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DECAY RATES OF FOURIER TRANSFORMS OF CURVES

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ABSTRACT. Let $d\mu$ be a smooth measure on a nondegenerate curve in $\mathbf{R}^n$. This paper examines the decay rate of spherical averages of its Fourier transform $\widehat{d\mu}$. Thus estimates of the following form are considered:

$$\left(\int_{\Sigma_r} |\widehat{d\mu}(\xi)|^p \, d\xi\right)^{1/p} \leq Cr^{-\sigma}||f||$$

where $\Sigma_r = \{\xi \in \mathbf{R}^n : |\xi| = r\}$.

Let α be a C^n curve in $\mathbf{R}^n$ parametrized by Euclidean arclength ($|\alpha'(t)| \equiv 1$). If f is a function defined on the curve α then let F be the Fourier transform of $f\,dt$ considered as a measure on $\mathbf{R}^n$: $F(\xi) = \int \exp(-i\xi \cdot \alpha(t)) f(t)\,dt$. This paper considers the averages of F over spheres $\Sigma_r = \{\xi \in \mathbf{R}^n : |\xi| = r\}$ of radius r in $\mathbf{R}^n$. Specifically we look for estimates of the form

$$\left(\int_{\Sigma_r} |F(\xi)|^p \, d\xi\right)^{1/p} \leq Cr^{-\sigma}||f|| \quad \text{for all } r > 0 \tag{1}$$

where σ depends on p and $||f||$ is usually $||f||_1 + ||f'||_1$, the norm of L^1_1.

This is closely related to the problem of restricting the Fourier transform of functions to curves. In this case one obtains estimates of the form

$$\left(\int |\hat{g}(\alpha(t))|^q \, dt\right)^{1/q} \leq C_{p,q}||g||_{L^p(\mathbf{R}^n)}. \tag{2}$$

This problem for nondegenerate compact curves was resolved by Drury in [**2**]. Earlier results on this problem include [**11**, **9**, and **1**]. For degenerate compact curves the problem was studied by S. Drury and the author in [**3** and **4**]. The results are complete for curves parametrized by Euclidean arclength but for those parametrized by affine arclength certain "end point" results remain open. The inequality dual to (2) is

$$||F||_{L^{p'}(\mathbf{R}^n)} \leq C_{p,q}||f||_{q'}.$$

The inequality (1) is a mixed norm estimate of similar form.

If in (1) the curve α were replaced by an $(n-1)$-dimensional surface then $F(\xi)$ has an asymptotic expansion similar to that of the Bessel functions and an estimate of the type (1) is easily obtained. For surfaces with nonvanishing curvature this asymptotic expansion was obtained by Hlawka [**6**]. In this case $\sigma = (n-1)/2$.

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Estimates for surfaces with vanishing curvature were studied by Littmann [**7**] and by the author in [**8**].

Unlike the case of nondegenerate surfaces, when α is a curve the decay rate of F changes dramatically with the direction $\xi/|\xi|$. An interesting consequence of this is that there are critical values of p at which the behavior of the decay rate σ changes. For small values of p however the curvature of α is unimportant:

THEOREM 1. *If α is a compact C^1 curve in $\mathbf{R}^n$ parametrized by Euclidean arc length then*

$$\left(\int_{\Sigma_r} |F(\xi)|^2\,d\xi\right)^{1/2} \le Cr^{-1/2}\|f\|_\infty, \qquad r > 1,$$

where $F(\xi) = \int \exp(-i\xi\cdot\alpha(t))f(t)\,dt$.

PROOF. Let $\xi = r\xi'$ where $|\xi| = r$. Then

$$\begin{aligned}\int_{\Sigma_r} |F(\xi)|^2\,d\xi' &= \int_{\Sigma_r}\int_0^l\int_0^l \overline{f(t_1)f(t_2)}e^{-i\xi\cdot(\alpha(t_1)-\alpha(t_2))}\,dt_1\,dt_2 \\ &= \int_0^l\int_0^l f(t_1)\overline{f(t_2)}\frac{J_\nu(r|\alpha(t_1)-\alpha(t_2)|)}{(r|\alpha(t_1)-\alpha(t_2)|)^\nu}\,dt_1\,dt_2\end{aligned}$$

where J_ν is the Bessel function of order $\nu = (n-2)/2$. Since $|s^{-\nu}J_\nu(s)| \le Cs^{-\nu-1/2} = Cs^{-(n-1)/2}$ then

$$\begin{aligned}\int_{\Sigma_r} |F(\xi)|^2\,d\xi &\le \int_0^l\int_0^l |f(t_1)\overline{f(t_2)}|C(1+r|\alpha(t_1)-\alpha(t_2)|)^{-(n-1)/2}\,dt_1\,dt_2 \\ &\le C\|f\|_\infty^2\int_0^l\int_0^l (1+r|t_1-t_2|)^{-(n-1)/2}\,dt_1\,dt_2 \\ &= C\|f\|_\infty^2 r^{-1}.\end{aligned}$$

This proves Theorem 1.

A curve α is called nondegenerate if the function $\det(\alpha^{(1)},\dots,\alpha^{(n)})$ is bounded and bounded away from zero, where the derivatives of α have been written as columns of the matrix. In fact it will be convenient to reparametrize α so that $\det(\alpha^{(1)},\dots,\alpha^{(n)}) \equiv 1$. The curve is then said to be parametrized by affine arclength. For nondegenerate curves this is equivalent to Euclidean arclength ($|\alpha'|$ and $|\alpha'|^{-1}$ are bounded).

THEOREM 2. *Let α be a compact nondegenerate C^n curve in $\mathbf{R}^n$ parametrized by Euclidean arclength. Then*

$$\left(\int_{\Sigma_r} |F(\xi)|^p\,d\xi\right)^{1/p} \le Cr^{-\sigma}(\|f\|_1 + \|f'\|_1), \qquad r > 1,$$

where

(i) *for* $n = 2$, $\sigma = \frac{1}{2}$,

(ii) *for* $n = 3$,

$$\sigma = \begin{cases} \frac{1}{2} & \text{for } p \le 4, \\ \frac{1}{3} + \frac{2}{3p} & \text{for } 4 \le p \le \infty, \end{cases}$$

(iii) *for* $n = 4$,

$$\sigma = \begin{cases} \frac{1}{2(n-3)} + \frac{n-4}{(n-3)p} & \text{for } 2 \le p \le 2n-4, \\ \frac{1}{n-1} + \frac{2}{(n-1)p} & \text{for } 2n-4 \le p \le 3n-5, \\ \frac{1}{n} + \frac{5}{np} & \text{for } 3n-5 < p \le \infty, \end{cases}$$

when $p = 3n-5$, $r^{-\sigma} = r^{-3/p}$ *is replaced by* $r^{-3/p}(1+\log r)^{1/p}$.

The estimate (i) is of course well known (for example, [**5** and **6**]). Since the calculations use only high order derivatives of $\varphi(s) = \alpha(s)\cdot\xi$, it should be possible to improve σ in the estimates of (iii) when $n \ge 5$, especially in the range $2 < p < 2n-4$.

PROOF. Assume that α is parametrized so that

$$\det(\alpha^{(1)}(s), \ldots, \alpha^{(n)}(s)) \equiv 1. \tag{3}$$

Let $\vec{E}$ be the column "vector" whose components are e_i, the unit vectors along the coordinate axes. Define formally

$$v_j(s) = (-1)^{j+1}\det(\vec{E}, \alpha^{(1)}(s), \ldots, \widehat{\alpha^{(j)}(s)}, \ldots, \alpha^{(n)}(s)) \tag{4}$$

where the caret denotes that the column corresponding to $\alpha^{(j)}$ is omitted. If the determinant is expanded in cofactors down the first column then v_j is written as a linear combination of the vectors e_i. If $n = 3$ then v_j is a cross product; for example, $v_1 = \alpha^{(2)} \times \alpha^{(3)}$. In general since α is parametrized by affine arclength then $\{v_j\}$ is the dual basis for $\{\alpha^{(j)}\}$. That is,

$$v_j(s)\cdot\alpha^{(k)}(s) \equiv \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \ne k. \end{cases}$$

A vector $\xi \in R^n$ will be written in terms of the coordinates $(\rho, s_0, \eta_1, \ldots, \eta_{n-2})$ where

$$\xi = \rho\Big(v_n(s_0) + \sum_{j=1}^{n-2} \eta_j v_j(s_0)\Big). \tag{5}$$

The Jacobian for the change of coordinates from $(\rho, \ldots, \eta_{n-2})$ to $(\xi_1, \ldots, \xi_n)$ is

$$\begin{aligned} D &= \det(v_n + \Sigma\eta_j v_j, \rho(\dot{v}_n + \Sigma\eta_j\dot{v}_j), \rho\cdot v_1, \ldots, \rho\cdot v_{n-2}) \\ &= \rho^{n-1}\det(v_n, \dot{v}_n + \Sigma\eta_j\dot{v}_j, v_1, \ldots, v_{n-2}). \end{aligned} \tag{6}$$

But differentiating (4) and ignoring terms with duplicated columns gives

$$\dot{v}_1 = \det(E, \alpha^{(2)}, \ldots, \widehat{\alpha^{(n)}}, \alpha^{(n+1)}),$$

$$\dot{v}_j = -v_{j-1} + (-1)^{j+1}\det(E, \alpha^{(1)}, \ldots, \widehat{\alpha^{(j)}}, \ldots, \widehat{\alpha^{(n)}}, \alpha^{(n+1)}), \qquad 1 < j < n,$$

and $\dot{v}_n = -v_{n-1}$. If $\alpha^{(n+1)} = \sum c_j\alpha^{(j)}$ then differentiating (3) shows that $c_n = 0$. Hence $\dot{v}_1 = -c_1 v_n$ and $\dot{v}_j = -v_{j-1} - c_j v_n$. Thus only $\dot{v}_n$ has a nonzero coefficient in v_{n-1}. Substituting in (6) gives

$$D = \rho^{n-1}\det(v_n, -v_{n-1}, v_1, \ldots, v_{n-2}) = \rho^{n-1}\det(v_1, \ldots, v_n). \tag{7}$$

Differentiating shows that $\det(v_1, \ldots, v_n)$ is a constant.

Consider the part of the curve near $\alpha(0)$. The decay of $F(\xi)$ will be slowest in the directions $\pm v_n(0)$, which are normal to the osculating plane at $\alpha(0)$. Define

$$\Gamma' = \{\xi : |v_n(0)\cdot\xi| > (1-\varepsilon_1)|v_n(0)|\,|\xi|\}$$

where ε_1 is small enough that the change of coordinates given in (5) is one-to-one in Γ'. Let ε_0 be so small that

$$|v_n(s) - v_n(0)| \le (.01)/(3|\alpha^{(n)}(0)|) \quad \text{for } |s| < \varepsilon_0 \tag{8}$$

and

$$|\alpha^{(n)}(s) - \alpha^{(n)}(0)| \le (.01)/(3|v_n(0)|) \quad \text{for } |s| < \varepsilon_0. \tag{9}$$

Define Γ to be the part of Γ' where $|s_0| < \varepsilon_0$. $\mathbf{R}^n - \Gamma$ can be written as a finite disjoint union of cones Γ_j such that for each cone there exists $\varepsilon_j > 0$, $c_j > 0$, and $k = k(j) < n$ such that

$$|\alpha^{(k)}(s)\cdot\xi| \ge c_j|\xi| \quad \text{for } |s| < \varepsilon_j \text{ and } \xi \in \Gamma_j.$$

In addition suppose that ε_j and Γ_j are small enough that the change of coordinates (5) is one-to-one in Γ_j, and for a fixed $\tilde{\xi} \ne 0$ in Γ_j with $\rho(\tilde{\xi}_j) = 1$,

$$|\alpha^{(k)}(s)\cdot\xi - \rho(\xi)A_j| \le (.01)A_j \quad \text{for } |s| < \varepsilon_j \text{ and } \varepsilon \in \Gamma_j$$

where $A_j = \alpha^{(k)}(0)\cdot\tilde{\xi}_j$. Let $0 < \varepsilon < \varepsilon_0$ and $\varepsilon \le \varepsilon_j$ for all j. By using a partition of unity we may assume that f is supported in $(-\varepsilon/2, \varepsilon/2)$.

Let $\varphi(s) = \alpha(s)\cdot\xi$. Suppose that ξ is written in the form (5). Then

$$\begin{aligned}\varphi^{(k)}(s_0) &= \rho\left(\alpha^{(k)}(s_0)\cdot v_n(s_0) + \sum_{j=1}^{n-2}\eta_j\alpha^{(k)}(s_0)\cdot v(s_0)\right)\\ &= \rho\left(\delta_{kn} + \sum_{j=1}^{n-2}\eta_j\delta_{jk}\right), \qquad k = 1,\dots,n.\end{aligned}$$

Hence Taylor's theorem with remainder gives

$$\varphi^{(k)}(s) = \sum_{j=0}^{n-k-2}\rho\eta_{j+k}(s-s_0)^j/j! + \rho E(s-s_0)^{n-k}/(n-k)!$$

where E is $\varphi^{(n)}/\rho$ evaluated at some point between s_0 and s. Since s will be restricted to the interval $|s| < \varepsilon$, the conditions (8) and (9) imply that

$$|\varphi^{(n)}(s) - \rho|\,|\alpha^{(n)}(s)\cdot v_n(s_0) - \alpha^{(n)}(0)\cdot v_n(0)| < \rho(.01) \quad \text{for } \xi \in \Gamma.$$

Consequently E is between .99 and 1.01. The dependence of E on s, s_0 and k will be suppressed and E may change from one occurrence to the next.

If

$$I(t) = \int_0^t e^{-i\xi\cdot\alpha(s)}\,ds$$

then

$$F(\xi) = \int_{-\varepsilon}^{\varepsilon} e^{-i\xi\cdot\alpha(t)}f(t)\,dt = -\int_{-\varepsilon}^{\varepsilon} I(t)f'(t)\,dt \tag{10}$$

since f is supported in $(-\varepsilon, \varepsilon)$. This fixes our attention on the integral $I(t)$. The following form of van der Corput's lemma will be used to estimate $I(t)$: if $|g^{(n)}(s)| \geq \lambda > 0$ for $a \leq s \leq b$ then

$$\left| \int_a^b \exp\{ig(s)\}\, dx \right| \leq C_n \lambda^{-1/n}$$

where C_n is a constant depending only on n. The cases $n = 1$ and $n = 2$ can be found in [**10**] and the general case is proved by induction in the same way.

It will be convenient in the proof to extend φ so that it is defined on all R in such a way that φ is unchanged on $(-\varepsilon, \varepsilon)$ and $\varphi^{(n)}(s)$ is between $\rho(.99)$ and $\rho(1.01)$ for all $x \in \mathbf{R}$. This extension will naturally depend on ξ, but will not affect $I(t)$ in the support of f.

The case $n = 3$. Let $\xi \in \Gamma$. Then ξ depends on ρ, s_0 and η_1. We may assume that $\xi \cdot v_n(0) > 0$. Different bounds will be obtained for $I(t)$ when η_1 is in each of the following three regions: $R_1 = (\rho^{-2/3}, \infty)$, $R_2 = (-\infty, -\rho^{-2/3})$, and $R_3 = (-\rho^{-2/3}, \rho^{-2/3})$. These bounds are obtained in (11), (12) and (13) below.

When $n = 3$

$$\varphi'(s) = \alpha'(s) \cdot \xi = \rho\eta_1 + \rho E(s - s_0)^2/2.$$

If $\eta_1 > 0$ then φ' has no zero and $\varphi'(s) \geq \rho\eta_1$. Therefore van der Corput's lemma gives

$$|I(t)| \leq C(\rho|\eta_1|)^{-1}. \tag{11}$$

If $\eta_1 < 0$ then the zeros of φ' are at $z = s_0 \pm \sqrt{-2\eta_1/E}$. Let J_1 and J_2 be the two intervals of length $\delta = \rho^{-1/2}|\eta_1|^{-1/4}/3$ centered at the zeros of φ'. Note that $\delta < |z - s_0|/4$ if $\eta_1 \in R_2$. Since $\varphi''(s) = \rho E(s - s_0)$ then $|\varphi''(s)| \geq c\rho|\eta_1|^{1/2}$ for $s \in J_1 \cup J_2$. Therefore

$$|\varphi'(s)| \geq \frac{\delta}{2}(c\rho|\eta_1|^{1/2}) = c'\rho^{1/2}|\eta_1|^{1/4} \quad \text{for } s \notin J_1 \cup J_2.$$

Now van der Corput's lemma implies that

$$\begin{aligned} |I(t)| &= \left| \int_{J_1 \cup J_2} + \int_{(J_1 \cup J_2)^c} \right| \leq |J_1 \cup J_2| + C(\min|\varphi'|)^{-1} \\ &\leq C\rho^{-1/2}|\eta_1|^{-1/4} \quad \text{for } \eta_1 \in R_3. \end{aligned} \tag{12}$$

Also $\varphi''' = \rho E \geq (.99)\rho$ implies that

$$|I(t)| \leq C\rho^{-1/3}. \tag{13}$$

Now we integrate $|F(\xi)|^p$ over $\Gamma \cap \Sigma_r = \Gamma \cap \{|\xi| = r\}$. Equation (7) shows that (ρ, s_0, η_1) is comparable to polar coordinates. Also ρ is equivalent to r. Therefore by (10)

$$\int_{\Sigma_r \cap \Gamma} |F(\xi)|^p\, d\xi \leq C \int ||f'||_1^p ||I(t)||_\infty^p\, ds_0\, d\eta_1$$

where $||I(t)||_\infty = \sup\{|I(t)|: |t| \le \varepsilon\}$. Note that s_0 and η_1 are restricted to a bounded set B and the estimates for $I(t)$ are independent of s_0. Therefore

$$\begin{aligned}
&\int_{\Sigma_r \cap \Gamma} |F(\xi)|^p \, d\xi \\
&\le C||f'||_1^p \left\{ \int_{R_1 \cap B} (r|\eta|)^{-p} \, d\eta + \int_{R_2 \cap B} r^{-p/2} |\eta|^{-p/4} \, d\eta + \int_{R_3} r^{-p/3} \, d\eta \right\} \qquad (14)\\
&\le C||f'||_1^p (r^{-1/2} + r^{-1/3-2/3p})^p.
\end{aligned}$$

If $\xi_0 \notin \Gamma$ then ξ_0 belongs to some cone Γ_j such that $|\alpha^{(1)}(s) \cdot \xi| \ge c|\xi|$ or $|\alpha^{(2)}(s) \cdot \xi| \ge c|\xi|$ for all $|s| < \varepsilon$ and $\xi \in \Gamma_j$. For these two cases van der Corput's lemma implies that $|I(t)| \le C|\xi|^{-1} = Cr^{-1}$ or $|I(t)| \le Cr^{-1/2}$. Thus the integral of $|F|^p$ over $\Sigma_r \cap \Gamma$ is dominated by the right-hand side of (14). Therefore

$$\left(\int_{\Sigma_r} |F(\xi)|^p \, d\xi \right)^{1/p} \le C||f'||_1 (r^{-1/2} + r^{-1/3-2/3p}).$$

The case $n \ge 4$. Let $\xi \in \Gamma$. By construction s_0 is the sole root of $\varphi^{(n-1)}$. If $\varphi^{(n-2)}(s_0) = \rho\eta_{n-2} < 0$ then $\varphi^{(n-2)}$ has two real roots t_1 and t_2. Order these roots so that $|\varphi^{(n-3)}(t_1)| < |\varphi^{(n-3)}(t_2)|$. Define $\beta = \varphi^{(n-3)}(t_1)$.

Although ξ depends on $\rho, s_0, \eta_1, \dots, \eta_{n-2}$ the estimates obtained for $I(t)$ will involve only $\rho, \eta = \eta_{n-2}, \varsigma = \eta_{n-3}$. The $(\eta, \varsigma) = (\eta_{n-2}, \eta_{n-3})$ plane will be divided into five regions as follows:

$$\begin{aligned}
R_1 &= \{\eta \ge 0, |\varsigma| \le |\eta|^{3/2}, \eta \ge 100\rho^{-2/n}\} \cup \{\eta < 0, 2|\varsigma| \le |\eta|^{3/2}, |\eta| \ge 100\rho^{-2/n}\},\\
R_2 &= \{|\eta|^{3/2} < |\varsigma|, |\varsigma| \ge 1000\rho^{-3/n}\},\\
R_3 &= \{\eta < 0, \rho^2|\beta|^{n-1} \le |\eta|^{(n-3)/2}, |\eta| > 100\rho^{-2/n}\},\\
R_4 &= \{\eta < 0, \rho^2|\beta|^{n-1} > |\eta|^{(n-3)/2}, |\eta|^{3/2}/2 < |\varsigma| < |\eta|^{3/2}, |\eta| \ge 100\rho^{-2/n}\},\\
R_5 &= \{|\eta| < 100\rho^{-2/n}, |\varsigma| < 1000\rho^{-3/n}\}.
\end{aligned}$$

The following estimates will be obtained for $I(t)$:

$$|I(t)| \le C(\rho|\eta|)^{-1/(n-2)}, \qquad (\eta, \varsigma) \in R_1, \qquad (15)$$

$$|I(t)| \le C(\rho|\varsigma|^{2/3}))^{-1/(n-2)}, \qquad (\eta, \varsigma) \in R_2, \qquad (16)$$

$$|I(t)| \le C(\rho|\eta|^{1/2})^{-1/(n-1)}, \qquad (\eta, \varsigma) \in R_3, \qquad (17)$$

$$|I(t)| \le C(\rho|\eta|^{1/4}|\beta|^{1/2})^{-1/(n-2)}, \qquad (\eta, \varsigma) \in R_4, \qquad (18)$$

$$|I(t)| \le C\rho^{-1/n}, \qquad (\eta, \varsigma) \in R_5. \qquad (19)$$

Consider $\eta = \eta_{n-2} > 0$. In this case

$$\varphi^{(n-2)}(s) = \rho\eta + \rho E(s - s_0)^2/2$$

has no real zeros and $\varphi^{(n-2)}(s) \ge \rho\eta$. This implies that

$$|I(t)| \le C(\rho\eta)^{-1/(n-2)} \qquad (20)$$

for $|t| \le \varepsilon$, $\xi \in \Gamma$. This proves (15) for the region $R_1 \cap \{\eta > 0\}$.

Now suppose that $\eta < 0$. Since $\varphi^{(n-2)}(s_0) = \rho\eta < 0$ and $\varphi^{(n)} > 0$ then $\varphi^{(n-3)}$ can have three zeros. In fact $\varphi^{(n-3)}$ has three zeros if and only if the values of

$\varphi^{(n-3)}$ at the zeros of $\varphi^{(n-2)}$ have opposite signs. The zeros of $\varphi^{(n-2)}$, t_1 and t_2, are at $t_j = s_0 \pm \sqrt{-2\eta/E}$ and the values of $\varphi^{(n-3)}$ at these points are

$$\varphi^{(n-3)}(t_j) = \rho\varsigma \pm \sqrt{\frac{2}{E}}|\eta|^{1/2}\eta\left(1 - \frac{E}{3E}\right).$$

Let $t_1 > t_2$ and define $b_j = b_j(\xi) > 0$ so that

$$\varphi^{(n-3)}(t_j) = \rho(\varsigma \pm b_j\eta|\eta|^{1/2}). \tag{21}$$

Since $0.99 < E < 1.01$ then $.92 < b_j < .96$.

If $(\eta, \varsigma) \in R_1 \cap \{\eta < 0\}$ then $|\varsigma| \leq (.50)|\eta|^{3/2} \leq (.55)b_j|\eta|^{3/2}$. Therefore $\varphi^{(n-3)}(t_1)\varphi^{(n-3)}(t_2) < 0$ and $\varphi^{(n-3)}$ has three zeros. Also

$$|\varphi^{(n-3)}(t_j)| \geq \rho(b_j|\eta|^{3/2} - |\varsigma|) \geq \rho(.42)|\eta|^{3/2}. \tag{22}$$

Let z be one of the zeros of $\varphi^{(n-3)}$. Choose t_3 to be either t_1 or t_2 so that $t_3 - s_0$ and $z - s_0$ have the same sign. Since $\varphi^{(n-3)}(z) = \varphi^{(n-2)}(t_3) = \varphi^{(n-1)}(s_0) = 0$ then

$$\begin{aligned} |\varphi^{(n-3)}(t_3)| &= \left|\int_z^{t_3}\int_{t_3}^{s}\int_{s_0}^{t}\varphi^{(n)}(u)\,du\,dt\,ds\right| \\ &\leq \rho(1.01)\left|\int_z^{t_3}\int_{t_3}^{s}(t - s_0)\,dt\,ds\right| \\ &= \rho(1.01)|t_3 - z|^2|(z - s_0) + 2(t_3 - s_0)|/6. \end{aligned} \tag{23}$$

Since $|(z - s_0) + 2(t_3 - s_0)| \leq 2.86|\eta|^{1/2}$ then (22) and (23) imply that $|t_3 - z| \geq (.93)|\eta|^{1/2}$.

For each zero z_j of $\varphi^{(n-3)}$ let J_j be the interval of length $\delta = (\rho\eta)^{-1/(n-2)}/4$ centered at z_j. For $(\eta, \varsigma) \in R_1 \cap \{\eta > 0\}$,

$$\delta \leq |\eta|^{1/2}/4 \leq (.27)|t_3 - z|.$$

If $t_3 - s_0$ and $z - s_0$ have the same sign and $s \in J_j(z)$ then

$$\begin{aligned} |\varphi^{(n-2)}(s)| = \left|\int_{t_3}^{s}\int_{s_0}^{t}\varphi^{(n)}(u)\,du\,dt\right| &\geq (.49)\rho|t_3 - s|\,|(s - s_0) + (t_3 - s_0)| \\ &\geq (.49)\rho\{(.93)|\eta|^{1/2} - |\eta|^{1/2}/8\}|t_3 - s_0| > (.55)\rho|\eta|. \end{aligned}$$

As a result, outside $\bigcup J_j$,

$$|\varphi^{(n-3)}(s)| \geq (.55)\rho|\eta|(\delta/2) \geq C(\rho|\eta|)^{(n-3)/(n-2)}.$$

Consequently, for $(\eta, \varsigma) \in R_1 \cap \{\eta < 0\}$

$$\begin{aligned} |I(t)| &\leq \left|\int_{\bigcap J_j} + \int_{(\bigcap J_j)^c}\right| \leq \left|\bigcup J_j\right| + C(\min|\varphi^{(n-3)}|)^{-1/(n-3)} \\ &\leq C(\rho|\eta|)^{-1/(n-2)}. \end{aligned} \tag{24}$$

This proves (15).

If $(\eta, \varsigma) \in R_2$ then

$$|\varsigma|^2 > |\eta|^3 > (.96)^2|\eta|^3.$$

Therefore $\varphi^{(n-3)}$ has only one real zero by (21). Suppose first that $\eta < 0$. Then $\varphi^{(n-3)}$ has either a positive local minimum or a negative local maximum. Since the two cases are similar we will assume that $\varphi^{(n-3)}$ has a positive local minimum. This occurs at the point $t_1 = s_0 + \sqrt{2|\eta|/E}$.

We will split up the integral into two parts by considering separately the integrals $I_1(t)$ and $I_2(t)$ over the segments $U_1 = \{s \leq t_2\}$ and $U_2 = \{s \geq t_2\}$:

$$I_j(t) = \int_{t_2}^{t} \exp(-i\varphi(s))\, ds \quad \text{for } t \in U_j.$$

The function $\varphi^{(n-3)}$ has one zero in U_1. Call it z. Let J be the interval centered at z of length $\delta = (\rho\varsigma)^{-1/(n-2)}/10$. Since $(\eta, \varsigma) \in R_2$, $\delta < |\varsigma|^{1/3}/10$. Because

$$\varphi^{(n-3)}(z) = \rho\varsigma + \rho\eta(z - s_0) + \rho E(z - s_0)^3/6 = 0$$

and $\eta(z - s_0) > 0$ then

$$(s_0 - z)^3 > 6|\varsigma|/E > (5.9)|\varsigma|.$$

If $s \in J$ then $|s_0 - s| > |s_0 - z| - |s - z| > ((5.9)^{1/3} - 1/20)|\varsigma|^{1/3} > (1.75)|\varsigma|^{1/3}$. Therefore $S \in J$

$$\varphi^{(n-2)}(s) = \rho\eta + \rho E(s - s_0)^2 > \rho((.99)(1.75)^2 - 1)|\varsigma|^{2/3} > 2\rho|\varsigma|^{2/3}.$$

Hence for $s \notin J$

$$|\varphi^{(n-3)}(s)| > (2\rho|\varsigma|^{2/3})(\delta/2) = C(\rho|\varsigma|^{2/3})^{(n-3)/(n-2)}.$$

Consequently,

$$|I_1(t)| \leq |J| + \min_{J^c} |\varphi^{(n-3)}|^{-1/(n-3)} \leq C(\rho|\varsigma|^{2/3})^{-1/(n-2)}. \tag{25}$$

Since $\varphi^{(n-3)}$ has its local minimum at t_1 then (21) shows that for $s \in U_2$

$$|\varphi^{(n-3)}(s)| \geq \rho(1 - (.96))|\varsigma| = (.04)\rho|\varsigma|.$$

Therefore

$$|I_2(t)| \leq C(\rho|\varsigma|)^{-1/(n-3)}.$$

Using the fact that $\varsigma > \rho^{-3/n}$ shows that the right-hand side is dominated by the same term as in (25). These estimates combine to produce a bound for $I(t)$; for example, if $0 < t_2 < t$ then

$$|I(t)| \leq |I_1(0)| + |I_2(t)| \leq C(\rho|\varsigma|^{2/3})^{-1/(n-2)}.$$

This proves the estimate (16) subject to the condition that $\eta < 0$. If on the other hand $\eta > 0$ then $\varphi^{(n-3)}$ would not have a local minimum and the entire integral $I(t)$ could be treated as $I_1(t)$. This gives the same bound.

Estimates (17) and (19) in R_3 and R_4 deal with the case where the value of $\varphi^{(n-3)}$ is small at its local minimum. We assume again that $t_2 < t_1$ and $|\varphi^{(n-3)}(t_1)| \leq |\varphi^{(n-3)}(t_2)|$. The case where $|\varphi^{(n-3)}(t_2)| < |\varphi^{(n-3)}(t_1)|$ is similar. Define $\beta = \varphi^{(n-3)}(t_1)/\rho$ and let w be the other zero of $\varphi^{(n-3)}(s) - \rho\beta$. Since $\varphi^{(n-3)}(s) - \rho\beta$ has a double zero at t_1 then

$$\varphi^{(n-3)}(s) = (s - t_1)^2\psi(s)/6 + \rho\beta \tag{26}$$

where $\psi(s) = 3\varphi^{(n-1)}(t_1) + \rho E(s - t_1)$ has a zero at w. We consider separately the two regions $U_1 = \{s < t_2\}$ and $U_2 = \{s \geq t_2\}$. In $R_3 \cup R_4$, $\frac{1}{2}|\eta|^{3/2} < |\varsigma| < |\eta|^{3/2}$. Hence by (21), $\varphi^{(n-3)}(t_2) > 0$. In U_2, $\varphi^{(n-3)}$ has either no zeros, one zero or two zeros depending on the sign of β. Suppose there are *two* zeros z_1 and z_2.

Equation (21) also implies that

$$|\beta| < \tfrac{1}{2}|\eta|^{3/2} < |\varsigma| \tag{27}$$

for $(\eta, \varsigma) \in R_3 \cup R_4$. Since

$$|\varphi^{(n-3)}(t_1)| = |\rho\beta| < |\rho\varsigma| = \varphi^{(n-3)}(s_0)$$

then $\varphi^{(n-3)}(s_0)$ must be positive and

$$|z_j - t_1| < |s_0 - t_1| < (1.43)|\eta|^{1/2}. \tag{28}$$

Hence

$$\psi(z_j) = 3\rho E(t_1 - s_0) + \rho E(z_j - t_1)$$

is between $(2.8)\rho|\eta|^{1/2}$ and $(5.8)\rho|\eta|^{1/2}$. By (26), $|z_j - t_1| = \sqrt{-6\rho\beta/\psi(z_j)}$. Now

$$(1.01)|\beta|^{1/2}|\eta|^{-1/4} \leq |z_j - t_1| \leq (1.47)|\beta|^{1/2}|\eta|^{-1/4}. \tag{29}$$

Also, (27) implies that

$$|z_j - t_1| \leq (1.47)|\eta|^{1/2}/\sqrt{2} < (1.04)|\eta|^{1/2} < (.74)|s_0 - t_1|. \tag{30}$$

Let J_1 and J_2 be the two intervals of length 2δ centered at z_1 and z_2, the zeros of $\varphi^{(n-3)}$. For $(\eta, \varsigma) \in R_3$ we will take $\delta = (1.5)(\rho|\eta|^{1/2})^{-1/(n-1)}$. Then using $\rho^2|\beta|^{n-1} \leq |\eta|^{(n-3)/2}$ and (29) gives

$$\delta = (1.5)(\rho|\eta|^{1/2})^{-1/(n-1)} > (1.47)|\beta|^{1/2}|\eta|^{1/4} > |z_j - t_1|. \tag{31}$$

This means that the two intervals J_1 and J_2 overlap. Also, in R_3, $\rho|\eta|^{n/2} > 10^n$ implies that

$$\delta = (1.5)(\rho|\eta|^{1/2})^{-1/(n-1)} \leq (.02)|\eta|^{1/2} \leq (.02)|t_1 - s_0|. \tag{32}$$

Thus the interval $J_1 \cup J_2$ lies inside U_2.

When $(\eta, \varsigma) \in R_4$ we take $\delta = (\rho|\beta|^{1/2}|\eta|^{1/4})^{-1/(n-2)}/4$. Then by $\rho^2|\beta|^{n-2} > |\eta|^{(n-3)/2}$ and (29),

$$\delta < \tfrac{1}{4}(1.01)|\beta|^{1/2}|\eta|^{-1/4} < \tfrac{1}{4}|z_j - t_1|. \tag{33}$$

In this case the two intervals J_1 and J_2 are disjoint.

First we consider the region R_3. The minimum value of $|\varphi^{(n-3)}|$ in $U_2 - J_1 \cup J_2$ will occur at one of the two endpoints of $J_1 \cup J_2$. Suppose that $J_1 \cup J_2 = [t_1 - a, t_1 + a]$. Consider

$$\varphi^{(n-3)}(t_1 \pm a) = \varphi^{(n-3)}(t_1) + \varphi^{(n-1)}(t_1)a^2/2 \pm \rho E a^3/6. \tag{34}$$

Since $a < 2\delta$ then by (32) the last term is dominated by $(.03)\rho|\eta|^{1/2}\delta^2$. On the other hand by (31),

$$|\varphi^{(n-1)}(t_1)a^2/2| = \rho|E(t_1 - s_0)|a^2/2 \geq \rho(.69)|\eta|^{1/2}\delta^2 \geq (1.49)\rho|\beta|.$$

Therefore since the first term of (34) equals β,

$$|\varphi^{(n-3)}(t_1 \pm a)| \geq \left\{ \frac{(.49)}{(1.49)}(.69) - (.03) \right\} |\eta|^{1/2}\delta^2$$
$$\geq (.19)|\eta|^{1/2}\delta^2 \geq (.19)(1.5)(\rho|\eta|^{1/2})^{-1/(n-1)}|\eta|^{1/2}.$$

Since $\rho|\eta|^{n/2} \geq (10)^n$ this shows that the minimum of $|\varphi^{(n-3)}|$ in $(J_1 \cup J_2)^c \cap U_2$ is bounded below by $c(\rho|\eta|^{1/2})^{(n-3)/(n-1)}$. Consequently

$$(35) \qquad |I(t)| \leq |J_1 \cup J_2| + \min |\varphi^{(n-3)}|^{-1/(n-3)} \leq C(\rho|\eta|^{1/2})^{-1/(n-1)}.$$

If $(\eta, \varsigma) \in R_4$ then $J_1 \cup J_2$ has four endpoints to be considered. In this case, the first term of

$$(36) \qquad \varphi^{(n-3)}(z_j \pm \delta) = \varphi^{(n-2)}(z_j)\delta + \varphi^{(n-1)}(z_j)\delta^2/2 + \rho E\delta^3/6$$

will turn out to be dominant. For this first term

$$|\varphi^{(n-2)}(z_j)\delta| = \delta|\varphi^{(n-1)}(t_1)(z_j - t_1) + \tfrac{1}{2}\rho E(z_j - t_1)^2|$$
$$\geq \delta\rho|z_j - t_1|(|E(t_1 - s_0)| - \tfrac{1}{2}|E(z_j - t_1)|)$$
$$\geq (.61)\delta\rho|z_j - t_1|\,|t_1 - s_0|$$

by (30). Also inequalities (30) and (33) show that the other two terms are small:

$$\rho|E(z_j - s_0)\delta^2/2 + E\delta^3/6|$$
$$\leq \delta\rho\left(\frac{1}{4}|z_j - t_1|\right)\left(\frac{|E|}{2}(|z_j - t_1| + |t_1 - s_0|) + \frac{|E|}{6}\left(\frac{1}{4}|z_j - t_1|\right)\right)$$
$$\leq (.23)\delta\rho|z_j - t_1|\,|t_1 - s_0|.$$

As a result, by (29)

$$|\varphi^{(n-3)}(z_j \pm \delta)| \geq (.61 - .23)(\rho|\beta|^{1/2}|\eta|^{1/4})^{-1/(n-2)}\rho(1.01)|\beta|^{1/2}|\eta|^{-1/4}\sqrt{2|\eta|/E}$$
$$\geq c(\rho|\beta|^{1/2}|\eta|^{1/4})^{(n-3)/(n-2)}.$$

Therefore as in (35)

$$(37) \qquad |I(t)| \leq 4\delta + \min |\varphi^{(n-3)}|^{-1/(n-3)} \leq C(\rho|\beta|^{1/2}|\eta|^{1/4})^{-1/(n-2)}.$$

Estimates (17) and (18) have now been verified for the case where $\beta < 0$ and I is restricted to U_2. If $\beta \geq 0$ then the argument leading to (36) still holds since the zeros z_j were not used. If $\beta \geq 0$ and $(\eta, \varsigma) \in R_4$ then we use the fact that $\varphi^{(n-3)}$ has a local minimum at t_1. Now for $t \in U_2$

$$|\varphi^{(n-3)}(t)| \geq |\varphi^{(n-3)}(t_1)| = \rho|\beta| \geq (\rho|\beta|^{1/2}|\eta|^{1/4})^{(n-3)/(n-2)}.$$

The last inequality holds because in R_4, $\rho^2|\beta| > |\eta|^{(n-3)/2}$. This completes the case $\beta \geq 0$. The analysis of $I(t)$ in the region U_1 is entirely similar. The only difference is that the corresponding value of β is larger, giving a better estimate in (37).

Estimate (19) is a simple consequence of van der Corput's lemma using the fact that $|\varphi^{(n)}| > (.99)\rho$.

As in the three-dimensional case the object now is to calculate the integral of $||I(t)||_\infty^p = \sup\{|I(t)|^p : |t| < \varepsilon\}$ over $\Gamma \cap \{|\xi| = r\}$. This means integrating with respect to $s_0, \eta_1, \ldots, \eta_{n-2}$. Because estimates (15)–(19) depend only on $(\eta, \varsigma) = (\eta_{n-2}, \eta_{n-3})$ this reduces to a double integral. In addition since many parts of the regions are similar we need only consider the reduced regions $R_1' = R_1 \cap \{\eta > 0\}$, $R_2' = R_2 \cap \{\varsigma > 0\}$, $R_3' = R_3 \cap \{\beta > 0\}$, $R_4' = R_4 \cap \{\beta > 0\}$, and $R_5' = R_5$. Now

$$\int_{R_1'} \|I\|_\infty^p \le C \int_{a_1}^{c} (\rho|\eta|)^{-p/(n-2)} 2|\eta|^{3/2}\, d\eta,$$

$$\int_{R_2'} \|I\|_\infty^p \le C \int_{a_2}^{c} (\rho|\varsigma|^{2/3})^{-p/(n-2)} 2|\varsigma|^{2/3}\, d\varsigma,$$

$$\int_{R_3'} \|I\|_\infty^p \le C \int_{a_1}^{c} (\rho|\eta|^{1/2})^{-p/(n-1)} (\rho^{-2/(n-1)}|\eta|^{(n-3)/2(n-1)})\, d\eta,$$

$$\int_{R_4'} \|I\|_\infty^p \le C \int_{a_1}^{c} \int_{a_3}^{a_4} (\rho|\eta|^{1/4}|\beta|^{1/2})^{-p/(n-2)}\, d\beta\, d\eta,$$

$$\int_{R_5'} \|I\|_\infty^p \le C\rho^{-p/n}\rho^{-2/n}\rho^{-3/n},$$

where $a_1 = 100\rho^{-2/n}$, $a_2 = 1000\rho^{-3/n}$, $a_3 = \rho^{-2/(n-1)}|\eta|^{(n-3)/1(n-1)}$, $a_4 = 2|\eta|^{3/2}$. Some further explanation is needed to explain the integrals over R_3' and R_4'. Recall that

$$\varphi^{(n-3)}(s) = \rho\varsigma + \rho\eta(s - s_0) + \rho E(s - s_0)^3/6$$

and β is the minimum value of $|\varphi^{(n-3)}|$ at the points where $\varphi^{(n-2)} = 0$. Therefore for fixed η, β is a piecewise linear function of ς with derivative ± 1. Consequently in R_3' β is a translate of ς, and $|\{\varsigma\colon \rho^2|\beta|^{n-1} \le |\eta|^{(n-3)/2}\}| = |\{\beta\colon \rho^2|\beta|^{n-1} \le |\eta|^{(n-3)/2}\}| = 2a_3$. Also by (21)

$$|\beta| = |\varsigma \pm b(\xi)\eta|\eta|^{1/2}| \le 2|\eta|^{3/2} = a_4 \quad \text{in } R_4$$

since $.92 < b < .96$.

Straightforward calculations show that except for $p = 3n - 5$ the integrals of $\|I\|_\infty^p$ are all bounded by $C\rho^{-\sigma p}$ where

$$\sigma = \min\left\{\frac{1}{n} + \frac{5}{np}, \frac{1}{n-1} + \frac{2}{p(n-1)}, \frac{1}{n-2}\right\}.$$

When $p = 3n - 5$ the integrals are bounded by $\rho^{-3}\log\rho = \rho^{-\sigma p}\log\rho$ (from regions R_3' and R_4'). The other logarithms that appear are in higher order terms and so are dominated by $C\rho^{-\sigma\rho}$. This proves that

$$\left(\int_{\Sigma_r \cap \Gamma} |f(\xi)|^p\, d\xi\right)^{1/p} \le C\rho^{-\sigma} \le Cr^{-\sigma} \tag{38}$$

since ρ and r are comparable.

For the integral of $|F|^p$ over any of the cones Γ_j recall that

$$|\alpha^{(k)}(s) \cdot \varsigma| \ge c_j|\xi| \quad \text{for } |s| < \varepsilon_j \text{ and } \xi \in \Gamma_j$$

and

$$|\varphi^{(k)}(s) - \rho A_j| \le (.01)A_j \quad \text{for } |s| < \varepsilon,\ \xi \in \Gamma_j$$

where $A_j = \alpha^{(k)}(0) \cdot \tilde{\xi}_j$. Thus the estimation of $I(t)$ in this region is similar and we obtain a bound for $|F|^p$ as in (38) but with $\sigma(n)$ replaced by $\sigma(k)$. Since σ is a decreasing function of n the integrals over $\Sigma_r \cap \Gamma_j$ will decrease faster than (38), $\sigma(k) > \sigma(n)$. Now

$$\left(\int_{\Sigma_r} |F|^p \, d\xi \right)^{1/p} \leq C r^{-\sigma}.$$

An interpolation between $p = 2n - 4$ and $p = 2$ with Theorem 1 completes the proof of Theorem 2.

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