TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 309, Number 1, September 1988

THE FUNDAMENTAL MODULE OF A NORMAL LOCAL DOMAIN OF DIMENSION 2

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ABSTRACT. The fundamental module E of a normal local domain $(R, \mathfrak{m})$ of dimension 2 is defined by the nonsplit exact sequence $0 \to K \to E \to \mathfrak{m} \to 0$, where K is the canonical module of R. We prove that, if R is complete with $R/\mathfrak{m} \simeq \mathbf{C}$, then E is decomposable if and only if R is a cyclic quotient singularity. Various other properties of fundamental modules will be discussed.

0. Introduction. Let $(R, \mathfrak{m})$ be a normal local domain of dimension 2 which possesses the canonical module K. Let $\mathrm{C}(R)$ denote the category of finitely generated reflexive R-modules. Note that a finitely generated R-module is an object in $\mathrm{C}(R)$ if and only if it is a maximal Cohen-Macaulay module over R. By definition, K is a reflexive module of rank 1 and it satisfies $\mathrm{Ext}_R^2(R/\mathfrak{m}, K) \simeq R/\mathfrak{m}$. (See Herzog and Kunz [8] for the details.) We denote the duality with respect to R (resp. K) by * (resp. $'$), that is, $(\ )^* = \mathrm{Hom}_R(\ , R)$ and $(\ )' = \mathrm{Hom}_R(\ , K)$. Remark that a finitely generated R-module M lies in $\mathrm{C}(R)$ if and only if $M^{**} \simeq M$, or equivalently $M'' \simeq M$.

We make the following

DEFINITION (0.1). Since $\mathrm{Ext}_R^1(\mathfrak{m}, K) \simeq \mathrm{Ext}_R^2(R/\mathfrak{m}, K) \simeq R/\mathfrak{m}$, there uniquely exists the nonsplit exact sequence $0 \to K \to E \to \mathfrak{m} \to 0$, which is called the fundamental sequence in [3]. In particular the module E appearing in the middle term of this sequence is also unique up to isomorphism. We call E the fundamental module of R. E is said to be decomposable if there is an isomorphism $E \simeq \mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are nontrivial ideals of R. Otherwise E is indecomposable.

In this paper we are interested in the properties of the fundamental modules. In particular we are mostly concerned with its decomposability. In fact we can prove that if R is complete with the residue field $\mathbf{C}$, then R has the decomposable fundamental module if and only if R is a cyclic quotient singularity. See Theorem (2.1). And if this is the case, then the AR-quiver of R will be easily described.

In §1 we summarize some elementary properties of fundamental modules. In particular we can prove that if R is a hypersurface, then the fundamental module of R is isomorphic to the third syzygy of k.

In §2 our main theorem (2.1) stated above will be proved. The reader will notice that the keys of the proof are the first author's theorem [**11**, Theorem (1.1)] and the theorem of Herzog-Auslander (Auslander [**3**, Theorem 4.9], Herzog [**7**]).

Received by the editors June 26, 1986 and, in revised form, July 29, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13H10, 13C13; Secondary 14B05, 14J17.

Key words and phrases. Local rings, reflexive modules, quotient singularities, Cohen-Macaulay modules, canonical modules, Auslander-Reiten quivers.

In §3 we will describe the AR-quivers for the rings with decomposable fundamental module, while a remark in positive characteristic cases is given in §4, where we restrict ourselves to considering hypersurfaces.

The authors wish to thank the referee for various comments on the first version of this paper.

1. Elementary properties of fundamental modules. Let $(R, \mathfrak{m}, k)$ be a normal local domain of dimension 2 which has the canonical module K and let us denote the fundamental module by E. The following lemma will be easy to show and is well known (see [**3**, §§3 and 6] for the proof).

LEMMA (1.1). *The fundamental module E is a reflexive module of rank 2, which is generated by at most $\mathrm{r}(R) + \mathrm{emb}(R)$ elements, where $\mathrm{r}(R)$ is the Cohen-Macaulay type of R and $\mathrm{emb}(R)$ denotes the embedding dimension of R.*

LEMMA (1.2). *There is an isomorphism of R-modules $(\bigwedge^2 E)^{**} \simeq K$.*

PROOF. Taking the divisor classes attached to the modules in the sequence $0 \to K \to E \to \mathfrak{m} \to 0$, we obtain the equality $\mathrm{c}(E) = \mathrm{c}(K)$ in the divisor class group of R (see Bourbaki [**5**, §7]). Then by definition it holds that $\mathrm{c}(\bigwedge^2 E) = \mathrm{c}(K)$ (cf. [**5**, Exercise 12 of §4]). This gives the isomorphism in the lemma.

COROLLARY (1.3). *If E is decomposable as $E \simeq \mathfrak{a} \oplus \mathfrak{b}$, then there is an isomorphism $\mathfrak{b} \simeq ((\mathfrak{a}^{-1}K)^{-1})^{-1}$.*

PROOF. Since $\bigwedge^2 E \simeq \mathfrak{a} \cdot \mathfrak{b}$ and since $\mathfrak{a}$ and $\mathfrak{b}$ are divisorial ideals, the corollary is easily obtained from the lemma.

In the rest of this section we consider a hypersurface $R = S/(f)$, where S is a regular local ring of dimension 3 with a regular system of parameters $\{x, y, z\}$. We assume that R is normal as above. Take f_x, f_y and f_z so that they satisfy $f = x \cdot f_x + y \cdot f_y + z \cdot f_z$. Then the minimal free resolution of the residue field k is given as follows:

$$\cdots \to R^4 \xrightarrow{C} R^4 \xrightarrow{D} R^4 \xrightarrow{C} R^4 \xrightarrow{B} R^3 \xrightarrow{A} R \to k \to 0, \tag{1.4}$$

where

$$A = {}^t(x, y, z), \qquad B = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \\ f_x & f_y & f_z \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & f_z & -f_y & x \\ -f_z & 0 & f_x & y \\ f_y & -f_x & 0 & z \\ -x & -y & -z & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & -z & y & -f_x \\ z & 0 & -x & -f_y \\ -y & x & 0 & -f_z \\ f_x & f_y & f_z & 0 \end{pmatrix}.$$

(See, for example, Tate [**9**] for free resolutions.) The third syzygy of k is the module which is the image of C or equivalently the cokernel of D. Under these circumstances one can prove the following

LEMMA 1.5. *The fundamental module E is isomorphic to the third syzygy of k.*

PROOF. Notice that $K \simeq R$ for R is Gorenstein. Setting the column vector $F = {}^t(f_x, f_y, f_z, 0)$ in R^4, it follows from the exact sequence (1.4) that the class of

${}^tF$ in $(R^4)^*$ generates the homology group $\mathrm{Ext}^2_R(k, R)$. Now define the module E by the following push-out diagram:

$$(1.5.1)\qquad \begin{array}{ccc} R & \xrightarrow{\alpha} & E \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle \beta} \\ R^4 & \xrightarrow[B]{} & R^3 \end{array}$$

Then it is easily observed that there is a nonsplit exact sequence $0 \to R \xrightarrow{\alpha} E \to \mathfrak{m} \to 0$ and that E is the fundamental module of R. By (1.5.1) one sees that E is the cokernel of the map $F \oplus B$ which is nothing but D. Hence E is the third syzygy of k.

2. The main theorem. In this section $(R, \mathfrak{m})$ is always a complete normal domain of dimension 2 with $R/\mathfrak{m} \simeq \mathbb{C}$. Our goal here is to prove the following

THEOREM (2.1). *Let R be as above. Then the following conditions are equivalent.*

(i) *The fundamental module E is decomposable, say $E \simeq \mathfrak{a} \oplus \mathfrak{b}$.*

(ii) *R is a cyclic quotient singularity.*

Moreover if this is the case, the divisor class group $\mathrm{Cl}(R)$ is generated by the class of $\mathfrak{a}$, and $\mathfrak{b}$ is isomorphic to $((\mathfrak{a}^{-1}K)^{-1})^{-1}$.

Combining this with Lemma (1.5) we will obtain the following

COROLLARY (2.2). *Assume that R is a complete normal hypersurface domain with the residue field $\mathbb{C}$. Then R is a cyclic quotient singularity if and only if the third syzygy of $\mathbb{C}$ is decomposable.*

We first consider the implication from (ii) to (i) in the theorem. For this let $S = \mathbb{C}[[x, y]]$ and let G be a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ which has no pseudo-reflections and which linearly acts on S. Consider the Koszul complex with respect to x and y:

$$\mathfrak{K}\colon 0 \to S(dx \wedge dy) \to S\,dx \oplus S\,dy \to S \to 0.$$

Recall that $\mathfrak{K}$ is the complex of SG-modules by the G-action given by the injection $G \subset \mathrm{GL}(\mathbb{C}\,dx \oplus \mathbb{C}\,dy)$. Hence the action of $g \in G$ on $dx \wedge dy$ is given by the multiplication of $\det(g)$. Taking the G-invariant part of $\mathfrak{K}$ we get the following exact sequence of S^G-modules;

$$\mathfrak{K}^G\colon 0 \to (S(dx \wedge dy))^G \to (S\,dx \oplus S\,dy)^G \to S^G.$$

The following is observed in [**3**, §3] (see also [**10**]).

LEMMA (2.3). *Let S and G be as above and let R be the invariant subring S^G. Then the canonical module K (resp. the fundamental module E) of R is isomorphic to $(S(dx \wedge dy))^G$ (resp. $(S\,dx \oplus S\,dy)^G$).*

Now we can prove the implication from (ii) to (i) in the theorem. Let G be a cyclic group in the above. Then we may assume that any elements in G are diagonal matrices in $\mathrm{GL}(2, \mathbb{C})$. Then by (2.3) the fundamental module E of R is isomorphic to $(S\,dx \oplus S\,dy)^G \simeq (S\,dx)^G \oplus (S\,dy)^G$, which is certainly decomposed. Moreover in this case the divisor class group $\mathrm{Cl}(R)$ is isomorphic to the character group $\mathrm{Hom}(G, \mathbb{C}^*)$ by the Galois descent and is known to be generated by the class of $(S\,dx)^G$. (See Fossum [**6**, Theorem 16.1 and Example 16.5] for the details.)

There remains to prove the implication from (i) to (ii) in Theorem (2.1). For this purpose we need some preliminaries from [**3**] and [**11**].

Let $(*)$ $0 \to K \to E \to R$ be the nonsplit exact sequence. For any indecomposable reflexive module M which is not isomorphic to R, the following sequence induced by $(*)$ is exact:

$$0 \to \mathrm{Hom}_R(M^*, K) \to \mathrm{Hom}_R(M^*, E) \to \mathrm{Hom}_R(M^*, R) \to 0.$$

Thus we obtain the sequence:

$$(**) \qquad 0 \to (M^*)' \to \mathrm{Hom}_R(M^*, E) \to M \to 0,$$

where each term in the sequence lies in $\mathrm{C}(R)$. It is known by [**3**, Theorem 6.6] that $(M^*)'$ is also indecomposable and that the sequence $(**)$ is almost split in the sense of Auslander-Reiten. The translation $M \to (M^*)'$ is called the AR-translation and is denoted by τ. The AR-graph of $\mathrm{C}(R)$ (or simply the AR-graph of R) is the directed graph such that the vertex set consist of all the isomorphic classes of indecomposable objects in $\mathrm{C}(R)$ and that there is an arrow from the class of M to that of N only if M is a direct summand of $\mathrm{Hom}_R(N^*, E)$, which is known to be equivalent to that $M \simeq \tau(L)$ for some $L \in \mathrm{C}(R)$ and N is a direct summand of $\mathrm{Hom}_R(L^*, E)$ (see [**3**] for more details). We say that $\mathrm{C}(R)$ is of finite representation type if the AR-graph of R is finite.

Let $E \simeq \mathfrak{a} \oplus \mathfrak{b}$ be the fundamental module of R which is decomposable and let Γ^0 be the connected component of the AR-graph Γ. The following claim is easily seen from the definition of AR-graphs.

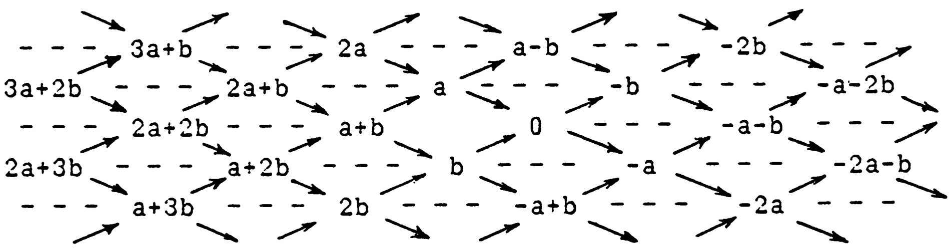
(2.4) Under the assumption in (i) in Theorem (2.1) any class of module in Γ^0 has rank 1.

By virtue of (2.4) and [**11**, Theorem (1.1)] one sees that, under the assumption of Theorem (2.1)(i), $\mathrm{C}(R)$ is of finite representation type. Then it can be concluded by [**3**, Theorem 4.9] that R is a quotient singularity, that is, there are $S = \mathbb{C}[[x, y]]$ and a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ such that $R = S^G$. Here by the usual argument as in [**10**] we may assume that G contains no pseudo-reflections. We want to prove that G is a cyclic group. By [**3**, Theorem 4.6] we know that there is a one-to-one correspondence between indecomposable reflexive R-modules and irreducible representations of G. Therefore by (2.4) we may obtain that all irreducible representations of G have dimension one, that is, G is an abelian group. Since it has no pseudo-reflections, G must be a cyclic group.

REMARK (2.5). The above proof shows that, if R is a normal local domain of dimension 2 which is essentially of finite type over $\mathbb{C}$ and if the fundamental module of R is decomposable, then the completion of R is a cyclic quotient singularity.

3. AR-graphs. Let R be a complete normal local domain of dimension 2 and let E be the fundamental module which is decomposed into the sum of ideals $\mathfrak{a}$ and $\mathfrak{b}$. Then the AR-graph of R can be easily described in the manner as in the proof of Theorem (2.1). In general it is figured in the following.

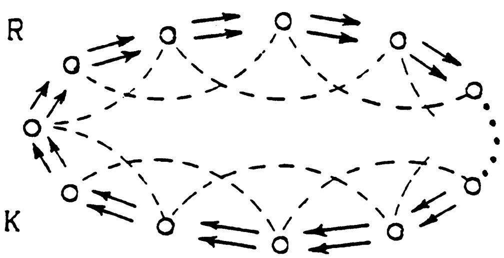
(3.1)



where $na+mb$ denotes the ideal class of $((\mathfrak{a}^n\mathfrak{b}^m)^{-1})^{-1}$ in $\mathrm{Cl}(R)$. Since this diagram must be finite by [**11**, Theorem (1.1)], some vertices in (3.1) should be identified. For example, if a has order n in $\mathrm{Cl}(R)$, then $0 = na = 2na = \cdots$ and so on. The AR-graph of R is then obtained from the graph (3.1) by dividing it by the divisor class group. We limit ourselves to describing some typical examples of these diagrams below (see [**4**] for more details).

(3.2) Let R be the Veronese subring $k[[x^n, x^{n-1}y, \ldots, y^n]]$ of $k[[x,y]]$ of degree n. Then the AR-graph of R is the following:

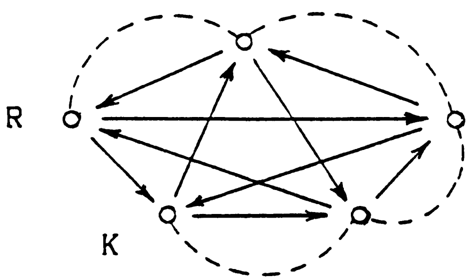
(3.2.1)



with n vertices.

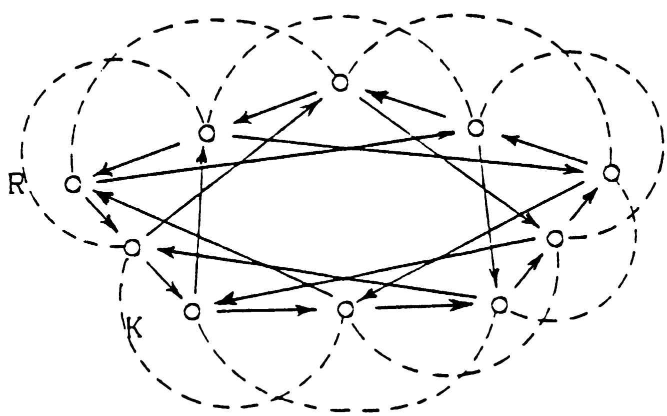
(3.3) Let R be the subring $k[[x^5, x^3y, xy^2, y^5]]$ of $k[[x,y]]$. Then the AR-graph of R is:

(3.3.1)



(3.4) Let R be the subring $k[[x^{10}, x^7y, x^4y^2, xy^3, y^{10}]]$ of $k[[x,y]]$. Then the AR-graph of R looks like:

(3.4.1)



4. A remark on positive characteristic case. Let R be a hypersurface $k[[x,y,z]]/(f)$ where k is an algebraically closed field of arbitrary characteristic p. We assume that R is a normal domain which is not regular. If the fundamental module of R is decomposable, then as in the proof of Theorem (2.1) one can show that the AR-graph of R is a finite graph, i.e. R is of finite representation type which implies R is a rational double point. On the other hand, it is known that the AR-graph Γ of a rational double point is a Euclidean graph, and by definition the fundamental module is decomposable if and only if the vertex $[R]$ in Γ is connected with two different vertices. Thus the decomposition of the fundamental module implies that Γ is the graph of type $\tilde{A}_n$. Since it is known by Artin and Verdier [**2**] that the graph obtained from Γ by deleting $[R]$ is isomorphic to the graph of the desingularization, we see that R must be a singularity of type A_n. Therefore we proved the following

PROPOSITION (4.1). *Let R be a normal hypersurface of dimension* 2. *Then the fundamental module of R is decomposable if and only if R is a rational double point of type A_n.*

By this remark and by the result of Artin [**1**] it will be easy to have a complete classification of hypersurfaces in any characteristic on which the fundamental modules are decomposed.

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