SEIFERT MATRICES AND 6-KNOTS

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Abstract. A new classification of simple \( Z \)-torsion-free \( 2q \)-knots, \( q \geq 3 \), is given in terms of Seifert matrices modulo an equivalence relation. As a result the classification of such \( 2q \)-knots, \( q \geq 4 \), in terms of \( F \)-forms is extended to the case \( q = 3 \).

0. Introduction. An \( n \)-knot \( k \) is an oriented locally flat PL sphere-pair \((S^{n+2}, S^n)\); equally, one could consider oriented smooth pairs where the embedded sphere is allowed to carry an exotic differentiable structure. Let \( K \) denote the closed complement of a regular neighbourhood of \( S^n \), often called the exterior of \( k \). Then \( k \) is simple if \( K \) has the homotopy \([ (n - 1)/2 ]\)-type of a circle; this is the most that can be asked without making \( k \) unknotted. With the exception of \( n = 4 \) and 6, the simple \( n \)-knots \((n \geq 3)\) have been classified in various ways during the past twenty years. The first such result is due to J. Levine \([L2]\), who classified the simple \((2q - 1)\)-knots \((q \geq 2)\) in terms of the Seifert matrix and \( S \)-equivalence. These knots were then classified in terms of the Blanchfield duality pairing in \([T1, T2, \text{and } K1]\). Results for certain classes of simple \( 2q \)-knots \((q \geq 4)\) may be found in \([K2, Ko1, Ko2, \text{and } K3]\); the general case is given by M. Sh. Farber in \([F2]\). One should also mention here the pioneering work of M. A. Kervaire \([Ke]\), who characterised the homology modules which can occur, in terms of presentation matrices.

Let \( K \to K \) be the infinite cyclic (= universal) cover of \( K \), where \( k \) is a simple \( 2q \)-knot, \( q \geq 3 \). Then \( k \) is \( Z \)-torsion-free if \( H_q(K) \) has no \( Z \)-torsion. In this paper we classify such knots in terms of Seifert matrices and an equivalence relation which we call \( F \)-equivalence. These matrices yield a presentation of the \( F \)-form of \( k \), which is used in \([K3]\) to classify these knots for \( q \geq 4 \). It is essentially the same as the \( \Lambda \)-quintet which Farber uses in \([F2]\), although his result is not restricted to the \( Z \)-torsion-free case. By the geometric results for \( q \geq 4 \), there is a one-one correspondence between \( F \)-equivalence classes of Seifert matrices and \( F \)-forms, and so the results of \([K3]\) also hold for \( q = 3 \).

1. The Seifert linking form. Let \( k \) be a simple \( Z \)-torsion-free \( 2q \)-knot, \( q \geq 3 \); then by Theorem 2 and Lemma 5 of \([L2]\) there exists a Seifert surface \( V \) of \( k \) which is \((q - 1)\)-connected and for which \( H_q(V) \cong \pi_q(V) \) has no \( Z \)-torsion.

Let \( u \in H_q(V) \), \( v \in H_{q+1}(V) \), and let \( i_+ : H_*(V) \to H_*(S^{2q+2} - V) \) denote the map induced by “pushing off” in the positive normal direction. Then \( u \) and \( i_+(v) \) are represented by disjoint cycles in \( S^{2q+2} \), hence have a linking number taking
values in $\mathbb{Z}$. Thus we obtain a linking pairing $L: H_{q+1}(V) \times H_q(V) \to \mathbb{Z}$ given by $L(v, u) = \text{link}(i_+(v), u)$.

Similarly, let $i_+: i_*(V) \to i_*(S^{2q+2} - V)$ denote the map obtained by “pushing off” in the positive normal direction. (The notation will cause no difficulty.) Any two elements $\mu, \nu \in i_{q+1}(V)$ can be represented by embedded spheres, and any $S^{q+1}$ is unknotted in $S^{2q+2}$ because the codimension is at least 3. Thus $\mu$ and $i_+(\nu)$ are represented by disjoint embeddings in $S^{2q+2}$, and $i_+(\nu)$ is an element of $i_{q+1}(S^{2q+2} - \text{Im} \mu) \cong i_{q+1}(S^q) \cong \mathbb{Z}/2\mathbb{Z}$. Thus we have a homotopy linking $\mathcal{L}: i_{q+1}(V) \times i_{q+1}(V) \to \mathbb{Z}/2\mathbb{Z}$ given by $(v, \mu) = \text{homotopy link}(i_+(v), \mu)$.

By a result of Whitehead [W, p. 555], there is a short exact sequence

$$\mathcal{E}: H_q(V)/2H_q(V) \xrightarrow{\sigma} i_{q+1}(V) \xrightarrow{\eta} H_{q+1}(V)$$

since $H_q(V) \cong i_q(V)$ by the Hurewicz theorem. Let $\tau: H_q(V) \to H_q(V)/2H_q(V)$ denote the quotient map. Then it is easy to see that

$$\mathcal{L}(v, \sigma \tau(u)) = L(\tau(v), u) \pmod{2}.$$

All this data is a Seifert linking form: $(H_q(V), H_{q+1}(V), L, \mathcal{L}, \mathcal{E}, \tau)$. Two such are isomorphic if there are isomorphisms of the groups which commute with all the appropriate maps and preserve $L$ and $\mathcal{L}$.

**Proposition 1.1.** The Seifert linking form determines $V$ up to ambient isotopy.

**Proof.** Suppose that $k, k'$ are two such knots with $(q-1)$-connected Seifert surfaces $V, V'$ giving rise to isomorphic Seifert linking forms. Let $\varphi: H_*(V) \to H_*(V')$ be the isomorphism on homology, so that $L(v, u) = L'(\varphi(v), \varphi(u))$. Choose a basis $u_1, \ldots, u_n$ for $H_q(V)$ and set $u'_i = \varphi(u_i)$ for $1 \leq i \leq n$ to obtain a basis for $H_q(V')$. Now let $B^{2q+1}$ be a closed $(2q+1)$-ball in the interior of $V$; then $H_q(V) \cong H_q(V, B^{2q+1})$ by the long exact sequence of homology, and so we obtain a basis $\tilde{u}_1, \ldots, \tilde{u}_n$ for $H_q(V, B^{2q+1})$. By handlebody theory, there is a handle decomposition of $V$ based on $B^{2q+1}$ involving only handles of index $q$ and $q+1$, say $h^q_1, \ldots, h^q_n, h^{q+1}_n, \ldots, h^{q+1}_n$, such that the core $C_i$ of $h^q_i$ represents $\tilde{u}_i$. Now $\partial C_1, \ldots, \partial C_n$ are a set of disjoint $(q-1)$-spheres embedded in $\partial B^{2q+1} \cong S^{2q}$; since $q \geq 3$, they are unlinked and isotopic to a standard set of such spheres. That is, they are isotopic to the boundaries of a set of $n$ disjoint $q$-balls embedded in $S^{2q}$. Thus we can ambient isotop $V'$ so that it coincides with $V$ on the closed ball $B^{2q+1}$ in its interior, and then (in the obvious notation for a corresponding handle decomposition of $V'$) we can isotope $\partial C'_i$ to coincide with $\partial C_i$ in $\partial B^{2q+1}$ for $1 \leq i \leq n$. Now we would like to isotope the core $C'_i$ to coincide with $C_i$; standard isotopy theorems enable us to do this, for we can thicken up $B^{2q+1}$ to $B^{2q+1} \times B^1 \subset S^{2q+2}$, and consider the $C_i$ as embedded in $A = \text{cl}[S^{2q+1} - (B^{2q+1} \times B^1)]$. Then $C'_i$ can be homotoped to $C_i$ keeping the boundary fixed by Theorem 10.1 of [H]. If $A_1$ is $A$ with an open neighbourhood of $C_1$ excised, then $A_1$ is $q$-connected, and so $C'_2$ can be homotoped to $C_2$ in $A_1$ keeping the boundary fixed, and hence isotoaped to $C_2$ in $A_1$ keeping the boundary fixed. Continuing in this way, we can isotope $C'_1 \cup \cdots \cup C'_n$ onto $C_1 \cup \cdots \cup C_n$ keeping the boundary fixed. By the argument of Levine [L2, §16], the obstruction to isotoping the $i$th $q$-handle of $V'$ onto $h^q_i$ lies in $\pi_2(S^{q+1}) = 0$, and so we can do this for each $i$. 

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Now we denote the \((q + 1)\)-handles of \(V\) (respectively \(V'\)) by \(h_1, \ldots, h_n\) (respectively \(h'_1, \ldots, h'_n\)). Our object is to isotop \(h'_i\) onto \(h_i\), but first we must select our handles carefully. The basis \(u_1, \ldots, u_n\) of \(H_q(V)\) yields a dual basis \(v_1, \ldots, v_n\) of \(H_{q+1}(V)\). Set \(v'_i = \varphi(v_i), 1 \leq i \leq n\). Let \(M = B^{2q+1} \cup h_1^2 \cup \cdots \cup h_n^2\), and similarly for \(M'\); note that \(M\) coincides with \(\partial V\). Since \((V, M)\) is \(q\)-connected, we have \(H_{q+1}(V) \cong H_{q+1}(V, M) \cong \pi_{q+1}(V, M)\). We can choose \(h_1, \ldots, h_n\) so that the core \(c_i\) of \(h_i\) realises the element \(\hat{v}_i \in \pi_{q+1}(V, M)\) corresponding to \(v_i\) under this isomorphism. And we make the corresponding choice for \(h'_1, \ldots, h'_n\), so that \(c'_i\) represents the element \(\hat{v}'_i \in \pi_{q+1}(V', M')\).

Because the \(q\)-handles are unknotted, we see that \(M \cong \#(S^q \times S^{q+1})_i\), the boundary connected sum taken over \(i = 1\) to \(n\), and so \(\partial M \cong (S^q \times S^q)_i\). Clearly \(u_i\), regarded as an element of \(H_q(V)\) or \(H_q(M)\) or \(H_q(\partial M)\), is represented by \((S^q \times \text{point})_i\). The homology class of \(d_i = (\text{point} \times S^q)_i\) being denoted by \(w_i \in H_q(\partial M)\), and \(H_q(\partial M)\) being identified with \(\pi_q(\partial M)\) by the Hurewicz isomorphism, it follows at once that \(\partial \hat{v}_i \in \pi_q(\partial M)\) lies in the subgroup \(\langle w_1, \ldots, w_n \rangle\), for otherwise the attaching sphere of \(h_i\) would represent a nonzero element of \(\pi_q(M) \cong H_q(M)\), and so \(H_q(V)\) would not be a free abelian group of rank \(n\). In fact, since \(v_1, \ldots, v_n\) is dual to \(u_1, \ldots, u_n\), it follows that \(\partial \hat{v}_i = w_i\).

Thus the attaching sphere \(\partial c_1\) of \(h_1\) is homotopic to \(d_1\) in \(\partial M\). But \(d_1 \cong S^q\) is \((q - 1)\)-connected, \(\partial M\) is \((q - 1)\)-connected, and \(q \geq 3\), so we can apply Theorem 10.1 of [H] to show that \(\partial c_1\) is ambient isotopic to \(d_1\) in \(\partial M\). The attaching sphere \(\partial c_2\) of \(h_2\) is homotopic to \(d_2\) in \(\partial M\); we need to show that \(\partial c_2\) is homotopic to \(d_2\) in \(\partial M - d_1\). Clearly \(\pi_q(\partial M - d_1) \cong \pi_q(\partial M)/\langle u_1 \rangle \cong \langle u_2, \ldots, u_n, w_1, \ldots, w_n \rangle\), regarding \(u_1\) as an element of \(\pi_q(\partial M)\). Hence \(\partial c_2\) is indeed homotopic to \(d_2\) in \(\partial M - d_1\). And hence \(\partial c_2\) is isotopic to \(d_2\) in \(\partial M - d_1\), using Hudson's result again. Continuing in this way we can isotop \(\partial c_1 \cup \cdots \cup \partial c_n\) onto \(d_1 \cup \cdots \cup d_n\). The same applies to \(\partial c'_1 \cup \cdots \cup \partial c'_n\).

Next we wish to isotop \(h_i\) onto \(h'_i\), keeping the attaching sphere fixed. Begin with \(h_1\): the difference between the homotopy class of \(c_1 \rel \partial\) in \(N_1 = c[\Sigma 2q+2 - M \times B^1]\) and that of \(c'_1\) can be interpreted in terms of \(L - L'\). Since the linking forms are isomorphic, the difference is zero, and so \(c_1\) is homotopic \(\rel \partial\) to \(c'_1\), and by Hudson's result the homotopy can be realised by an ambient isotopy. Let \(N_2\) be the closed complement of a regular neighbourhood of \(M \cup h_1\) (\(\rel \partial\)); then the difference in the homotopy classes of \(c_2\) and \(c'_2\) (\(\rel \partial\)) in \(N_2\) can be interpreted in terms of \(L - L'\) and \(L - L'\). Hence \(c_2\) is homotopic, and so isotopic, to \(c'_2\) \(\rel \partial\) in \(N_2\). Continuing in this way we obtain an ambient isotopy taking each \(c_i\) onto \(c'_i\). Finally we can isotop \(h_i\) onto \(h'_i\) as in §16 of [L2], the obstruction being \(\mathcal{L}(\delta_i, \delta'_i) - \mathcal{L}'(\delta'_i, \delta'_i) = 0 \in \pi_{q+1}(S^q), \delta_i \in \pi_{q+1}(V)\), where \(\delta_i \in \pi_{q+1}(V)\) is represented by \(c_i \cup (\text{point} \times B^{q+1})_i \cong S^{q+1}\). Q.E.D.

2. Nice bases. Let \(V\) be a \((q - 1)\)-connected Seifert surface of the \(Z\)-torsion-free simple \(2q\)-knot \(k\), with \(H_q(V)\) torsion-free. Consider \(u = \{u_1, \ldots, u_n\} \subset H_q(V), \nu = \{v_1, \ldots, v_n\} \subset H_{q+1}(V), \nu = \{v_1, \ldots, v_n\} \subset H_{q+1}(V)\). We say that \(u\) and \(v\) are nice bases, and that \(v\) lies over \(v\) if the following properties hold.

(i) \(u\) and \(v\) are dual bases of \(H_q(V)\) and \(H_{q+1}(V);\)
(ii) the \(u_i\) are represented by disjoint embedded spheres \(S^q_i;\)
(iii) the \(v_i\) are represented by disjoint embedded spheres \(S^{q+1}_i;\)
(iv) $S^q_i$ meets $S^{q+1}_j$ in exactly $\delta_{ij}$ points;
(v) $\nu_i$ is mapped onto $\nu_i$ by the Hurewicz homomorphism.

**Proposition 2.1.** Let $u$ be a basis of $H_q(V)$. If $q \geq 3$, then there exist $v$ and $\nu$ such that $u$ and $v$ are nice bases, and $\nu$ lies over $v$.

This result is implicit in the proof of Proposition 1.1.

Let $\zeta$ be the nontrivial element of $\pi_{q+1}(S^q)$, where $q \geq 3$. By the Hurewicz theorem, $H_q(V) \cong \pi_q(V)$, so thinking of $u_i$ as an element of $\pi_q(V)$, we obtain an element $u_i \circ \zeta \in \pi_{q+1}(V)$.

**Proposition 2.2.** Let $q \geq 3$, and suppose that $u$ and $v$ are nice bases with $v$ lying over $v$. Then $\nu$ can be modified in either of the following two ways to obtain $\nu'$, also lying over $v$.

(i) Replace $\nu_i$ by $\nu'_i = \nu_i + u_i \circ \zeta$.
(ii) For $i \neq j$, replace $\nu_i$ by $\nu'_i = \nu_i + u_j \circ \zeta$ and $\nu_j$ by $\nu'_j = \nu_j + u_i \circ \zeta$.

**Proof.** (i) The homotopy class $\nu'_i$ can be represented by an embedding in the complement of the other $S^{q+1}_i \cup S^q_i$, using Theorem 8.1 of [H]. Then $S^q_i \cap (\text{new } S^{q+1}_i)$ can be reduced to one point by the Whitney trick.

(ii) Let $A = S^q_i \cap S^{q+1}_j$, $B = S^q_j \cap S^{q+1}_i$, and choose a path from $A$ to $B$ which misses the spheres apart from its endpoints. Take a regular neighbourhood $N$ of this path, meeting the spheres regularly. Regard $N$ as $B^{2q} \times I$, so that $\partial N \cap (S^q_i \cup S^{q+1}_i)$ is a pair of once linked spheres in $B^{2q} \times 0$, and $\partial N \cap (S^q_j \cup S^{q+1}_j)$ is another such pair in $B^{2q} \times 1$.

Let $f : S^q_i \cup S^q_j \to \partial N \cap (S^q_i \cup S^{q+1}_i)$ take $S^q_i$ homeomorphically onto $\partial N \cap S^{q+1}_i$, and be such that its restriction to $S^q_j$ is a map $S^q_j \to \partial N - S^{q+1}_i \simeq S^{q-1}$ representing a generator of $\pi_q(S^{q-1})$. Note that if $q = 3$ then $\pi_q(S^{q-1}) \cong \mathbb{Z}$, and if $q > 3$ then it is $\mathbb{Z}/2\mathbb{Z}$. Since $q \geq 3$, we can homotop $f|_{S^q_i} : S^q_i \to \partial N - S^{q+1}_i$ to an embedding, using Theorem 8.1 of [H]. Extend $f$ to an embedding $f : S^q_i \times I \to B^{2q} \times I$ so that $f|_{S^q_i \times 1}$ maps $S^q_i \times 1$ homeomorphically onto $\partial N \cap S^{q+1}_i$. Note that $B^{2q} \times I - f(S^q_i \times I) \cong (B^{2q} - S^q) \times I$ since $q \geq 3$, and so $f$ extends to an embedding $f : (S^q_i \cup S^q_j) \times I \to B^{2q} \times I$ such that $f|_{S^q_i \times 1}$ represents a generator of $\pi_q(S^{q-1}) \cong \pi_q(B^{2q} \times 1 - f(S^q_i \times 1))$.

Now $N \cap S^q_i$ is a q-ball, and hence so is $B^q_i = \text{cl}[S^q_i \cap N \cap S^q_i]$. Let $B^{2q+1}_i$ be a regular neighbourhood of $B^q_i$ in $\text{cl}[V - N]$, meeting the boundary regularly. We can assume that $f(S^q_i \times 0) \subset B^{2q+1}_i \cap N$, since $B^{2q} \times 0 - S^{q+1}$ deformation retracts onto $B^{2q+1}_i \cap N$. Now we can extend $f|_{S^q_i \times I}$ to $f : S^{q+1}_i \to V$ by coning on $S^q_i \times 0$ to get a map $B^{q+1} \to B^{2q+1}_i$, and by coning on $S^q_j \times 1$ to get a map $B^{q+1} \to S^{q+1}_j \cap \text{cl}[V - N]$. Clearly $f : S^{q+1}_i \to V$ is an embedding which represents $\nu'_i$. Similarly we can extend $f|_{S^q_j \times 1}$ to obtain an embedding representing $\nu'_j$, and $f : S^{q+1}_j \cup S^{q+1}_i \to V$ is an embedding.

Finally we use the Whitney trick to ensure that $S^q_i$ meets $f(S^{q+1}_j)$ transversely in just one point, and similarly for $S^q_j$ and $f(S^{q+1}_i)$.

Q.E.D.
PROPOSITION 2.3. Let \( q \geq 3 \), and let \( u, v \) be nice bases of \( H_q(V), H_{q+1}(V) \) respectively, with \( v \) lying over \( v \) and \( v' \) lying over \( v \). Then

\[
\nu'_i = \nu_i + \sum_{j=1}^{n} \lambda_{ij} u_j \circ \varsigma, \quad 1 \leq i \leq n,
\]

where \( \lambda_{ij} = \lambda_{ji} \pmod{2} \) for all \( i, j \).

PROOF. By the exact sequence of \([W, \text{p. 555}]\), it is clear that \( \nu'_i = \nu_i + \sum \lambda_{ij} u_j \circ \varsigma \). Using Proposition 2.2(i), we can replace \( \nu'_i \) by \( \nu'_i + \lambda_{ii} u_i \circ \varsigma \), and hence we can assume that \( \lambda_{ii} = 0 \) for all \( i \). Now use Proposition 2.2(ii) to modify \( \nu'_1, \nu'_2, \ldots, \nu'_{n-1} \) until

\[
\nu'_1 = \nu_1,
\nu'_2 = \mu_{21} u_1 \circ \varsigma + \nu_2,
\vdots
\nu'_n = \mu_{n1} u_1 \circ \varsigma + \cdots + \mu_{n(n-1)} u_{n-1} \circ \varsigma + \nu_n
\]

where

\[
\mu_{21} = \lambda_{21} + \lambda_{12} \pmod{2},
\vdots
\mu_{nj} = \lambda_{nj} + \lambda_{jn} \pmod{2}.
\]

Let \( V_1 = \text{cl}[V - N(v_1)] \), that is, the closed complement in \( V \) of a regular neighbourhood of the embedded \((q+1)\)-sphere representing \( v_1 \). Then \( V = V_1 \cup \hat{h}^q \cup \hat{h}^{2q+1} \), and \( H_q(V) = H_q(V_1) \oplus \langle u_1 \rangle \), \( \pi_{q+1}(V) = \pi_{q+1}(V_1) \oplus \langle u_1 \circ \varsigma \rangle \). Since \( \nu'_2 \in \pi_{q+1}(V_1) \), we see that \( \mu_{21} = 0 \), and hence \( \lambda_{12} = \lambda_{21} \pmod{2} \). A similar argument shows that each \( \mu_{ij} = 0 \) and hence \( \lambda_{ij} = \lambda_{ji} \pmod{2} \). Q.E.D.

3. Seifert matrices. Let \( V \) be a \((q-1)\)-connected Seifert surface of the simple \( 2q \)-knot \( k, q \geq 3 \), with \( H_q(V) \) \( \mathbb{Z} \)-torsion-free.

Let \( u_1, \ldots, u_n \) be a basis of the group \( H_q(V) \), and \( v_1, \ldots, v_n \) the dual basis of \( H_{q+1}(V) \) under the Poincaré duality pairing; thus

\[
(v_i, u_j) = \delta_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n.
\]

Let \( i_\pm \) denote the map on the homology induced by pushing a cycle off \( V \) in the ±ve direction, and define

\[
L(i_+(v_i), u_j) = a_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n,
\]

\[
L(i_-(v_i), u_j) = b_{ij},
\]

where \( L: H_{q+1}(S^{2q+2} - V) \times H_q(V) \to \mathbb{Z} \) is the linking pairing.

Note that

\[
\delta_{ij} = (v_i, u_j) = L(i_+(v_i) - i_-(v_i), u_j) = L(i_+(v_i), u_j) - L(i_-(v_i), u_j) = a_{ij} - b_{ij}.
\]

So \( A - B = I \).
If we make a change of basis \( u'_j = p_{jk}u_k, \ v'_i = q_{il}v_l \), then it is easily checked that \( P' = Q^{-1} \) and the new matrices are \( A_1 = QAQ^{-1}, \ B_1 = QBQ^{-1} \). Of course, \( P \) and \( Q \) are unimodular \( n \times n \) matrices over \( \mathbb{Z} \).

Now assume that \( u \) and \( v \) are nice dual bases of \( H_q(V) \) and \( H_{q+1}(V) \). Let \( \nu_1, \ldots, \nu_n \) be elements of \( \pi_{q+1}(V) \) lying over \( v_1, \ldots, v_n \).

Define matrices \( C, D \) over \( \mathbb{Z}/2\mathbb{Z} \) by

\[
\mathcal{L}(i_+(\nu_i), \nu_j) = c_{ij}, \quad \mathcal{L}(i_-(\nu_i), \nu_j) = d_{ij},
\]

where

\[
\mathcal{L} : \pi_{q+1}(S^{2q+2} - V) \times \pi_{q+1}(V) \to \mathbb{Z}/2\mathbb{Z}
\]
is the homotopy linking pairing. Clearly

\[
\mathcal{L}(i_+(\nu_i), \nu_j) = \mathcal{L}(\nu_i, i_-(\nu_j)) = \mathcal{L}(i_-(\nu_j), \nu_i).
\]

Hence we have \( c_{ij} = d_{ji} \).

For \( i \neq j, i_+(\nu_j) \) is homotopic to \( i_-(\nu_j) \) in the complement of \( \nu_i \), and so

\[
\mathcal{L}(i_-(\nu_j), \nu_i) = \mathcal{L}(i_+(\nu_j), \nu_i).
\]

Thus we have \( d_{ji} = c_{ji} \) for \( i \neq j \). Therefore \( C \) is symmetric, and \( D = C \).

Suppose that \( \nu'_1, \ldots, \nu'_n \) is another set lying over \( v_1, \ldots, v_n \); so that \( \nu'_i = \nu_i + \lambda_{ij}u_j \circ \zeta \), where \( \lambda_{ij} = \lambda_{ji} \). Then

\[
\mathcal{L}(i_+(\nu'_i), \nu'_j) = \mathcal{L}(i_+(\nu_i), \nu_j) + \mathcal{L}(i_+(\lambda_{ik}u_k \circ \zeta), \nu_j) + \mathcal{L}(i_+(\nu_i), \lambda_{jl}u_l \circ \zeta)
\]

\[
= c_{ij} + \lambda_{ik}L(i_+(u_k), v_j) + \lambda_{jl}L(i_+(u_l), v_i)
\]

\[
= c_{ij} + \lambda_{ik}L(u_k, i_-(v_j)) + \lambda_{jl}L(i_+(u_l), v_i)
\]

\[
= c_{ij} + \lambda_{ik}b_{jk} + \lambda_{jl}a_{il}.
\]

Thus \( C_1 = C + AB' + AA' \).

Any unimodular integer matrix \( Q \) can be written as a product of elementary integer matrices: this is proved using the Euclidean algorithm column by column. What happens if we make such a change of basis in \( H_q(V) \)? Say \( u'_1 = u_1 + u_2, \ u'_i = u_i \) for \( 2 \leq i \leq n \). Then \( v'_2 = v_2 - v_1, \ v'_i = v_i \) for \( i \neq 2 \). We claim that \( \nu'_1, \ldots, \nu'_n \) lies over \( v'_1, \ldots, v'_n \) where \( v'_2 = v_2 - v_1, \ v'_i = v_i \) for \( i \neq 2 \).

For \( v'_2 \) can be represented by an embedded \( (q + 1) \)-sphere, simply by taking the connected sum of \( \nu_1 \) and \( \nu_2 \) with suitable orientations. A similar statement holds for \( u'_1 \). The Whitney trick can then be used to make \( v'_2 \) and \( u'_1 \) disjoint.

Thus if \( u'_j = p_{jk}u_k, \ v'_i = q_{il}v_l \), then \( v'_i = q_{il}v_l \) lies over \( v'_i \). Hence

\[
\mathcal{L}(i_+(\nu'_i), \nu'_j) = \mathcal{L}(i_+(q_{il}v_l), q_{jk}v_k) = q_{il}c_{lk}q_{jk}
\]

and so \( C_1 = QCQ' \).

It follows that any change of basis in \( H_q(V) \), represented by a unimodular integer matrix \( P = Q'^{-1} \), can be realised geometrically by embedded spheres.

**Proposition 3.1.** Let \( k, k_1 \) be simple \( \mathbb{Z} \)-torsion-free \( 2q \) knots, \( q \geq 3 \), with \( (q - 1) \)-connected Seifert surfaces \( V, V_1 \) respectively, such that \( H_q(V), H_q(V_1) \) are torsion-free. Let \( (A, B, C), (A_1, B_1, C_1) \) be Seifert matrices arising from \( V, V_1 \). Then \( k \) is ambient isotopic to \( k_1 \) if and only if \( (A, B, C) \) is related to \( (A_1, B_1, C_1) \) as above.

**Proof.** The “only if” part has already been established. So assume that the two sets of matrices are related as above. Then by a change of basis in \( H_q(V_1) \) and
We conclude this section with a realisation result.

**Proposition 3.2.** Let $A, B$ be $n \times n$ integer matrices satisfying $A - B = I$, and $C$ a symmetric $n \times n$ matrix over $\mathbb{Z}/2\mathbb{Z}$. Then for $q \geq 3$, there exists a simple $\mathbb{Z}$-torsion-free $2q$-knot $k$ with $A, B, C$ as Seifert matrices.

**Proof.** It is implicit in the proof of Theorem II.2 of [Ke] that there is a simple $\mathbb{Z}$-torsion-free $2q$-knot $k_0$ realising the matrices $A, B$ by means of a $(q-1)$-connected Seifert surface $V$, with $H_q(V)$ $\mathbb{Z}$-torsion-free. Let $C_0$ be the $\mathbb{Z}/2\mathbb{Z}$ matrix associated with the handle decomposition of this matrix, that is, with the basis $u, v, \nu$. By altering the twisting and homotopy linking of the $(q+1)$-handles of $V$, we can change $C_0$ to $C$ without altering $A$ or $B$. With this new embedding of $V$, we obtain the desired knot $k = \partial V$. Q.E.D.

4. Ambient surgery. Let $k$ be a simple, $\mathbb{Z}$-torsion-free $2q$-knot $2q$-knot, $q \geq 3$. Assume that $V_0, V_1$ are two Seifert surfaces of $k$ which are each $(q-1)$-connected and have no $\mathbb{Z}$-torsion in homology. By results of Levine [L1] or Farber [F2], there exists a submanifold $W \subset S^{2q+2} \times I$ such that if $W_t = W \cap (S^{2q+2} \times t)$, then $W_0 = V_0, W_1 = V_1$, and $\partial W = V_0 \cup (S^{2q} \times I) \cup V_1$. Moreover we can by ambient surgery ensure that $W$ is $(q-1)$-connected and that $\pi_q(W)$ is $\mathbb{Z}$-torsion-free.

Take a handle decomposition of $(W, W_0)$ with handles of index $q, q+1, q+2$; this is possible because $(W, W_0)$ is $(q-1)$-connected and so is $(W, W_1)$, and $\dim W = 2q + 2 \geq 8$. Let $W_0 \times I$ be the collar of $W_0$ to which the handles are added; then $H_q(W_0 \times I \cup q$-handles) is free abelian. In these dimensions the standard handle moves can all be realised geometrically, and so because $H_q(W)$ is $\mathbb{Z}$-torsion-free we can arrange that for each $(q+1)$-handle, either its attaching sphere is part of a basis for a direct summand of $H_q(W_0 \times I \cup q$-handles) or is 0 in $H_q(W_0 \times I)$. In the latter case the core of the $(q+1)$-handle yields a basis element for $H_{q+1}(W)$.

Let $N$ be a regular neighbourhood of $(S^{2q+2} \times 0) \cup W$, and take a handle decomposition of $S^{2q+2} \times I$ based on $N$. Now $S^{2q+2} \times I - W$ is $(q-1)$-connected, and hence so is the pair $(S^{2q+2} \times I, W)$. Thus we can arrange that the handle decomposition involves only handles of index $q, q+1, q+2$, and $q+3$.

Each handle of $(W, W_0 \times I)$ yields a handle of $(N, S^{2q+2} \times I')$, where $S^{2q+2} \times I'$ is a collar neighbourhood of $S^{2q+2} \times 0$. Denote the $r$-handles of $N$ by $H^r$, and of $(S^{2q+2} \times I, N)$ by $H^r$. We have say $h^q_1, \ldots, h^q_m, H^q_1, \ldots, H^q_n$. These must be cancelled (as a set) by the $(q+1)$-handles, and so if we add trivial $(q+1, q+2)$-pairs of handles, we can move the new $(q+1)$-handles (all $H^{q+1}$'s) over the existing $(q+1)$-handles to obtain a set $H^{q+1}_1, \ldots, H^{q+1}_r$ which cancels the $q$-handles. Of course, $r \geq m + n$. From the attaching spheres we obtain an $r \times (m + n)$ matrix of integers $U$, with one column for each $q$-handle and one row for each $H^{q+1}_i$:

$$
U = \begin{pmatrix}
\begin{array}{c}
h^q_1 \\
h^q_2 \\
\vdots \\
h^q_m \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
H^{q+1}_1 \\
H^{q+1}_2 \\
\vdots \\
H^{q+1}_n \\
\end{pmatrix}.
$$

Moving one $(q+1)$-handle over another adds one row to another, and similarly for $q$-handles and columns. Of course, we cannot move an $h^q$ over an $H^q$, but otherwise there are no restrictions. Because all the $q$-handles are cancelled, the hcf
of the last column of \( U \) is 1. Repeated application of the Euclidean algorithm to the final column enables us, by moving the \( H^q_{r+1} \) over each other, to put \( U \) in the form \( \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \) where \( R \) is an \( (r-1) \times (m+n-1) \) matrix. Repeatedly moving \( H^q_n \) around puts \( U \) in for the form \( \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \). Now we can cancel \( H^q_n \) and \( H^{q+1}_r \), replacing \( U \) by the smaller matrix \( R \).

Continue in this way until all the \( H^q_i \) have been cancelled. Repeat the performance to obtain a matrix

\[
\left( \begin{array}{c} h^q_s \\ 0 \\ 0 \\ 1 \end{array} \right) \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}.
\]

Now we can "cancel" \( h^q_m \) and \( H^{q+1}_{r-n} \); what this gives us is a critical level decomposition involving a \( q \)-handle added to a neighbourhood of \( W_0 \), in other words an ambient surgery of index \( q \) performed on \( V_0 = W_0 \).

Repeat until all the \( h^q_i \) have been "cancelled" in this way.

Using the dual handle decomposition of \( (W, W) \), we can repeat the procedure above. Once this has been done, we are left with a manifold (which by abuse of notation we still call \( W \)) having handles only of index \( q + 1 \). Moreover, if \( N \) is a regular neighbourhood of \( (S^{2q+2} \times 0) \cup W \), then there is a handle decomposition of \( S^{2q+2} \times I \) based on \( N \) which has handles only of index \( q + 1, q + 2 \).

The attaching spheres of the \((q+2)\)-handles yield a square integer matrix \( U \) as above

\[
U = \left( \begin{array}{c} h^{q+1} \\ H^{q+1} \end{array} \right) \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}.
\]

Of course, \( U \) is unimodular, and so may be written as a product of elementary matrices. The argument above goes through to show that \( W \) is the trace of ambient surgeries of index \( q + 1 \).

5. The effect of ambient surgery. In this section we consider the effect of ambient surgery, of index \( q, q + 1, \) and \( q + 2 \), on a \((q-1)\)-connected Seifert surface \( V \) of the \( \mathbb{Z}\)-torsion-free simple \( 2q \)-knot \( k \), where \( q \geq 3 \) and \( H_q(V) \) is torsion-free.

(i) INDEX \( q \). Since \( V \) is \((q-1)\)-connected, the attaching sphere is null-homotopic, and so we have a new basis element \( u_{n+1} \) of \( H_q(U) \), where \( U \) is the new Seifert surface. Thus \( H_q(U) = H_q(V) \oplus \langle u_{n+1} \rangle \). The belt sphere of the surgery supplies a new \( v_1 \), so that \( v_1, \ldots, v_{n+1} \) is dual to \( u_1, \ldots, u_{n+1} \). Indeed, the belt sphere supplies \( v_{n+1} \in \pi_{q+1}(U) \) which lies over \( u_{n+1} \). Depending on which side of \( V \) the surgery is performed, \( A \) is replaced by

\[
\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} A & \alpha \\ 0 & 0 \end{pmatrix}
\]

where \( \alpha \) is a column vector of integers.

The matrix \( B \) is therefore replaced by

\[
\begin{pmatrix} B & \alpha \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} B & \alpha \\ 0 & -1 \end{pmatrix}
\]
The matrix \( C \) is replaced by \( \left( \begin{array}{cc} C & 0 \\ 0 & 0 \end{array} \right) \) in either case.

(ii) INDEX \( q + 1 \). There are two possibilities, since \( H_q(U) \) must be \( \mathbb{Z} \)-torsion-free. Either the attaching sphere of the surgery represents 0 in \( H_q(V) \), or else it represents a primitive element, i.e., one that forms part of a basis of \( H_q(V) \). These two possibilities are in fact dual to each other, for in the former case \( H_r(U) = H_r(U) \oplus \mathbb{Z} \) for \( r = q, q + 1 \), and in the latter case \( H_r(V) = H_r(U) \oplus \mathbb{Z} \) for \( r = q, q + 1 \).

Thus we need only investigate the former case, for the latter will induce the inverse effect on \( A, B, \) and \( C \). Let \( u_{n+1} \) be the basis element of \( H_q(U) \) represented by the belt sphere of the surgery, and \( v_{n+1} \) be the element of \( \pi_{q+1}(U) \) represented by \( B^{q+1} \times * \subset B^{q+1} \times \partial B^{q+1} \) together with a null-homotopy of \( \partial B^{q+1} \times * \). We can assume that this null-homotopy misses \( v_1, \ldots, v_n \), for \( V - \bigcup_{i=1}^n v_i \) is \((q-1)\)-connected (the fibre of the sphere bundle over \( v_i \) associated with the normal bundle is null-homotopic, using \( u_i \)) and

\[
\pi_q \left( V - \bigcup_{i=1}^n v_i \right) \cong H_q \left( V - \bigcup_{i=1}^n v_i \right) \cong H_q(V).
\]

Thus we can choose a null-homotopy in \( V - \bigcup_{i=1}^n v_i \). Since \( q \geq 3 \) and \( U - \bigcup_{i=1}^n v_i \) \((q-1)\)-connected, \( v_{n+1} \) may be homotoped to an embedding, and indeed we now have \( v_1, \ldots, v_{n+1} \) lying over \( v_1, \ldots, v_{n+1} \), a basis for \( H_{q+1}(U) \).

The effect on \( A \) is

\[
\begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix}
\]

where \( \alpha \) is a row vector of integers.

The matrix \( B \) is therefore replaced by

\[
\begin{pmatrix} B & 0 \\ \alpha & -1 \end{pmatrix}
\]

The matrix \( C \) is replaced by \( \left( \begin{array}{cc} C & \beta' \\ \beta & \gamma \end{array} \right) \) where \( \beta \) is a row vector with entries in \( \mathbb{Z}/2\mathbb{Z} \), and \( \gamma \in \mathbb{Z}/2\mathbb{Z} \).

(iii) INDEX \( q + 2 \). This is the inverse of a surgery of index \( q \), and so the effect on the matrices is the inverse of (i).

REMARK. In (ii) we could choose \( \gamma \) to be 0, or 1. For replace the element \( v_{n+1} \) by \( v_{n+1}' = v_{n+1} + u_{n+1} \circ \zeta \). This can be represented by an embedded sphere, and the effect on \( C \) is to replace \( \gamma \) by \( \gamma + 1 \) (mod 2).

PROPOSITION 5.1. For \( q \geq 3 \), each of the above algebraic operations on \( A, B, \) \( C, \) may be realised geometrically by an ambient surgery on \( V \).

PROOF. (i) INDEX \( q \). Suppose we have to realise the matrices

\[
\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} B & \alpha \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}
\]

Fatten \( V \) to a tubular neighbourhood \( V \times I \), and embed a sphere \( S^q \) in \( S^{2q+2} - V \times I \) so that its linking number with \( v_i \) is \( \alpha_i \) (\( 1 \leq i \leq n \)). Because the codimension is at least 3, \( S^q \) is unknotted in \( S^{2q+2} \) and so has trivial normal bundle \( S^q \times B^{q+2} \).
Take the boundary connected sum of $V \times I$ and $S^q \times B^{q+2}$, using the positive side of $V \times I$. This is the trace of the ambient surgery.

(ii) INDEX $q + 1$. Suppose that we have to realise the matrices

\[
\begin{pmatrix}
A & 0 \\
\alpha & 0
\end{pmatrix}, \quad
\begin{pmatrix}
B & 0 \\
\alpha & -1
\end{pmatrix}, \quad
\begin{pmatrix}
C & \beta' \\
\beta & \gamma
\end{pmatrix}.
\]

Once again, fatten $V$ to $V \times I$, and embed a sphere $S^{q+1}$ in $S^{2q+2} - V \times I$ so that it has linking number $\alpha_i$ with $u_i$ ($1 \leq i \leq n$), and homotopy linking $\beta_i$ with $\nu_i$ ($1 \leq i \leq n$). Note that there is no difficulty about obtaining an embedding: we just use Theorem 8.1 of [H]. Take a regular neighborhood $S^{q+1} \times B^{q+1}$ of $S^{q+1}$, and then take the boundary connected sum of this with $V \times I$, attaching it on the positive side. The new basis elements are realised by $S^{q+1} \times \text{point}$ and $\text{point} \times S^q$, both contained in $S^{q+1} \times \partial B^{q+1}$, and clearly we realise the desired matrices, except possibly for $\gamma$. But by the remark above, if we can realise $\gamma + 1$, then we can realise $\gamma$.

Of course, similar arguments hold in the case of the negative side of $V$. To realise the dual case of index $q + 1$, we realise the reduced matrices by a new knot and Seifert surface $V_1$, using Proposition 3.2, perform ambient surgery on $V_1$ to realise the original matrices, and then appeal to Proposition 3.1 to show that the new Seifert surface is ambient isotopic to $V$. Hence we have an ambient surgery which realises the algebraic move. A similar argument applies to index $q + 2$, which is dual to index $q$.

6. Presentation of the $F$-form. Recall that $u_1, \ldots, u_n$ is a basis of $H_q(V)$ and $v_1, \ldots, v_n$ the dual basis of $H_{q+1}(V)$. Let $\nu_1, \ldots, \nu_n$ be elements of $\pi_{q+1}(V)$ lying over $v_1, \ldots, v_n$. Then $S^{2q+2} - V$ is $(q - 1)$-connected, and we can choose bases $\alpha_1, \ldots, \alpha_n$ of $H_q(S^{2q+2} - V)$ and $\beta_1, \ldots, \beta_n$ of $H_{q+1}(S^{2q+2} - V)$ as follows. The bases $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are represented by spheres $S^q_1, \ldots, S^q_n$ and $S^{q+1}_1, \ldots, S^{q+1}_n$ embedded in $V$, such that $S^q_i$ meets $S^{q+1}_j$ transversely in $\delta_{ij}$ points, and otherwise these spheres are disjoint. Each sphere is unknotted in $S^{2q+2}$, having codimension at least $q + 1 > 3$, and so has trivial normal bundle. Let $\alpha_i$ be represented by the boundary of a fibre of the bundle over $S^{q+1}_i$, and $\beta_i$ by the boundary of a fibre of the bundle over $S^q_i$. Let $\gamma_i \in \pi_{q+1}(S^{2q+2} - V)$ lie over $\beta_i$, and note that these elements can be chosen so that

\[L(\beta_i, u_j) = \delta_{ij} = L(v_i, \alpha_j),\]

\[L(\gamma_i, \nu_j) = 0,\]

Let $i_+(v_i) = h_{ij} \beta_j$; then

\[a_{ij} = L(i_+(v_i), u_j) = L(h_{ik} \beta_k, u_j) = h_{ik} L(\beta_k, u_j) = h_{ik} \delta_{kj} = h_{ij}.\]

Thus $i_+(v_i) = a_{ij} \beta_j$, and similarly $i_-(v_i) = b_{ij} \beta_j$.

We can write $i_+(v_i) = a_{ij} \gamma_j + e_{ij} \alpha_j \circ \zeta$, and using the fact that $(\alpha_k \circ \zeta, \nu_j) \equiv L(\alpha_k, v_j) \equiv \delta_{kj}$ (mod 2), we see that

\[c_{ij} = L(i_+(v_i), \nu_j) = L(a_{ik} \gamma_k + e_{ik} \alpha_k \circ \zeta, \nu_j) = a_{ik} L(\gamma_k, \nu_j) + e_{ik} L(\alpha_k \circ \zeta, \nu_j) = e_{ik} \delta_{kj} = e_{ij}.\]
Thus \( i_+(\nu_i) = a_{ij} \gamma_j + c_{ij} \alpha_j \circ \zeta \), and similarly \( i_-(\nu_i) = b_{ij} \gamma_j + c_{ij} \alpha_j \circ \zeta \).

By standard arguments, \( tA - B \) is a presentation matrix for \( H_{q+1}(\tilde{K}) \) as a \( \Lambda \)-module; that is,

\[
H_{q+1}(\tilde{K}) \cong (\beta_1, \ldots, \beta_n; (ta_{ij} - b_{ij})\beta_j, \ 1 \leq i \leq n).
\]

Let \( i_-(\nu_i) = f_{ij} \alpha_j \); then

\[
a_{ij} = L(i_+(\nu_i), u_j) = L(v_i, i_-(u_j)) = L(v_i, f_{jk} \alpha_k) = f_{jk} \delta_{ik} = f_{ji}
\]

so that \( i_-(\nu_i) = a_{ji} \alpha_j \). And similarly \( i_+(\nu_i) = b_{ji} \alpha_j \). Thus \( H_q(\tilde{K}) \) is presented as a \( \Lambda \)-module by \( tB' - A' \). Allowing \( \alpha_i, \beta_j \) to represent their images in \( H_q(\tilde{K}), H_{q+1}(\tilde{K}) \), respectively, the Blanchfield pairing is given (up to sign) by the formula

\[
(\beta_i, \alpha_j) \equiv (t-1)(tA-B)^{-1}_{ij} \pmod{\Lambda}.
\]

There is a map of \( \Gamma \)-modules

\[
(\alpha_i \circ \zeta, \ldots, \alpha_n \circ \zeta, \gamma_1, \ldots, \gamma_n; \\
(t_i^+(\nu_i) - i_-(\nu_i)) \circ \zeta, t_i^+(\nu_i) - i_-(\nu_i), \ 1 \leq i \leq n) \rightarrow \Pi_{q+1}(\tilde{K}),
\]

that is,

\[
(\alpha_1 \circ \zeta, \ldots, \alpha_n \circ \zeta, \gamma_1, \ldots, \gamma_n; \\
(tb_{ji} - a_{ji}) \alpha_j \circ \zeta, (ta_{ij} - b_{ij}) \gamma_j + (t-1)c_{ij} \alpha_j \circ \zeta) \rightarrow \Pi_{q+1}(\tilde{K}).
\]

Denoting this presentation \( \Gamma \)-module by \( N \), and the \( \Gamma \)-module

\[
(\alpha_1 \circ \zeta, \ldots, \alpha_n \circ \zeta; (tb_{ji} - a_{ji}) \alpha_j \circ \zeta, \ 1 \leq i \leq n)
\]

by \( M \), and the \( \Gamma \)-module

\[
(\beta_1, \ldots, \beta_n; (ta_{ij} - b_{ij}) \beta_j, \ 1 \leq i \leq n)
\]

by \( P \), we see that there is a commutative diagram

\[
\begin{array}{ccc}
M & \rightarrow & N \\
\downarrow & & \downarrow \\
\mathcal{H}_q(\tilde{K}) & \rightarrow & \Pi_{q+1}(\tilde{K})
\end{array}
\]

\[
\begin{array}{ccc}
P & \rightarrow & \mathcal{H}_{q+1}(\tilde{K}) \\
\downarrow & & \downarrow \\
\Pi_{q+1}(\tilde{K}) & \rightarrow & \mathcal{H}_{q+1}(\tilde{K})
\end{array}
\]

of \( \Gamma \)-modules, both rows being short exact sequences.

The first and third vertical arrows are isomorphisms, and so by the five-lemma is the middle one. Hence we have a presentation for \( \Pi_{q+1}(\tilde{K}) \) as a \( \Gamma \)-module. As in [K] the hermitian pairing is given by

\[
[\gamma_i, \gamma_i] \equiv (t-1)(tA-B)^{-1}(t^{-1}C' - C)(B' - t^{-1}A')_{ij}.
\]

7. Seifert matrices and \( F \)-forms. Let \( A, B, C \) be the Seifert matrices of a simple \( \mathbb{Z} \)-torsion-free \( 2q \)-knot \( k, q \geq 3 \), arising from a choice of basis \( u \) of \( H_q(V) \) where \( V \) is a \((q - 1)\)-connected Seifert surface of \( k \), with \( H_q(V) \) torsion free. Of course, we also have in mind a choice of \( \nu \in \pi_{q+1}(V) \) lying over \( v \), the dual basis of \( u \). In §3 we investigated the way in which \( A, B, C \) change when \( u \) and \( \nu \) are changed, and in §5 the way an ambient surgery on \( V \) affects them. Call the equivalence relation generated by these changes \( F \)-equivalence.
THEOREM 7.1. Let $A, B$ be $n \times n$ integer matrices satisfying $A - B = I$, and $C$ a symmetric $n \times n$ matrix over $\mathbb{Z}/2\mathbb{Z}$; and let $A_1, B_1, C_1$ be another such set of $m \times m$ matrices. Then $(A, B, C)$ is $F$-equivalent to $(A_1, B_1, C_1)$ if and only if they present isomorphic $F$-forms.

PROOF. Choose an integer $q \geq 4$, and by Proposition 3.1 realise $A, B, C$ as Seifert matrices of a simple $\mathbb{Z}$-torsion-free $2q$-knot $k$, and $A_1, B_1, C_1$ as Seifert matrices of a similar knot $k_1$. Then if the $F$-forms of $k$ and $k_1$ are isomorphic, by Corollary 11.3 of [K], $k$ and $k_1$ are ambient isotopic. Hence the Seifert surfaces, $V$ and $V_1$ say, are related by a sequence of ambient surgeries and so $(A, B, C)$ is $F$-equivalent to $(A_1, B_1, C_1)$. Conversely, if the matrices are $F$-equivalent we can realise the algebraic moves by ambient surgeries; hence $k$ and $k_1$ are ambient isotopic and so the $F$-forms are isometric. Q.E.D.

The $F$-form of a simple $\mathbb{Z}$-torsion-free $2q$-knot is defined in [K3] for $q > 4$. We are in a position to extend this to the case $q = 3$.

THEOREM 7.2. Every simple $\mathbb{Z}$-torsion-free $2q$-knot, $q \geq 3$, gives rise to an $F$-form which is an invariant of the knot.

PROOF. The case $q > 3$ is dealt with in [K]. For $q = 3$, the Whitehead exact sequence

$$2H_q(\tilde{K}) \rightarrow H_q(\tilde{K}) \rightarrow \pi_{q+1}(\tilde{K}) \rightarrow H_{q+1}(\tilde{K})$$

holds, and so we have the necessary modules and sequences. (Recall that $\tilde{K} \rightarrow K$ is the infinite-cyclic cover of the exterior $K$ of the knot $k$.)

In [K, §1], the pairing $\{ \ , \}$ is well defined except for the choice of $\alpha$. If we take $\alpha = \Delta \Delta$ as in [K, Corollary 1.3], then this is a canonical choice, being a knot invariant, and so we obtain a homotopy linking which is well defined for $q = 3$.

Alternatively, we could define the $F$-form by means of the presentation in terms of $A, B, C$, and use Theorem 7.1. Q.E.D.

An axiomatic description of $F$-forms is given in [K3], together with a realisation theorem for $q \geq 4$. Again, we extend that result to the case $q = 3$.

THEOREM 7.3. Every $F$-form can be realised by a simple $\mathbb{Z}$-torsion-free $2q$-knot, $q \geq 3$.

PROOF. For $q > 3$, the $F$-form can be realised by a simple $\mathbb{Z}$-torsion-free $2q$-knot. From this we obtain Seifert matrices $A, B, C$ which present the $F$-form. And Proposition 3.2 yields a simple $\mathbb{Z}$-torsion-free $6$-knot with these as Seifert matrices, hence with $F$-form isometric to the given one. Q.E.D.

Finally we show that $F$-form is a complete invariant of the knot when $q \geq 3$.

THEOREM 7.4. Two simple $\mathbb{Z}$-torsion-free $2q$-knots, $q \geq 3$, with isometric $F$-forms, are ambient isotopic.

PROOF. By [K, Theorem 11.1] the result is true for $q \geq 4$. For $q = 3$, let $(A, B, C), (A_1, B_1, C_1)$ be Seifert matrices arising from Seifert surfaces $V, V_1$ of the knots $k, k_1$. Since the $F$-forms are isometric, the Seifert matrices are $F$-equivalent by Theorem 7.1. The algebraic moves on the matrices can be realised geometrically, by Proposition 5.1, and so $k$ and $k_1$ are isotopic by Proposition 3.1. Q.E.D.

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