l_0 \). Note that we may assume that \( \lambda_0 = l_0|n \in \Omega_\alpha \). Let \( \nu \in \tilde{N} \) such that \( \lambda_0 \in O_\nu \). By Lemma 4.3, there is \( \nu_n \in \tilde{N} \) such that \( O_{\nu_n} \subset O_{\rho_n}|n, n = 1, 2, 3, \ldots \), and such that the sequence \( \{\nu_n\}_{n=1}^\infty \) converges to \( \nu \). Thus we have \( \lambda_n \in O_{\nu_n} \subset O_{\rho_n}|n, n = 1, 2, 3, \ldots \), such that \( \{\lambda_n\} \) converges to \( \lambda_0 \). Now by restriction to a subsequence, we may assume that \( \dim(h_n) = m, n = 0, 1, 2, \ldots \), and since \( \Omega_\alpha \) is open in \( \bigcup_{\beta \geq \alpha} \Omega_\beta \) and \( \{\lambda_n\} \subset \bigcup_{\beta \geq \alpha} \Omega_\alpha \), we may assume that \( \lambda_n \in \Omega_\alpha \) for all \( n \). We proceed by induction on \( \dim(h_n/n) = m - p \).

The case \( m - p = 0 \) is now trivial due to the above, so assume that \( m > p \) and that the theorem is valid for sequences in \( S_N(G) \) whose subgroup have dimension less than \( m \). Let \( \{\lambda_k\}_{k=1}^\infty \) be a subsequence of \( \{\lambda_n\}_{n=1}^\infty \) such that for some subalgebra \( J_0 \), the sequence \( j_k = R(\lambda_k, h_k) + n, k = 1, 2, 3, \ldots \), converges to \( j_0 \). Let \( J_k = \exp(j_k) \), \( k \geq 0 \). By Lemma 4.3, we have \( \sigma_k \in \tilde{J}_k, k = 0, 1, 2, \ldots \), such that \( l_0|0 \in O_{\sigma_0}, O_{\sigma_k} \subset O_{\rho_k}|k \), \( k \geq 1 \), and the sequence \( \{\sigma_k, J_k\}_{k=1}^\infty \) converges to \( (\sigma_0, J_0) \). Suppose that \( \dim(J_0) < m \). By induction there is \( l_k \in g^* \) such that \( l_k|l_k \in O_{\sigma_k}, k = 1, 2, 3, \ldots \), and such that the sequence \( \{l_k\}_{k=1}^\infty \) converges to \( l_0 \). Now if \( \rho_k : h_k^* \to j_k^* \) is the restriction mapping, then \( \rho_k^{-1}(O_{\sigma_k}) \subset O_{\rho_k}, k \geq 1 \) (cf. [1, Chapter II, §4.2]). Therefore \( l_k|h_k \in O_{\rho_k} \), and we are done. Hence by induction we have reduced to the case \( j_k = h_k \), \( k \geq 0 \).

For each \( k \), let \( \{A_1^{(k)}, A_2^{(k)}, \ldots, A_{r(k)}^{(k)}\} \subset \rho_k \) be a basis for \( h_k \) such that for each \( 1 \leq j \leq r, A_j^{(0)} = \lim_k A_j^{(k)} \). Extending \( \lambda_k \) to \( h_k \) by setting \( \lambda_k(A_j^{(k)}) = 0, 1 \leq j \leq r \), let \( t_1^{(k)}, t_2^{(k)}, \ldots, t_{r(k)}^{(k)} \) be real numbers such that \( \lambda_k + \sum_{j=1}^r t_j^{(k)} A_j^{(k)*} \in O_{\rho_k} \) (where \( A_j^{(k)*} \) is defined by \( A_j^{(k)*}(A_j^{(k)}) = \delta_{ij}, A_j^{(k)*}|n \equiv 0 \)). For each \( j, i \leq j \leq r \), apply Lemma 4.6 to the sequence \( \{(\rho_k|\exp(\rho A_j^{(k)} + N), \exp(\rho A_j^{(k)} + N))\}_{k=1}^\infty \) which converges to \( (\rho_0|\exp(\rho A_j^{(0)} + N), \exp(\rho A_j^{(0)} + N)) \), and we obtain \( t_j^{(0)} = \lim_k t_j^{(k)} \). Since \( A_j^{(k)} \to A_j^{(0)}, 1 \leq j \leq q \), it is clear that we may extend \( \lambda_k + \sum_{j=1}^q t_j^{(k)} A_j^{(k)*} \) to an element \( l_k \in g^* \) such that the sequence \( \{l_k\}_{k=1}^\infty \) converges to \( l_0 \). This finishes the proof. Q.E.D.

The following corollary is immediate.

COROLLARY 4.8. \( \eta_G|_{U_\alpha} : U_\alpha \to V_\alpha \) is a homeomorphism, for each \( \alpha \).

Note that for each \( \alpha \in E \), the dimensions of the orbits in \( U_\alpha \) may vary, and the relative topology in \( U_\alpha \) may not be \( T_1 \). Indeed, if \( N \) is abelian, then \( E = \{a_0\} \) and \( U_{a_0} = g^* \). Examples indicate that for each \( \alpha \in E \), there is a finite partition of \( U_\alpha \), each element of which is \( T_2 \). Finally, the subsets \( U_\alpha \) may be describable as Zariski-open subsets of algebraic varieties in \( g^* \).

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