SEIFERT MATRICES AND 6-KNOTS

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ABSTRACT. A new classification of simple \( Z \)-torsion-free \( 2q \)-knots, \( q \geq 3 \), is given in terms of Seifert matrices modulo an equivalence relation. As a result the classification of such \( 2q \)-knots, \( q \geq 4 \), in terms of \( F \)-forms is extended to the case \( q = 3 \).

0. Introduction. An \( n \)-knot \( k \) is an oriented locally flat PL sphere-pair \((S^{n+2}, S^n)\); equally, one could consider oriented smooth pairs where the embedded sphere is allowed to carry an exotic differentiable structure. Let \( K \) denote the closed complement of a regular neighbourhood of \( S^n \), often called the exterior of \( k \). Then \( k \) is simple if \( K \) has the homotopy \( [(n - 1)/2] \)-type of a circle; this is the most that can be asked without making \( k \) unknotted. With the exception of \( n = 4 \) and \( 6 \), the simple \( n \)-knots \( (n \geq 3) \) have been classified in various ways during the past twenty years. The first such result is due to J. Levine \([L2]\), who classified the simple \((2q - 1)\)-knots \( (q \geq 2) \) in terms of the Seifert matrix and \( S \)-equivalence. These knots were then classified in terms of the Blanchfield duality pairing in \([T1, T2, \text{and } K1]\). Results for certain classes of simple \( 2q \)-knots \( (q \geq 4) \) may be found in \([K2, Ko1, Ko2, \text{and } K3]\); the general case is given by M. Sh. Farber in \([F2]\). One should also mention here the pioneering work of M. A. Kervaire \([Ke]\), who characterised the homology modules which can occur, in terms of presentation matrices.

Let \( K \to K \) be the infinite cyclic (= universal) cover of \( K \), where \( k \) is a simple \( 2q \)-knot, \( q \geq 3 \). Then \( k \) is \( Z \)-torsion-free if \( H_0(K) \) has no \( Z \)-torsion. In this paper we classify such knots in terms of Seifert matrices and an equivalence relation which we call \( F \)-equivalence. These matrices yield a presentation of the \( F \)-form of \( k \), which is used in \([K3]\) to classify these knots for \( q \geq 4 \). It is essentially the same as the \( A \)-quintet which Farber uses in \([F2]\), although his result is not restricted to the \( Z \)-torsion-free case. By the geometric results for \( q \geq 4 \), there is a one-one correspondence between \( F \)-equivalence classes of Seifert matrices and \( F \)-forms, and so the results of \([K3]\) also hold for \( q = 3 \).

1. The Seifert linking form. Let \( k \) be a simple \( Z \)-torsion-free \( 2q \)-knot, \( q \geq 3 \); then by Theorem 2 and Lemma 5 of \([L2]\) there exists a Seifert surface \( V \) of \( k \) which is \( (q - 1) \)-connected and for which \( H_q(V) \cong \pi_q(V) \) has no \( Z \)-torsion.

Let \( u \in H_q(V), v \in H_{q+1}(V) \), and let \( i_+: H_*(V) \to H_*(S^{2q+2} - V) \) denote the map induced by "pushing off" in the positive normal direction. Then \( u \) and \( i_+(v) \) are represented by disjoint cycles in \( S^{2q+2} \), hence have a linking number taking

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values in $\mathbb{Z}$. Thus we obtain a linking pairing $L: H_{q+1}(V) \times H_q(V) \to \mathbb{Z}$ given by $L(v, u) = \text{link}(i_+(v), u)$.

Similarly, let $i_+: \pi_* (V) \to \pi_*(S^{2q+2} - V)$ denote the map obtained by “pushing off” in the positive normal direction. (The notation will cause no difficulty.) Any two elements $\mu, \nu \in \pi_{q+1}(V)$ can be represented by embedded spheres, and any $S^{q+1}$ is unknotted in $S^{2q+2}$ because the codimension is at least 3. Thus $\mu$ and $i_+(\nu)$ are represented by disjoint embeddings in $S^{2q+2}$, and $i_+(\nu)$ is an element of $\pi_{q+1}(S^{2q+2} - \text{Im} \mu) \cong \pi_{q+1}(S^q) \cong \mathbb{Z}/2\mathbb{Z}$. Thus we have a homotopy linking $\mathcal{L} : \pi_{q+1}(V) \times \pi_{q+1}(V) \to \mathbb{Z}/2\mathbb{Z}$ given by $(\nu, \mu) = \text{homotopy link}(i_+(\nu), \mu)$.

By a result of Whitehead [W, p. 555], there is a short exact sequence

$$
\mathcal{E} : H_q(V)/2H_q(V) \xrightarrow{\sigma} \pi_{q+1}(V) \xrightarrow{\eta} H_{q+1}(V)
$$

since $H_q(V) \cong \pi_q(V)$ by the Hurewicz theorem. Let $\tau : H_q(V) \to H_q(V)/2H_q(V)$ denote the quotient map. Then it is easy to see that

$$
\mathcal{L}(\nu, \sigma \tau(\nu)) = L(\eta(\nu), u) \quad (\text{mod } 2).
$$

All this data is a Seifert linking form: $(H_q(V), H_{q+1}(V), L, \mathcal{L}, \mathcal{E}, \tau)$. Two such are isomorphic if there are isomorphisms of the groups which commute with all the appropriate maps and preserve $L$ and $\mathcal{L}$.

**Proposition 1.1.** The Seifert linking form determines $V$ up to ambient isotopy.

**Proof.** Suppose that $k, k'$ are two such knots with $(q-1)$-connected Seifert surfaces $V, V'$ giving rise to isomorphic Seifert linking forms. Let $\varphi : H_* (V) \to H_* (V')$ be the isomorphism on homology, so that $L(v, u) = L' (\varphi(v), \varphi(u))$. Choose a basis $u_1, \ldots, u_n$ for $H_q(V)$ and set $u'_i = \varphi(u_i)$ for $1 \leq i \leq n$ to obtain a basis for $H_q(V')$. Now let $B^{2q+1}$ be a closed $(2q + 1)$-ball in the interior of $V$; then $H_q(V) \cong H_q(V, B^{2q+1})$ by the long exact sequence of homology, and so we obtain a basis $\tilde{u}_1, \ldots, \tilde{u}_n$ for $H_q(V, B^{2q+1})$. By handlebody theory, there is a handle decomposition of $V$ based on $B^{2q+1}$ involving only handles of index $q$ and $q + 1$, say $h^q_1, \ldots, h^q_n, h^{q+1}_{n+1}, \ldots, h^{q+1}_{2q},$ such that the core $C_i$ of $h^q_i$ represents $\tilde{u}_i$. Now $\partial C_i, \ldots, \partial C_n$ are a set of disjoint $(q-1)$-spheres embedded in $\partial B^{2q+1} \cong S^{2q}$; since $q \geq 3$, they are unlinked and isotopic to a standard set of such spheres. That is, they are isotopic to the boundaries of a set of $n$ disjoint $q$-balls embedded in $S^{2q}$. Thus we can ambient isotop $V'$ so that it coincides with $V$ on the closed ball $B^{2q+1}$ in its interior, and then (in the obvious notation for a corresponding handle decomposition of $V'$) we can isotope $\partial C'_i$ to coincide with $\partial C_i$ in $\partial B^{2q+1}$ for $1 \leq i \leq n$. Now we would like to isotope the core $C'_i$ to coincide with $C_i$; standard isotopy theorems enable us to do this, for we can thicken up $B^{2q+1}$ to $B^{2q+1} \times B^1 \subset S^{2q+2}$, and consider the $C_i$ as embedded in $A = \text{cl}[S^{2q+1} - (B^{2q+1} \times B^1)]$. Then $C'_i$ can be homotoped to $C_i$ keeping the boundary fixed by Theorem 10.1 of [H]. If $A_1$ is $A$ with an open neighbourhood of $C_1$ excised, then $A_1$ is $q$-connected, and so $C'_1$ can be homotoped to $C_2$ in $A_1$ keeping the boundary fixed, and hence isotoped to $C_2$ in $A_1$ keeping the boundary fixed. Continuing in this way, we can isotope $C'_1 \cup \cdots \cup C'_n$ onto $C_1 \cup \cdots \cup C_n$ keeping the boundary fixed. By the argument of Levine [L2, §16], the obstruction to isotoping the $i$th $q$-handle of $V'$ onto $h^q_i$ lies in $\pi_q(S^{q+1}) = 0$, and so we can do this for each $i$. 

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Now we denote the \((q + 1)\)-handles of \(V\) (respectively \(V'\)) by \(h_1, \ldots, h_n\) (respectively \(h'_1, \ldots, h'_n\)). Our object is to isotop \(h'_i\) onto \(h_i\), but first we must select our handles carefully. The basis \(u_1, \ldots, u_n\) of \(H_q(V)\) yields a dual basis \(v_1, \ldots, v_n\) of \(H_{q+1}(V)\) under the intersection pairing \(H_{q+1}(V) \times H_q(V) \rightarrow \mathbb{Z}\). Set \(v'_i = \varphi(u_i), 1 \leq i \leq n\). Let \(M = B^{2q+1} \cup h'_1 \cup \cdots \cup h'_n\), and similarly for \(M'\); note that \(M\) coincides with \(\Delta P\). Since \((V, M)\) is \(q\)-connected, we have \(H_{q+1}(V) \cong H_q(V, M) \cong \pi_{q+1}(V, M)\). We can choose \(h_1, \ldots, h_n\) so that the core \(c_i\) of \(h_i\) realises the element \(v_i \in \pi_{q+1}(V, M)\) corresponding to \(u_i\) under this isomorphism. And we make the corresponding choice for \(h'_1, \ldots, h'_n\), so that \(c'_i\) represents the element \(v'_i \in \pi_{q+1}(V', M')\).

Because the \(q\)-handles are unknotted, we see that \(M \cong \# \partial(\Sigma^q \times B^{q+1})\), the boundary connected sum taken over \(i = 1\) to \(n\), and so \(\partial M \cong \# (\Sigma^q \times \text{point})\). Clearly \(u_i\), regarded as an element of \(H_q(V)\) or \(H_q(M)\) or \(H_q(\partial M)\), is represented by \((\Sigma^q \times \text{point})\). The homology class of \(d_i = (\text{point} \times \Sigma^q)\), being denoted by \(w_i \in H_q(\partial M)\), and \(H_q(\partial M)\) being identified with \(\pi_q(\partial M)\) by the Hurewicz isomorphism, it follows at once that \(\partial v_i \in \pi_q(\partial M)\) lies in the subgroup \(\langle w_1, \ldots, w_n \rangle\), for otherwise the attaching sphere of \(h_i\) would represent a nonzero element of \(\pi_q(M) \cong H_q(M)\), and so \(H_q(V)\) would not be a free abelian group of rank \(n\). In fact, since \(v_1, \ldots, v_n\) is dual to \(u_1, \ldots, u_n\), it follows that \(\partial v_i = w_i\).

Thus the attaching sphere \(\partial c_1\) of \(h_1\) is homotopic to \(d_1\) in \(\partial M\). But \(d_1 \cong (\Sigma^q \times \text{point})\) is \((q - 1)\)-connected, \(\partial M\) is \((q - 1)\)-connected, and \(q \geq 3\), so we can apply Theorem 10.1 of [H] to show that \(\partial c_1\) is ambient isotopic to \(d_1\) in \(\partial M\). The attaching sphere \(\partial c_2\) of \(h_2\) is homotopic to \(d_2\) in \(\partial M\); we need to show that \(\partial c_2\) is homotopic to \(d_2 - d_1\) in \(\partial M - d_1\). Clearly \(\pi_q(\partial M - d_1) \cong \pi_q(\partial M)/(\langle u_1 \rangle) \cong \langle u_2, \ldots, u_n, w_1, \ldots, w_n \rangle\), regarding \(u_1\) as an element of \(\pi_q(\partial M)\). Hence \(\partial c_2\) is indeed homotopic to \(d_2\) in \(\partial M - d_1\). And hence \(\partial d_2\) is isotopic to \(d_2 - d_1\) in \(\partial M - d_1\), using Hudson’s result again. Continuing in this way we can isotop \(\partial c_1 \cup \cdots \cup \partial c_n\) onto \(d_1 \cup \cdots \cup d_n\). The same applies to \(\partial c'_1 \cup \cdots \cup \partial c'_n\).

Next we wish to isotop \(h_i\) onto \(h'_i\), keeping the attaching sphere fixed. Begin with \(h_1\): the difference between the homotopy class of \(c_1\) rel \(\partial\) in \(N_1 = \partial(S^q \times B^{q+1})\) and that of \(c'_1\) can be interpreted in terms of \(L - L'\). Since the linking forms are isomorphic, the difference is zero, and so \(c_1\) is homotopic rel \(\partial\) to \(c'_1\), and by Hudson's result the homotopy can be realised by an ambient isotopy. Let \(N_2\) be the closed complement of a regular neighbourhood of \(M \cup h_1\) (rel \(\partial\)); then the difference in the homotopy classes of \(c_2\) and \(c'_2\) (rel \(\partial\)) in \(N_2\) can be interpreted in terms of \(L - L'\) and \(\mathcal{L} - \mathcal{L}'\). Hence \(c_2\) is homotopic, and so isotopic, to \(c'_2\) rel \(\partial\) in \(N_2\). Continuing in this way we obtain an ambient isotopy taking each \(c_i\) onto \(c'_i\). Finally we can isotop \(h_i\) onto \(h'_i\) as in §16 of [L2], the obstruction being \(\mathcal{L}(\delta_i, \delta_i) - \mathcal{L}'(\delta'_i, \delta'_i) = 0 \in \pi_{q+1}(S^q)\), where \(\delta_i \in \pi_{q+1}(V)\) is represented by \(c_i \cup (\text{point} \times B^{q+1})\) i.e. \(S^{q+1}\). Q.E.D.

2. Nice bases. Let \(V\) be a \((q - 1)\)-connected Seifert surface of the \(\mathbb{Z}\)-torsion-free simple \(2q\)-knot \(k\), with \(H_q(V)\) torsion-free. Consider \(u = \{u_1, \ldots, u_n\} \subset H_q(V), v = \{v_1, \ldots, v_n\} \subset H_{q+1}(V), \nu = \{\nu_1, \ldots, \nu_n\} \subset H_{q+1}(V)\). We say that \(u\) and \(v\) are nice bases, and that \(v\) lies over \(v\) if the following properties hold:

(i) \(u\) and \(v\) are dual bases of \(H_q(V)\) and \(H_{q+1}(V)\);
(ii) the \(u_i\) are represented by disjoint embedded spheres \(S^q_i\);
(iii) the \(\nu_i\) are represented by disjoint embedded spheres \(S^{q+1}_i\);
(iv) $S_i^q$ meets $S_j^{q+1}$ in exactly $\delta_{ij}$ points;  
(v) $\nu_i$ is mapped onto $\nu_i$ by the Hurewicz homomorphism.

PROPOSITION 2.1. Let $u$ be a basis of $H_q(V)$. If $q \geq 3$, then there exist $v$ and $\nu$ such that $u$ and $v$ are nice bases, and $\nu$ lies over $v$.

This result is implicit in the proof of Proposition 1.1.

Let $\xi$ be the nontrivial element of $\pi_{q+1}(S^q)$, where $q \geq 3$. By the Hurewicz theorem, $H_q(V) \cong \pi_q(V)$, so thinking of $u_i$ as an element of $\pi_q(V)$, we obtain an element $u_i \circ \xi \in \pi_{q+1}(V)$.

PROPOSITION 2.2. Let $q \geq 3$, and suppose that $u$ and $v$ are nice bases with $\nu$ lying over $v$. Then $\nu$ can be modified in either of the following two ways to obtain $\nu'$, also lying over $v$.

(i) Replace $\nu_i$ by $\nu'_i = \nu_i + u_i \circ \xi$.

(ii) For $i \neq j$, replace $\nu_i$ by $\nu'_i = \nu_i + u_j \circ \xi$ and $\nu_j$ by $\nu'_j = \nu_j + u_i \circ \xi$.

PROOF. (i) The homotopy class $\nu'_i$ can be represented by an embedding in the complement of the other $S^{q+1}_i \cup S^q_i$, using Theorem 8.1 of [H]. Then $S^q_i \cap (\text{new } S^{q+1}_i)$ can be reduced to one point by the Whitney trick.

(ii) Let $A = S^q_i \cap S^{q+1}_j$, $B = S^q_j \cap S^{q+1}_i$, and choose a path from $A$ to $B$ which misses the spheres apart from its endpoints. Take a regular neighbourhood $N$ of this path, meeting the spheres regularly. Regard $N$ as $B^{2q} \times I$, so that $\partial N \cap (S^q_i \cup S^{q+1}_i)$ is a pair of once linked spheres in $B^{2q} \times 0$, and $\partial N \cap (S^q_j \cup S^{q+1}_j)$ is another such pair in $B^{2q} \times 1$.

Let $f : S^q_i \cup S^q_j \to \partial N \cap (S^q_i \cup S^{q+1}_i)$ take $S^q_i$ homeomorphically onto $\partial N \cap S^{q+1}_i$, and be such that its restriction to $S^q_j$ is a map $S^q_j \to \partial N - S^{q+1}_i \cong S^{q-1}$ representing a generator of $\pi_q(S^{q-1})$. Note that if $q = 3$ then $\pi_q(S^{q-1}) \cong \mathbb{Z}$, and if $q > 3$ then it is $\mathbb{Z}/2\mathbb{Z}$. Since $q \geq 3$, we can homotop $f|_{S^q_i} : S^q_i \to \partial N - S^{q+1}_i$ to an embedding, using Theorem 8.1 of [H]. Extend $f$ to an embedding $f : S^q_i \times I \to B^{2q} \times I$ so that $f|_{S^q_i \times 1}$ maps $S^q_i \times 1$ homeomorphically onto $\partial N \cap S^{q+1}_i$. Note that $B^{2q} \times I - f(S^q_i \times I) \cong (B^{2q} - S^q_i) \times I$ since $q \geq 3$, and so $f$ extends to an embedding $f : (S^q_i \cup S^q_j) \times I \to B^{2q} \times I$ such that $f|_{S^q_i \times 1}$ represents a generator of $\pi_q(S^{q-1}) \cong \pi_q(B^{2q} \times 1 - f(S^q_i \times 1))$.

Now $N \cap S^q_i$ is a q-ball, and hence so is $B^q_i = \text{cl}[S^q_i - N \cap S^q_i]$. Let $B^{2q+1}_i$ be a regular neighbourhood of $B^q_i$ in $\text{cl}[V - N]$, meeting the boundary regularly. We can assume that $f(S^q_i \times 0) \subset B^{2q+1}_i \cap N$, since $B^{2q} \times 0 - S^{q+1}_i$ deformation retracts onto $B^{2q+1}_i \cap N$. Now we can extend $f|_{S^q_i \times I}$ to $f : S^{q+1}_i \to V$ by coning on $S^q_i \times 0$ to get a map $B^{2q+1}_i \to B^{2q+1}_i$, and by coning on $S^q_i \times 1$ to get a map $B^{q+1}_i \to S^{q+1}_i \cap \text{cl}[V - N]$. Clearly $f : S^{q+1}_i \to V$ is an embedding which represents $\nu'_i$. Similarly we can extend $f|_{S^q_j \times 1}$ to obtain an embedding representing $\nu'_j$, and $f : S^{q+1}_i \cup S^{q+1}_j \to V$ is an embedding.

Finally we use the Whitney trick to ensure that $S^q_i$ meets $f(S^q_j \cup S^{q+1}_j)$ transversely in just one point, and similarly for $S^q_j$ and $f(S^q_i \cup S^{q+1}_i)$. Q.E.D.
Proposition 2.3. Let $q \geq 3$, and let $u, v$ be nice bases of $H_q(V), H_{q+1}(V)$ respectively, with $v$ lying over $v$ and $v'$ lying over $v$. Then

$$\nu'_i = \nu_i + \sum_{j=1}^{n} \lambda_{ij} u_j \circ \varsigma, \quad 1 \leq i \leq n,$$

where $\lambda_{ij} = \lambda_{ji} \pmod{2}$ for all $i, j$.

Proof. By the exact sequence of [W, p. 555], it is clear that $\nu'_i = \nu_i + \sum \lambda_{ij} u_j \circ \varsigma$. Using Proposition 2.2(i), we can replace $\nu_i'$ by $\nu_i' + \lambda_{ii} u_i \circ \varsigma$, and hence we can assume that $\lambda_{ii} = 0$ for all $i$. Now use Proposition 2.2(ii) to modify $\nu'_1, \nu'_2, \ldots, \nu'_{n-1}$ until

$$\nu'_1 = \nu_1,$$

$$\nu'_2 = \mu_{21} u_1 \circ \varsigma + \nu_2,$$

$$\vdots$$

$$\nu'_n = \mu_{n1} u_1 \circ \varsigma + \cdots + \mu_{n-1,n-1} u_{n-1} \circ \varsigma + \nu_n$$

where

$$\mu_{21} = \lambda_{21} + \lambda_{12} \pmod{2},$$

$$\vdots$$

$$\mu_{nj} = \lambda_{nj} + \lambda_{jn} \pmod{2}.$$
If we make a change of basis \( u'_j = p_{jk} u_k, \ v'_i = q_{il} v_l \), then it is easily checked that 
\( P' = Q^{-1} \) and the new matrices are \( A_1 = QAQ^{-1}, \ B_1 = QBQ^{-1} \). Of course, \( P \) and \( Q \) are unimodular \( n \times n \) matrices over \( \mathbb{Z} \).

Now assume that \( u \) and \( v \) are nice dual bases of \( H_q(V) \) and \( H_{q+1}(V) \). Let \( \nu_1, \ldots, \nu_n \) be elements of \( \pi_{q+1}(V) \) lying over \( v_1, \ldots, v_n \).

Define matrices \( C, D \) over \( \mathbb{Z}/2\mathbb{Z} \) by

\[
\mathcal{L}(i_+(\nu_i), \nu_j) = c_{ij}, \quad \mathcal{L}(i_-(\nu_i), \nu_j) = d_{ij},
\]

where

\[
\mathcal{L} : \pi_{q+1}(S^{2q+2} - V) \times \pi_{q+1}(V) \to \mathbb{Z}/2\mathbb{Z}
\]

is the homotopy linking pairing. Clearly

\[
\mathcal{L}(i_+(\nu_i), \nu_j) = \mathcal{L}(\nu_i, i_-(\nu_j)) = \mathcal{L}(i_-(\nu_j), \nu_i).
\]

Hence we have \( c_{ij} = d_{ji} \).

For \( i \neq j, i_+(\nu_j) \) is homotopic to \( i_-(\nu_j) \) in the complement of \( \nu_i \), and so

\[
\mathcal{L}(i_-(\nu_j), \nu_i) = \mathcal{L}(i_+(\nu_j), \nu_i).
\]

Thus we have \( d_{ji} = c_{ji} \) for \( i \neq j \). Therefore \( C \) is symmetric, and \( D = C \).

Suppose that \( \nu'_1, \ldots, \nu'_n \) is another set lying over \( v_1, \ldots, v_n \); so that \( \nu'_i = \nu_i + \lambda_{ij} u_j \circ \zeta \), where \( \lambda_{ij} = \lambda_{ji} \). Then

\[
\mathcal{L}(i_+(\nu'_i), \nu'_j) = \mathcal{L}(i_+(\nu_i), \nu_j) + \mathcal{L}(i_+ (\lambda_{ik} u_k \circ \zeta), \nu_j) + \mathcal{L}(i_+(\nu_i), \lambda_{jl} u_l \circ \zeta) = c_{ij} + \lambda_{ik} b_{kj} + \lambda_{jl} a_{li}.
\]

Thus \( C_1 = C + \Lambda B' + A A' \).

Any unimodular integer matrix \( Q \) can be written as a product of elementary integer matrices: this is proved using the Euclidean algorithm column by column. What happens if we make such a change of basis in \( H_q(V) \)? Say \( u'_1 = u_1 + u_2 \), \( u'_i = u_i \) for \( 2 \leq i \leq n \). Then \( \nu'_2 = \nu_2 - \nu_1 \), \( v'_i = v_i \) for \( i \neq 2 \). We claim that \( \nu'_1, \ldots, \nu'_n \) lies over \( v'_1, \ldots, v'_n \) where \( \nu'_2 = \nu_2 - \nu_1 \), \( \nu'_i = v_i \) for \( i \neq 2 \).

For \( \nu'_2 \) can be represented by an embedded \( (q + 1) \)-sphere, simply by taking the connected sum of \( \nu_1 \) and \( \nu_2 \) with suitable orientations. A similar statement holds for \( \nu'_1 \). The Whitney trick can then be used to make \( \nu'_2 \) and \( u'_1 \) disjoint.

Thus if \( v'_j = p_{jk} u_k, \ v'_i = q_{il} v_l \), then \( v'_i = q_{il} \nu_l \) lies over \( v'_i \). Hence

\[
\mathcal{L}(i_+(\nu'_i), \nu'_j) = \mathcal{L}(i_+(q_{il} \nu_l), q_{jk} \nu_k) = q_{il} c_{lk} q_{jk}
\]

and so \( C_1 = QCQ' \).

It follows that any change of basis in \( H_q(V) \), represented by a unimodular integer matrix \( P = Q'^{-1} \), can be realised geometrically by embedded spheres.

**Proposition 3.1.** Let \( k, k_1 \) be simple \( \mathbb{Z} \)-torsion-free \( 2q \)-knots, \( q \geq 3 \), with \( (q - 1) \)-connected Seifert surfaces \( V, V_1 \) respectively, such that \( H_q(V), H_q(V_1) \) are torsion-free. Let \( (A, B, C), (A_1, B_1, C_1) \) be Seifert matrices arising from \( V, V_1 \). Then \( k \) is ambient isotopic to \( k_1 \) if and only if \( (A, B, C) \) is related to \( (A_1, B_1, C_1) \) as above.

**Proof.** The "only if" part has already been established. So assume that the two sets of matrices are related as above. Then by a change of basis in \( H_q(V_1) \) and
We conclude this section with a realisation result.

**PROPOSITION 3.2.** Let $A, B$ be $n \times n$ integer matrices satisfying $A - B = I$, and $C$ a symmetric $n \times n$ matrix over $\mathbb{Z}/2\mathbb{Z}$. Then for $q \geq 3$, there exists a simple $\mathbb{Z}$-torsion-free $2q$-knot $k$ with $A, B, C$ as Seifert matrices.

**PROOF.** It is implicit in the proof of Theorem II.2 of [Ke] that there is a simple $\mathbb{Z}$-torsion-free $2q$-knot $k_0$ realising the matrices $A, B$ by means of a $(q-1)$-connected Seifert surface $V$, with $H_q(V)$ $\mathbb{Z}$-torsion-free. Let $C_0$ be the $\mathbb{Z}/2\mathbb{Z}$ matrix associated with the handle decomposition of this matrix, that is, with the basis $u, v, \nu$. By altering the twisting and homotopy linking of the $(q+1)$-handles of $V$, we can change $C_0$ to $C$ without altering $A$ or $B$. With this new embedding of $V$, we obtain the desired knot $k = \partial V$. Q.E.D.

4. **Ambient surgery.** Let $k$ be a simple, $\mathbb{Z}$-torsion-free $2q$-knot, $q \geq 3$. Assume that $V_0, V_1$ are two Seifert surfaces of $k$ which are each $(q-1)$-connected and have no $\mathbb{Z}$-torsion in homology. By results of Levine [L1] or Farber [F2], there exists a submanifold $W \subset S^{2q+2} \times I$ such that if $W_t = W \cap (S^{2q+2} \times t)$, then $W_0 = V_0, W_1 = V_1$, and $\partial W = V_0 \cup (S^{2q} \times I) \cup V_1$. Moreover we can by ambient surgery ensure that $W$ is $(q-1)$-connected and that $\pi_q(W)$ is $\mathbb{Z}$-torsion-free.

Take a handle decomposition of $(W, W_0)$ with handles of index $q, q+1, q+2$; this is possible because $(W, W_0)$ is $(q-1)$-connected and so is $(W, W_1)$, and $\dim W = 2q + 2 \geq 8$. Let $W_0 \times I$ be the collar of $W_0$ to which the handles are added; then $H_q(W_0 \times I \cup q$-handles) is free abelian. In these dimensions the standard handle moves can all be realised geometrically, and so because $H_q(W)$ is $\mathbb{Z}$-torsion-free we can arrange that for each $(q+1)$-handle, either its attaching sphere is part of a basis for a direct summand of $H_q(W_0 \times I \cup q$-handles) or is 0 in $H_q(W_0 \times I)$. In the latter case the core of the $(q+1)$-handle yields a basis element for $H_{q+1}(W)$.

Let $N$ be a regular neighbourhood of $(S^{2q+2} \times 0) \cup W$, and take a handle decomposition of $S^{2q+2} \times I$ based on $N$. Now $S^{2q+2} \times I - W$ is $(q-1)$-connected, and hence so is the pair $(S^{2q+2} \times I, W)$. Thus we can arrange that the handle decomposition involves only handles of index $q, q+1, q+2, q+3$.

Each handle of $(W, W_0 \times I)$ yields a handle of $(N, S^{2q+2} \times I')$, where $S^{2q+2} \times I'$ is a collar neighbourhood of $S^{2q+2} \times 0$. Denote the $r$-handles of $N$ by $H^r$, and of $(S^{2q+2} \times I, N)$ by $H^r$. We have say $h_1^q, \ldots, h_m^q, H_1^q, \ldots, H_n^q$. These must be cancelled (as a set) by the $(q+1)$-handles, and so if we add trivial $(q+1, q+2)$-pairs of handles, we can move the new $(q+1)$-handles (all $H^{q+1}$'s) over the existing $(q+1)$-handles to obtain a set $H_1^{q+1}, \ldots, H_n^{q+1}$ which cancels the $q$-handles. Of course, $r \geq m + n$. From the attaching spheres we obtain an $r \times (m + n)$ matrix of integers $U$, with one column for each $q$-handle and one row for each $H_i^{q+1}$.

$$U = \left( \begin{array}{c|c} h^q & H^2 \end{array} \right)_{H^{q+1}}.$$  

Moving one $(q+1)$-handle over another adds one row to another, and similarly for $q$-handles and columns. Of course, we cannot move an $h^q$ over an $H^q$, but otherwise there are no restrictions. Because all the $q$-handles are cancelled, the hcf
of the last column of $U$ is 1. Repeated application of the Euclidean algorithm to
the final column enables us, by moving the $H_i^{q+1}$ over each other, to put $U$ in the
form $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$ where $R$ is an $(r - 1) \times (m + n - 1)$ matrix. Repeatedly moving $H_i^q$
around puts $U$ in for the form $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$. Now we can cancel $H_i^q$ and $H_{r+1}^{q+1}$, replacing
$U$ by the smaller matrix $R$.

Continue in this way until all the $H_i^q$ have been cancelled. Repeat the perfor-
mance to obtain a matrix

$$
\begin{pmatrix}
  h^q \\
  \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}
\end{pmatrix} H_i^{q+1}.
$$

Now we can “cancel” $h_m^q$ and $H_{r-h}^{q+1}$; what this gives us is a critical level decom-
position involving a $q$-handle added to a neighbourhood of $W_0$, in other words an
ambient surgery of index $q$ performed on $V_0 = W_0$.

Repeat until all the $h_i^q$ have been “cancelled” in this way.

Using the dual handle decomposition of $(W, W)$, we can repeat the procedure
above. Once this has been done, we are left with a manifold (which by abuse of
notation we still call $W$) having handles only of index $q + 1$. Moreover, if $N$ is a
regular neighbourhood of $(S^{2q+2} \times 0) \cup W$, then there is a handle decomposi-
tion of $S^{2q+2} \times I$ based on $N$ which has handles only of index $q + 1, q + 2$.

The attaching spheres of the $(q + 2)$-handles yield a square integer matrix $U$ as
above

$$
U = \begin{pmatrix}
  h^{q+1} \\
  H^{q+1}
\end{pmatrix}
$$

Of course, $U$ is unimodular, and so may be written as a product of elementary
matrices. The argument above goes through to show that $W$ is the trace of ambient
surgeries of index $q + 1$.

5. The effect of ambient surgery. In this section we consider the effect of
ambient surgery, of index $q, q + 1$, and $q + 2$, on a $(q - 1)$-connected Seifert surface
$V$ of the $Z$-torsion-free simple $2q$-knot $k$, where $q \geq 3$ and $H_q(V)$ is torsion-free.

(i) INDEX $q$. Since $V$ is $(q-1)$-connected, the attaching sphere is null-homotopic,
and so we have a new basis element $u_{n+1}$ of $H_q(U)$, where $U$ is the new Seifert
surface. Thus $H_q(U) = H_q(V) \oplus \langle u_{n+1} \rangle$. The belt sphere of the surgery supplies
a new $v_{n+1}$, so that $v_1, \ldots, v_{n+1}$ is dual to $u_1, \ldots, u_{n+1}$. Indeed, the belt sphere
supplies $v_{n+1} \in \pi_{q+1}(U)$ which lies over $v_{n+1}$. Depending on which side of $V$ the
surgery is performed, $A$ is replaced by

$$
\begin{pmatrix}
  A & \alpha \\
  0 & 1
\end{pmatrix}
$$

where $\alpha$ is a column vector of integers.

The matrix $B$ is therefore replaced by

$$
\begin{pmatrix}
  B & \alpha \\
  0 & 0
\end{pmatrix}
$$

where $\alpha$ is a column vector of integers.
The matrix $C$ is replaced by $\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ in either case.

(ii) INDEX $q+1$. There are two possibilities, since $H_q(U)$ must be $\mathbb{Z}$-torsion-free. Either the attaching sphere of the surgery represents 0 in $H_q(V)$, or else it represents a primitive element, i.e., one that forms part of a basis of $H_q(V)$. These two possibilities are in fact dual to each other, for in the former case $H_r(U) = H_r(V) \oplus \mathbb{Z}$ for $r = q, q+1$, and in the latter case $H_r(V) = H_r(U) \oplus \mathbb{Z}$ for $r = q, q+1$.

Thus we need only investigate the former case, for the latter will induce the inverse effect on $A$, $B$, and $C$. Let $u_{n+1}$ be the basis element of $H_q(U)$ represented by the belt sphere of the surgery, and $\nu_{n+1}$ be the element of $\pi_{q+1}(U)$ represented by $B^{q+1} \times \ast \subset B^{q+1} \times \partial B^{q+1}$ together with a null-homotopy of $\partial B^{q+1} \times \ast$. We can assume that this null-homotopy misses $\nu_1, \ldots, \nu_n$, for $V - \bigcup_{i=1}^n \nu_i$ is $(q-1)$-connected (the fibre of the sphere bundle over $\nu_i$ associated with the normal bundle is null-homotopic, using $u_i$) and

$$\pi_q \left( V - \bigcup_{i=1}^n \nu_i \right) \cong H_q \left( V - \bigcup_{i=1}^n \nu_i \right) \cong H_q(V).$$

Thus we can choose a null-homotopy in $V - \bigcup_{i=1}^n \nu_i$. Since $q \geq 3$ and $U - \bigcup_{i=1}^n \nu_i$ $(q-1)$-connected, $\nu_{n+1}$ may be homotoped to an embedding, and indeed we now have $\nu_1, \ldots, \nu_{n+1}$ lying over $v_1, \ldots, v_{n+1}$, a basis for $H_{q+1}(U)$.

The effect on $A$ is

$$\begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ \alpha & 1 \end{pmatrix}$$

where $\alpha$ is a row vector of integers.

The matrix $B$ is therefore replaced by

$$\begin{pmatrix} B & 0 \\ \alpha & -1 \end{pmatrix}, \begin{pmatrix} B & 0 \\ \alpha & 0 \end{pmatrix}.$$

The matrix $C$ is replaced by $\begin{pmatrix} C & \beta' \\ \beta & \gamma \end{pmatrix}$ where $\beta$ is a row vector with entries in $\mathbb{Z}/2\mathbb{Z}$, and $\gamma \in \mathbb{Z}/2\mathbb{Z}$.

(iii) INDEX $q+2$. This is the inverse of a surgery of index $q$, and so the effect on the matrices is the inverse of (i).

REMARK. In (ii) we could choose $\gamma$ to be 0, or 1. For replace the element $\nu_{n+1}$ by $\nu'_{n+1} = \nu_{n+1} + u_{n+1} \circ \zeta$. This can be represented by an embedded sphere, and the effect on $C$ is to replace $\gamma$ by $\gamma + 1$ (mod 2).

PROPOSITION 5.1. For $q \geq 3$, each of the above algebraic operations on $A$, $B$, $C$, may be realised geometrically by an ambient surgery on $V$.

PROOF. (i) INDEX $q$. Suppose we have to realise the matrices

$$\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} B & \alpha \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Fatten $V$ to a tubular neighbourhood $V \times I$, and embed a sphere $S^q$ in $S^{2q+2} - V \times I$ so that its linking number with $v_i$ is $\alpha_i$ ($1 \leq i \leq n$). Because the codimension is at least 3, $S^q$ is unknotted in $S^{2q+2}$ and so has trivial normal bundle $S^q \times B^{q+2}$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Take the boundary connected sum of \( V \times I \) and \( S^q \times B^{q+2} \), using the positive side of \( V \times I \). This is the trace of the ambient surgery.

(ii) INDEX \( q + 1 \). Suppose that we have to realise the matrices

\[
\begin{pmatrix}
A & 0 \\
\alpha & 0
\end{pmatrix}, \quad \begin{pmatrix}
B & 0 \\
\alpha & -1
\end{pmatrix}, \quad \begin{pmatrix}
C & \beta' \\
\beta & \gamma
\end{pmatrix}.
\]

Once again, fatten \( V \) to \( V \times I \), and embed a sphere \( S^{q+1} \) in \( S^{2q+2} - V \times I \) so that it has linking number \( \alpha_i \) with \( u_i \) (1 \( \leq i \leq n \)), and homotopy linking \( \beta_i \) with \( \nu_i \) (1 \( \leq i \leq n \)). Note that there is no difficulty about obtaining an embedding: we just use Theorem 8.1 of [H]. Take a regular neighbourhood \( S^{q+1} \times B^{q+1} \) of \( S^{q+1} \), and then take the boundary connected sum of this with \( V \times I \), attaching it on the positive side. The new basis elements are realised by \( S^{q+1} \times \text{point} \) and \( \text{point} \times S^q \), both contained in \( S^{q+1} \times \partial B^{q+1} \), and clearly we realise the desired matrices, except possibly for \( \gamma \). But by the remark above, if we can realise \( \gamma + 1 \), then we can realise \( \gamma \).

Of course, similar arguments hold in the case of the negative side of \( V \). To realise the dual case of index \( q + 1 \), we realise the reduced matrices by a new knot and Seifert surface \( V_1 \), using Proposition 3.2, perform ambient surgery on \( V_1 \) to realise the original matrices, and then appeal to Proposition 3.1 to show that the new Seifert surface is ambient isotopic to \( V \). Hence we have an ambient surgery which realises the algebraic move. A similar argument applies to index \( q + 2 \), which is dual to index \( q \).

6. Presentation of the F-form. Recall that \( u_1, \ldots, u_n \) is a basis of \( H_q(V) \) and \( v_1, \ldots, v_n \) the dual basis of \( H_{q+1}(V) \). Let \( \nu_1, \ldots, \nu_n \) be elements of \( \pi_{q+1}(V) \) lying over \( v_1, \ldots, v_n \). Then \( S^{2q+2} - V \) is \((q - 1)\)-connected, and we can choose bases \( \alpha_1, \ldots, \alpha_n \) of \( H_q(S^{2q+2} - V) \) and \( \beta_1, \ldots, \beta_n \) of \( H_{q+1}(S^{2q+2} - V) \) as follows. The bases \( u_1, \ldots, u_n \) and \( \nu_1, \ldots, \nu_n \) are represented by spheres \( S_1^q, \ldots, S_n^q \) and \( S_1^{q+1}, \ldots, S_n^{q+1} \) embedded in \( V \), such that \( S_i^q \) meets \( S_j^{q+1} \) transversely in \( \delta_{ij} \) points, and otherwise these spheres are disjoint. Each sphere is unknotted in \( S^{2q+2} \), having codimension at least \( q + 1 > 3 \), and so has trivial normal bundle. Let \( \alpha_i \) be represented by the boundary of a fibre of the bundle over \( S_i^{q+1} \), and \( \beta_i \) by the boundary of a fibre of the bundle over \( S_i^q \). Let \( \gamma_i \in \pi_{q+1}(S^{2q+2} - V) \) lie over \( \beta_i \), and note that these elements can be chosen so that

\[
L(\beta_i, u_j) = \delta_{ij} = L(v_i, \alpha_j),
\]

\[
\mathcal{L}(\gamma_i, \nu_j) = 0,
\]

Let \( i_+(v_i) = h_{ij} \beta_j \); then

\[
a_{ij} = L(i_+(v_i), u_j) = L(h_{ik} \beta_k, u_j) = h_{ik} L(\beta_k, u_j) = h_{ik} \delta_{kj} = h_{ij}.
\]

Thus \( i_+(v_i) = a_{ij} \beta_j \), and similarly \( i_-(v_i) = b_{ij} \beta_j \).

We can write \( i_+(v_i) = a_{ij} \gamma_j + e_{ij} \alpha_j \circ \varsigma \), and using the fact that \( (\alpha_k \circ \varsigma, \nu_j) \equiv L(\alpha_k, v_j) \equiv \delta_{kj} \) (mod 2), we see that

\[
e_{ij} = \mathcal{L}(i_+(v_i), \nu_j) = \mathcal{L}(a_{ik} \gamma_k + e_{ik} \alpha_k \circ \varsigma, \nu_j)
= a_{ik} \mathcal{L}(\gamma_k, \nu_j) + e_{ik} \mathcal{L}(\alpha_k \circ \varsigma, \nu_j) = e_{ik} \delta_{kj} = e_{ij}.
\]
Thus $i_+(v_i) = a_{ij}\gamma_j + c_{ij}\alpha_j \circ \varsigma$, and similarly $i_-(v_i) = b_{ij}\gamma_j + c_{ij}\alpha_j \circ \varsigma$.

By standard arguments, $tA - B$ is a presentation matrix for $H_{q+1}(\tilde{K})$ as a $\Lambda$-module; that is,

$$H_{q+1}(\tilde{K}) \cong (\beta_1, \ldots, \beta_n; (ta_{ij} - b_{ij})\beta_j, 1 \leq i \leq n).$$

Let $i_-(u_i) = f_{ij}\alpha_j$; then

$$a_{ij} = L(i_+(u_i), u_j) = L(v_i, i_-(u_j)) = L(v_i, f_{jk}\alpha_k) = f_{jk}L(v_i, \alpha_k) = f_{jk}\delta_{ik} = f_{ji}$$
so that $i_-(u_i) = a_{ji}\alpha_j$. And similarly $i_+(u_i) = b_{ji}\alpha_j$. Thus $H_q(\hat{K})$ is presented as a $\Lambda$-module by $tB' - A'$. Allowing $\alpha_i, \beta_j$ to represent their images in $H_q(\hat{K})$, $H_{q+1}(\hat{K})$, respectively, the Blanchfield pairing is given (up to sign) by the formula

$$\langle \beta_i, \alpha_j \rangle \equiv (t - 1)(tA - B)^{-1}_{ij} \pmod{\Lambda}.$$

There is a map of $\Gamma$-modules

$$(\alpha_1 \circ \varsigma, \ldots, \alpha_n \circ \varsigma; \gamma_1, \ldots, \gamma_n;$$

$$(ti_+(u_i) - i_-(u_i)) \circ \varsigma, ti_+(v_i) - i_-(v_i), 1 \leq i \leq n) \to \Pi_{q+1}(\hat{K})$$

that is,

$$(\alpha_1 \circ \varsigma, \ldots, \alpha_n \circ \varsigma; \gamma_1, \ldots, \gamma_n;$$

$$(tb_{ji} - a_{ji})\alpha_j \circ \varsigma, (ta_{ij} - b_{ij})\gamma_j + (t - 1)c_{ij}\alpha_j \circ \varsigma \to \Pi_{q+1}(\hat{K}).$$

Denoting this presentation $\Gamma$-module by $N$, and the $\Gamma$-module

$$(\alpha_1 \circ \varsigma, \ldots, \alpha_n \circ \varsigma; (tb_{ji} - a_{ji})\alpha_j \circ \varsigma, 1 \leq i \leq n)$$

by $M$, and the $\Gamma$-module

$$(\beta_1, \ldots, \beta_n; (ta_{ij} - b_{ij})\beta_j, 1 \leq i \leq n)$$

by $P$, we see that there is a commutative diagram

$$\begin{array}{ccc}
M & \to & N \\
downarrow & & \downarrow \\
\mathcal{H}_q(\hat{K}) & \to & \Pi_{q+1}(\hat{K}) \\
& & \downarrow \\
& & \mathcal{H}_{q+1}(\hat{K})
\end{array}$$

of $\Gamma$-modules, both rows being short exact sequences.

The first and third vertical arrows are isomorphisms, and so by the five-lemma is the middle one. Hence we have a presentation for $\Pi_{q+1}(\hat{K})$ as a $\Gamma$-module. As in [K] the hermitian pairing is given by

$$[\gamma_i, \gamma_i] \equiv (t - 1)[(tA - B)^{-1}(t^{-1}C' - C)(B' - t^{-1}A')]_{ij}.$$

7. Seifert matrices and F-forms. Let $A, B, C$ be the Seifert matrices of a simple $\mathbb{Z}$-torsion-free $2q$-knot $k$, $q \geq 3$, arising from a choice of basis $u$ of $H_q(V)$ where $V$ is a $(q - 1)$-connected Seifert surface of $k$, with $H_q(V)$ torsion free. Of course, we also have in mind a choice of $\nu \in \pi_{q+1}(V)$ lying over $v$, the dual basis of $u$. In §3 we investigated the way in which $A, B, C$ change when $u$ and $\nu$ are changed, and in §5 the way an ambient surgery on $V$ affects them. Call the equivalence relation generated by these changes $F$-equivalence.
THEOREM 7.1. Let $A, B$ be $n \times n$ integer matrices satisfying $A - B = I$, and $C$ a symmetric $n \times n$ matrix over $\mathbb{Z}/2\mathbb{Z}$; and let $A_1, B_1, C_1$ be another such set of $m \times m$ matrices. Then $(A, B, C)$ is $F$-equivalent to $(A_1, B_1, C_1)$ if and only if they present isomorphic $F$-forms.

PROOF. Choose an integer $q \geq 4$, and by Proposition 3.1 realise $A, B, C$ as Seifert matrices of a simple $\mathbb{Z}$-torsion-free $2q$-knot $k$, and $A_1, B_1, C_1$ as Seifert matrices of a similar knot $k_1$. Then if the $F$-forms of $k$ and $k_1$ are isomorphic, by Corollary 11.3 of [K], $k$ and $k_1$ are ambient isotopic. Hence the Seifert surfaces, $V$ and $V_1$ say, are related by a sequence of ambient surgeries and so $(A, B, C)$ is $F$-equivalent to $(A_1, B_1, C_1)$. Conversely, if the matrices are $F$-equivalent we can realise the algebraic moves by ambient surgeries; hence $k$ and $k_1$ are ambient isotopic and so the $F$-forms are isometric. Q.E.D.

The $F$-form of a simple $\mathbb{Z}$-torsion-free $2q$-knot is defined in [K3] for $q \geq 4$. We are in a position to extend this to the case $q = 3$.

THEOREM 7.2. Every simple $\mathbb{Z}$-torsion-free $2q$-knot, $q \geq 3$, gives rise to an $F$-form which is an invariant of the knot.

PROOF. The case $q > 3$ is dealt with in [K]. For $q = 3$, the Whitehead exact sequence

$$2H_q(\tilde{K}) \to H_q(\tilde{K}) \to \pi_{q+1}(\tilde{K}) \to H_{q+1}(\tilde{K})$$

holds, and so we have the necessary modules and sequences. (Recall that $\tilde{K} \to K$ is the infinite-cyclic cover of the exterior $K$ of the knot $k$.)

In [K, §1], the pairing $\{ , \}$ is well defined except for the choice of $\alpha$. If we take $\alpha = \Delta \Delta$ as in [K, Corollary 1.3], then this is a canonical choice, being a knot invariant, and so we obtain a homotopy linking which is well defined for $q = 3$.

Alternatively, we could define the $F$-form by means of the presentation in terms of $A, B, C$, and use Theorem 7.1. Q.E.D.

An axiomatic description of $F$-forms is given in [K3], together with a realisation theorem for $q \geq 4$. Again, we extend that result to the case $q = 3$.

THEOREM 7.3. Every $F$-form can be realised by a simple $\mathbb{Z}$-torsion-free $2q$-knot, $q \geq 3$.

PROOF. For $q > 3$, the $F$-form can be realised by a simple $\mathbb{Z}$-torsion-free $2q$-knot. From this we obtain Seifert matrices $A, B, C$ which present the $F$-form. And Proposition 3.2 yields a simple $\mathbb{Z}$-torsion-free $6$-knot with these as Seifert matrices, hence with $F$-form isometric to the given one. Q.E.D.

Finally we show that $F$-form is a complete invariant of the knot when $q \geq 3$.

THEOREM 7.4. Two simple $\mathbb{Z}$-torsion-free $2q$-knots, $q \geq 3$, with isometric $F$-forms, are ambient isotopic.

PROOF. By [K, Theorem 11.1] the result is true for $q \geq 4$. For $q = 3$, let $(A, B, C), (A_1, B_1, C_1)$ be Seifert matrices arising from Seifert surfaces $V, V_1$ of the knots $k, k_1$. Since the $F$-forms are isometric, the Seifert matrices are $F$-equivalent by Theorem 7.1. The algebraic moves on the matrices can be realised geometrically, by Proposition 5.1, and so $k$ and $k_1$ are isotopic by Proposition 3.1. Q.E.D.
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