TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 309, Number 2, October 1988

SEIFERT MATRICES AND 6-KNOTS

J. A. HILLMAN AND C. KEARTON

ABSTRACT. A new classification of simple $\mathbf{Z}$-torsion-free $2q$-knots, $q \geq 3$, is given in terms of Seifert matrices modulo an equivalence relation. As a result the classification of such $2q$-knots, $q \geq 4$, in terms of F-forms is extended to the case $q = 3$.

0. Introduction. An *n-knot* k is an oriented locally flat PL sphere-pair (S^{n+2}, S^n); equally, one could consider oriented smooth pairs where the embedded sphere is allowed to carry an exotic differentiable structure. Let K denote the closed complement of a regular neighbourhood of S^n, often called the *exterior* of k. Then k is *simple* if K has the homotopy $[(n-1)/2]$-type of a circle; this is the most that can be asked without making k unknotted. With the exception of $n = 4$ and 6, the simple n-knots ($n \geq 3$) have been classified in various ways during the past twenty years. The first such result is due to J. Levine [**L2**], who classified the simple $(2q-1)$-knots ($q \geq 2$) in terms of the Seifert matrix and S-equivalence. These knots were then classified in terms of the Blanchfield duality pairing in [**T1**, **T2**, and **K1**]. Results for certain classes of simple $2q$-knots ($q \geq 4$) may be found in [**K2**, **Ko1**, **Ko2**, and **K3**]; the general case is given by M. Sh. Farber in [**F2**]. One should also mention here the pioneering work of M. A. Kervaire [**Ke**], who characterised the homology modules which can occur, in terms of presentation matrices.

Let $\tilde{K} \to K$ be the infinite cyclic ($=$ universal) cover of K, where k is a simple $2q$-knot, $q \geq 3$. Then k is $\mathbf{Z}$*-torsion-free* if $H_q(\tilde{K})$ has no $\mathbf{Z}$-torsion. In this paper we classify such knots in terms of Seifert matrices and an equivalence relation which we call F-equivalence. These matrices yield a presentation of the F-form of k, which is used in [**K3**] to classify these knots for $q \geq 4$. It is essentially the same as the Λ-quintet which Farber uses in [**F2**], although his result is not restricted to the $\mathbf{Z}$-torsion-free case. By the geometric results for $q \geq 4$, there is a one-one correspondence between F-equivalence classes of Seifert matrices and F-forms, and so the results of [**K3**] also hold for $q = 3$.

1. The Seifert linking form. Let k be a simple $\mathbf{Z}$-torsion-free $2q$-knot, $q \geq 3$; then by Theorem 2 and Lemma 5 of [**L2**] there exists a Seifert surface V of k which is $(q-1)$-connected and for which $H_q(V) \cong \pi_q(V)$ has no $\mathbf{Z}$-torsion.

Let $u \in H_q(V)$, $v \in H_{q+1}(V)$, and let $i_+ \colon H_*(V) \to H_*(S^{2q+2} - V)$ denote the map induced by "pushing off" in the positive normal direction. Then u and $i_+(v)$ are represented by disjoint cycles in S^{2q+2}, hence have a linking number taking

Received by the editors August 17, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57Q45.
This paper was conceived while the first author was an S.E.R.C. Visiting Fellow at Durham University.

values in $\mathbf{Z}$. Thus we obtain a linking pairing $L: H_{q+1}(V) \times H_q(V) \to \mathbf{Z}$ given by $L(v, u) = \mathrm{link}(i_+(v), u)$.

Similarly, let $i_+: \pi_*(V) \to \pi_*(S^{2q+2} - V)$ denote the map obtained by "pushing off" in the positive normal direction. (The notation will cause no difficulty.) Any two elements $\mu, \nu \in \pi_{q+1}(V)$ can be represented by embedded spheres, and any S^{q+1} is unknotted in S^{2q+2} because the codimension is at least 3. Thus μ and $i_+(\nu)$ are represented by disjoint embeddings in S^{2q+2}, and $i_+(\nu)$ is an element of $\pi_{q+1}(S^{2q+2} - \mathrm{Im}\,\mu) \cong \pi_{q+1}(S^q) \cong \mathbf{Z}/2\mathbf{Z}$. Thus we have a homotopy linking $\mathscr{L}: \pi_{q+1}(V) \times \pi_{q+1}(V) \to \mathbf{Z}/2\mathbf{Z}$ given by $(\nu, \mu) =$ homotopy link $(i_+(\nu), \mu)$.

By a result of Whitehead [**W**, p. 555], there is a short exact sequence

$$\mathscr{E}: H_q(V)/2H_q(V) \xrightarrow{\sigma} \pi_{q+1}(V) \xrightarrow{\eta} H_{q+1}(V)$$

since $H_q(V) \cong \pi_q(V)$ by the Hurewicz theorem. Let $\tau: H_q(V) \to H_q(V)/2H_q(V)$ denote the quotient map. Then it is easy to see that

$$\mathscr{L}(\nu, \sigma\tau(u)) = L(\eta(\nu), u) \pmod 2.$$

All this data is a *Seifert linking form*: $(H_q(V), H_{q+1}(V), L, \mathscr{L}, \mathscr{E}, \tau)$. Two such are *isomorphic* if there are isomorphisms of the groups which commute with all the appropriate maps and preserve L and $\mathscr{L}$.

PROPOSITION 1.1. *The Seifert linking form determines V up to ambient isotopy.*

PROOF. Suppose that k, k' are two such knots with $(q-1)$-connected Seifert surfaces V, V' giving rise to isomorphic Seifert linking forms. Let $\varphi: H_*(V) \to H_*(V')$ be the isomorphism on homology, so that $L(v, u) = L'(\varphi(v), \varphi(u))$. Choose a basis $u_1, \ldots, u_n$ for $H_q(V)$ and set $u_i' = \varphi(u_i)$ for $1 \leq i \leq n$ to obtain a basis for $H_q(V')$. Now let B^{2q+1} be a closed $(2q+1)$-ball in the interior of V; then $H_q(V) \cong H_q(V, B^{2q+1})$ by the long exact sequence of homology, and so we obtain a basis $\tilde{u}_1, \ldots, \tilde{u}_n$ for $H_q(V, B^{2q+1})$. By handlebody theory, there is a handle decomposition of V based on B^{2q+1} involving only handles of index q and $q+1$, say $h_1^q, \ldots, h_n^q, h_n^{q+1}, \ldots, h_n^{q+1}$, such that the core C_i of h_i^q represents $\tilde{u}_i$. Now $\partial C_1, \ldots, \partial C_n$ are a set of disjoint $(q-1)$-spheres embedded in $\partial B^{2q+1} \cong S^{2q}$; since $q \geq 3$, they are unlinked and isotopic to a standard set of such spheres. That is, they are isotopic to the boundaries of a set of n disjoint q-balls embedded in S^{2q}. Thus we can ambient isotop V' so that it coincides with V on the closed ball B^{2q+1} in its interior, and then (in the obvious notation for a corresponding handle decomposition of V') we can isotop $\partial C_i'$ to coincide with ∂C_i in ∂B^{2q+1} for $1 \leq i \leq n$. Now we would like to isotop the core C_i' to coincide with C_i; standard isotopy theorems enable us to do this, for we can thicken up B^{2q+1} to $B^{2q+1} \times B^1 \subset S^{2q+2}$, and consider the C_i as embedded in $A = \mathrm{cl}[S^{2q+1} - (B^{2q+1} \times B^1)]$. Then C_1' can be homotoped to C_1 keeping the boundary fixed by Theorem 10.1 of [**H**]. If A_1 is A with an open neighourhood of C_1 excised, then A_1 is q-connected, and so C_2' can be homotoped to C_2 in A_1 keeping the boundary fixed, and hence isotoped to C_2 in A_1 keeping the boundary fixed. Continuing in this way, we can isotop $C_1' \cup \cdots \cup C_n'$ onto $C_1 \cup \cdots \cup C_n$ keeping the boundary fixed. By the argument of Levine [**L2**, §16], the obstruction to isotoping the ith q-handle of V' onto h_i^q lies in $\pi_q(S^{q+1}) = 0$, and so we can do this for each i.

Now we denote the $(q+1)$-handles of V (respectively V') by $h_1, \ldots, h_n$ (respectively $h'_1, \ldots, h'_n$). Our object is to isotop h'_i onto h_i, but first we must select our handles carefully. The basis $u_1, \ldots, u_n$ of $H_q(V)$ yields a dual basis $v_1, \ldots, v_n$ of $H_{q+1}(V)$ under the intersection pairing $H_{q+1}(V) \times H_q(V) \to \mathbf{Z}$. Set $v'_i = \varphi(v_i)$, $1 \leq i \leq n$. Let $M = B^{2q+1} \cup h_1^q \cup \cdots \cup h_n^q$, and similarly for M'; note that M coincides with M'. Since (V, M) is q-connected, we have $H_{q+1}(V) \cong H_{q+1}(V, M) \cong \pi_{q+1}(V, M)$. We can choose $h_1, \ldots, h_n$ so that the core c_i of h_i realises the element $\tilde{v}_i \in \pi_{q+1}(V, M)$ corresponding to v_i under this isomorphism. And we make the corresponding choice for $h'_i, \ldots, h'_n$, so that c'_i represents the element $\tilde{v}'_i \in \pi_{q+1}(V', M')$.

Because the q-handles are unknotted, we see that $M \cong \#\partial(S^q \times B^{q+1})_i$, the boundary connected sum taken over $i = 1$ to n, and so $\partial M \cong \#(S^q \times S^q)_i$. Clearly u_i, regarded as an element of $H_q(V)$ or $H_q(M)$ or $H_q(\partial M)$, is represented by $(S^q \times \text{point})_i$. The homology class of $d_i = (\text{point} \times S^q)_i$ being denoted by $w_i \in H_q(\partial M)$, and $H_q(\partial M)$ being identified with $\pi_q(\partial M)$ by the Hurewicz isomorphism, it follows at once that $\partial\tilde{v}_i \in \pi_q(\partial M)$ lies in the subgroup $\langle w_1, \ldots, w_n\rangle$, for otherwise the attaching sphere of h_i would represent a nonzero element of $\pi_q(M) \cong H_q(M)$, and so $H_q(V)$ would not be a free abelian group of rank n. In fact, since $v_1, \ldots, v_n$ is dual to $u_1, \ldots, u_n$, it follows that $\partial\tilde{v}_i = w_i$.

Thus the attaching sphere ∂c_1 of h_1 is homotopic to d_1 in ∂M. But $d_1 \cong S^q$ is $(q-1)$-connected, ∂M is $(q-1)$-connected, and $q \geq 3$, so we can apply Theorem 10.1 of **[H]** to show that ∂c_1 is ambient isotopic to d_1 in ∂M. The attaching sphere ∂c_2 of h_2 is homotopic to d_2 in ∂M; we need to show that ∂c_2 is homotopic to d_2 in $\partial M - d_1$. Clearly $\pi_q(\partial M - d_1) \cong \pi_q(\partial M)/\langle u_1\rangle \cong \langle u_2, \ldots, u_n, w_1, \ldots, w_n\rangle$, regarding u_1 as an element of $\pi_q(\partial M)$. Hence ∂c_2 is indeed homotopic to d_2 in $\partial M - d_1$. And hence ∂_2 is isotopic to d_2 in $\partial M - d_1$, using Hudson's result again. Continuing in this way we can isotop $\partial c_1 \cup \cdots \cup \partial c_n$ onto $d_1 \cup \cdots \cup d_n$. The same applies to $\partial c'_1 \cup \cdots \cup \partial c'_n$.

Next we wish to isotop h_i onto h'_i, keeping the attaching sphere fixed. Begin with h_1: the difference between the homotopy class of $c_1 \operatorname{rel} \partial$ in $N_1 = \operatorname{cl}[S^{2q+2} - M \times B^1]$ and that of c'_1 can be interpreted in terms of $L - L'$. Since the linking forms are isomorphic, the difference is zero, and so c_1 is homotopic $\operatorname{rel} \partial$ to c'_1, and by Hudson's result the homotopy can be realised by an ambient isotopy. Let N_2 be the closed complement of a regular neighbourhood of $M \cup h_1$ $(\operatorname{rel} \partial)$; then the difference in the homotopy classes of c_2 and c'_2 $(\operatorname{rel} \partial)$ in N_2 can be interpreted in terms of $L - L'$ and $\mathscr{L} - \mathscr{L}'$. Hence c_2 is homotopic, and so isotopic, to $c'_2 \operatorname{rel} \partial$ in N_2. Continuing in this way we obtain an ambient isotopy taking each c_i onto c'_i. Finally we can isotop h_i onto h'_i as in §16 of **[L2]**, the obstruction being $\mathscr{L}(\delta_i, \delta_i) - \mathscr{L}'(\delta'_i, \delta'_i) = 0 \in \pi_{q+1}(S^q)$, where $\delta_i \in \pi_{q+1}(V)$ is represented by $c_i \cup (\text{point} \times B^{q+1})_i \cong S^{q+1}$. Q.E.D.

2. Nice bases. Let V be a $(q-1)$-connected Seifert surface of the $\mathbf{Z}$-torsion-free simple $2q$-knot k, with $H_q(V)$ torsion-free. Consider $\mathbf{u} = \{u_1, \ldots, u_n\} \subset H_q(V)$, $\mathbf{v} = \{v_1, \ldots, v_n\} \subset H_{q+1}(V)$, $\boldsymbol{\nu} = \{\nu_1, \ldots, \nu_n\} \subset \pi_{q+1}(V)$. We say that $\mathbf{u}$ and $\mathbf{v}$ are *nice bases*, and that $\boldsymbol{\nu}$ *lies over* $\mathbf{v}$ if the following properties hold.

(i) $\mathbf{u}$ and $\mathbf{v}$ are dual bases of $H_q(V)$ and $H_{q+1}(V)$;

(ii) the u_i are represented by disjoint embedded spheres S_i^q;

(iii) the ν_i are represented by disjoint embedded spheres S_i^{q+1};

(iv) S_i^q meets S_j^{q+1} in exactly δ_{ij} points;

(v) ν_i is mapped onto v_i by the Hurewicz homomorphism.

PROPOSITION 2.1. *Let* $\mathbf{u}$ *be a basis of* $H_q(V)$. *If* $q \geq 3$, *then there exist* $\mathbf{v}$ *and* $\boldsymbol{\nu}$ *such that* $\mathbf{u}$ *and* $\mathbf{v}$ *are nice bases, and* $\boldsymbol{\nu}$ *lies over* $\mathbf{v}$.

This result is implicit in the proof of Proposition 1.1.

Let ς be the nontrivial element of $\pi_{q+1}(S^q)$, where $q \geq 3$. By the Hurewicz theorem, $H_q(V) \cong \pi_q(V)$, so thinking of u_i as an element of $\pi_q(V)$, we obtain an element $u_i \circ \varsigma \in \pi_{q+1}(V)$.

PROPOSITION 2.2. *Let* $q \geq 3$, *and suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are nice bases with* $\boldsymbol{\nu}$ *lying over* $\mathbf{v}$. *Then* $\boldsymbol{\nu}$ *can be modified in either of the following two ways to obtain* $\boldsymbol{\nu}'$, *also lying over* $\mathbf{v}$.

(i) *Replace* ν_i *by* $\nu_i' = \nu_i + u_i \circ \varsigma$.

(ii) *For* $i \neq j$, *replace* ν_i *by* $\nu_i' = \nu_i + u_j \circ \varsigma$ *and* ν_j *by* $\nu_j' = \nu_j + u_i \circ \varsigma$.

PROOF. (i) The homotopy class ν_i' can be represented by an embedding in the complement of the other $S_j^{q+1} \cup S_j^q$, using Theorem 8.1 of [**H**]. Then$S_i^q \cap (\text{new } S_i^{q+1})$ can be reduced to one point by the Whitney trick.

(ii) Let $A = S_i^q \cap S_i^{q+1}$, $B = S_j^q \cap S_j^{q+1}$, and choose a path from A to B which misses the spheres apart from its endpoints. Take a regular neighbourhood N of this path, meeting the spheres regularly. Regard N as $B^{2q} \times I$, so that $\partial N \cap (S_j^q \cup S_i^{q+1})$ is a pair of once linked spheres in $B^{2q} \times 0$, and $\partial N \cap (S_i^q \cup S_j^{q+1})$ is another such pair in $B^{2q} \times 1$.

Let $f: S_1^q \cup S_2^q \to \partial N \cap (S_i^q \cup S_i^{q+1})$ take S_2^q homeomorphically onto $\partial N \cap S_i^{q+1}$, and be such that its restriction to S_1^q is a map $S_1^q \to \partial N - S_i^{q+1} \simeq S^{q-1}$ representing a generator of $\pi_q(S^{q-1})$. Note that if $q = 3$ then $\pi_q(S^{q-1}) \cong \mathbf{Z}$, and if $q > 3$ then it is $\mathbf{Z}/2\mathbf{Z}$. Since $q \geq 3$, we can homotop $f|_{S_1^q}: S_1^q \to \partial N - S_i^{q+1}$ to an embedding, using Theorem 8.1 of [**H**]. Extend f to an embedding $f: S_1^q \times I \to B^{2q} \times I$ so that $f|_{S_1^q \times 1}$ maps $S_1^q \times 1$ homeomorphically onto $\partial N \cap S_j^{q+1}$. Note that $B^{2q} \times I - f(S_1^q \times I) \cong (B^{2q} - S^q) \times I$ since $q \geq 3$, and so f extends to an embedding $f: (S_1^q \cup S_2^q) \times I \to B^{2q} \times I$ such that $f|_{S_2^q \times 1}$ represents a generator of $\pi_q(S^{q-1}) \cong \pi_q(B^{2q} \times 1 - f(S_1^q \times 1))$.

Now $N \cap S_i^q$ is a q-ball, and hence so is $B_i^q = \mathrm{cl}[S_i^q - N \cap S_i^q]$. Let B_i^{2q+1} be a regular neighbourhood of B_i^q in $\mathrm{cl}[V - N]$, meeting the boundary regularly. We can assume that $f(S_1^q \times 0) \subset B_i^{2q+1} \cap N$, since $B^{2q} \times 0 - S_i^{q+1}$ deformation retracts onto $B_i^{2q+1} \cap N$. Now we can extend $f|_{S_1^q \times I}$ to $f: S_1^{q+1} \to V$ by coning on $S_1^q \times 0$ to get a map $B^{q+1} \to B_i^{2q+1}$, and by coning on $S_1^q \times 1$ to get a map $B^{q+1} \to S_j^{q+1} \cap \mathrm{cl}[V - N]$. Clearly $f: S_1^{q+1} \to V$ is an embedding which represents ν_j'. Similarly we can extend $f|_{S_2^q \times 1}$ to obtain an embedding representing ν_i', and $f: S_1^{q+1} \cup S_2^{q+1} \to V$ is an embedding.

Finally we use the Whitney trick to ensure that S_i^q meets $f(S_2^{q+1})$ transversely in just one point, and similarly for S_j^q and $f(S_1^{q+1})$. Q.E.D.

PROPOSITION 2.3. *Let $q \geq 3$, and let* $\mathbf{u}, \mathbf{v}$ *be nice bases of* $H_q(V)$, $H_{q+1}(V)$ *respectively, with* $\boldsymbol{\nu}$ *lying over* $\mathbf{v}$ *and* $\boldsymbol{\nu}'$ *lying over* $\mathbf{v}$. *Then*

$$\nu_i' = \nu_i + \sum_{j=1}^{n} \lambda_{ij} u_j \circ \varsigma, \qquad 1 \leq i \leq n,$$

where $\lambda_{ij} = \lambda_{ji} \pmod 2$ *for all* i, j.

PROOF. By the exact sequence of [**W**, p. 555], it is clear that $\nu_i' = \nu_i + \sum \lambda_{ij} u_j \circ \varsigma$. Using Proposition 2.2(i), we can replace ν_i' by $\nu_i' + \lambda_{ii} u_i \circ \varsigma$, and hence we can assume that $\lambda_{ii} = 0$ for all i. Now use Proposition 2.2(ii) to modify $\nu_1', \nu_2', \dots, \nu_{n-1}'$ until

$$\begin{aligned} \nu_1' &= \nu_1, \\ \nu_2' &= \mu_{21} u_1 \circ \varsigma + \nu_2, \\ &\vdots \\ \nu_n' &= \mu_{n1} u_1 \circ \varsigma + \cdots + \mu_{n\,n-1} u_{n-1} \circ \varsigma + \nu_n \end{aligned}$$

where

$$\begin{aligned} \mu_{21} &= \lambda_{21} + \lambda_{12} \pmod 2, \\ &\vdots \\ \mu_{nj} &= \lambda_{nj} + \lambda_{jn} \pmod 2. \end{aligned}$$

Let $V_1 = \mathrm{cl}[V - N(\nu_1)]$, that is, the closed complement in V of a regular neighbourhood of the embedded $(q+1)$-sphere representing ν_1. Then $V = V_1 \cup h^q \cup h^{2q+1}$, and $H_q(V) = H_q(V_1) \oplus \langle u_1 \rangle$, $\pi_{q+1}(V) = \pi_{q+1}(V_1) \oplus \langle u_1 \circ \varsigma \rangle$. Since $\nu_2' \in \pi_{q+1}(V_1)$, we see that $\mu_{21} = 0$, and hence $\lambda_{12} = \lambda_{21} \pmod 2$. A similar argument shows that each $\mu_{ij} = 0$ and hence $\lambda_{ij} = \lambda_{ji} \pmod 2$. Q.E.D.

3. Seifert matrices. Let V be a $(q-1)$-connected Seifert surface of the simple $2q$-knot k, $q \geq 3$, with $H_q(V)$ $\mathbf{Z}$-torsion-free.

Let $u_1, \dots, u_n$ be a basis of the group $H_q(V)$, and $v_1, \dots, v_n$ the dual basis of $H_{q+1}(V)$ under the Poincaré duality pairing; thus

$$(v_i, u_j) = \delta_{ij}, \qquad 1 \leq i \leq n,\ 1 \leq j \leq n.$$

Let $i_\pm$ denote the map on the homology induced by pushing a cycle off V in the $\pm$ve direction, and define

$$\left.\begin{aligned} L(i_+(v_i), u_j) &= a_{ij}, \\ L(i_-(v_i), u_j) &= b_{ij}, \end{aligned}\right. \qquad 1 \leq i \leq n,\ 1 \leq j \leq n,$$

where $L\colon H_{q+1}(S^{2q+2} - V) \times H_q(V) \to \mathbf{Z}$ is the linking pairing.

Note that

$$\begin{aligned} \delta_{ij} = (v_i, u_j) &= L(i_+(v_i) - i_-(v_i), u_j) \\ &= L(i_+(v_i), u_j) - L(i_-(v_i), u_j) = a_{ij} - b_{ij}. \end{aligned}$$

So $A - B = I$.

If we make a change of basis $u'_j = p_{jk}u_k$, $v'_i = q_{il}v_l$, then it is easily checked that $P' = Q^{-1}$ and the new matrices are $A_1 = QAQ^{-1}$, $B_1 = QBQ^{-1}$. Of course, P and Q are unimodular $n \times n$ matrices over $\mathbf{Z}$.

Now assume that $\mathbf{u}$ and $\mathbf{v}$ are nice dual bases of $H_q(V)$ and $H_{q+1}(V)$. Let $\nu_1, \dots, \nu_n$ be elements of $\pi_{q+1}(V)$ lying over $v_1, \dots, v_n$.

Define matrices C, D over $\mathbf{Z}/2\mathbf{Z}$ by

$$\mathscr{L}(i_+(\nu_i), \nu_j) = c_{ij}, \qquad \mathscr{L}(i_-(\nu_i), \nu_j) = d_{ij},$$

where

$$\mathscr{L}\colon \pi_{q+1}(S^{2q+2} - V) \times \pi_{q+1}(V) \to \mathbf{Z}/2\mathbf{Z}$$

is the homotopy linking pairing. Clearly

$$\mathscr{L}(i_+(\nu_i), \nu_j) = \mathscr{L}(\nu_i, i_-(\nu_j)) = \mathscr{L}(i_-(\nu_j), \nu_i).$$

Hence we have $c_{ij} = d_{ji}$.

For $i \neq j, i_+(\nu_j)$ is homotopic to $i_-(\nu_j)$ in the complement of ν_i, and so

$$\mathscr{L}(i_-(\nu_j), \nu_i) = \mathscr{L}(i_+(\nu_j), \nu_i).$$

Thus we have $d_{ji} = c_{ji}$ for $i \neq j$. Therefore C is symmetric, and $D = C$.

Suppose that $\nu'_1, \dots, \nu'_n$ is another set lying over $v_1, \dots, v_n$; so that $\nu'_i = \nu_i + \lambda_{ij}u_j \circ \varsigma$, where $\lambda_{ij} = \lambda_{ji}$. Then

$$\begin{aligned}\mathscr{L}(i_+(\nu'_i), \nu'_j) &= \mathscr{L}(i_+(\nu_i), \nu_j) + \mathscr{L}(i_+(\lambda_{ik}u_k \circ \varsigma), \nu_j) + \mathscr{L}(i_+(\nu_i), \lambda_{jl}u_l \circ \varsigma) \\ &= c_{ij} + \lambda_{ik}L(i_+(u_k), v_j) + \lambda_{jl}L(i_+(v_i), u_l) \\ &= c_{ij} + \lambda_{ik}L(u_k, i_-(v_j)) + \lambda_{jl}L(i_+(v_i), u_l) \\ &= c_{ij} + \lambda_{ik}b_{jk} + \lambda_{jl}a_{il}.\end{aligned}$$

Thus $C_1 = C + \Lambda B' + A\Lambda'$.

Any unimodular integer matrix Q can be written as a product of elementary integer matrices: this is proved using the Euclidean algorithm column by column. What happens if we make such a change of basis in $H_q(V)$? Say $u'_1 = u_1 + u_2$, $u'_i = u_i$ for $2 \leq i \leq n$. Then $v'_2 = v_2 - v_1$, $v'_i = v_i$ for $i \neq 2$. We claim that $\nu'_1, \dots, \nu'_n$ lies over $v'_1, \dots, v'_n$ where $\nu'_2 = \nu_2 - \nu_1$, $\nu'_i = \nu_i$ for $i \neq 2$.

For ν'_2 can be represented by an embedded $(q+1)$-sphere, simply by taking the connected sum of ν_1 and ν_2 with suitable orientations. A similar statement holds for u'_1. The Whitney trick can then be used to make ν'_2 and u'_1 disjoint.

Thus if $u'_j = p_{jk}u_k$, $v'_i = q_{il}v_l$, then $\nu'_i = q_{il}\nu_l$ lies over v'_i. Hence

$$\mathscr{L}(i_+(\nu'_i), \nu'_j) = \mathscr{L}(i_+(q_{il}\nu_l), q_{jk}\nu_k) = q_{il}c_{lk}q_{jk}$$

and so $C_1 = QCQ'$.

It follows that any change of basis in $H_q(V)$, represented by a unimodular integer matrix $P = Q'^{-1}$, can be realised geometrically by embedded spheres.

PROPOSITION 3.1. *Let k, k_1 be simple $\mathbf{Z}$-torsion-free $2q$-knots, $q \geq 3$, with $(q-1)$-connected Seifert surfaces V, V_1 respectively, such that $H_q(V)$, $H_q(V_1)$ are torsion-free. Let (A, B, C), (A_1, B_1, C_1) be Seifert matrices arising from V, V_1. Then k is ambient isotopic to k_1 if and only if (A, B, C) is related to (A_1, B_1, C_1) as above.*

PROOF. The "only if" part has already been established. So assume that the two sets of matrices are related as above. Then by a change of basis in $H_q(V_1)$ and

$\pi_{q+1}(V_1)$ we can assume that $A_1 = A$, $B_1 = B$, $C_1 = C$. The Seifert linking forms are therefore isomorphic, and Proposition 1.1 completes the proof. Q.E.D.

We conclude this section with a realisation result.

PROPOSITION 3.2. *Let A, B be $n \times n$ integer matrices satisfying $A - B = I$, and C a symmetric $n \times n$ matrix over $\mathbf{Z}/2\mathbf{Z}$. Then for $q \geq 3$, there exists a simple $\mathbf{Z}$-torsion-free $2q$-knot k with A, B, C as Seifert matrices.*

PROOF. It is implicit in the proof of Theorem II.2 of [**Ke**] that there is a simple $\mathbf{Z}$-torsion-free $2q$-knot k_0 realising the matrices A, B by means of a $(q-1)$-connected Seifert surface V, with $H_q(V)$ $\mathbf{Z}$-torsion-free. Let C_0 be the $\mathbf{Z}/2\mathbf{Z}$ matrix associated with the handle decomposition of this matrix, that is, with the basis $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\nu}$. By altering the twisting and homotopy linking of the $(q+1)$-handles of V, we can change C_0 to C without altering A or B. With this new embedding of V, we obtain the desired knot $k = \partial V$. Q.E.D.

4. Ambient surgery. Let k be a simple, $\mathbf{Z}$-torsion-free $2q$-knot $2q$-knot, $q \geq 3$. Assume that V_0, V_1 are two Seifert surfaces of k which are each $(q-1)$-connected and have no $\mathbf{Z}$-torsion in homology. By results of Levine [**L1**] or Farber [**F2**], there exists a submanifold $W \subset S^{2q+2} \times I$ such that if $W_t = W \cap (S^{2q+2} \times t)$, then $W_0 = V_0$, $W_1 = V_1$, and $\partial W = V_0 \cup (S^{2q} \times I) \cup V_1$. Moreover we can by ambient surgery ensure that W is $(q-1)$-connected and that $\pi_q(W)$ is $\mathbf{Z}$-torsion-free.

Take a handle decomposition of (W, W_0) with handles of index $q, q+1, q+2$; this is possible because (W, W_0) is $(q-1)$-connected and so is (W, W_1), and $\dim W = 2q+2 \geq 8$. Let $W_0 \times I$ be the collar of W_0 to which the handles are added; then$H_q(W_0 \times I \cup q$-handles) is free abelian. In these dimensions the standard handle moves can all be realised geometrically, and so because $H_q(W)$ is $\mathbf{Z}$-torsion-free we can arrange that for each $(q+1)$-handle, either its attaching sphere is part of a basis for a direct summand of $H_q(W_0 \times I \cup q$-handles) or is 0 in $H_q(W_0 \times I)$. In the latter case the core of the $(q+1)$-handle yields a basis element for $H_{q+1}(W)$.

Let N be a regular neighourhood of $(S^{2q+2} \times 0) \cup W$, and take a handle decomposition of $S^{2q+2} \times I$ based on N. Now $S^{2q+2} \times I - W$ is $(q-1)$-connected, and hence so is the pair $(S^{2q+2} \times I, W)$. Thus we can arrange that the handle decomposition involves only handles of index q, $q+1$, $q+2$, and $q+3$.

Each handle of $(W, W_0 \times I)$ yields a handle of $(N, S^{2q+2} \times I')$, where $S^{2q+2} \times I'$ is a collar neighbourhood of $S^{2q+2} \times 0$. Denote the r-handles of N by h^r, and of $(S^{2q+2} \times I, N)$ by H^r. We have say $h_1^q, \ldots, h_m^q, H_1^q, \ldots, H_n^q$. These must be cancelled (as a set) by the $(q+1)$-handles, and so if we add trivial $(q+1, q+2)$-pairs of handles, we can move the new $(q+1)$-handles (all H^{q+1}'s) over the existing $(q+1)$-handles to obtain a set $H_1^{q+1}, \ldots, H_r^{q+1}$ which cancels the q-handles. Of course, $r \geq m+n$. From the attaching spheres we obtain an $r \times (m+n)$ matrix of integers U, with one column for each q-handle and one row for each H_i^{q+1}

$$U = \Big(\overbrace{\qquad\qquad}^{h^q} \Big| \overbrace{\qquad\qquad}^{H^q} \Big) \Big\} H^{q+1}.$$

Moving one $(q+1)$-handle over another adds one row to another, and similarly for q-handles and columns. Of course, we cannot move an h^q over an H^q, but otherwise there are no restrictions. Because all the q-handles are cancelled, the hcf of the last column of U is 1. Repeated application of the Euclidean algorithm to the final column enables us, by moving the H_i^{q+1} over each other, to put U in the form $\left(\begin{smallmatrix} R & 0 \\ * & 1 \end{smallmatrix}\right)$ where R is an $(r-1)\times(m+n-1)$ matrix. Repeatedly moving H_n^q around puts U in for the form $\left(\begin{smallmatrix} R & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Now we can cancel H_n^q and H_r^{q+1}, replacing U by the smaller matrix R.

Continue in this way until all the H_i^q have been cancelled. Repeat the performance to obtain a matrix

$$\overbrace{\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}}^{h^q}\Big\} H^{q+1}.$$

Now we can "cancel" h_m^q and H_{r-n}^{q+1}; what this gives us is a critical level decomposition involving a q-handle added to a neighbourhood of W_0, in other words an ambient surgery of index q performed on $V_0 = W_0$.

Repeat until all the h_i^q have been "cancelled" in this way.

Using the dual handle decomposition of (W, W), we can repeat the procedure above. Once this has been done, we are left with a manifold (which by abuse of notation we still call W) having handles only of index $q+1$. Moreover, if N is a regular neighbourhood of $(S^{2q+2}\times 0)\cup W$, then there is a handle decomposition of $S^{2q+2}\times I$ based on N which has handles only of index $q+1$, $q+2$.

The attaching spheres of the $(q+2)$-handles yield a square integer matrix U as above

$$U = (\overbrace{\qquad}^{h^{q+1}} \mid \overbrace{\qquad}^{H^{q+1}})\}H^{q+2}.$$

Of course, U is unimodular, and so may be written as a product of elementary matrices. The argument above goes through to show that W is the trace of ambient surgeries of index $q+1$.

5. The effect of ambient surgery. In this section we consider the effect of ambient surgery, of index q, $q+1$, and $q+2$, on a $(q-1)$-connected Seifert surface V of the **Z**-torsion-free simple $2q$-knot k, where $q \geq 3$ and $H_q(V)$ is torsion-free.

(i) INDEX q. Since V is $(q-1)$-connected, the attaching sphere is null-homotopic, and so we have a new basis element u_{n+1} of $H_q(U)$, where U is the new Seifert surface. Thus $H_q(U) = H_q(V)\oplus\langle u_{n+1}\rangle$. The belt sphere of the surgery supplies a new v_{n+1}, so that $v_1, \ldots, v_{n+1}$ is dual to $u_1, \ldots, u_{n+1}$. Indeed, the belt sphere supplies $\nu_{n+1}\in\pi_{q+1}(U)$ which lies over v_{n+1}. Depending on which side of V the surgery is performed, A is replaced by

$$\underset{+\text{ve side}}{\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}} \qquad \underset{-\text{ve side}}{\begin{pmatrix} A & \alpha \\ 0 & 0 \end{pmatrix}}$$

where α is a column vector of integers.

The matrix B is therefore replaced by

$$\underset{+\text{ve side}}{\begin{pmatrix} B & \alpha \\ 0 & 0 \end{pmatrix}} \qquad \underset{-\text{ve side}}{\begin{pmatrix} B & \alpha \\ 0 & -1 \end{pmatrix}}.$$

The matrix C is replaced by $\left(\begin{smallmatrix} C & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in either case.

(ii) INDEX $q+1$. There are two possibilities, since $H_q(U)$ must be $\mathbf{Z}$-torsion-free. Either the attaching sphere of the surgery represents 0 in $H_q(V)$, or else it represents a primitive element, i.e., one that forms part of a basis of $H_q(V)$. These two possibilities are in fact dual to each other, for in the former case $H_r(U) = H_r(V) \oplus \mathbf{Z}$ for $r = q, q+1$, and in the latter case $H_r(V) = H_r(U) \oplus \mathbf{Z}$ for $r = q, q+1$.

Thus we need only investigate the former case, for the latter will induce the inverse effect on A, B, and C. Let u_{n+1} be the basis element of $H_q(U)$ represented by the belt sphere of the surgery, and ν_{n+1} be the element of $\pi_{q+1}(U)$ represented by $B^{q+1} \times * \subset B^{q+1} \times \partial B^{q+1}$ together with a null-homotopy of $\partial B^{q+1} \times *$. We can assume that this null-homotopy misses $\nu_1, \ldots, \nu_n$, for $V - \bigcup_{i=1}^n \nu_i$ is $(q-1)$-connected (the fibre of the sphere bundle over ν_i associated with the normal bundle is null-homotopic, using u_i) and

$$\pi_q\left(V - \bigcup_{i=1}^n \nu_i\right) \cong H_q\left(V - \bigcup_{i=1}^n \nu_i\right) \cong H_q(V).$$

Thus we can choose a null-homotopy in $V - \bigcup_{i=1}^n \nu_i$. Since $q \geq 3$ and $U - \bigcup_{i=1}^n \nu_i$ $(q-1)$-connected, ν_{n+1} may be homotoped to an embedding, and indeed we now have $\nu_1, \ldots, \nu_{n+1}$ lying over $v_1, \ldots, v_{n+1}$, a basis for $H_{q+1}(U)$.

The effect on A is

$$\underset{+\text{ve side}}{\begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix}} \qquad \underset{-\text{ve side}}{\begin{pmatrix} A & 0 \\ \alpha & 1 \end{pmatrix}}$$

where α is a row vector of integers.

The matrix B is therefore replaced by

$$\underset{+\text{ve side}}{\begin{pmatrix} B & 0 \\ \alpha & -1 \end{pmatrix}} \qquad \underset{-\text{ve side}}{\begin{pmatrix} B & 0 \\ \alpha & 0 \end{pmatrix}}.$$

The matrix C is replaced by $\left(\begin{smallmatrix} C & \beta' \\ \beta & \gamma \end{smallmatrix}\right)$ where β is a row vector with entries in $\mathbf{Z}/2\mathbf{Z}$, and $\gamma \in \mathbf{Z}/2\mathbf{Z}$.

(iii) INDEX $q+2$. This is the inverse of a surgery of index q, and so the effect on the matrices is the inverse of (i).

REMARK. In (ii) we could choose γ to be 0, or 1. For replace the element ν_{n+1} by $\nu'_{n+1} = \nu_{n+1} + u_{n+1} \circ \varsigma$. This can be represented by an embedded sphere, and the effect on C is to replace γ by $\gamma + 1 \pmod 2$.

PROPOSITION 5.1. *For $q \geq 3$, each of the above algebraic operations on A, B, C, may be realised geometrically by an ambient surgery on V.*

PROOF. (i) INDEX q. Suppose we have to realise the matrices

$$\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} B & \alpha \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Fatten V to a tubular neighbourhood $V \times I$, and embed a sphere S^q in $S^{2q+2} - V \times I$ so that its linking number with v_i is α_i $(1 \leq i \leq n)$. Because the codimension is at least 3, S^q is unknotted in S^{2q+2} and so has trivial normal bundle $S^q \times B^{q+2}$.

Take the boundary connected sum of $V \times I$ and $S^q \times B^{q+2}$, using the positive side of $V \times I$. This is the trace of the ambient surgery.

(ii) INDEX $q+1$. Suppose that we have to realise the matrices

$$\begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix}, \qquad \begin{pmatrix} B & 0 \\ \alpha & -1 \end{pmatrix}, \qquad \begin{pmatrix} C & \beta' \\ \beta & \gamma \end{pmatrix}.$$

Once again, fatten V to $V \times I$, and embed a sphere S^{q+1} in $S^{2q+2} - V \times I$ so that it has linking number α_i with u_i $(1 \le i \le n)$, and homotopy linking β_i with ν_i $(1 \le i \le n)$. Note that there is no difficulty about obtaining an embedding: we just use Theorem 8.1 of [**H**]. Take a regular neighbourhood $S^{q+1} \times B^{q+1}$ of S^{q+1}, and then take the boundary connected sum of this with $V \times I$, attaching it on the positive side. The new basis elements are realised by $S^{q+1} \times$ point and point $\times\, S^q$, both contained in $S^{q+1} \times \partial B^{q+1}$, and clearly we realise the desired matrices, except possibly for γ. But by the remark above, if we can realise $\gamma + 1$, then we can realise γ.

Of course, similar arguments hold in the case of the negative side of V. To realise the dual case of index $q+1$, we realise the reduced matrices by a new knot and Seifert surface V_1, using Proposition 3.2, perform ambient surgery on V_1 to realise the original matrices, and then appeal to Proposition 3.1 to show that the new Seifert surface is ambient isotopic to V. Hence we have an ambient surgery which realises the algebraic move. A similar argument applies to index $q+2$, which is dual to index q.

6. Presentation of the F-form. Recall that $u_1, \dots, u_n$ is a basis of $H_q(V)$ and $v_1, \dots, v_n$ the dual basis of $H_{q+1}(V)$. Let $\nu_1, \dots, \nu_n$ be elements of $\pi_{q+1}(V)$ lying over $v_1, \dots, v_n$. Then $S^{2q+2} - V$ is $(q-1)$-connected, and we can choose bases $\alpha_1, \dots, \alpha_n$ of $H_q(S^{2q+2} - V)$ and $\beta_1, \dots, \beta_n$ of $H_{q+1}(S^{2q+2} - V)$ as follows. The bases $u_1, \dots, u_n$ and $\nu_1, \dots, \nu_n$ are represented by spheres $S^q_1, \dots, S^q_n$ and $S^{q+1}_1, \dots, S^{q+1}_n$ embedded in V, such that S^q_i meets S^{q+1}_j transversely in δ_{ij} points, and otherwise these spheres are disjoint. Each sphere is unknotted in S^{2q+2}, having codimension at least $q+1 > 3$, and so has trivial normal bundle. Let α_i be represented by the boundary of a fibre of the bundle over S^{q+1}_i, and β_i by the boundary of a fibre of the bundle over S^q_i. Let $\gamma_i \in \pi_{q+1}(S^{2q+2} - V)$ lie over β_i, and note that these elements can be chosen so that

$$\left.\begin{aligned} L(\beta_i, u_j) = \delta_{ij} = L(v_i, \alpha_j), \\ \mathscr{L}(\gamma_i, \nu_j) = 0, \end{aligned}\right. \qquad 1 \le i, j \le n.$$

Let $i_+(v_i) = h_{ij}\beta_j$; then

$$a_{ij} = L(i_+(v_i), u_j) = L(h_{ik}\beta_k, u_j) = h_{ik}L(\beta_k, u_j) = h_{ik}\delta_{kj} = h_{ij}.$$

Thus $i_+(v_i) = a_{ij}\beta_j$, and similarly $i_-(v_i) = b_{ij}\beta_j$.

We can write $i_+(\nu_i) = a_{ij}\gamma_j + e_{ij}\alpha_j \circ \varsigma$, and using the fact that $(\alpha_k \circ \varsigma, \nu_j) \equiv L(\alpha_k, v_j) \equiv \delta_{kj} \pmod 2$, we see that

$$\begin{aligned} c_{ij} &= \mathscr{L}(i_+(\nu_i), \nu_j) = \mathscr{L}(a_{ik}\gamma_k + e_{ik}\alpha_k \circ \varsigma, \nu_j) \\ &= a_{ik}\mathscr{L}(\gamma_k, \nu_j) + e_{ik}\mathscr{L}(\alpha_k \circ \varsigma, \nu_j) = e_{ik}\delta_{kj} = e_{ij}. \end{aligned}$$

Thus $i_+(\nu_i) = a_{ij}\gamma_j + c_{ij}\alpha_j \circ \varsigma$, and similarly $i_-(\nu_i) = b_{ij}\gamma_j + c_{ij}\alpha_j \circ \varsigma$.

By standard arguments, $tA - B$ is a presentation matrix for $H_{q+1}(\tilde{K})$ as a Λ-module; that is,

$$H_{q+1}(\tilde{K}) \cong (\beta_1, \ldots, \beta_n : (ta_{ij} - b_{ij})\beta_j,\ 1 \le i \le n).$$

Let $i_-(u_i) = f_{ij}\alpha_j$; then

$$\begin{aligned} a_{ij} &= L(i_+(v_i), u_j) = L(v_i, i_-(u_j)) = L(v_i, f_{jk}\alpha_k) \\ &= f_{jk}L(v_i, \alpha_k) = f_{jk}\delta_{ik} = f_{ji} \end{aligned}$$

so that $i_-(u_i) = a_{ji}\alpha_j$. And similarly $i_+(u_i) = b_{ji}\alpha_j$. Thus $H_q(\tilde{K})$ is presented as a Λ-module by $tB' - A'$. Allowing α_i, β_j to represent their images in $H_q(\tilde{K})$, $H_{q+1}(\tilde{K})$, respectively, the Blanchfield pairing is given (up to sign) by the formula

$$\langle \beta_i, \alpha_j \rangle \equiv (t-1)(tA - B)^{-1}_{ij} \pmod{\Lambda}.$$

There is a map of Γ-modules

$$\begin{aligned} &(\alpha_1 \circ \varsigma, \ldots, \alpha_n \circ \varsigma, \gamma_1, \ldots, \gamma_n : \\ &\qquad (ti_+(u_i) - i_-(u_i)) \circ \varsigma, ti_+(\nu_i) - i_-(\nu_i),\ 1 \le i \le n) \to \Pi_{q+1}(\tilde{K}), \end{aligned}$$

that is,

$$\begin{aligned} &(\alpha_1 \circ \varsigma, \ldots, \alpha_n \circ \varsigma, \gamma_1, \ldots, \gamma_n : \\ &\qquad (tb_{ji} - a_{ji})\alpha_j \circ \varsigma, (ta_{ij} - b_{ij})\gamma_j + (t-1)c_{ij}\alpha_j \circ \varsigma) \to \Pi_{q+1}(\tilde{K}). \end{aligned}$$

Denoting this presentation Γ-module by N, and the Γ-module

$$(\alpha_1 \circ \varsigma, \ldots, \alpha_n \circ \varsigma : (tb_{ji} - a_{ji})\alpha_j \circ \varsigma,\ 1 \le i \le n)$$

by M, and the Γ-module

$$(\beta_1, \ldots, \beta_n : (ta_{ij} - b_{ij})\beta_j,\ 1 \le i \le n)$$

by P, we see that there is a commutative diagram

$$\begin{array}{ccccc} M & \to & N & \to & P \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{H}_q(\tilde{K}) & \to & \Pi_{q+1}(\tilde{K}) & \to & \mathscr{H}_{q+1}(\tilde{K}) \end{array}$$

of Γ-modules, both rows being short exact sequences.

The first and third vertical arrows are isomorphisms, and so by the five-lemma is the middle one. Hence we have a presentation for $\Pi_{q+1}(\tilde{K})$ as a Γ-module. As in [**K**] the hermitian pairing is given by

$$[\gamma_i \gamma_i] \equiv (t-1)[(tA - B)^{-1}(t^{-1}C' - C)(B' - t^{-1}A')]_{ij}.$$

7. Seifert matrices and F-forms. Let A, B, C be the Seifert matrices of a simple $\mathbf{Z}$-torsion-free $2q$-knot k, $q \ge 3$, arising from a choice of basis $\mathbf{u}$ of $H_q(V)$ where V is a $(q-1)$-connected Seifert surface of k, with $H_q(V)$ torsion free. Of course, we also have in mind a choice of $\boldsymbol{\nu} \in \pi_{q+1}(V)$ lying over $\mathbf{v}$, the dual basis of $\mathbf{u}$. In §3 we investigated the way in which A, B, C change when $\mathbf{u}$ and $\boldsymbol{\nu}$ are changed, and in §5 the way an ambient surgery on V affects them. Call the equivalence relation generated by these changes *F-equivalence*.

THEOREM 7.1. *Let A, B be $n \times n$ integer matrices satisfying $A - B = I$, and C a symmetric $n \times n$ matrix over $\mathbf{Z}/2\mathbf{Z}$; and let A_1, B_1, C_1 be another such set of $m \times m$ matrices. Then (A, B, C) is F-equivalent to (A_1, B_1, C_1) if and only if they present isomorphic F-forms.*

PROOF. Choose an integer $q \geq 4$, and by Proposition 3.1 realise A, B, C as Seifert matrices of a simple $\mathbf{Z}$-torsion-free $2q$-knot k, and A_1, B_1, C_1 as Seifert matrices of a similar knot k_1. Then if the F-forms of k and k_1 are isomorphic, by Corollary 11.3 of [**K**], k and k_1 are ambient isotopic. Hence the Seifert surfaces, V and V_1 say, are related by a sequence of ambient surgeries and so (A, B, C) is F-equivalent to (A_1, B_1, C_1). Conversely, if the matrices are F-equivalent we can realise the algebraic moves by ambient surgeries; hence k and k_1 are ambient isotopic and so the F-forms are isometric. Q.E.D.

The F-form of a simple $\mathbf{Z}$-torsion-free $2q$-knot is defined in [**K3**] for $q \geq 4$. We are in a position to extend this to the case $q = 3$.

THEOREM 7.2. *Every simple $\mathbf{Z}$-torsion-free $2q$-knot, $q \geq 3$, gives rise to an F-form which is an invariant of the knot.*

PROOF. The case $q > 3$ is dealt with in [**K**]. For $q = 3$, the Whitehead exact sequence

$$2H_q(\tilde{K}) \to H_q(\tilde{K}) \to \pi_{q+1}(\tilde{K}) \twoheadrightarrow H_{q+1}(\tilde{K})$$

holds, and so we have the necessary modules and sequences. (Recall that $\tilde{K} \to K$ is the infinite-cyclic cover of the exterior K of the knot k.)

In [**K**, §1], the pairing $\{\ ,\ \}$ is well defined except for the choice of α. If we take $\alpha = \Delta\overline{\Delta}$ as in [**K**, Corollary 1.3], then this is a canonical choice, being a knot invariant, and so we obtain a homotopy linking which is well defined for $q = 3$.

Alternatively, we could define the F-form by means of the presentation in terms of A, B, C, and use Theorem 7.1. Q.E.D.

An axiomatic description of F-forms is given in [**K3**], together with a realisation theorem for $q \geq 4$. Again, we extend that result to the case $q = 3$.

THEOREM 7.3. *Every F-form can be realised by a simple $\mathbf{Z}$-torsion-free $2q$-knot, $q \geq 3$.*

PROOF. For $q > 3$, the F-form can be realised by a simple $\mathbf{Z}$-torsion-free $2q$-knot. From this we obtain Seifert matrices A, B, C which present the F-form. And Proposition 3.2 yields a simple $\mathbf{Z}$-torsion-free 6-knot with these as Seifert matrices, hence with F-form isometric to the given one. Q.E.D.

Finally we show that F-form is a complete invariant of the knot when $q \geq 3$.

THEOREM 7.4. *Two simple $\mathbf{Z}$-torsion-free $2q$-knots, $q \geq 3$, with isometric F-forms, are ambient isotopic.*

PROOF. By [**K**, Theorem 11.1] the result is true for $q \geq 4$. For $q = 3$, let (A, B, C), (A_1, B_1, C_1) be Seifert matrices arising from Seifert surfaces V, V_1 of the knots k, k_1. Since the F-forms are isometric, the Seifert matrices are F-equivalent by Theorem 7.1. The algebraic moves on the matrices can be realised geometrically, by Proposition 5.1, and so k and k_1 are isotopic by Proposition 3.1. Q.E.D.

REFERENCES

[**F1**] M. Sh. Farber, *Isotopy types of knots of codimension two*, Trans. Amer. Math. Soc. **261** (1980), 185–209.

[**F2**] ——, *The classification of simple knots*, Russian Math. Surveys **38** (1983), 63–117.

[**H**] J. F. P. Hudson, *Piecewise linear topology*, Benjamin, New York, 1969.

[**K1**] C. Kearton, *Classification of simple knots by Blanchfield duality*, Bull. Amer. Math. Soc. **79** (1973), 952–955.

[**K2**] ——, *An algebraic classification of some even-dimensional simple knots*, Topology **15** (1976), 363–373.

[**K3**] ——, *An algebraic classification of certain simple even-dimensional knots*, Trans. Amer. Math. Soc. **276** (1983), 1–53.

[**Ke**] M. A. Kervaire, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. France **93** (1965), 225–271.

[**Ko1**] S. Kojima, *A classification of some even-dimensional fibred knots*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 671–683.

[**Ko2**] ——, *Classification of simple knots by Levine pairings*, Comment. Math. Helv. **54** (1979), 356–367.

[**L1**] J. Levine, *Unknotting spheres in codimension two*, Topology **4** (1965), 9–16.

[**L2**] ——, *An algebraic classification of some knots of codimension two*, Comment. Math. Helv. **45** (1970), 185–198.

[**T1**] H. F. Trotter, *On S-equivalence of Seifert matrices*, Invent. Math. **20** (1973), 173–207.

[**T2**] ——, *Knot modules and Seifert matrices*, Lecture Notes in Math., vol. 685, Springer-Verlag, Berlin and New York, 1978, pp. 291–299.

[**W**] G. W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, Berlin and New York, 1978.

SCHOOL OF MATHEMATICS, PHYSICS, COMPUTING AND ELECTRONICS, MACQUARIE UNIVERSITY, N.S.W. 2109 AUSTRALIA

DEPARTMENT OF MATHEMATICS, SCIENCE LABORATORIES, DURHAM UNIVERSITY, DURHAM DH1 3LE, ENGLAND