CORRECTION TO "DIFFERENTIAL IDENTITIES IN PRIME RINGS WITH INVOLUTION"

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An example of Chuang [1] shows that the main results of [2] are false as stated. The purpose of this note is to state the correct versions of these theorems. We shall use the notation in [2], and all references to results are from that paper. We begin by noting that all of the results in [2] before Theorem 4 are correct as stated, and that the correction needed in Theorem 4 requires a subsequent change in Theorem 7 and in Theorem 9. All other results in the paper are correct.

The statement of Theorem 4 concerns a linear $G^*\text{-DI } f$, all of whose exponents come from $W$, the ordered collection of $k$-tuples of outer derivations which are independent modulo the inner derivations. For any exponent $w$ appearing in $f$ and coming from $W$, let $f_w$ be the sum of all monomials in $f$ with exponent $w$. The error in the proof of Theorem 4 is the assumption that if $f_w(x,y)$ is a $G^*\text{-PI}$ for $R$, then $f_w(x^w,y^w)$ is also an identify for $R$. This is true when no involution is present, or equivalently, when $y$ does not appear in $f$. However, given an exponent $w$ appearing in $f$, a relation between $r$ and $r^*$ will not in general hold for $r^w$ and $(r^*)^w$, unless $*$ commutes in $\text{End}(R)$ with $w$. Thus the induction used in Theorem 4 fails. The most important feature of Theorem 4 can be salvaged, using essentially the proof given.

For any $w$ coming from $(d_1,\ldots,d_k) \in W$, let $k$ be the length of $w$. If $f \in F$ is linear and has all its exponents coming from $W$, an exponent $w$ appearing in $f$ is said to be of longest length if no other exponent of $f$ has longer length. The conclusion of Theorem 4 is correct for all exponents of longest length, and the following is what the statement of the theorem should be.

**THEOREM 4.** Let $R$ be a prime ring with $*$, and let $f \in F$ be linear and have all its exponents coming from $W$, so that $f = \sum_h \sum_i a_{hi}x^h b_{hi} + \sum_k \sum_j c_{kj} y^j d_{kj}$ with all $h$ and $k$ coming from $W$ and all coefficients in $N$. Suppose that for some nonzero ideal $I$ of $R$, $f(I) = 0$. Then for each exponent $w$ appearing in $f$ and of longest length, $f_w(x) = \sum_i a_{w_i}x b_{w_i} + \sum_j c_{w_j} y^j d_{w_j}$ is a $G^*\text{-PI}$ for $R$. In addition, if no $y$ appears in $f$, or if each exponent appearing in $f$ commutes with $*$ in $\text{End}(R)$, then $f_w(x)$ is a $G^*\text{-PI}$ for $R$ for every exponent $w$ appearing in $f$.

The proof proceeds as in [2], except that one uses induction on the longest length of exponents appearing in $f$. One may still assume that $R$ satisfies a GPI by Theorem 1, and if $0$ is the longest length then $f = f_1$ is a $G^*\text{-PI}$ for $R$. As in [2], the expression $g(x) = f(cx) - cf(x)$ is a linear $G^*\text{-DI}$ which contains no basis monomial appearing in $f$, and has its exponents of longest length at most
one less than the longest length for $f$. Thus induction can be applied to $g$. In the case that the longest length for $f$ is 1, $f = f_1(x, y) + \sum f_d(x^d, y^d)$, $g = \sum c_d f_d(x)$, and the Vandermonde type argument given in [2] shows that each $f_d(x)$ is a $G^*$-PI for $R$. For the general case, let $w_1$ be any exponent of $f$ of longest length, and let $w_1$ come from $(d_1, m_2, \ldots, m_k) \in W$. Write $w_1 = d_1 v$ where $v$ comes from $(m_2, \ldots, m_k)$. As in [2], by induction on $k$, $g_v(x)$ is a $G^*$-PI for $R$, and as in [2] one sees that $g_v(x) = \sum q_s c_d^s f_{w_s}(x)$ where $w_s$ represents any exponent of $f$ of length $k$ which comes from a $k$-tuple having some $d_s$ inserted in the appropriate place in the ordered $k - 1$ tuple $(m_2, \ldots, m_k)$. Also, $q_s$ counts the number of occurrences of $d_s$ in the $k$-tuple from which $w_s$ comes. Since the collection of $d_s$ appearing is independent modulo the inner derivations, the Vandermonde type argument shows again that each $f_{w_s}$ is a $G^*$-PI for $R$, so in particular, $f_{w}(x)$ is.

To see how Theorem 7 needs to be changed in light of the change to Theorem 4, we recall that for $f \in F$ which is multilinear and homogeneous of degree $n$, and having all exponents coming from $W$, $W(f)$ is the set of all $n$-tuples $\bar{w} = (w_1, \ldots, w_n)$ for which there is a monomial in $f$ having each $w_i$ as the exponent of $x_i$ or $y_i$. Then for any $\bar{w} \in W(f)$, $f_{\bar{w}}(x_1^{w_1}, \ldots, x_n^{w_n}, y_1^{w_1}, \ldots, y_n^{w_n})$ is the sum of all such monomials. Theorem 7 asserts that if $f$ is a $G^*$-DI for an ideal $I$, then each $f_{\bar{w}}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a $G^*$-PI for $R$. Using the correct statement of Theorem 4 above requires a restriction on which $\bar{w} \in W(f)$ one can use. Call $\bar{w} \in W(f)$ special if after some reordering of subscripts, the length of $w_1$ is maximal among the lengths of exponents in $f$ appearing with either $x_1$ or $y_1$, the length of $w_2$ is maximal among the lengths of exponents of either $x_2$ or $y_2$, appearing in any monomial in which the exponent of $x_1$, or of $y_1$, is $w_1$, and in general, the length of $w_i$ is maximal among the lengths of exponents of $x_i$ or $y_i$ which appear in monomials for which $(w_1, \ldots, w_{i-1})$ is the exponent sequence of the variables with subscript smaller than $i$. The proof of Theorem 7 is valid for all $\bar{w} \in W(f)$ which are special, and the following is the correct statement.

**THEOREM 7.** Let $R$ be a prime ring with involution, $*$, and let $f \in F$ be multilinear and homogeneous of degree $n$ with all exponents coming from $W$ and all subscripts of variables in \{1, 2, \ldots, n\}. For any special $\bar{w} = (w_1, \ldots, w_n) \in W(f)$, let $f_{\bar{w}}(x_1^{w_1}, \ldots, x_n^{w_n}, y_1^{w_1}, \ldots, y_n^{w_n})$ denote the sum of all monomials in $f$ in which $x_i$ or $y_i$ appears with exponent $w_i$. If $f$ is a $G^*$-DI for some nonzero ideal $I$ of $R$, then $f_{\bar{w}}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a $G^*$-PI for $R$, and $R$ satisfies a GPI, unless $f = 0$ in $F$. Furthermore, the same conclusion holds for every $\bar{w} \in W(f)$ if either no $y$ appears in $f$, or if each exponent appearing in $f$ commutes with $*$ in $\text{End}(R)$.

We note for Theorem 7 that in the case when every exponent appearing in $f$ is either a derivation or is 1, then an exponent sequence is special if it contains a maximal number of derivations, although other sequences may be special. For example, if $W(f)$ consists of the sequences $(d, 1, d)$, $(1, d, 1)$, and $(h, 1, 1)$, then each would be special. Finally, it is important to observe that the applications we have made of Theorem 4 and Theorem 7 [3 and 4] are valid as given, using the corrected versions of these two theorems as they appear here.

The last correction needed in [2] is to change the hypothesis of Theorem 9 to assume that for each $i$, $h_i$ is inner on $Q$ exactly when $k_i$ is inner on $Q$. With this modification and the comments above, the proof as given in [2] holds.
References


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