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THE CLASSIFYING TOPOS OF A CONTINUOUS GROUPOID. I

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ABSTRACT. We investigate some properties of the functor B which associates to any continuous groupoid G its classifying topos BG of equivariant G-sheaves. In particular, it will be shown that the category of toposes can be obtained as a localization of a category of continuous groupoids.

If G is a group, the category of G-sets (sets equipped with a right G-action) is a topos BG, which classifies principal G-bundles: for instance, if X is a topological space there is an equivalence between topos morphisms Sheaves$(X) \to BG$ and principal G-bundles over X.

The construction of BG also applies to the case where G is a topological group, or more generally, a topological groupoid. It is a rather surprising result that this essentially exhausts the range of toposes: Joyal and Tierney **(1984)** have shown that any topos is equivalent to one of the form BG for a topological group G, *provided* one works with a slightly generalized notion of topological space, by taking the lattice of open sets as the primitive notion, rather than the set of points (one sometimes speaks of "pointless" spaces). The continuous groupoids of this paper are the groupoid objects in this category of generalized spaces.

The aim of this paper is threefold. First, $G \mapsto BG$ is a functor, and we wish to investigate how the properties of the topos BG depend on those of the continuous groupoid G, and more generally how the properties of a geometric morphism $BG \xrightarrow{B\phi} BH$ depend on those of the map of continuous groupoids $G \xrightarrow{\phi} H$. The second aim is to extend the Joyal-Tierney result, and not only represent toposes in terms of continuous groupoids, but also the geometric morphisms from one topos to another. There are several possible solutions to this problem. In this paper, I present one approach, and show that the category of toposes can be obtained as a category of fractions from a category of continuous groupoids. Another approach, somewhat similar in spirit to the Morita theorems for categories of modules, will be presented elsewhere. The third aim of this paper is of a more methodological nature: in presenting many arguments concerning generalized, "pointless" spaces, I have tried to convey the idea that by using change-of-base techniques *and* exploiting the internal logic of a Grothendieck topos, point-set arguments are perfectly suitable for dealing with pointless spaces (at least as long as one stays within the "stable" part of the theory). Although the general underlying idea is very clear (see e.g. the discussion in 5.3 below), it is a challenging open problem to express this

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as a general metatheorem which allows one to transfer (constructively valid) results concerning topological spaces immediately to the context of these generalized spaces.

Let me outline the contents of this paper in more detail. The construction of BG for a continuous groupoid G is a particular case of a colimit of toposes: one takes the nerve of G, which is a simplicial space NG, then one takes sheaves to obtain a simplicial topos $\mathrm{Sh}(NG)$, and BG is simply the colimit of this simplicial topos (in the appropriate bicategorical sense). Before describing this construction in more detail in §4, I will first consider the general construction of colimits of toposes, and prove the following theorem.

THEOREM 1. *All (small) indexed colimits of Grothendieck toposes exist, and are computed as indexed limits of the underlying categories and inverse image functors.*

In §3 the special case of a simplicial topos is discussed.

In §5, I will take a slightly different point of view, and regard BG as a category of spaces equipped with a G-action. The proofs in this section are also intended to serve the third, methodological, aim just mentioned. The results are of the following kind: sufficient conditions are given on homomorphisms $G \xrightarrow{\phi} H$ of continuous groupoids to imply that the induced geometric morphism $BG \xrightarrow{B\phi} BH$ is of a specific type. For instance, one can give a meaningful definition of when a map of continuous groupoids is open, full, faithful, and essentially surjective respectively (cf. 5.5), and prove (see 5.1, 5.15):

THEOREM 2. *If $G \xrightarrow{\phi} H$ is open and full then $BG \xrightarrow{B\phi} BH$ is an atomic map of toposes.*

THEOREM 3. *If $G \xrightarrow{\phi} H$ is open, full and faithful, and essentially surjective, then $BG \xrightarrow{B\phi} BH$ is an equivalence of toposes.*

In §6, we will show that the construction of the topos BG is stable under change of base, at least when the domain and codomain maps are open. Writing $B(\mathscr{E}, G)$ for the classifying $\mathscr{E}$-topos of a continuous groupoid G in $\mathscr{E}$, this can be expressed as follows:

THEOREM 4. *Let $\mathscr{F} \xrightarrow{p} \mathscr{E}$ be a geometric morphism, and let G be an open continuous groupoid in $\mathscr{E}$. Then there is an equivalence of toposes $B(\mathscr{F}, p^{\#}(G)) \to \mathscr{F} \times_{\mathscr{E}} B(\mathscr{E}, G)$.*

Theorem 4 allows us to use point-set arguments in the context of these classifying toposes BG. A consequence of Theorem 4 is used to prove the results of §5, such as Theorems 2 and 3.

It is Theorem 3 that leads to our second aim, namely that of obtaining the category of toposes as a category of fractions from a category of continuous groupoids. Imitating the definition of an equivalence of categories, we call a map ϕ of continuous groupoids an essential equivalence if it satisfies the hypotheses of Theorem 3.

The following result will be proved in §7.

THEOREM 5. *The class of essential equivalences admits a right calculus of fractions, and the category of toposes is equivalent to the localization of a category of continuous groupoids obtained by inverting the essential equivalences.*

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1. Preliminaries. In this section I will recall some basic definitions and facts.

1.1. *Spaces and locales.* Our terminology concerning spaces and locales follows Joyal and Tierney **(1984)**, in this section referred to as [**JT**]. So a *locale* is a complete Heyting algebra (a "frame"), and a morphism of locales $A \xrightarrow{H} B$ is a function which preserves finite meets and arbitrary sups. The category of (*generalized*) *spaces* is the dual of the category of locales. It contains the category of sober topological spaces as a full subcategory. If X is a space, the corresponding locale is denoted by $\mathscr{O}(X)$, and elements of $\mathscr{O}(X)$ are called *opens* of X. So a map of spaces, or a *continuous map* $X \xrightarrow{f} Y$ is by definition given as a locale morphism $f^{-1}: \mathscr{O}(Y) \to \mathscr{O}(X)$.

A *point* of a space is a map $1 \to X$, where 1 is the terminal space, $\mathscr{O}(1) = \mathscr{P}(\{*\}) = \Omega$. A neighbourhood of a point $1 \xrightarrow{x} X$ is an open $U \in \mathscr{O}(X)$ such that $* \in x^{-1}(U)$, i.e. $1 \xrightarrow{x} X$ factors through the subspace $U \subset X$.

We suppose that the reader is familiar with the basic properties of spaces [**JT**].

A *presentation* of a space X is a poset $\mathbf{P}$ equipped with a stable system of covering families, such that $\mathscr{O}(X)$ is isomorphic to the set of downwards closed subsets of $\mathbf{P}$ which are closed for the system of covers, i.e. $\mathscr{O}(X) \simeq \{S \subseteq \mathbf{P} | (p \leq q \in S \Rightarrow p \in S)$ and $(T$ covers p, $T \subseteq S \Rightarrow p \in S)\}$; this is equivalent to saying that $\mathbf{P}$ is a site for the topos of sheaves on X. The opens in the image of the canonical order-preserving map $\mathbf{P} \to \mathscr{O}(X)$ are also called *basic opens* of X.

1.2. *Open maps, étale maps* (see [**JT**]). A map $X \xrightarrow{f} Y$ of spaces is open if f^{-1} has a left-adjoint $f(-): \mathscr{O}(X) \to \mathscr{O}(Y)$ such that the identity $f(U \wedge f^{-1}(V)) = f(U) \wedge V$ holds. f is an open surjection if f^{-1} is moreover 1-1, i.e. $ff^{-1}(U) = U$. Open maps and open surjections are stable under composition and pullback; and if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps such that gf is open and f is a surjection, then g is open.

Recall that $U \in \mathscr{O}(X)$ is called *positive* if every cover of U contains at least one element. $X \to 1$ is open iff X has a presentation consisting of positive opens. A map $X \xrightarrow{f} Y$ is étale if it is a local homeomorphism (i.e. f is open and there is an open cover $X = \bigvee_i U_i$ such that $f|U_i: U_i \xrightarrow{\sim} f(U_i)$ is an isomorphism). This is equivalent to requiring that $X \xrightarrow{f} Y$ and the diagonal $X \to X \times_Y X$ are open. In particular, X is discrete iff $X \to 1$ and $X \to X \times X$ are open. The usual equivalence between étale maps into X and sheaves on X also holds in the context of these generalized spaces.

1.3. *Quotients.* Colimits of spaces are computed as limits of the corresponding *sets* of opens. In particular, given maps $X \overset{f}{\rightrightarrows} Y$ of spaces, their coequalizer $Y \xrightarrow{q} Q$

is described by $\mathscr{O}(Q) = \{U \in \mathscr{O}(Y) | f^{-1}(U) = g^{-1}(U)\}$, and q^{-1} is the inclusion $\mathscr{O}(Q) \to \mathscr{O}(Y)$. It is easy to see that if f and g are open, then so is q. Open surjections are coequalizers of their kernel pairs (Moerdijk **(1986)**, p. 66). It follows that such coequalizers are stable (a coequalizer $X \rightrightarrows Y \to Q$ is stable if for any space T, $T \times X \rightrightarrows T \times Y \to T \times Q$ is again a coequalizer). It seems to be a hard problem to describe the stable coequalizers of spaces (the corresponding problem for *topological* spaces is discussed in Day, Kelly **(1970)**). But at least we conclude that if $X \to 1$ is open and $R \subset X \times X$ is an open subspace which is an equivalence relation, then the coequalizer

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \xrightarrow{q} X/R$$

is stable. (*Sketch of proof*: q is an open surjection since r_1 and r_2 are, and X/R is discrete because $X/R \to 1$ is open, and so is the diagonal $X/R \xrightarrow{\Delta} X/R \times X/R$, as follows by considering the square

$$\begin{tikzcd} R \arrow[r, "\text{open}"] \arrow[d] & X \times X \arrow[d, "\text{open}"] \\ X/R \arrow[r, "\Delta"] & X/R \times X/R \end{tikzcd}$$

and using 1.2. To see that R is the kernelpair of $X \xrightarrow{q} X/R$, it suffices to consider the case where $X/R = 1$, by writing $X = \coprod_{t \in X/R} q^{-1}(t)$. But if $X/R = 1$, then for any two positive (cf. 1.2) opens U and V of X, $r_1 r_2^{-1}(U) = q^{-1}q(U) = q^{-1}q(V) = r_1 r_2^{-1}(V)$, from which it easily follows that $R = X \times X$.)

1.4. *Toposes.* In this paper, topos means Grothendieck topos. We fix one such topos $\mathscr{S}$ as our base throughout, and work with the comma category of toposes over $\mathscr{S}$. If $\mathscr{E}$ and $\mathscr{F}$ are two such toposes, $\mathrm{Hom}_{\mathscr{S}}(\mathscr{F}, \mathscr{E})$ is the set of geometric morphisms $\mathscr{F} \xrightarrow{f} \mathscr{E}$ over $\mathscr{S}$. These form a category denoted by $\underline{\mathrm{Hom}}_{\mathscr{S}}(\mathscr{F}, \mathscr{E})$, where for $f, g \colon \mathscr{F} \to \mathscr{E}$, the maps $\alpha \colon f \Rightarrow g$ are the natural transformations $f^* \to g^*$ over $\mathscr{S}$. I will often omit the subscript $\mathscr{S}$, and just write $\mathrm{Hom}(\mathscr{F}, \mathscr{E})$, $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{E})$. Moreover, I will often tacitly work inside $\mathscr{S}$, and abuse the language as if $\mathscr{S}$ = Sets, in the usual way. We recall that (2-categorical) pullbacks of Grothendieck toposes, which are used throughout this paper, exist (see e.g. [**TT**, p. 131]), as well as filtered inverse limits (see Moerdijk **(1986)**).

1.5. *Change of base.* We will often work with the category of internal spaces in a topos $\mathscr{E}$, $(\text{spaces})_{\mathscr{E}}$. If $\mathscr{F} \xrightarrow{p} \mathscr{E}$ is a geometric morphism, p induces an adjunction

$$(\text{spaces})_{\mathscr{E}} \underset{p^{\#}}{\overset{p_!}{\leftrightarrows}} (\text{spaces})_{\mathscr{F}}, \qquad p_! \dashv p^{\#}.$$

$p_!$ is defined by $\mathscr{O}(p_! Y) = p_*(\mathscr{O}(Y))$. $p^{\#}$ is most easily described in terms of presentations: if $\mathbf{P}$ is a presentation of a space X in $\mathscr{E}$, then the poset $p^*(\mathbf{P})$ together with the p^*-images of the covers in $\mathbf{P}$ give a presentation of $p^{\#}(X)$.

1.6. *Sheaves and spatial reflection* ([**JT**], *Johnstone* **(1981)**). A spatial topos is a topos of the form $\mathrm{Sh}(X)$ = sheaves on X, where X is a (generalized) space. A geometric morphism $\mathscr{F} \xrightarrow{f} \mathscr{E}$ is called *spatial* if it is equivalent to one of the canonical form $\mathrm{Sh}_{\mathscr{E}}(X) \to \mathscr{E}$, where X is a space in $\mathscr{E}$ and $\mathrm{Sh}_{\mathscr{E}}(X)$ is the category of $\mathscr{E}$-internal sheaves on X. If $\mathscr{F} \xrightarrow{f} \mathscr{E}$ is a geometric morphism, there is a reflection into spatial toposes over $\mathscr{E}$, which is stable under pullback along an arbitrary $\mathscr{E}' \to \mathscr{E}$, and which preserves products [**JT**, §VI.5].

1.7. *Open maps.* A geometric morphism $\mathscr{F} \xrightarrow{f} \mathscr{E}$ is *open* if f^* preserves $\forall$ (i.e. $f^*(\forall_\alpha(S)) \cong \forall_{f^*(\alpha)} f^*(S)$ for any diagram $S \subset X \xrightarrow{\alpha} Y$ in $\mathscr{E}$). See the references [**JT**], Johnstone **(1980)** for equivalent characterizations. If $\mathscr{G} \xrightarrow{g} \mathscr{F} \xrightarrow{f} \mathscr{E}$ are geometric morphisms then $f \circ g$ is open if f and g are, and f is open if fg is and g is a surjection. Open maps and open surjections are stable under pullback ([**JT**], Johnstone **(1980)**); open surjections are stable under filtered inverse limits (Moerdijk **(1986)**, Theorem 5.1(ii)).

If $\mathscr{F} \xrightarrow{f} \mathscr{E}$ is open, then the canonical maps $f^*(X^Y) \to f^*(X)^{f^*(Y)}$ and $f^*(\Omega_{\mathscr{E}}) \to \Omega_{\mathscr{F}}$ are mono.

1.8. *Locally connected maps.* $\mathscr{F} \xrightarrow{f} \mathscr{E}$ is locally connected if f^* has an $\mathscr{E}$-indexed left adjoint. This is equivalent to requiring that f^* commutes with Π-functors (or that $f^*/X \colon \mathscr{E}/X \to \mathscr{F}/f^*(X)$ preserves exponentials, for every $X \in \mathscr{E}$). Locally connected maps are stable under composition and pullback. (See Barr and Paré **(1980)**, and Moerdijk **(1986)**, Appendix.)

1.9. *Atomic maps.* A geometric morphism $\mathscr{F} \xrightarrow{f} \mathscr{E}$ is *atomic* if f^* preserves exponentials and the subobject classifier; see Barr-Diaconescu **(1980)**. This is equivalent to requiring $\mathscr{F} \xrightarrow{f} \mathscr{E}$ and the diagonal $\mathscr{F} \to \mathscr{F} \times_{\mathscr{E}} \mathscr{F}$ to be open [**JT**]. Atomic maps are closed under pullback and composition.

1.10. LEMMA. *Let $\mathscr{G} \xrightarrow{g} \mathscr{F} \xrightarrow{f} \mathscr{E}$ be geometric morphisms. If g is an open surjection and fg is locally connected (resp. atomic) then so is f.*

(This follows easily from the last-mentioned property of open maps in 1.7.)

2. Colimits of toposes. The aim of this section is to show that colimits of toposes exist, and to give an explicit way to construct them.

2.1. *Coequalizers.* The main example will be that of a (pseudo-) coequalizer. (I will usually omit the prefix "pseudo", following the common convention in topos theory, cf. [**TT**, p. 5].) Suppose we are given a parallel pair of geometric morphisms $\mathscr{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathscr{E}$ over the base $\mathscr{S}$. Let $\mathscr{D}$ be the subcategory of $\mathscr{E}$ whose objects are pairs (X, θ), X an object of $\mathscr{E}$ and $\theta \colon f^*(X) \xrightarrow{\sim} g^*(X)$ an isomorphism in $\mathscr{F}$, and whose maps $(X, \theta) \to (Y, \xi)$ are $\mathscr{E}$-morphisms $X \xrightarrow{\alpha} Y$ which are compatible with these isomorphisms, i.e. $g^*(\alpha) \circ \theta = \xi \circ f^*(\alpha)$. Write $q* \colon \mathscr{D} \to \mathscr{E}$ for the forgetful functor $(X, \theta) \mapsto X$; q^* is faithful.

THEOREM. *$\mathscr{D}$ is a Grothendieck topos over $\mathscr{S}$, and q^* defines a geometric morphism $\mathscr{E} \xrightarrow{q} \mathscr{D}$ making $\mathscr{F} \rightrightarrows \mathscr{E} \xrightarrow{q} \mathscr{D}$ into a coequalizer of toposes over $\mathscr{S}$: for any other topos $\mathscr{H}$, q induces an equivalence of categories* $\mathrm{Eq}(\mathscr{H}, f, g) \xrightarrow{\sim}$ $\underline{\mathrm{Hom}}_{\mathscr{S}}(\mathscr{D}, \mathscr{H})$, *where the left-hand side is the categories of pairs (h, μ), $\mathscr{E} \xrightarrow{h} \mathscr{H}$ a geometric morphism with $\mu \colon hf \xrightarrow{\sim} hg$.*

PROOF. To see that $\mathscr{D}$ is a topos and q^* defines a geometric morphism, we use Giraud's criterion [**SGA 4**, exposé IV; **TT**, p. 16]. So we have to check the following:

(a) $\mathscr{D}$ has finite limits,

(b) $\mathscr{D}$ has all set-indexed coproducts, and these are disjoint and universal,

(c) every equivalence relation in $\mathscr{D}$ has a universal coequalizer,

(d) every equivalence relation in $\mathscr{D}$ is effective, and every epimorphism in $\mathscr{D}$ is a coequalizer,

(e) $\mathscr{D}$ has small hom-sets,

(f) $\mathscr{D}$ has a set of generators.

Now (a)–(e) follow from the corresponding facts for $\mathscr{E}$, using that f^* and g^* preserve colimits and finite limits. For example in the case of (d), if $(R, \theta) \rightarrowtail (X, \mu) \times (X, \mu)$ is an equivalence relation in $\mathscr{D}$, then $R \rightarrowtail X \times X$ is an equivalence relation in $\mathscr{E}$, and μ induces an isomorphism

$$\mu/R\colon f^*(X/R) \cong f^*(X)/f^*(R) \to g^*(X)/g^*(R) \cong g^*(X/R),$$

making X/R into an object of $\mathscr{D}$.

In particular, we see that q^* preserves finite limits and arbitrary colimits, and thus defines a geometric morphism $\mathscr{E} \xrightarrow{q} \mathscr{D}$.

Condition (f), however, requires some argument. Suppose $\mathbf{C}$ is a site for $\mathscr{E}$, $\mathbf{D}$ a site for $\mathscr{F}$, full subcategories of $\mathscr{E}$ resp. $\mathscr{F}$ and both closed under finite limits, and that $f^*, g^*\colon \mathscr{E} \to \mathscr{F}$ are induced by left exact continuous functors $F, G\colon \mathbf{C} \to \mathbf{D}$ by left Kan extension. For $X \in \mathrm{Sh}(\mathbf{C})$, $P(X)$ is the presheaf

$$P(X)(D) = \varinjlim_{\substack{x \in X(C) \\ C \in \mathbf{C}}} \mathbf{D}(D, FC),$$

and $f^*(X)$ is the associated sheaf of $P(X)$, and similarly $g^*(X)$ is the associated sheaf of the presheaf $Q(X) = \varinjlim \mathbf{D}(D; GC)$.

Now define an increasing sequence of (small) full subcategories of $\mathscr{E}$, $\langle \mathbf{C}_n : n \in \mathbf{N}\rangle$, by

$$\mathbf{C}_0 = \mathbf{C},$$
$$\mathbf{C}_{n+1} = \text{ objects of the form } \coprod_{j \in \mathscr{J}} C_j, \text{ where } C_j \in \mathbf{C}_n \text{ and } \mathscr{J}$$
$$\text{is the index set of some cover of } \mathbf{D}.$$

Let $\mathbf{C}_\infty$ be the full subcategory whose objects are of the form $\coprod_{n \in \mathbf{N}} C_n$, $C_n \in \mathbf{C}_n$, and let for $0 \le n \le \infty$, $\hat{\mathbf{C}}_n$ be the category whose objects are quotients of objects of $\mathbf{C}_n$ (as a full subcategory of $\mathscr{E}$). I claim that $\mathscr{D}$ is generated by the set of objects $(C, \mu) \in \mathscr{D}$ with $C \in \hat{\mathbf{C}}_\infty$, $\mu\colon f^*(C) \xrightarrow{\sim} g^*(C)$.

To see this, suppose $(X, \theta) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} (Y, \xi)$ are maps in $\mathscr{D}$ such that $\alpha u = \beta u$ for every $(C, \mu) \xrightarrow{u} (X, \theta)$ with $C \in \hat{\mathbf{C}}_\infty$. To show that $\alpha = \beta$, take $x_0 \in X(C_0)$, $C_0 \in \mathbf{C}$. x_0 determines an element $f^*(x_0) \in f^*(X)(FC_0)$, corresponding to the element of $P(X)(FC_0)$ given by id_{FC_0} at the vertex (x_0, C_0). $\theta_{FC_0}(x_0)$ is given by a family of elements $y_j \in Q(X)(D_j)$, for a cover $\{D_j \to FC\}_{j \in J}$ in $\mathbf{D}$. Say y_j is represented by $g_j\colon D_j \to G(C_j)$ at the vertex (x_j, C_j). Let $C_1 = \coprod_{j \in J} C_j$. Then we have a map $x_1 = \{x_j\}\colon C_1 \to X$, with image $U_1 \subset X$ say, and $\theta \circ f^*(x_0)$ factors through $g^*(U_1)$:

$$\begin{tikzcd} & f^*(C_0) \arrow[r, "f^*(x_0)"] \arrow[d, dashed] & f^*(X) \arrow[d, "\theta"] \\ g^*(C_1) \arrow[r, two heads] & g^*(U_1) \arrow[r, tail] & g^*(X). \end{tikzcd}$$

Now consider $x_j \in X(C_j)$. As before, $\theta(f^*(x_j))$ is given by $\{D_{ji} \to F_0 C_j\}_{i \in I_j}$ and elements $g_{ji}\colon D_{ji} \to G(C_{ji})$ at the vertex (x_{ji}, C_{ji}). Let

$$U_j = \text{image}\left(\coprod_{i \in I_j} C_{ji} \xrightarrow{\{x_{ji}\}} X\right),$$

so as to get a factorization

$$\begin{tikzcd} f^*(C_j) \arrow[r, tail, "f^*(x_j)"] \arrow[d, dashed] & f^*(X) \arrow[d, "\theta"] \\ g^*(U_j) \arrow[r, tail] & g^*(X) \end{tikzcd}$$

and let $C_2 = \coprod_{j \in J} \coprod_{i \in I_j} C_{ji} \in \mathbf{C}_2$, $C_2 \xrightarrow{x_2} X$ the map given by the x_{ji}, and $U_2 = \text{image}(C_2 \xrightarrow{x_2} X)$. Then

$$\begin{tikzcd} f^*(C_1) \arrow[r, "x_1"] \arrow[d, dashed] & f^*(X) \arrow[d] \\ g^*(U_2) \arrow[r] & g^*(X). \end{tikzcd}$$

Proceed by induction to produce a sequence $C_n \xrightarrow{x_n} X$, $C_n \in \mathbf{C}_n$, with image $U_n \subset X$, such that there are factorizations

$$\begin{tikzcd} f^*(U_n) \arrow[r, tail] \arrow[d, dashed] & f^*(X) \arrow[d, "\theta"] \\ g^*(U_{n+1}) \arrow[r, tail] & g^*(X), \end{tikzcd}$$

and let $U = \bigcup_n U_n \in \hat{\mathbf{C}}_\infty$. Then θ restricts to a map $f^*(U) \to g^*(U)$, making (U, θ) a subobject of X in $\mathscr{D}$. Since $\alpha | U = \beta | U$, also $\alpha(x_0) = \beta(x_0)$. Since $x_0 \in X(C_0)$ was arbitrary, $\alpha = \beta$.

This proves the claim. The other statements in the theorem are obvious.

2.2. *Coproducts of toposes.* Let $(\mathscr{E}_i)_{i \in I}$ be a set of toposes. The coproduct $\coprod_{i \in I} \mathscr{E}_i$ as a topos is the product $\prod_i \mathscr{E}_i$ as a category. It is well known that $\coprod \mathscr{E}_i$ is a topos. In fact, if $\mathbf{C}_i$ is a site for $\mathscr{E}_i$, then the coproduct $\coprod \mathbf{C}_i$ of the categories $\mathbf{C}_i$, equipped with the smallest Grothendieck topology which makes all the inclusions $\mathbf{C}_j \hookrightarrow \coprod \mathbf{C}_i$ continuous, is a site for $\coprod \mathscr{E}_i$.

2.3. *Tensors.* Let $\mathscr{E}$ be a topos, and let $\mathbf{C}$ be a small category. The tensor $\mathscr{E} \otimes \mathbf{C}$ is a topos such that there is a natural equivalence

$$\underline{\mathrm{Hom}}_{\mathscr{S}}(\mathscr{E} \otimes \mathbf{C}, \mathscr{X}) \simeq \underline{\mathrm{Cat}}(\mathbf{C}, \underline{\mathrm{Hom}}_{\mathscr{S}}(\mathscr{E}, \mathscr{X}))$$

for any $\mathscr{S}$-topos $\mathscr{X}$ (Cat$(-,?)$ is the category of functors from $-$ to ?; see Street **(1976)**, Kelly **(1982)** for the general definition of this tensor). It is easily verified that we can explicitly define $\mathscr{E} \otimes \mathbf{C}$ to be the topos $\mathscr{E}^{\mathbf{C}}$ of $\mathscr{E}$-valued functors on $\mathbf{C}$.

2.4. *Indexed colimits.* The general setting is as follows (see Street **(1980)**). Given a small category $\mathbf{K}$ and two pseudofunctors

$$\mathscr{G}\colon \mathbf{K} \to (\text{toposes}), \quad \mathbf{K}^{\mathrm{op}} \xrightarrow{w} \underline{\mathrm{Cat}},$$

we wish to construct the indexed colimit $w * \mathscr{G}$, which is to be a topos such that there is an equivalence

$$\underline{\mathrm{Hom}}(w * \mathscr{G}, \mathscr{X}) \simeq \underline{\mathrm{Nat}}(w, \underline{\mathrm{Hom}}(\mathscr{G}-, \mathscr{X})), \tag{1}$$

natural in the parameter topos $\mathscr{X}$. ($\underline{\mathrm{Hom}}$ is the category of geometric morphisms, and $\underline{\mathrm{Nat}}$ is the category of (pseudo) natural transformations between functors $\mathbf{K} \to \underline{\mathrm{Cat}}$. If $\mathscr{G}$ and $\mathscr{X}$ are over the base $\mathscr{S}$, then so is $w * \mathscr{G}$, and (1) will hold over $\mathscr{S}$.)

For instance, in the case of 2.1 one takes $\mathbf{K} = 0\cdot \overset{u}{\underset{v}{\rightrightarrows}} \cdot 1$ (two parallel nonidentity arrows u and v), $\mathscr{G}(0) = \mathscr{F}$, $\mathscr{G}(1) = \mathscr{E}$, $\mathscr{G}(u) = \alpha$, $\mathscr{G}(v) = \beta$, $w(1) = 1$, $w(0) = \cdot \xrightarrow{\sim} \cdot$ (one nonidentity isomorphism). In the case of coproducts, cf. 2.2, one takes $\mathbf{K}$ to be the discrete category I, $\mathscr{G}(i) = \mathscr{E}_i$, and w the ccnstant functor 1. In the case of the tensor $\mathscr{E} \otimes \mathbf{C}$, one takes $\mathbf{K} = 1$, $\mathscr{G}$ has value $\mathscr{E}$, and w has value $\mathbf{C}$.

In the general case, $w * \mathscr{G}$ can be described *as a category* in the following way. The *objects* of $w * \mathscr{G}$ are pairs $\langle D_{(-)}, u_{(-)}\rangle$ where D_K is a diagram of type $w(K)$ in the topos $\mathscr{G}(K)$, and for $K \xrightarrow{\alpha} K'$ in $\mathbf{K}$, u_α is a natural isomorphism $D_K \circ w(\alpha) \xrightarrow{\sim} \mathscr{G}(\alpha)^* \circ D_{K'}\colon w(K') \to \mathscr{G}(K)$ of diagrams of type $w(K')$ in the topos $\mathscr{G}(K)$. Moreover, the u_α are required to be coherent, in the sense that for $K \xrightarrow{\alpha} K' \xrightarrow{\beta} K''$ in $\mathbf{K}$, $u_{\beta\circ\alpha} \cong u_\beta \circ u_\alpha$ in the only way that makes sense; that is, if we suppress the isomorphisms which tell us that w and $\mathscr{G}$ are pseudofunctors rather than functors, we require the natural transformation

$$\begin{tikzcd}
 & w(K') \arrow[r, "w(\alpha)"] \arrow[ddr, phantom, "\Downarrow u_{\alpha\beta}"] & w(K) \arrow[dr, "D_K"] & \\
w(K'') \arrow[ur, "w(\beta)"] \arrow[dr, "D_{K''}"'] & & & \mathcal{G}(K) \arrow[dl, "\mathcal{G}(\alpha)^*"] \\
 & \mathcal{G}(K'') \arrow[r, "\mathcal{G}(\beta)^*"'] & \mathcal{G}(K') &
\end{tikzcd}$$

to be the same as the composite

$$\begin{tikzcd}
 & & w(K) \arrow[d, Rightarrow, "u_\alpha"] \arrow[dr, "D_K"] & \\
w(K'') \arrow[r, "w(\beta)"] \arrow[dr, "D_{K''}"'] & w(K') \arrow[ur, "w(\alpha)"] \arrow[r, "D_{K'}"] \arrow[d, Rightarrow, "u_\beta"] & \mathscr{G}(K') \arrow[r, "\mathscr{G}(\alpha)^*"] & \mathscr{G}(K) \\
 & \mathscr{G}(K'') \arrow[ur, "\mathscr{G}(\beta)^*"'] & &
\end{tikzcd}$$

The *morphisms* $(D,u) \xrightarrow{\tau} (D',u')$ of $w * \mathscr{G}$ are families $(\tau_K \colon K \in \mathbf{K})$ of morphisms of diagrams in $\mathscr{G}(K)$

$$\begin{tikzcd}
w(K) \arrow[r, bend left, "D_K"] \arrow[r, bend right, "D'_K"'] \arrow[r, phantom, "\Downarrow \tau_K"] & \mathscr{G}(K)
\end{tikzcd}$$

which are natural in K, i.e. for $K_1 \xrightarrow{\alpha} K_2$,

$$\begin{tikzcd}
\mathscr{G}(\alpha)^* \circ D_{K_1} \arrow[r, Rightarrow, "\mathscr{G}(\alpha)^* \cdot \tau_{K_1}"] & \mathscr{G}(\alpha)^* \circ D'_{K_1} \\
D_{K_2} \circ w(\alpha) \arrow[u, Rightarrow, "u_\alpha"] \arrow[r, Rightarrow, "\tau_{K_2} \circ F(\alpha)"] & D'_{K_2} \circ w(\alpha) \arrow[u, Rightarrow, "u'_\alpha"']
\end{tikzcd}$$

This defines a category $w * \mathscr{G}$. The reader may wish to check but is advised to believe that colimits and finite limits in $w * \mathscr{G}$ are computed by just taking the corresponding colimits and finite limits "pointwise" in each of the toposes $\mathscr{G}(K)$, $K \in \mathbf{K}$, in the obvious way. Then the isomorphism (1) above is easily checked; in fact it is induced by the (pseudo) natural transformation

$$w \xrightarrow{\pi} \underline{\mathrm{Hom}}(\mathscr{G}-, w * \mathscr{G}),$$

whose components

$$\pi_K \colon w(K) \to \underline{\mathrm{Hom}}(\mathscr{G}(K), w * \mathscr{G})$$

are defined as follows: taking inverse images, the functor π_K is the same as an inverse image functor of a geometric morphism $\mathscr{G}(K)^{w(K)} \to w * \mathscr{G}$, and for this we can just take the projection $(D_-, u_-) \mapsto D_K$.

I suppose that it is possible to show directly that $w * \mathscr{G}$ has generators. But fortunately we do not have to go through this, because a result of Street **(1980)** says that all indexed colimits can be constructed from coequalizers (2.1), coproducts (2.2), and tensors (2.3). Thus the following theorem is proved.

2.5. THEOREM. *All small indexed colimits in the bicategory of Grothendieck toposes exist, and are computed as indexed (bi-)limits of the underlying categories and inverse image functors.*

REMARK. The reader must have noticed that given Street's result, the only work involved in the proof of 2.5 is to show the existence of coequalizers. This was proved independently by several people, among whom M. Tierney, P. Freyd, and the present author. An elegant approach to the existence of colimits is that via accessible categories, as demonstrated by recent work of Makkai and Paré (to appear).

3. Simplicial toposes and descent. In this section we consider a special type of colimit, namely that of a simplicial topos. Simplicial toposes occur naturally in a variety of circumstances; for instance, sheaves on a simplicial topological space, the étale topos of a simplicial scheme, etc. To each topos, one can associate the singular complex, which is a simplicial topos (cf. Moerdijk and Wraith **(1986)**, and 3.10 below).

3.1. *Simplicial toposes.* Let Δ be the usual category of finite nonempty sets $[n] = \{0, \dots, n\}$ $(n \geq 0)$ and $\leq$-preserving maps. A *simplicial topos* $\mathscr{E}.$ is a pseudo-functor from Δ^{op} into toposes (over the base topos $\mathscr{S}$), i.e. $\mathscr{E}.$ is a sequence of toposes $\mathscr{E}_n$ $(n \geq 0)$, together with geometric morphisms $\overline{\alpha}\colon \mathscr{E}_m \to \mathscr{E}_n$ for $[n] \xrightarrow{\alpha} [m]$, functorial up to a specified coherent isomorphism $\tau = \tau_{\alpha,\beta}\colon \overline{\alpha} \circ \overline{\beta} \overset{\sim}{\Longrightarrow} \overline{\beta\alpha}$ (i.e. τ is a natural isomorphism of inverse image functors $\overline{\beta}^* \circ \overline{\alpha}^* \xrightarrow{\sim} (\overline{\beta\alpha})^*$). So this is the usual definition of a simplicial object in a category (see e.g. Gabriel and Zisman **(1967)**, May **(1968)**), except that we have to take into account that the category of toposes and geometric morphisms can only be usefully considered as a 2-category. Thus, a simplicial topos $\mathscr{E}.$ may alternatively be described as a sequence of toposes $\mathscr{E}_n$, $n \geq 0$, together with geometric morphisms $\mathscr{E}_n \xrightarrow{d_i} \mathscr{E}_{n-1}$ $(i = 0, \dots, n)$ and $\mathscr{E}_{n-1} \xrightarrow{s_j} \mathscr{E}_n$ $(j = 1, \dots, n-1)$ satisfying the usual simplicial identities, but only up to a coherent isomorphism τ.

A (pseudo) cocone $\mathscr{E}. \xrightarrow{f.} \mathscr{F}$ under a given simplicial topos is a sequence of geometric morphisms $\mathscr{E}_n \xrightarrow{f_n} \mathscr{F}$ (over the base $\mathscr{S}$) into a given topos $\mathscr{F}$, together with natural isomorphisms $\sigma_\alpha\colon f_n \circ \overline{\alpha} \Rightarrow f_m$ which are compatible with the τ's. That is, σ_{id} is the identity for each $[n] \xrightarrow{\mathrm{id}} [n]$, and for $[n] \xrightarrow{\alpha} [m] \xrightarrow{\beta} [k]$ the square

$$\begin{tikzcd}
f_n\overline{\alpha}\overline{\beta} \arrow[r, Rightarrow, "f_n\cdot\tau"] \arrow[d, Rightarrow, "\sigma_\alpha\cdot\overline{\beta}"'] & f_n\overline{\beta\alpha} \arrow[d, Rightarrow, "\sigma_{\beta\alpha}"] \\
f_m\overline{\beta} \arrow[r, Rightarrow, "\sigma_\beta"] & f_k
\end{tikzcd}$$

commutes.

The universal such cocone is the (pseudo) colimit of the simplicial topos $\mathscr{E}.$ denoted by $\mathscr{E}. \xrightarrow{q.} L(\mathscr{E}.)$. The topos $L(\mathscr{E}.)$ can explicitly be described as the category whose objects are sequences $((X_n)_n, \xi_\alpha)$, X_n an object of $\mathscr{E}_n$ and $\xi_\alpha\colon X_m \xrightarrow{\sim} \overline{\alpha}^*(X_n)$ an isomorphism in $\mathscr{E}_n$ for each $[n] \xrightarrow{\alpha} [m]$, compatible with the τ's. We do not need this description of the topos $L(\mathscr{E}.)$, but only its existence, cf. 3.3 below.

3.2. *Descent.* Given a simplicial topos $\mathscr{E}.$, consider that part of the data which only uses the maps

$$\begin{tikzcd} \mathscr{E}_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & \mathscr{E}_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & \mathscr{E}_0 \arrow[l, "s_0" description] \end{tikzcd}$$

and the τ's between composites of these. A *descent cocone* into a topos $\mathscr{F}$ is a pair (g, μ), where $\mathscr{E}_0 \xrightarrow{g} \mathscr{F}$ is a geometric morphism, and $\mu\colon gd_1 \Rightarrow gd_0$ is a 2-cell satisfying

(i) *unit condition*:

$$\begin{tikzcd} gd_1s_0 \arrow[rr, Rightarrow, "\mu\cdot s_0"] \arrow[dr, Rightarrow, "g\cdot\tau"'] & & gd_0s_0 \arrow[dl, Rightarrow, "g\cdot\tau"] \\ & g & \end{tikzcd}$$

commutes, and

(ii) *cocycle condition*:

$$\begin{tikzcd} & gd_1d_2 \arrow[rr, Rightarrow, dashed, "\mu d_2"] \arrow[dl, Rightarrow, "g_\tau"'] & & gd_0d_2 \arrow[dr, Rightarrow, "\tau"] & \\ gd_1d_1 \arrow[dr, Rightarrow, "\mu d_1"'] & & & & gd_1d_0 \arrow[dl, Rightarrow, "\mu d_0"] \\ & gd_0d_1 \arrow[rr, Rightarrow, dashed, "\tau"'] & & gd_0d_0 & \end{tikzcd}$$

commutes.

In other words, $\mu\colon d_1g \xrightarrow{\sim} d_0g$ is a natural isomorphism such that $s_0^*(\mu) \cong \mathrm{id}$ and $d_0^*(\mu) \circ d_2^*(\mu) \cong d_1^*(\mu)$, provided one plugs in enough τ-isomorphisms for this to make sense.

The *universal* descent cocone is denoted by $(p, \theta)\colon \mathscr{E}. \to \mathrm{Desc}(\mathscr{E}.)$. $\mathrm{Desc}(\mathscr{E}.)$ exists, cf. 3.3 below, and can explicitly be described as the category whose objects are pairs (X, θ), where X is an object of $\mathscr{E}_0$ and $\theta\colon d_1^*(X) \xrightarrow{\sim} d_0^*(X)$ is an isomorphism such that $s_0^*(\theta) \cong \mathrm{id}$ and $d_0^*(\theta) \circ d_2^*(\theta) \cong d_1^*(\theta)$, provided we plug in some τ's as before.

This category $\mathrm{Desc}(\mathscr{E}.)$ plays an important role in the representation of toposes by groupoids, cf. Joyal and Tierney **(1984)**, and below.

3.3. *Existence of the toposes $L(\mathscr{E}.)$ and* $\mathrm{Desc}(\mathscr{E}.)$. These have been defined as categories, but we have to show that they are Grothendieck toposes, and that the obvious "forgetful" functors $L(\mathscr{E}.) \xrightarrow{q_n^*} \mathscr{E}_n$ and $\mathrm{Desc}(\mathscr{E}.) \xrightarrow{p^*} \mathscr{E}_0$ indeed define inverse images of universal geometric morphisms $\{\mathscr{E}_n \xrightarrow{q_n} L(\mathscr{E}.)\}_n$ and $\mathscr{E}_0 \xrightarrow{p} \mathrm{Desc}(\mathscr{E}.)$.

These are all cases of indexed colimits of toposes, however, and therefore this follows from the general result in §2.

3.4. PROPOSITION. *The toposes $L(\mathscr{E}.)$ and* $\mathrm{Desc}(\mathscr{E}.)$ *are equivalent, and hence so are the 2-categories of pseudo-cocones and descent cocones, for any simplicial topos $\mathscr{E}$.*

As a sketch of proof, let me indicate how to pass from a cocone $(f.,\sigma.)\colon \mathscr{E}. \to \mathscr{F}$ to a descent cocone $(g,\mu)\colon \mathscr{E}. \to \mathscr{F}$ and vice versa, by functors T and U respectively.

Given $(f_n)_n$ and $(\sigma_\alpha)_\alpha$ as in 2.2, define

$$T(f_n,\sigma_\alpha) = (f_0, fd_1 \overset{\tilde{\sigma}}{\Rightarrow} fd_0)$$

where $\tilde{\sigma}$ is the composite

$$f_0d_1 = f_0 \circ \overline{0} \overset{\sigma_0}{\Rightarrow} f_1 \overset{(\sigma_1)^{-1}}{\Longleftarrow} f_0 \circ \overline{1} = f_0 d_0$$

(here $0,1\colon [0] \to [1]$ denote the maps in Δ with the corresponding value). Given (g,μ), define $U(g,\mu) = (\hat{g}_n, \hat{\sigma}_n)$, where $g_n = gd_0\cdots d_0 = g\bar{n}$ ($n\colon [0] \to [n]$ has $n(0) = n$), and for $[n] \xrightarrow{\alpha} [m]$, $\hat{\sigma}_\alpha\colon g_n \circ \overline{\alpha} \Rightarrow g_m$ is the composite

$$\begin{aligned} g_n\overline{\alpha} = g\circ\bar{n}\circ\bar{\alpha} &\overset{\tau}{\Longrightarrow} g\circ\overline{\alpha(n)} \overset{\tau}{\Longrightarrow} g\circ\overline{0}\circ\overline{(\alpha(n),m)} \\ &\overset{\mu\overline{(\alpha(n),m)}}{\Longrightarrow} g\circ\overline{1}\circ\overline{(\alpha(n),m)} \overset{\tau}{\Longrightarrow} g\circ\bar{m}, \end{aligned}$$

where

$$[0] \underset{1}{\overset{0}{\rightrightarrows}} [1] \xrightarrow{(\alpha(n),m)} [m]$$

denote the obvious maps in Δ.

A tedious argument shows that these functorial operations T and U are, up to isomorphism, mutually inverse.

3.5. *Localization.* Let $\mathscr{E}.$ be a truncated simplicial topos as in 3.2(1), with colimit $\mathrm{Desc}(\mathscr{E}.)$

(1) $$\mathscr{E}_2 \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \mathscr{E}_1 \mathrel{\substack{\rightarrow\\\leftarrow\\\rightarrow}} \mathscr{E}_0 \to \mathrm{Desc}(\mathscr{E}.),$$

and let (X,θ) be an object of $\mathrm{Desc}(\mathscr{E}.)$. We obtain an induced truncated simplicial topos $\mathscr{E}./(X,\theta)$, namely

(2) $$\mathscr{E}_2/d_0^*d_0^*(X) \;\; \begin{matrix} \xrightarrow{\bar{d}_0} \\ \xrightarrow{\bar{d}_1} \\ \xrightarrow[\bar{d}_2]{} \end{matrix} \;\; \mathscr{E}_1/d_0^*(X) \;\; \begin{matrix} \xrightarrow{\bar{d}_0} \\ \xleftarrow{\bar{s}} \\ \xrightarrow[\bar{d}_1]{} \end{matrix} \;\; \mathscr{E}_0/X;$$

the $\bar{d}_i$ and $\bar{s}$ are defined from d_i and s as follows: $\bar{d}_0^* = d_0^*/X$, i.e.

$$\bar{d}_0^*(Y \xrightarrow{\alpha} X) = d_0^*(Y) \xrightarrow{d_0^*(\alpha)} d_0^*(X),$$

$$\bar{d}_1^*(Y \xrightarrow{\alpha} X) = d_1^*(Y) \xrightarrow{d_1^*(\alpha)} d_1^*(X) \xrightarrow{\theta} d_0^*(X),$$

and

$$\bar{s}^*(Z \xrightarrow{\beta} d_0^*(X)) = s^*(Z) \xrightarrow{s^*(\beta)} s^*d_0^*(X) \xrightarrow{\tau} X.$$

The $\bar{d}_i\colon \mathscr{E}_2/d_0^*d_0^*(X) \to \mathscr{E}_1/d_0^*(X)$ are defined similarly, just by inserting enough τ's and θ's:

$$\bar{d}_0^*(Y \xrightarrow{\alpha} d_0^*(X)) = d_0^*(Y) \xrightarrow{d_0^*(\alpha)} d_0^*d_0^*(X),$$
$$\bar{d}_1^*(Y \xrightarrow{\alpha} d_0^*(X)) = d_1^*(Y) \xrightarrow{d_1^*(\alpha)} d_1^*d_0^*(X) \xrightarrow{\tau} d_0^*d_0^*(X),$$
$$\bar{d}_2^*(Y \xrightarrow{\alpha} d_0^*(X)) = d_2^*(Y) \xrightarrow{d_2^*(\alpha)} d_2^*d_0^*(X) \xrightarrow{\tau} d_0^*d_1^*(X) \xrightarrow{\theta} d_0^*d_0^*(X).$$

Let $\mathrm{Desc}(\mathscr{E}_\cdot/(X,\theta))$ be the colimit of this diagram (2). Then we have the following result.

LOCALIZATION LEMMA. *Let $\mathscr{E}_\cdot$ be a truncated simplicial topos, and (X,θ) an object of $\mathrm{Desc}(\mathscr{E}_\cdot)$, as above. Then there is a canonical equivalence of toposes*

$$\mathrm{Desc}(\mathscr{E}_\cdot/(X,\theta)) \xrightarrow{\sim} \mathrm{Desc}(\mathscr{E}_\cdot)/(X,\theta).$$

The proof is by direct inspection.

3.6. THEOREM. *Let $\mathscr{E}_\cdot = (\mathscr{E}_2 \mathrel{\substack{\to\\\to\\\to}} \mathscr{E}_1 \mathrel{\substack{\to\\\leftarrow\\\to}} \mathscr{E}_0)$ be a truncated simplicial topos as in* (1) *of* 3.2, *with universal morphism $\mathscr{E}_0 \xrightarrow{p} \mathrm{Desc}(\mathscr{E}_\cdot)$. p is obviously a surjection.*

(a) *If $d_0, d_1\colon \mathscr{E}_1 \rightrightarrows \mathscr{E}_0$ are both open, then so is p.*

(b) *If $d_0, d_1\colon \mathscr{E}_1 \rightrightarrows \mathscr{E}_0$ are both locally connected, and $\mathscr{E}_2 \mathrel{\substack{\to\\\to\\\to}} \mathscr{E}_1$ are all open, then p is locally connected.*

(c) *If $d_0, d_1\colon \mathscr{E}_1 \rightrightarrows \mathscr{E}_0$ are both atomic and $\mathscr{E}_2 \mathrel{\substack{\to\\\to\\\to}} \mathscr{E}_1$ are all open, then p is atomic.*

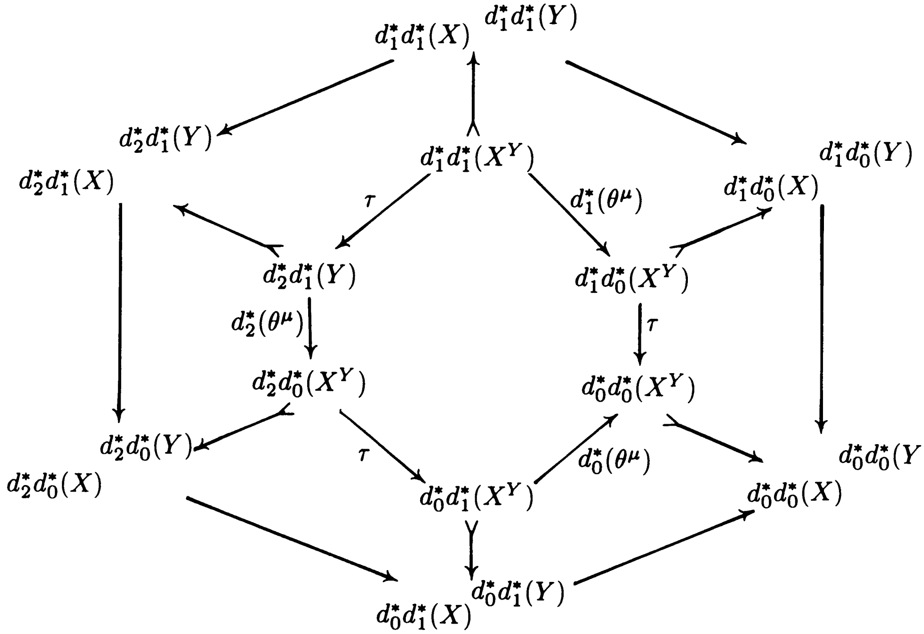
PROOF. (a) Suppose d_0^*, d_1^* both preserve $\forall$ (cf. 1.7), and consider a diagram $(S,\theta) \rightarrowtail (X,\theta) \xrightarrow{\alpha} (Y,\mu)$ in $\mathrm{Desc}(\mathscr{E}_\cdot)$. To compute $\forall_\alpha(S,\theta)$ in $\mathrm{Desc}(\mathscr{E}_\cdot)$, a first approximation would be to take $T = \forall_\alpha(S) \subset Y$ in $\mathscr{E}$. In general, the problem is that T need not be "closed" under the action μ, but if d_0^*, d_1^* preserve $\forall$, then $d_1^*(Y) \xrightarrow{\mu} d_0^*(Y)$ maps $d_1^*(T) \subset d_1^*(Y)$ into $d_0^*(T) \subset d_0^*(Y)$, as is easy to see. It is then clear that $(T,\mu) = \forall_\alpha(S,\theta)$ in $\mathrm{Desc}(\mathscr{E}_\cdot)$.

(b) We have to show that p^* preserves Π-functors if the assumptions of (b) hold (cf. 1.8). Since the assumptions of (b), however, are all stable under slicing by an object of $\mathrm{Desc}(\mathscr{E}_\cdot)$, cf. 3.5, it is enough to check that p^* preserves exponentials if d_0^* and $d_1^*\colon \mathscr{E}_0 \rightrightarrows \mathscr{E}_1$ do, and $d_0, d_1, d_2\colon \mathscr{E}_2 \rightrightarrows \mathscr{E}_1$ are open.

Given (X,θ) and (Y,μ) in $\mathrm{Desc}(\mathscr{E}_\cdot)$, define the map $d_1^*(X^Y) \xrightarrow{\theta^\mu} d_0^*(X^Y)$ as the composite

$$d_1^*(X^Y) \xrightarrow{\sim} d_1^*(X)^{d_1^*(Y)} \xrightarrow{\theta^{d_1^*(Y)}} d_0^*(X)^{d_1^*(Y)} \xrightarrow{d_0^*(X)^{\mu^{-1}}} d_0^*(X)^{d_0^*(Y)} \xrightarrow{\sim} d_0^*(X^Y).$$

We need to verify that θ satisfies the unit and cocycle conditions. The first is a straightforward diagram chase argument, using maps on test objects $T \to X^Y$. For the cocycle condition, we use that the $d_i\colon \mathscr{E}_2 \to \mathscr{E}_1$ are all open, so that for any objects $A, B \in \mathscr{E}_1$ the canonical map $d_i^*(A^B) \to d_i^*(A)^{d_i^*(B)}$ is mono (cf. 1.7). Thus, we obtain monomorphisms $d_2^*d_1^*(X^Y) \to d_2^*d_1^*(X)^{d_2^*d_1^*(Y)}$, etc., and we can therefore "embed" the to-be-commutative hexagon (the inner one) into the outer hexagon, which commutes as a simple consequence of the cocycle condition for θ and μ:



(c) To prove that p is atomic under the given assumptions, we only need to check that p^* preserves the subobject classifier, by (b). Write Ω_i for the subobject classifier of $\mathscr{E}_i$. Since $\mathscr{E}_1 \rightrightarrows \mathscr{E}_0$ are atomic, the canonical maps σ_i are isomorphisms $d_0^*(\Omega_0) \xrightarrow{\sigma_0} \Omega_1 \xleftarrow{\sigma_1} d_1^*(\Omega_0)$, so we can define the composed map $\sigma = \sigma_0^{-1}\sigma_1 \colon d_1^*(\Omega_0) \to d_0^*(\Omega_0)$. We claim that (Ω_0, σ) is the subobject classifier of $\mathrm{Desc}(\mathscr{E}.)$. It is rather obvious that (Ω_0, σ) would indeed classify subobjects in $\mathrm{Desc}(\mathscr{E}.)$, provided we show that it is an object of $\mathrm{Desc}(\mathscr{E}.)$ in the first place. One easily sees that σ satisfies the unit condition. The cocycle condition can be verified by using that the geometric morphisms $\mathscr{E}_2 \xrightarrow{d_i} \mathscr{E}_1$ are open, which implies that the canonical maps $d_i^*(\Omega_1) \to \Omega_2$ are mono (cf. 1.7): it is enough to show that each of the inner triangles in

$$\begin{tikzcd}
& d_1^*d_1^*(\Omega_0) \arrow[dl, "\tau"'] \arrow[dr, "d_1^*(\sigma)"] \arrow[d] & \\
d_2^*d_1^*(\Omega_0) \arrow[r] \arrow[dd, "d_2^*(\sigma)"'] & \Omega_2 & d_1 d_0(\Omega_0) \arrow[l] \arrow[dd, "\tau"] \\
& & \\
d_2^*d_0^*(\Omega_0) \arrow[ur] \arrow[dr, "\tau"'] & & d_0^*d_0^*(\Omega_0) \arrow[ul] \\
& d_0^*d_1^*(\Omega_0) \arrow[uuu] \arrow[ur, "d_0^*(\sigma)"'] &
\end{tikzcd}$$

commutes. For those with a $d_i(\sigma)$ as edge this is obvious, and for those with a τ it is straightforwardly checked.

This completes the proof of Theorem 3.6.

3.7. REMARK. A similar but easier proof gives an analogous result for coequalizers $\mathscr{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathscr{E} \xrightarrow{p} \mathscr{D}$ as in 2.1, namely

THEOREM. *If f and g are open (resp. locally connected, atomic) then so is p.*

3.8. COROLLARY. *Let $\mathscr{E}_2 \rightrightarrows \mathscr{E}_1 \rightleftarrows \mathscr{E}_0 \to$ Desc$(\mathscr{E}.)$ be a colimit diagram, as before.*

(a) *If $\mathscr{E}_0 \to \mathscr{S}$ is open, so is* Desc$(\mathscr{E}.)$.

(b) *If $\mathscr{E}_0 \to \mathscr{S}$ is locally connected and $\mathscr{E}_1 \rightrightarrows \mathscr{E}_0$ are open, then* Desc$(\mathscr{E}.)$ *is locally connected.*

(c) *If $\mathscr{E}_0 \to \mathscr{S}$ is atomic and $\mathscr{E}_1 \rightrightarrows \mathscr{E}_0$ are open, then so is* Desc$(\mathscr{E}.)$.

PROOF. Apply 1.10 to the triangle

$$\begin{tikzcd} \mathscr{E}_0 \arrow[rr] \arrow[dr] & & \mathrm{Desc}(\mathscr{E}.) \arrow[dl] \\ & \mathscr{S} & \end{tikzcd}$$

and use 3.6(a) for (b) and (c).

3.9. COROLLARY. *Let $\mathscr{E}_2 \rightrightarrows \mathscr{E}_1 \rightleftarrows \mathscr{E}_0 \to$ Desc$(\mathscr{E}.)$ be as in 3.8, and suppose that $\mathscr{E}_0 \to \mathscr{S}$, $\mathscr{E}_1 \xrightarrow{(d_0,d_1)} \mathscr{E}_0 \times_{\mathscr{S}} \mathscr{E}_0$ are open (respectively open surjections). Then* Desc$(\mathscr{E}.)$ *is atomic (resp. atomic connected).*

PROOF. We use 1.9. Desc$(\mathscr{E}.) \to \mathscr{S}$ is open by 3.8(a); moreover by considering the diagram

$$\begin{tikzcd} \mathscr{E}_1 \arrow[r, "(d_0,d_1)"] \arrow[d] & \mathscr{E}_0 \times_{\mathscr{S}} \mathscr{E}_0 \arrow[d] \\ \mathrm{Desc}(\mathscr{E}.) \arrow[r, "\Delta"] & \mathrm{Desc}(\mathscr{E}.) \times_{\mathscr{S}} \mathrm{Desc}(\mathscr{E}.) \end{tikzcd}$$

it follows that the diagonal Δ is open (an open surjection) if $\mathscr{E}_1 \to \mathscr{E}_0 \times_{\mathscr{S}} \mathscr{E}_0$ is open (an open surjection).

3.10. EXAMPLE (taken from Moerdijk and Wraith **(1986)**). Let $\mathscr{E}$ be a topos, and let

$$1 = \Delta_0 \rightrightarrows I = \Delta_1 \rightrightarrows \Delta_2 \cdots$$

be the standard cosimplicial topological space (Δ_n is the standard n-simplex), but defined as a cosimplicial *locale* if $\mathscr{S} \neq$ Sets. By taking sheaves and exponentiating, we obtain the *singular complex* $\mathscr{E}^{\mathrm{Sh}(\Delta.)}$, which is a simplicial topos. The colimit $L(\mathscr{E}.)$ "is" the fundamental group of $\mathscr{E}$. For instance, if $\mathscr{E}$ is connected and locally connected, then $\mathscr{E}^I \to \mathscr{E} \times \mathscr{E}$ is an open surjection (Moerdijk and Wraith **(1986)**), and hence by 3.9 and 3.4 $L(\mathscr{E}.)$ is an atomic connected topos. A point of $\mathscr{E}$ gives a point of $L(\mathscr{E}.)$, and it follows that $L(\mathscr{E}.)$ is continuous G-sets for a continuous group G, by a result of Joyal-Tierney **(1984)**, see also 4.3). On some other occasion, I hope to come back to this construction of a fundamental group of $\mathscr{E}$, and to its relation to Grothendieck's construction of the fundamental group of a topos, in some more detail.

4. The classifying topos of a continuous groupoid. In this section we will describe a functor B which associates a topos BG to any groupoid G in the category of spaces, as an indexed colimit of a simplicial topos.

4.1. *Continuous groupoids.* A continuous groupoid is a groupoid in the category of (generalized) spaces (cf. 1.1). So such a groupoid G consists of two spaces G_0 (the space of objects) and G_1 (the space of arrows), together with domain and codomain maps d_0 and $d_1 \colon G_1 \rightrightarrows G_0$, respectively, a unit map $G_0 \xrightarrow{s} G_1$, and a multiplication or composition map $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1$ (in point-set notation: $m(g,f) = g \circ f$, i.e. $G_1 \times_{G_0} G_1$ is the pullback of $G_1 \xrightarrow{d_0} G_0$ on the left and $G_1 \xrightarrow{d_1} G_0$ on the right). These structure maps are supposed to satisfy the usual identities. The existence of an inverse $G_1 \xrightarrow{(\cdot)^{-1}} G_1$ can be expressed by requiring $G_1 \times_{G_0} G_1 \xrightarrow{(m,\pi_2)} G_1 \times_{G_0} G_1$, as well as (π_1, m), to be isomorphisms over G_1.

A *continuous group* is a continuous groupoid with $G_0 = 1$; in this case we write G for G_1, as usual.

A *continuous homomorphism*, or just a *map*, of continuous groupoids $G \xrightarrow{\phi} H$ is a pair $G_1 \xrightarrow{\phi_1} H_1$, $G_0 \xrightarrow{\phi_0} H_0$ of maps of spaces which satisfy the usual equations.

In the sequel, we will often just work with continuous groupoids G having the property that $G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0$ are open maps. Notice that this implies that m is open, since $m = \pi_1 \circ (m, \pi_2)$ and (m, π_2) is an isomorphism, as just mentioned.

4.2. Definition of BG. Given a continuous groupoid G, let $N.(G)$ be the nerve of G, so $N.(G)$ is a simplicial space. (We number the faces and degeneracies of $N.(G)$ in such a way that d_0 and $d_1 \colon G_1 \rightrightarrows G_0$ remain the faces d_0 and $d_1 \colon N_1(G) \rightrightarrows N_0(G)$, respectively.)

Applying sheaves, we obtain a simplicial topos $\mathrm{Sh}(N.(G)) \simeq N.(\mathrm{Sh}(G))$. The topos BG is by definition the (pseudo-) colimit $L(\mathrm{Sh}(N.(G))$ of this simplicial topos, i.e. we have a universal augmentation

$$\mathrm{Sh}(N.G) \to BG. \tag{1}$$

By universality, BG is obviously a (pseudo) functor of G. If $G \xrightarrow{\phi} H$ is a map of continuous groupoids, the corresponding geometric morphism is denoted by $B\phi \colon BG \to BH$; sometimes we will just write ϕ^* for the inverse image $(B\phi)^*$.

When we wish to make the base topos explicit in the notation, we will write $B(\mathscr{S}, G)$ for BG.

By 3.4, BG can alternatively be described as a descent-topos, as indicated by the diagram

$$\mathrm{Sh}(G_1 \times_{G_0} G_1) \quad \underset{\pi_2}{\overset{\xrightarrow{\pi_1}}{\underset{m}{\rightrightarrows}}} \quad \mathrm{Sh}(G_1) \quad \underset{d_1}{\overset{\xrightarrow{d_0}}{\underset{s}{\leftrightarrows}}} \quad \mathrm{Sh}(G_0) \twoheadrightarrow BG. \tag{2}$$

4.3. *Facts from Joyal-Tierney* (**1984**). Recall that it is shown in loc. cit. that every topos $\mathscr{E}$ over $\mathscr{S}$ is equivalent to one of the form BG for some continuous groupoid G. In fact, one may assume that $G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0$ are open, or even connected and locally connected. If $\mathscr{E} \to \mathscr{S}$ is open, one may take G_0 to be an open space (i.e. $G_0 \to 1$ is an open map). Special cases include: that every atomic connected topos with a point is equivalent to BG for a continuous group G (i.e. $G_0 = 1$), and that every étendue is equivalent to BG for a continuous groupoid G with $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0$ all étale maps (an "étale groupoid").

We list some elementary properties of the functor $G \to BG$ which follow from the results in §3.

4.4. PROPOSITION. *Let G be a continuous groupoid.*

(a) *If $d_0, d_1 \colon G_1 \rightrightarrows G_0$ are open, $\mathrm{Sh}(G_0) \to BG$ is open.*

(b) *If $d_0, d_1 \colon G_1 \rightrightarrows G_0$ are locally connected, $\mathrm{Sh}(G_0) \to BG$ is locally connected.*

(c) *If $d_0, d_1 \colon G_1 \rightrightarrows G_0$ are étale, then $\mathrm{Sh}(G_0) \to BG$ is atomic.*

PROOF. Immediate from 3.6.

4.5. PROPOSITION. *Let G be a continuous groupoid.*

(a) *If G_0 is open, so is BG.*

(b) *If G_0 is locally connected and $G_1 \rightrightarrows G_0$ are open, then BG is locally connected.*

(c) *If G_0 is discrete and G_1 is an open space, then BG is atomic. In particular, if G is an open continuous group, BG is an atomic topos.*

PROOF. Immediate from 3.8.

4.6. PROPOSITION. *Let $G \xrightarrow{f} H$ be a map of continuous groupoids, and let $BG \xrightarrow{Bf} BH$ be the induced geometric morphism.*

(a) *If $G_0 \xrightarrow{f_0} H_0$ is open and $H_1 \rightrightarrows H_0$ are open, then $BG \xrightarrow{Bf} BH$ is open.*

(b) *If $G_0 \xrightarrow{f_0} H_0$ and $H_1 \rightrightarrows H_0$ are locally connected, and if $G_1 \rightrightarrows G_0$ are open, then $BG \to BH$ is locally connected.*

(c) *If $G_0 \xrightarrow{f_0} H_0$ and $H_1 \rightrightarrows H_0$ are étale, and if $G_1 \rightrightarrows G_0$ are open, then $BG \to BH$ is atomic.*

PROOF. Use 4.5 and apply 1.10 to the square

$$\begin{tikzcd}
\mathrm{Sh}(G_0) \arrow[r, two heads] \arrow[d, "f_0"] & BG \arrow[d, "Bf"] \\
\mathrm{Sh}(H_0) \arrow[r, two heads] & BH
\end{tikzcd}$$

Proposition 4.5 also allows us to formulate the results of Joyal and Tierney (cf. 4.3) as if-and-only-if's; for (c), one uses 3.9.

4.7. COROLLARY. *Let $\mathscr{E} \to \mathscr{S}$ be a Grothendieck topos over $\mathscr{S}$. Then*

(a) *$\mathscr{E} \to \mathscr{S}$ is open iff $\mathscr{E}$ is equivalent to BG for a continuous groupoid G with $G_1 \rightrightarrows G_0 \to 1$ all open maps;*

(b) *$\mathscr{E} \to \mathscr{S}$ is (connected) locally connected iff $\mathscr{E}$ is equivalent to BG for a continuous groupoid G with $G_1 \rightrightarrows G_0 \to 1$ all (connected) locally connected maps;*

(c) *$\mathscr{E} \to \mathscr{S}$ is (connected) atomic iff $\mathscr{E}$ is equivalent to BG for a continuous groupoid G with $G_0 \to 1$ open (and surjective) and $G_1 \xrightarrow{(d_0,d_1)} G_0 \times G_0$ open (and surjective).*

(d) $\mathscr{E}$ *is equivalent to an étendue if* $\mathscr{E}$ *is equivalent to* BG *for an étale groupoid* G (*cf.* 4.3).

5. G-spaces and étale G-spaces. In this section we will extensively analyze the properties of the functor B, by viewing the topos BG as a category of spaces equipped with an action of G.

5.1. *Important convention.* Although some of the results that follow hold for arbitrary groupoids, we will from now on assume that the structure maps d_0 and $d_1 : G_1 \to G_0$ of any continuous groupoid are *open* (this implies that $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1$ is open). As said before, any topos $\mathscr{E}$ is equivalent to BG for such a groupoid G. For emphasis, we will sometimes call such a G an *open groupoid.*

5.2. *G-spaces.* Let G be a continuous groupoid. A G-space is a (generalized) space over G_0, $E \xrightarrow{p} G_0$, equipped with an action $E \times_{G_0} G_1 \xrightarrow{\cdot} E$ satisfying the usual axioms (the pullback here is along $G_1 \xrightarrow{d_1} G_0$). In "point-set notation" (cf. 5.3), we have for points x, y, z in G_0, $z \xrightarrow{h} y$ and $y \xrightarrow{g} x$ in G_1, and e in $E_x = p^{-1}(x)$,

(1) $p(e \cdot g) = y$,

(2) $e \cdot s(x) = e$,

(3) $(e \cdot g) \cdot h = e \cdot (g \circ h) = e \cdot m(g, h)$.

A *map* of G-spaces from $(E \xrightarrow{p} G_0, \cdot)$ to $(E' \xrightarrow{p'} G_0, \cdot)$ is a map of spaces $E \xrightarrow{f} E'$ over G_0 which commutes with the action

$$\begin{tikzcd} E \times_{G_0} G_1 \arrow[r, "\cdot"] \arrow[d, "f"] & E \arrow[d, "f"] \\ E' \times_{G_0} G_1 \arrow[r, "\cdot"] & E' \end{tikzcd}$$

This defines a category (G-spaces).

A G-space $E = (E \xrightarrow{p} G_0, \cdot)$ is called étale if p is an étale map (a local homeomorphism) of generalized spaces (recall that p is étale iff $E \xrightarrow{p} G_0$ and $E \xrightarrow{\Delta} E \times_{G_0} E$ are open, cf. 1.2). This gives a full subcategory (étale G-spaces) of (G-spaces).

By the equivalence between $\mathrm{Sh}(G_0)$ and étale spaces over G_0, we immediately conclude:

PROPOSITION. *The category of étale G-spaces is equivalent to the classifying topos BG.*

5.3. *Remark on point-set notation.* Of course, G_0, G_1, and E are generalized spaces, which may not have any points at all. Still, the notation in (1)–(3) of 5.2 is not merely suggestive, but can be taken to be literally the definition of an action, provided we interpret "points" in a sufficiently liberal way: it is standard practice in algebraic geometry to express conditions like (1)–(3) by means of test-spaces. So (1), for instance, becomes:

(1′) for any space T and any two maps $T \xrightarrow{e} E$ and $T \xrightarrow{g} G_1$ with $d_1 g = pe$, $p \cdot (e, g) = d_0 g$. Now a map $T \to E$ is just a point of E, if we change the base to $\mathrm{Sh}(T)$ and pull back our data along the map $\mathrm{Sh}(T) \to \mathscr{S}$ of toposes. So (1) makes sense in $\mathrm{Sh}(T)$, and if we interpret "point of E", etc. as point of E in any base extension $\mathrm{Sh}(T) \to \mathscr{S}$ (or $\mathscr{E} \to \mathscr{S}$ for a topos $\mathscr{E}$, for that matter), then (1) is equivalent to (1′).

Needless to say, this interpretation by change of base only gives the desired result if pulling back along $\mathrm{Sh}(T) \to \mathscr{S}$ also preserves *constructions* performed on the data (i.e. these constructions are stable under change of base). In the case of (1), for instance, this is the construction of the pullback $E \times_{G_0} G_1$, which is obviously preserved by change of base. One has to be very careful, however, when dealing with constructions which are not necessarily stable, such as the formation of quotients of spaces.

5.4. *Functoriality of G-spaces.* Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids. ϕ induces a functor $\phi^*\colon (H\text{-spaces}) \to (G\text{-spaces})$ by pullback: given $E \xrightarrow{p} H_0$ with action $E \times_{H_0} H_1 \xrightarrow{\mu} E$, $\phi^*(E)$ is the G_0-space $E \times_{H_0} G_0 \xrightarrow{\pi_2} G_0$ with G-action

$$(E \times_{H_0} G_0) \times_{G_0} G_1 \cong E \times_{H_0} G_1 \xrightarrow{(\mu\circ(E\times\phi),d_0)} E \times_{H_0} G_0;$$

i.e. in point-set notation, and writing E_y for the fiber $p^{-1}(y)$ (y a point of H_0), $\phi^*(E)_x = E_{\phi(x)}$; and for $x' \xrightarrow{g} x$ in G_1 and $e \in E_{\phi(x)}$, $e \cdot g = e \cdot \phi(g)$ defines the action of G on $\phi^*(E)$.

Obviously, ϕ^* is a functor. Moreover, ϕ^* maps étale H-spaces to étale G-spaces, so as to restrict to a functor $\phi^*\colon BH \to BG$. This is precisely the inverse image $(B\phi)^*$ of the geometric morphism $B\phi$:

$$\begin{array}{ccc} (H\text{-spaces}) & \xrightarrow{\phi^*} & (G\text{-spaces}) \\ \cup & & \cup \\ BH & \xrightarrow{(B\phi)^*} & BG \end{array}$$

Therefore, we will often write ϕ^*, rather than $(B\phi)^*$.

5.5. DEFINITIONS. Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids.

(a) ϕ is *open* if $\phi_1\colon G_1 \to H_1$ is open (it follows that $\phi_0\colon G_0 \to H_0$ is open).

(b) ϕ is *essentially surjective* if $G_0 \times_{H_0} H_1 \xrightarrow{d_1\pi_2} H_0$ ($(x, \phi(x) \xrightarrow{h} y) \mapsto y$) is an open surjection.

(c) ϕ is *full* if $G_1 \to H_1 \times_{(H_0\times H_0)} (G_0 \times G_0)$ is an open surjection.

(d) ϕ is *fully faithful* if

$$\begin{tikzcd} G_1 \arrow[r, "\phi_1"] \arrow[d, "(d_0,d_1)"'] & H_1 \arrow[d, "(d_0,d_1)"] \\ G_0 \times G_0 \arrow[r, "\phi_0\times\phi_0"] & H_0 \times H_0 \end{tikzcd}$$

is a pullback of spaces.

5.6. *Preview on stability.* Let G be a continuous groupoid, with d_0 and $d_1\colon G_1 \rightrightarrows G_0$ open (cf. 5.1). It is a consequence of the stability theorem, to be proved in the next section, that the coequalizer

$$G_1 \rightrightarrows G_0 \to \pi(G) \tag{1}$$

is stable (see 6.9). That is, if T is any space, then $T \times G_1 \rightrightarrows T \times G_0 \to T \times \pi(G)$ is again a coequalizer of spaces. In particular, if the structure-maps of the groupoid are all over some base-space X, $\pi(G)$ is a space over X in a unique way which makes (1) a coequalizer of spaces over X, and by interpreting what we just said in $\mathrm{Sh}(X)$, it follows that for any map $T \to X$ of spaces, $T \times_X G_1 \rightrightarrows T \times_X G_0 \to T \times_X \pi(G)$ is again a coequalizer.

5.7. THEOREM. *Let $G \xrightarrow{\phi} H$ be an open map of continuous groupoids. Then ϕ^* has a left adjoint $\phi_!$: (G-spaces) $\to$ (H-spaces); for a G-space $D = (D \xrightarrow{p} G_0, \cdot)$, $\phi_!(D)$ is defined as the coequalizer $D \otimes_G H$:*

$$D \times_{G_0} G_1 \times_{H_0} H_1 \underset{D\times m(\phi\times H_1)}{\overset{\cdot\times H_1}{\rightrightarrows}} D \times_{G_0} H_1 \to D \otimes_G H \tag{1}$$

which is a stable coequalizer of spaces over H_0.

REMARK. In point-set notation, the maps in (1) are $(d \in D_x, x' \xrightarrow{g} x, y \xrightarrow{h} \phi(x')) \mapsto (d \cdot g, h)$, resp. $(d, \phi(g) \circ h)$, so $D \otimes_G H$ can be thought of as the space of equivalence classes $[d,h]$ of pairs $(d \in D_x, y \xrightarrow{h} \phi(x))$. H acts on $D \otimes_G H$ by composition: $[d,h] \cdot h' = [d, h \circ h']$. This point-set notation can be interpreted as a definition, using change of base, provided the coequalizer (1) is stable.

PROOF. We just indicated how to define an action $(D \otimes_G H) \times_{H_0} H_1 \to (D \otimes_G H)$ making $D \otimes_G H$ into an H-space. However, this definition already uses the stability of (1), so as to conclude that

$$(D \times_{G_0} G_1 \times_{H_0} H_1) \times_{H_0} H_1 \rightrightarrows (D \times_{H_0} H_1) \times_{H_0} H_1 \to (D \otimes_G H) \times_{H_0} H_1 \tag{2}$$

is again a coequalizer. This is indeed the case by 5.6, since the parallel pair of maps $D \times_{G_0} G_1 \times_{H_0} H_1 \rightrightarrows D \times_{G_0} H_1$ are the d_0 and d_1 of a suitably defined groupoid, and are both open: ϕ_1 is open by assumption, and hence $D \times m(\phi \times H_1)$ is open; that $\cdot \times H_1$ is open follows (because for any groupoid, d_0 is open iff d_1 is), but can also be seen directly, by observing that any action $D \times_{G_0} G_1 \to D$ must be an open map (cf. 5.11).

For a G-space $D = (D \xrightarrow{p} G_0, \cdot)$ and an H-space $E = (E \xrightarrow{q} H_0, \cdot)$ the correspondence

$$\frac{D \xrightarrow{\alpha} \phi^*(E) = E \times_{H_0} G_0}{D \otimes_G H \xrightarrow{\beta} E}$$

goes as follows. Given α, define $\hat{\beta}\colon D \times_{H_0} H_1 \to E$ to be the composite

$$D \times_{H_0} H_1 \xrightarrow{\alpha} (E \times_{H_0} G_0) \times_{H_0} H_1 \to E \times_{H_0} H_1 \xrightarrow{\cdot} E$$

(in point-set notation: for $d \in D_x$, $y \xrightarrow{h} \phi(x)$, $\hat{\beta}_y(d,h) = \alpha_x(d) \cdot h$, where $\alpha_x(d) \in E_{\phi(x)} = \phi^*(E)_x$). Clearly $\hat{\beta}$ passes to the quotient and gives a map $D \otimes_G H \to E$ of H-spaces.

Conversely, given β, define α_1 to be the composite $D \xrightarrow{D\times s\phi} D \times_{H_0} H_1 \twoheadrightarrow D \otimes_G H \xrightarrow{\beta} E$, and let $\alpha = (\alpha_1, p)\colon D \to E \times_H G_0$ (in point-set notation, $\alpha_x(d) = \beta_{\phi(x)}([d, s(\phi(x))]) \in E_{\phi(x)} = \phi^*(E)_x$, for $d \in D_x$).

This proves Theorem 5.7.

In the sequel, we will often just define mappings by point-set notation, when this is justified by 5.3 (and 5.6).

5.8. PROPOSITION. *Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids.*
(i) *If ϕ is essentially surjective, then ϕ^*: (H-spaces) $\to$ (G-spaces) is faithful.*
(ii) *If ϕ is moreover full, then ϕ^* is fully faithful.*

PROOF. Let $E \overset{\alpha}{\underset{\beta}{\rightrightarrows}} E'$ be maps of H-spaces such that $\phi^*(\alpha) = \phi^*(\beta)\colon E \times_{H_0} G_0 \to E' \times_{H_0} G_0$. Since $d_1\pi_2$ is a stable surjection $G_0 \times_{H_0} H_1 \to H_0$, so is t in the pullback square

$$\begin{array}{ccc} E \times_{H_0} G_0 \times_{H_0} H_1 & \xrightarrow{t} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ G_0 \times_{H_0} H_1 & \xrightarrow{d_1\pi_2} & H_0 \end{array}$$

($t(e,g,h) = e$ for $e \in E_y$, $x \in G_0$, $\phi(x) \xrightarrow{h} y$ in H_1). Now consider the diagram

$$\begin{array}{ccccc} (E \times_{H_0} G_0) \times_{H_0} H_1 & \underset{u}{\overset{\sim}{\longleftarrow}} & E \times_{H_0} (G_0 \times_{H_0} H_1) & \xrightarrow{t} & E \\ {\scriptstyle \phi^*(\alpha)\times H_1}\downarrow\downarrow{\scriptstyle \phi^*(\beta)\times H_1} & & {\scriptstyle \alpha\times(G_0\times_{H_0}H_1)}\downarrow\downarrow{\scriptstyle \beta\times(C_0\times_{H_0}H_1)} & & {\scriptstyle\alpha}\downarrow\downarrow{\scriptstyle\beta} \\ (E' \times_{H_0} G_0) \times_{H_0} H_1 & \underset{u'}{\overset{\sim}{\longleftarrow}} & E' \times_{H_0} (G_0 \times_{H_0} H_1) & \xrightarrow{t'} & E' \end{array}$$

where u is the map defined by $u(e,x,h) = (e \cdot h, x, h)$, for $e \in E_y$, $x \in G_0$, $(\phi(x) \xrightarrow{h} y) \in H_1$; u' is defined similarly. Both right-hand squares (one with α, one with β) commute by naturality of t, and both left-hand squares commute since α and β preserve the action. Since t is a surjection, $\phi^*(\alpha) = \phi^*(\beta)$ implies $\alpha = \beta$.

(ii) Suppose given $\phi^*(E) \xrightarrow{\alpha} \phi^*(E')$. We define $E \xrightarrow{\beta} E'$ such that $\phi^*(\beta) = \alpha$. I will give two arguments, as an illustration of the "point-set method":

(a) In point-set language: for $y \in H_0$, there is a $\phi(x) \xrightarrow{h} y$ (because $G_0 \times_{H_0} H_1 \to H_0$ is an open surjection), and we define $\beta_y(e) = \alpha_{\phi(x)}(e \cdot h) \cdot h^{-1}$ for $e \in E_y$. This is well defined, for if $\phi(x') \xrightarrow{h'} y$ is another one, write $h'^{-1}h = \phi(g)$ for an $x \xrightarrow{g} x'$ (ϕ is full). Then

$$\begin{aligned} \alpha_{\phi(x')}(e \cdot h')h'^{-1} &= \alpha_{\phi(x)}(e \cdot h')\phi(gg^{-1})h'^{-1} = \alpha_{\phi(x)}(e \cdot h' \cdot \phi(g))(h'\phi(g))^{-1} \\ &= \alpha_{\phi(x)}(e \cdot h') \cdot h^{-1}. \end{aligned}$$

This argument actually makes sense, by change of base techniques; "there exists..." is interpreted as "there exists in some open surjective base extension", and "well defined on equivalence classes" corresponds to the fact that every open surjection is the (stable) coequalizer of its kernelpair.

(b) In the language of generalized spaces: if $G_0 \times_{H_0} H_1 \xrightarrow{d_1\pi_2} H_0$ is an open surjection, it is the coequalizer of its kernelpair

$$(G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1) \overset{\pi_1}{\rightrightarrows} G_0 \times_{H_0} H_1 \xrightarrow{d_1\pi_2} H_0,$$

and this still holds when we pull back along $E \to H_0$, i.e.

$$E \times_{H_0} (G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1) \rightrightarrows E \times_{H_0} (G_0 \times_{H_0} H_1) \xrightarrow{t} E$$

is a coequalizer. To define $E \xrightarrow{\beta} E'$, it is therefore enough to define

$$E \times_{H_0} (G_0 \times_{H_0} H_1) \xrightarrow{\hat{\beta}} E'$$

with $\hat\beta \circ \pi_{12} = \hat\beta \circ \pi_{13}$. Define $\hat\beta(e, x, \phi(x) \xrightarrow{h} g) = \alpha(e \cdot h)h^{-1}$ for $e \in E_y$, $x \in G_0$, $h \in H_1$. Then $\hat\beta \circ \pi_{12} = \hat\beta \circ \pi_{13}$ because ϕ is full, and therefore

$$H_1 \times_{G_0} G_1 \xrightarrow{w} (G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1)$$

$(\langle \phi(x) \xrightarrow{h} y,\ x' \xrightarrow{g} x\rangle \mapsto \langle h,\ h \circ \phi(g)\rangle)$ is an open surjection; that $\hat\beta\pi_{12}w = \hat\beta\pi_{13}w$ is clear from the fact that α is a map of G-spaces.

The aim of this section is to investigate how properties of $BG \xrightarrow{B\phi} BH$ depend on those of $G \xrightarrow{\phi} H$. In this direction we now have the following corollary. (a) comes from 4.6(a) and 5.1, (b) and (c) follow from 5.8.

5.9. COROLLARY. *Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids, and $BG \xrightarrow{B\phi} BH$ the induced geometric morphism.*

(a) *$B\phi$ is open if $G_0 \xrightarrow{\phi_0} H_0$ is open.*
(b) *$B\phi$ is surjective if ϕ is essentially surjective.*
(c) *$B\phi$ is connected if ϕ is essentially surjective and full.*

Our next aim is to see when $BG \xrightarrow{B\phi} BH$ is locally connected, i.e. when $(B\phi)^*$ has a BH-indexed left-adjoint. Since $B\phi^*$ is ϕ^* restricted to étale spaces, this comes down to asking (i) when $\phi_!$ is indexed over (H-spaces), and (ii) when $\phi_!$ maps étale G-spaces into étale H-spaces. We begin with the second question.

5.10. THEOREM. *Let $G \xrightarrow{\phi} H$ be an open map of continuous groupoids, with corresponding adjoint functors* (*cf.* 5.7).

$$(G\text{-}spaces) \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} (H\text{-}spaces)$$

Then $\phi_!$ restricts to a functor $BG \to BH$ if ϕ is full (*cf.* 5.1.5).

PROOF. Recall that for a G-space $D = (D \xrightarrow{p} G_0, \cdot)$, $\phi_!(D)$ is the (stable) coequalizer

$$D \times_{G_0} G_1 \times_{H_0} H_1 \underset{v}{\overset{u}{\rightrightarrows}} D \times_{H_0} H_1 \overset{r}{\twoheadrightarrow} D \otimes_G H,$$

where $u(d, g, h) = (d \cdot g, h)$, $v(d, g, h) = (d, \phi(g) \circ h)$ for $d \in D_x$, $x' \xrightarrow{g} x$, $y \xrightarrow{h} \phi(x')$; see 5.7.

First of all, $\phi_0 p \colon D \to H_0$ is open since ϕ_0 is, and hence so is its pullback $D \times_{H_0} H_1 \xrightarrow{\pi_2} H_1$ along d_1, and thus (cf. 5.1) $d_0\pi_2 \colon D \times_{H_0} H_1 \to H_0$ is open. Since $D \times_{H_0} H_1 \to D \otimes_G H$ is a surjection, the structure map $D \otimes_G H \xrightarrow{q} H_0$ of the H-space $\phi_!(D)$ is open.

Second, we have to show that q has an open diagonal $D \otimes_G H \to (D \otimes_G H) \times_{H_0} (D \otimes_G H)$. Since r is an open surjection, so is $r \times r$, and therefore it is enough to show that (u, v) is open, as in the diagram

$$\begin{tikzcd}
D \times_{G_0} G_1 \times_{H_0} H_1 \arrow[r, "{(u,v)}"] \arrow[d, "ru=rv"] & (D \times_{H_0} H_1) \times_{H_0} (D \times_{H_0} H_1) \arrow[d, "r \times r"] \\
D \otimes_G H \arrow[r] & (D \otimes_G H) \times_{H_0} (D \otimes_G H)
\end{tikzcd}$$

We give a point-set argument to show that (u,v) is open (i.e. we implicitly use test-spaces and change of base!).

First notice that since $D \to G_0$ is étale, the action $D \times_{G_0} G_1 \to D$ is not only open, but has the much stronger property that whenever $U_{d\cdot g}$ is a neighborhood of $d \cdot g$ ($d \in D_x$, $x' \xrightarrow{g} x$ in G_1), so small that $U_{d\cdot g} \xrightarrow{p} p(U_{d\cdot g})$ is an isomorphism, then there are small neighborhoods V_g and U_d of g and d such that $d_0(V) \subset p(U_{d\cdot g})$, and for $g' \in V_g$ and $d' \in U_d$ with $p(d') = d_1(g')$, $d' \cdot g'$ is the unique point x in $U_{d\cdot g}$ with $p(x) = d_0(g')$.

Now take a point $(d \in D_x,\ x' \xrightarrow{g} x,\ y \xrightarrow{h} \phi(x'))$ of $D \times_{G_0} G_1 \times_{H_0} H_1$, and take sufficiently small neighborhoods $V_g \subset G_1$ and $W_h, W_{\phi(g)\circ h} \subset H_1$. In fact, since ϕ is full, the map $G_1 \times_{G_0} H_1 \xrightarrow{M} (G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1)$ is an open surjection, where in point-set notation: $G_1 \times_{G_0} H_1$ has points $\langle x \xrightarrow{g} x', y \xrightarrow{h} \phi(x)\rangle$, $(G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1)$ has points $\langle x_1, z \xrightarrow{h_1} \phi(x_1),\ x_2, z \xrightarrow{h_2} \phi(x_2)\rangle$, and $M(g,h) = (x, h, x', \phi(g) \circ h)$. ($M$ comes from pulling back the open surjection $G_1 \xrightarrow{\phi_1} H_1 \times_{(H_0 \times H_0)} (G_0 \times G_0)$ of 5.5 along the map

$$(G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1) \to H_1 \times_{(H_0 \times H_0)} (G_0 \times G_0)$$

given by $(z \xrightarrow{h_1} \phi(x_1),\ z \xrightarrow{h_2} \phi(x_2)) \mapsto h_2 h_1^{-1}$.) So by taking $W_{\phi(g)\circ h}$ small enough, we may assume that

$$(1) \qquad W_h \times W_{\phi(g)\circ h} \subset M(V_g \times W_h).$$

Moreover, choose small neighborhoods $U_d, U_{dg} \subset D$ (on which p restricts to an isomorphism) such that $U_d \cdot V_g = U_{d\cdot g}$ (this can be done if we take V_g small enough, by the observation just made about the action $D \times_{G_0} G_1 \to D$.

We claim that

$$(u,v)(U_d \times_{G_0} V_g \times_{H_0} W_h) \geq (U_{dg} \times_{H_0} W_h) \times_{H_0} (U_d \times_{H_0} W_{\phi(g)h}).$$

Indeed, take a point $((d_1,h_1),(d_2,h_2))$ of the right-hand space, say $z \xrightarrow{h_1} \phi(x_1) \in W_h$, $z \xrightarrow{h_2} \phi(x_2) \in W_{\phi(g)\circ h}$, $d_1 \in U_{dg} \cap D_{x_1}$, $d_2 \in U_d \cap D_{x_2}$. Since ϕ is full, we can find (after extending the base by some open surjection of toposes!) a point $x_1 \xrightarrow{k} x_2 \in V_g$ such that $\phi(k) \circ h_1 = h_2$ (cf. (1)). But then

$$(u,v)(d_2,k,h_1) = ((d_2 \cdot k, h_1), (d_2, \phi(k) \circ h_1)) = ((d_1,h_1),(d_2,h_2)),$$

because $\phi(k) \circ h_1 = h_2$ by choice of k, and $d_2 \cdot k$ is an element of $U_{d\cdot g}$ over x_1, and hence cannot be anything else but d_1, by the remark above on the action $D \times_{G_0} G_1 \to D$.

This completes the proof of the theorem.

5.11. *Localization.* Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids, inducing $\phi^*\colon (H\text{-spaces}) \to (G\text{-spaces})$, and let $E \xrightarrow{p} H_0$ be an H-space. E gives rise to a groupoid $\hat{E} = E \times_{H_0} H_1$, i.e.

$$\hat{E}_0 = E, \qquad \hat{E}_1 = E \times_{H_0} H_1 \overset{\cdot}{\underset{\pi_1}{\rightrightarrows}} E$$

where $\cdot$ is the d_0 of $\hat{E}$ and π_2 the d_1. (So this is the "diagram" of E: the objects of $\hat{E}$ are elements of E, and the arrows from e' to e are H-maps $p(e') \xrightarrow{h} p(e)$ such that $e \cdot h = e'$.) $\hat{E}$ is again an open groupoid (cf. 5.1) if H is an open groupoid (in particular, the action $E \times_{H_0} H_1 \xrightarrow{\bullet} E$ is open).

There is a projection $\pi_E \colon \hat{E} \to H$ which is a map of continuous groupoids. $E \mapsto \hat{E}$ is a functor from H-spaces into continuous groupoids over H.

PROPOSITION. (a) *There is a canonical equivalence*

$$(H\text{-}spaces)/E \xrightarrow{\sim} (\hat{E}\text{-}spaces), \tag{1}$$

and the functor $(H\text{-}spaces) \to (H\text{-}spaces)/E$ *given by taking the fibered product over* H_0 *with* $E \to H_0$ *coincides—modulo the equivalence* (1)*—with* π_E^*.

(b) $G \xrightarrow{\phi} H$ *induces a map of continuous groupoids* $\phi_E \colon \widehat{\phi^*(E)} \to \hat{E}$ *such that*

$$\begin{tikzcd} \widehat{\phi^*(E)} \arrow[r, "\phi_E"] \arrow[d, "\pi_{\phi^*(E)}"] & \hat{E} \arrow[d, "\pi_E"] \\ G \arrow[r, "\phi"] & H \end{tikzcd} \tag{2}$$

commutes, and the functor ϕ^*/E *corresponds to* ϕ_E^* *under the equivalence* (1), *i.e. the diagram*

$$\begin{tikzcd} (H\text{-}spaces)/E \arrow[r, "\phi^*/E"] \arrow[d, "\wr"] & (G\text{-}spaces)/\phi^*(E) \arrow[d, "\wr"] \\ (\hat{E}\text{-}spaces) \arrow[r, "\phi_E^*"] & (\widehat{\phi^*(E)}\text{-}spaces) \end{tikzcd} \tag{3}$$

commutes, up to canonical isomorphism.

PROOF. Straightforward and omitted.

REMARK. Everything in the preceding proposition can be restricted to étale spaces, so as to get corresponding statements about localization of BG and BH: for an étale H-space E we have

$$BH/E \xrightarrow{\sim} B\hat{E}; \tag{1'}$$

the square

$$\begin{tikzcd} B(\widehat{\phi^*(E)}) \arrow[r, "B\phi_E"] \arrow[d, "\pi_{\phi^*(E)}"] & B\hat{E} \arrow[d, "\pi_E"] \\ BG \arrow[r, "B\phi"] & BH \end{tikzcd} \tag{2'}$$

commutes, and is moreover a pullback of toposes; and the square

$$\begin{tikzcd} BG/\phi^*(E) \arrow[r, "B\phi/E"] \arrow[d, "\wr"] & BH/E \arrow[d, "\wr"] \\ B(\widehat{\phi^*(E)}) \arrow[r, "B\phi_E"] & B\hat{E} \end{tikzcd} \tag{3'}$$

of geometric morphisms commutes (compare also 3.5).

Notice that if $G \to H$ is open (respectively full), then so is $\phi_E \colon \widehat{\phi^*(E)} \to \hat{E}$ for any H-space E. In particular, $\phi_{E!}$ restricts to a functor $B(\widehat{\phi^*(E)}) \to B(\hat{E})$ if ϕ is open and full, by Theorem 5.10.

5.12. THEOREM. *Let $G \xrightarrow{\phi} H$ be open and full, as in* 5.10. *Then for any map $E \xrightarrow{\alpha} E'$ of étale H-spaces, with corresponding map $\hat{E} \xrightarrow{\hat{\alpha}} \hat{E}'$ of continuous groupoids over H, the diagram*

$$\begin{array}{ccc} B(\widehat{\phi^*(E)}) & \xrightarrow{(\phi_E)_!} & B\hat{E} \\ \big\uparrow {\scriptstyle (\widehat{\phi^*(\alpha)})^*} & & \big\uparrow {\scriptstyle \hat{\alpha}^*} \\ B(\widehat{\phi^*(E')}) & \xrightarrow{(\phi_{E'})_!} & B\hat{E}' \end{array}$$

commutes (*up to canonical isomorphism*).

PROOF. By replacing $G \xrightarrow{\phi} H$ by $\widehat{\phi^*(E)} \xrightarrow{\phi_{E'}} \hat{E}'$, we may without loss assume that $E' = 1$; so it is enough to show that

$$\begin{array}{ccc} B(\widehat{\phi^*(E)}) & \xrightarrow{(\phi_E)_!} & B\hat{E} \\ \big\uparrow {\scriptstyle \phi^*(\alpha)^*} & & \big\uparrow {\scriptstyle \alpha^*} \\ BG & \xrightarrow{\phi!} & BH \end{array}$$

commutes. (In what follows, I will indicate what the points of spaces are and what the maps do to them, just to remind you which pullback is along which map, etc.)

Take a G-space $D = (D \xrightarrow{q} G_0, \cdot)$. $\phi_!(D)$ is the coequalizer

$$D \times_{G_0} G_1 \times_{H_0} H_1 \underset{v}{\overset{u}{\rightrightarrows}} D \times_{H_0} H_1 \xrightarrow{r} D \otimes_G H$$

(recall that the points of the left-hand space are triples $(d \in D_x, x' \xrightarrow{g} x, y \xrightarrow{h} \phi(x))$, and $u(d,g,h) = (d \cdot g, h)$, $v(d,g,h) = (d, \phi(g) \circ h))$. So by stability (cf. 5.7), $\alpha^* \phi_!(D)$ is the coequalizer

$$(1) \quad D \times_{G_0} G_1 \times_{G_0} H_1) \times_{H_0} E \underset{v \times G}{\overset{u \times E}{\rightrightarrows}} (D \times_{H_0} H_1) \times_{H_0} H_1 \xrightarrow{r \times E} (D \otimes_G H) \times_{H_0} E$$

(points of the left-hand space are of the form $(d, g, h, e \in E_y)$, with (d, g, h) as above). Notice that the action of H on $(D \otimes_G H) \times_{H_0} E$ is defined by acting on *both* the H- and the E-coordinate, i.e. for $d \in D_x$, $y \xrightarrow{h} \phi(x)$, $e \in E_y$, $y' \xrightarrow{k} y$, we have $([d,h], e) \cdot k = ([d, hk], e \cdot k)$.

The other way round, $\phi^*(\alpha)^*(D)$ is the $\widehat{\phi^*(E)}$-space $D \times_{G_0} (G_0 \times_{H_0} E) \cong D \times_{H_0} E$ (points are pairs $(d \in D_x, e \in E_{\phi(x)})$), and hence $(\phi_E)_! \phi^*(\alpha)^*(D)$ is the coequalizer $\phi^*(\alpha)^*(D) \otimes_{\widehat{\phi^*(E)}} \hat{E}$ of

$$(2) \quad (D \times_{G_0} G_0 \times_{H_0} E) \times_{\widehat{\phi^*(E)}_0} \widehat{\phi^*(E)}_1 \times_{\hat{E}_0} \hat{E}_1 \underset{v'}{\overset{u'}{\rightrightarrows}} D \times_{G_0} (G_0 \times_{H_0} E) \times_{\hat{E}_0} \hat{E}_1.$$

Unwinding the definitions, it is not difficult to check that the coequalizer of (2) is isomorphic to the coequalizer (3)

$$(3) \qquad D \times_{G_0} E \times_{G_0} G_1 \times_{H_0} H_1 \underset{\tilde{v}}{\overset{\tilde{u}}{\rightrightarrows}} D \times_{H_0} E \times_{H_0} H_1 \overset{s}{\twoheadrightarrow} \phi^*(\alpha)^*(D) \otimes_{\widehat{\phi^*(E)}} \hat{E}$$

(to remind you of where we are: points of the left-hand space in (3) are quadruples $(d \in D_x, e \in E_{\phi(x)}, x' \xrightarrow{g} x,\ y \xrightarrow{h} \phi(x'))$, and $\tilde{u}$ sends this to $(d \cdot g,\ e \cdot \phi(g),\ h)$ while $\tilde{v}$ sends it to $(d, e, \phi(g) \circ h)$.

But (1) and (3) are the same under the isomorphisms α and β,

$$\begin{tikzcd} (1)\colon (D \times_{G_0} G_1 \times_{G_0} H_1) \times_{H_0} E \arrow[r, shift left, "u \times E"] \arrow[r, shift right, "v \times E"'] & D \times_{H_0} H_1 \times_{H_0} E \\ (3)\colon D \times_{G_0} E \times_{G_0} G_1 \times_{G_0} H_1 \arrow[u, "\alpha"] \arrow[r, shift left, "\tilde{u}"] \arrow[r, shift right, "\tilde{v}"'] & D \times_{H_0} E \times_{H_0} H_1 \arrow[u, "\beta"'] \end{tikzcd}$$

where α and β are defined by $\alpha(d,e,g,h) = (d, g, h, e \cdot \phi(g) \cdot h)$, $\beta(d,e,h) = (d, h, e \cdot h)$. Then $\beta \circ \tilde{u} = (u \times E) \circ \alpha$, $\beta \circ \tilde{v} = (v \times E) \circ \alpha$, so (1) and (3) have isomorphic coequalizers.

This proves the theorem.

5.13. PROPOSITION. *Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids, and suppose ϕ is open and fully faithful* (5.5). *Then the composite $BG \xrightarrow{\phi_!} BH \xrightarrow{\phi^*} BG$ is canonically isomorphic to the identity on BG.*

PROOF. Take an étale G-space $(D \to G_0, \cdot)$. $\phi_!(D)$ is the coequalizer

$$D \times_{G_0} G_1 \times_{H_0} H_1 \underset{v}{\overset{u}{\rightrightarrows}} D \times_{H_0} H_1 \xrightarrow{r} D \otimes_G H$$

over H_0. This coequalizer is stable under pullback along $G_0 \to H_0$ (5.7), so $\phi^* \phi_!(D)$ is the coequalizer

$$(D \times_{G_0} G_1 \times_{H_0} H_1) \times_{H_0} G_0 \underset{v \times G_0}{\overset{u \times G_0}{\rightrightarrows}} D \times_{H_0} H_1 \times_{H_0} G_0 \to (D \otimes_G H) \times_{H_0} G_0$$

(so points of the left-hand space are quadruples $(d \in D_x, x' \xrightarrow{g} x,\ \phi(x'') \xrightarrow{h} \phi(x'),$ $x''))$, If ϕ is fully faithful, this is isomorphic to the coequalizer of

$$D \times_{G_0} G_1 \times_{G_0} G_1 \underset{D \times m}{\overset{\cdot \times G_1}{\rightrightarrows}} D \times_{G_0} G_1.$$

But

$$D \times_{G_0} G_1 \times_{G_0} G_1 \underset{D \times m}{\overset{\cdot \times G_1}{\rightrightarrows}} D \times_{G_0} G_1 \xrightarrow{\cdot} D$$

is a split coequalizer, for any G-space D.

5.14. PROPOSITION. *Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids, and suppose ϕ is open and full. Then $\phi_!$ induces an isomorphism of subobject lattices* $\mathrm{Sub}_{BG}(E) \xrightarrow{\sim} \mathrm{Sub}_{BH}(\phi_! E)$ *for each étale G-space E.*

PROOF. (a) Let us first note that $\phi_!$ maps monos to monos: Take $S \xrightarrow{u} E$ in BG, and consider the diagram

$$\begin{tikzcd} S \times_{G_0} G_1 \times_{H_0} H_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & S \times_{H_0} H_1 \arrow[r, "q"] \arrow[d, tail] & S \otimes_G H \arrow[d, dashed, "\phi_!(u)"] \\ E \times_{G_0} G_1 \times_{H_0} H_1 \arrow[r, shift left] \arrow[r, shift right] & E \times_{H_0} H_1 \arrow[r, "p"] & E \otimes_G H \end{tikzcd}$$

the rows of which are stable coequalizers (cf. 5.7). We use change of base and "point-set language". Suppose given points x, x' of the space $S \otimes_G H$ (in any extension of the base topos $\mathscr{E} \to \mathscr{S}$) such that $\phi_!(u)(x) = \phi_!(u)(x')$. Since q is an open surjection, we can find representing points $(s,h), (s',h') \in S \times_{H_0} H_1$ with $[s,h] = q(s,h) = x$, $[s',h'] = x'$ (in some further base extension $\mathscr{E}'$ where $\mathscr{E}' \to \mathscr{E}$ is an open surjection). Then $p(u(s),h) = p(u(s'),h')$, so (by going to an $\mathscr{E}''$ where $\mathscr{E}'' \to \mathscr{E}'$ is an open surjection) there is an $x \xrightarrow{g} x'$ such that $u(s') \cdot g = u(s)$ and $\phi(g) \circ h = h'$ (here we use that $E \times_{G_0} G_1 \times_{H_0} H_1 \to (E \times_{H_0} H_1) \times_{H_0} (E \times_{H_0} H_1)$ is open, cf. 5.10). Since u is mono, $s'g = s$, hence $x = [s,h] = [s',h'] = x'$ in $\mathscr{S} \otimes_G H$. (This latter identity holds in $\mathscr{E}''$, but since $\mathscr{E}'' \to \mathscr{E}$ is an open surjection, it follows that it must hold in $\mathscr{E}$, the base over which x and x' were originally defined). This proves that $\phi_!(x)$ is a monomorphism of spaces.

(b) Next, we verify that $\mathrm{Sub}_{BG}(E) \underset{\overline{\phi}}{\overset{\phi_!}{\rightleftarrows}} \mathrm{Sub}_{BH}(\phi_! E)$ are mutually inverse functors, where for $A \subset \phi_!(E)$, $\overline{\phi}(A)$ is defined as the pullback along the unit of the adjunction $\phi_! \dashv \phi^*$:

$$\begin{tikzcd}
\overline{\phi}(A) \arrow[r] \arrow[d] & E \arrow[d, "\eta"] \\
A \arrow[r] & \phi^*\phi_!(E).
\end{tikzcd}$$

For $S \subset E$ and $A \subset \phi_!(E)$, the inclusions $S \subset \overline{\phi}\phi_!(S)$ and $\phi_!\overline{\phi}(A) \subset A$ are obvious. To prove the inclusions $\overline{\phi}\phi_!(S) \subset S$ and $A \subset \phi_!\overline{\phi}(A)$ we use point-set arguments as in (a). This can be done since everything in sight is stable. Leaving the subsequent base extensions of the base topos $\mathscr{S}$ implicit, the proof is as follows. For $\overline{\phi}\phi_!(S) \subset S$, take $e \in \overline{\phi}\phi_!(S)_x \subset E_x$, i.e. $[e, s(\phi(x))] \in \phi_!(S)_{\phi(x)}$, $\phi_!(S)_{\phi(x)} \subset \phi_!(E)_{\phi(x)} = (E \otimes_G H)_{\phi(x)}$ (recall that $G_0 \xrightarrow{s} G_1$ "sends x to the indentity at x"). This means that there is a pair $[t,h] \in S \otimes_G H$ with $[e, s(\phi(x))] = [t,h]$, say with $\phi(x) \xrightarrow{h} \phi(x')$ and $t \in S_{x'}$. Then there is an $x' \xrightarrow{g} x$ such that $t = e \cdot g$ and $\phi(g) \circ h = s(\phi(x))$. So $e = t \cdot g^{-1} \in S_x$ since $t \in S_{x'}$. Thus $\overline{\phi}\phi_!(S) \subset S$.

For $A \subset \phi_!\overline{\phi}(A)$, take $a \in A_y \subset \phi_!(E)_y$ over a point y of H_0. So a is a class $a = [e_0, h_0]$, where $e_0 \in E_x$ and $y \xrightarrow{h_0} \phi(x)$. Now $\overline{\phi}(A)_x$ consists of points $e \in E_x$ such that $[e, s(\phi(x))] \in A_{\phi(x)}$, so $e_0 \in \overline{\phi}(A)_n$ because $[e_0, s(\phi(x))] = a \cdot h_0^{-1} \in A_{\phi(x)}$. Then $\phi_!(e_0) = [e_0 \cdot s(\phi(x))] \in \phi_!\overline{\phi}(A)_{\phi(x)}$, i.e. $a \cdot h_0^{-1} \in \phi_!\overline{\phi}(A)_{\phi(x)}$. So $a \in \phi_!\overline{\phi}(A)_y$. This shows $A \subset \overline{\phi}\phi_!(A)$.

We now reformulate the preceding results in terms of properties of the geometric morphism $BG \xrightarrow{\phi} BH$.

5.15. SUMMARY. *Let $G \xrightarrow{\phi} H$ be a continuous map of groupoids, with induced geometric morphism $BG \xrightarrow{B\phi} BH$.*

(i) *If $\phi_0 \colon G_0 \to H_0$ is open, then $BG \to BH$ is open.*

(ii) *If ϕ is essentially surjective then $B\phi$ is surjective.*

(iii) *If ϕ is essentially surjective and full then $B\phi$ is connected.*

(iv) *If ϕ is open and full, then $BG \xrightarrow{B\phi} BH$ is atomic.*

(v) *If ϕ is open, fully faithful and essentially surjective, then* $BG \xrightarrow{B\phi} BH$ *is an equivalence of toposes.*

PROOF. For (i)–(iii) see 5.9. For (iv), notice that $B\phi^*\colon BH \to BG$ has a BH-indexed left adjoint by 5.11 (so $B\phi$ is locally connected). $B\phi^*$ also preserves the subobject classifier: writing Ω_G, Ω_H for the subobject classifiers of BG and BH, we have for $E \in BG$,

$$\begin{array}{ll} \underline{E \to \Omega_G} & \\ \underline{A \subset E \quad \text{in } BG} & \\ \underline{S \subset \phi_!(E) \quad \text{in } BH} & \qquad \text{(by 5.14)} \\ \underline{\phi_!(E) \to \Omega_H} & \\ \underline{E \to (B\phi)^*(\Omega_H)} & \end{array}$$

so $\Omega_G \cong (B\phi)^*(\Omega_H)$. Finally, (v) follows from (iii) and 5.13.

6. The Stability Theorem. Recall that if G is a continuous groupoid in a topos $\mathscr{E}$, $B(\mathscr{E}, G)$ denotes the $\mathscr{E}$-topos of étale G-spaces over (or "in") $\mathscr{E}$. If $\mathscr{F} \xrightarrow{p} \mathscr{E}$ is a geometric morphism, we obtain a continuous groupoid $p^{\#}(G)$ in $\mathscr{F}$ (see 1.5). The aim of this section is to show there is a canonical equivalence of toposes $B(\mathscr{F}, p^{\#}(G)) \xrightarrow{\sim} \mathscr{F} \times_{\mathscr{E}} B(\mathscr{E}, G)$ for any (open!, cf. 5.1) continuous groupoid G in $\mathscr{E}$.

6.1. *Generators for BG.* Let G be a continuous groupoid (we still tacitly assume that d_0 and $d_1\colon G_1 \rightrightarrows G_0$ are open maps). The aim is to find a more manageable set of generators for BG than the one coming from the proof of the existence of colimits (cf. 2.1).

Let $E = (E \xrightarrow{p} G_0, \cdot)$ be an étale G-space, and let $U \xrightarrow{t} E$ be a section of p over some $U \in \mathscr{O}(G_0)$. Let $N_t \subset G_1$ be the subspace defined by the pullback

$$\begin{tikzcd} G_1 \cap d_1^{-1}(U) \arrow[r, "\hat{t}"] & E \\ N_t \arrow[u] \arrow[r] & U \arrow[u] \end{tikzcd}$$

where $\hat{t}$ is the composition $G_1 \cap d_1^{-1}(U) \xrightarrow{td_1 \times 1} E \times_{G_0} G_1 \xrightarrow{\cdot} E$ (i.e. "$N_t = \{x \xrightarrow{g} y \in G_1 | y \in U$ and $t(y) \cdot g = t(x)\}$"). Then

(i) N_t is an open subspace of G_1, and $d_0(N_t), d_1(N_t) \subset U$,

(ii) N_t contains all identities, and is closed under inverse and composition (in point-set notation: $x \in U \Rightarrow s(x) \in N_t$, $g \in N_t \Rightarrow g^{-1} \in N_t$, and $g, h \in N_t \Rightarrow g \circ h \in N_t$ when $d_0 g = d_1 h$).

Now consider the pullback of spaces

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "m(1\times r)"] & G_1 \\ R_t \arrow[u] \arrow[r] & N_t \arrow[u] \end{tikzcd}$$

where $G_1 \xrightarrow{r} G_1$ is the inverse, and $G_1 \times_{G_0} G_1$ is the pullback of $G_1 \xrightarrow{d_0} G_0 \xleftarrow{d_0} G_1$ (so $R_t = \{(x \xrightarrow{g} y, x \xrightarrow{h} z | gh^{-1} \in N_t\})$. R_t defines an equivalence relation on

$G_1 \cap d_1^{-1}(U)$ over G_0 (via $G_1 \xrightarrow{d_0} G_0$), and we write $G_1 \cap d_1^{-1}(U)/N_t$ for the quotient, i.e. there is a coequalizer of spaces over G_0:

$$\begin{tikzcd} R_t \arrow[r, shift left] \arrow[r, shift right] & G_1 \cap d_1^{-1}(U) \arrow[r] \arrow[dr, "d_0"'] & G_1 \cap d_1^{-1}(U)/N_t \arrow[d, "d_0"] \\ & & G_0 \end{tikzcd} \tag{1}$$

Notice that this coequalizer (1) is stable, because $R_t \to G_1 \times_{G_0} G_1$ is open, and $G_1 \cap d_1^{-1}(U)$ is an open space over G_0 (so we apply the last fact of 1.3 to spaces over G_0).

We claim that $G_1 \cap d_1^{-1}(U)/N_t$ is an étale G-space. Composition $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1$ induces an action $(G_1 \cap d_1^{-1}(U)/N_t) \times_{G_0} G_1 \to G_1 \cap d_1^{-1}(U)/N_t$ (in point-set notation: $[g] \cdot h = [gh]$ for a class $[g] \in G_1 \cap d_1^{-1}(U)/N_t$, $g \in G_1, d_1(g) \in U$). Moreover, $G_1 \cap d_1^{-1}(U)/N_t \xrightarrow{d_0} G_0$ is open because $G_1 \xrightarrow{d_0} G_0$ is open (by assumption 5.1), while the diagonal over G_0

$$G_1 \cap d_1^{-1}(U)/N_t \to (G_1 \cap d_1^{-1}(U)/N_t) \times_{G_0} (G_1 \cap d_1^{-1}(U)/N_t)$$

is open, as follows by considering the square

$$\begin{tikzcd} R_t \arrow[r, "\text{open}"] \arrow[d] & G_1 \cap d_1^{-1}(U) \times_{G_0} G_1 \cap d_1^{-1}(U) \arrow[d] \\ G_1 \cap d_1^{-1}(U)/N_t \arrow[r] & (G_1 \cap d_1^{-1}(U)/N_t) \times_{G_0} (G_1 \cap d_1^{-1}(U)/N_t). \end{tikzcd}$$

Thus $G_1 \cap d_1^{-1}(U)/N_t$ is an object of BG.

Furthermore, the section $t\colon U \to E$ induces a map of étale G-spaces $\tilde{t}$, defined by factoring the map $G_1 \cap d_1^{-1}(U) \xrightarrow{\hat{t}} E$, $(g \mapsto t(d_1 g) \cdot g)$ through the coequalizer (1):

$$\begin{tikzcd} G_1 \cap d_1^{-1}(U)/N_t \arrow[rr, "\tilde{t}"] \arrow[dr] & & E \arrow[dl] \\ & G_0 & \end{tikzcd} \tag{2}$$

$\tilde{t}$ contains the section t, in the sense that there is a commutative diagram

$$\begin{tikzcd} G_1 \cap d_1^{-1}(U)/N_t \arrow[rr, "\tilde{t}"] & & E \\ & U \arrow[ul, "{[s]}"] \arrow[ur, "t"'] & \end{tikzcd}$$

($[s]$ stands for the composite $U \xrightarrow{s} G_1 \cap d_1^{-1}(U) \twoheadrightarrow G_1 \cap d_1^{-1}(U)/N_t$.)

We conclude that the étale G-spaces of the form $G_1 \cap d_1^{-1}(U)/N_t \xrightarrow{d_0} G_0$ (with G-action defined by composition) generate BG.

DEFINITION. Let S_G be the full subcategory of BG whose objects are étale G-spaces of the form $G_1 \cap d_1^{-1}(U)/N$, where $U \subset G_0$ and $N \subset G_1$ are open subspaces such that $d_0(N), d_1(N) \subset U$, and $s(U) \subset N$, $m(N \times_{G_0} N) \subset N)$, $N^{-1} \subset N$ (cf. (i),

(ii) above). If we define the covering families of $\mathbf{S}_G$ to be the epimorphic families of BG, then $\mathbf{S}_G$ is a site for BG, i.e. there is a canonical equivalence

$$BG \xrightarrow{\sim} \mathrm{Sh}(\mathbf{S}_G).$$

Notation. We write $[G,U,N]$ for the G-space $(G_1 \cap d_1^{-1}(U)/N \xrightarrow{d_0} G_0, \cdot)$, where U and N are as in the preceding definition.

For later use, we note the following lemma.

6.2. LEMMA. *Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids, with corresponding geometric morphism $B\phi\colon BG \to BH$, and let $\mathbf{S}_G$, $\mathbf{S}_H$ be the sites for BG and BH as just defined. If ϕ is a fibration in the sense that $G_1 \xrightarrow{(d_0,\phi_1)} G_0 \times_{H_0} H_1$ is an open surjection, then $(B\phi)^*$ maps $\mathbf{S}_H$ into $\mathbf{S}_G$; in fact for $[H,V,N] \in \mathbf{S}_H$,*

$$(B\phi)^*[H,V,N] = [G, \phi_0^{-1}(V), \phi_1^{-1}(N)].$$

PROOF. Take $V \subset G_0$, $N \subset G_1$ such that $[H,V,N] = H_1 \cap d_1^{-1}(V)/N$ is an object of the site $\mathbf{S}_H$, and write $R_N \subset (H_1 \cap d_1^{-1}(V)/N) \times_{H_0} (H_1 \cap d_1^{-1}(V)/N)$ for the equivalence relation corresponding to N. Note that R_N is an open sublocale, and that $R_{\phi^{-1}(N)} = (\phi \times \phi)^{-1}(R_N)$. Since quotients by an open equivalence relation are stable (1.3), we find that

$$\begin{aligned}(B\phi)^*[H,V,N] &= G_0 \times_{H_0} (H_1 \cap d_1^{-1}(V)/R_N) \\ &\cong (G_0 \times_{H_0} H_1 \cap d_1^{-1}(V))/(G_0 \times_{H_0} R_N).\end{aligned}$$

Clearly, ϕ induces a map $G_1 \cap d_1^{-1}\phi_0^{-1}(V) \xrightarrow{(d_0,\phi)} G_0 \times_{H_0} (H_1 \cap d_1^{-1}(V))$, which passes to the quotient to give a map

$$G_1 \cap d_1^{-1}\phi_0^{-1}(V)/R_{\phi^{-1}(N)} \xrightarrow{\alpha} G_0 \times_{H_0} (H_1 \cap d_1^{-1}(V))/(G_0 \times_{H_0} R_N).$$

Also, since by assumption

$$G_1 \cap d_1^{-1}\phi_1^{-1}(V) \xrightarrow{(d_0,\phi_1)} G_0 \times_{H_0} (H_1 \cap d_1^{-1}(V))$$

is an open surjection, it is a coequalizer of its kernelpair $K \rightarrowtail (G_1 \cap d_1^{-1}\phi_0^{-1}(V)) \times_{G_0} (G_1 \cap d_1^{-1}\phi_0^{-1}(V))$. But $K \subset R_{\phi^{-1}(N)}$, so we obtain a factorization

$$\begin{tikzcd}
K \arrow[r, shift left] \arrow[r, shift right] & G_1 \cap d_1^{-1}\phi_0^{-1}(V) \arrow[r, "(d_0,\phi_1)"] \arrow[d, "\pi"] & G_0 \times_{H_0} (H_1 \cap d_1^{-1}(V)) \arrow[dl, dashed, "\overline{\beta}"] \\
& G_1 \cap d_1^{-1}\phi_0^{-1}(V)/\phi_1^{-1}(R_N) &
\end{tikzcd}$$

where π is the projection. It is easy to see that $\overline{\beta}$ passes to the quotient to give a map

$$G_0 \times_{H_0} H_1 \cap d_1^{-1}(V)/(G_0 \times_{H_0} R_N) \xrightarrow{\beta} G_1 \cap d_1^{-1}\phi_0^{-1}(V)/\phi_1^{-1}(R_N),$$

and that α and β are mutually inverse maps of G-spaces.

6.3. *Morphisms in* $\mathbf{S}_G$. It is clear that for any étale G-space E and any object $[G,U,N]$ of $\mathbf{S}_G$, there is a bijective correspondence between G-maps $[G,U,N] \xrightarrow{\bar{t}} E$ and sections $U \xrightarrow{t} E$ with $N_t \subset N$

$$(1) \qquad \frac{[G,U,N] \xrightarrow{\bar{t}} E}{U \xrightarrow{t} E,\, N \subset N_t};$$

given $\bar{t}$, $t = \bar{t} \circ [s]$, where $U \xrightarrow{[s]} G_1 \cap d_1^{-1}(U)/N$ is as in 6.1; and given t, the map $\bar{t}$ is defined as the projection $G_1 \cap d_1^{-1}(U)/N \to G_1 \cap d_1^{-1}(U)/N_t$ followed by the map $\tilde{t}$ as described in 6.1(2).

In particular, the maps $[G,U,N] \xrightarrow{\bar{t}} [G,V,M]$ of $\mathbf{S}_G$ correspond to sections t of $[G,V,M]$ over U such that (in point-set notation:) for each $x \xrightarrow{h} y \in N$, $t(y) \cdot h = t(x)$ in $[G,V,M]$.

6.4. *Covers in* $\mathbf{S}_G$. Let $[G,U,N] \xrightarrow{f} [G,V,M]$ be a map in $\mathbf{S}_G$; i.e. f is an étale map $G_1 \cap d_1^{-1}(U)/N \to G_1 \cap d_1^{-1}(V)/M$ of G-spaces, induced by a section $U \xrightarrow{t} G_1 \cap d_1^{-1}(V)/M$ say. Then $N \subset N_t$, and there is a factorization

$$\begin{tikzcd} G_1 \cap d_1^{-1}(U)/N \arrow[rr, "f"] \arrow[dr, "\pi"'] & & G_1 \cap d_1^{-1}(V)/M \\ & G_1 \cap d_1^{-1}(U)/N_t \arrow[ur, hook, "u"'] & \end{tikzcd}$$

π is the projection onto the quotient, and u is mono. We also write $\mathrm{Ker}(f)$ for N_t.

Now suppose $\{[G,U_i,N_i] \xrightarrow{f_i} [G,V,M]\}_i$ is a cover in $\mathbf{S}_G$, i.e. an epimorphic family of étale G-spaces. Factor each f_i as

$$[G,U_i,N_i] \overset{\pi_i}{\twoheadrightarrow} [G,U_i,\mathrm{Ker}(f_i)] \overset{u_i}{\hookrightarrow} [G,V,M].$$

Each π_i is a singleton covering family in $\mathbf{S}_G$.

Consider the pullback

(1)
$$\begin{tikzcd} G_1 \cap d_1^{-1}(U)/\mathrm{Ker}(f_i) \arrow[r, hook, "u_i"] & G_1 \cap d_1^{-1}(V)/M \\ V_i \arrow[u, hook, "r_i"] \arrow[r, hook] & V \arrow[u, hook, "{[s]}"'] \end{tikzcd}$$

The V_i form an open cover of V, inducing a cover

(2)
$$\{G_1 \cap d_1^{-1}(V_i)/(M|V_i) \overset{\eta_i}{\hookrightarrow} G_1 \cap d_1^{-1}(V)/M\}$$

in $\mathbf{S}_G$, where $M|V_i$ is the restriction of M to V_i ("$M|V_i = \{x \xrightarrow{g} y | g \in M, x, y \in V_i\}$"), and η_i is induced by the identity-section $[s]\colon V_i \to G_1 \cap d_1^{-1}(V)/M$. For each i, the triangle

$$\begin{tikzcd} G_1 \cap d_1^{-1}(V_i)/\mathrm{Ker}(f_i) \arrow[r, hook, "u_i"] & G_1 \cap d_1^{-1}(V)/M \\ G_1 \cap d_1^{-1}(V_i)/M|V_i \arrow[u, hook, "\bar{r}_i"] \arrow[ur, hook, "\eta_i"'] & \end{tikzcd}$$

commutes, i.e. the cover $\{\eta_i\}$ refines the cover $\{u_i\}$.

We conclude that the Grothendieck topology on $\mathbf{S}_G$ is generated by (is the smallest one containing) the covers of the following two types

(1) projections $[G,U,N] \twoheadrightarrow [G,U,M]$, $N \subset M$,

(2) covers $\{[G,U_i,M|U_i] \overset{\eta_i}{\rightarrowtail} [G,U,M]\}_i$ coming from open covers $\{U_i\}$ of U in G_0.

Notice that this is a stable system, i.e. the pullback of a cover of type (1) or (2) along a map $[G,V,L] \xrightarrow{f} [V,U,M]$ in $\mathbf{S}_G$ is again of type (1) or (2).

6.5. *Bases for G.* Let $\mathscr{B}_0$ be a basis for G_0, $\mathscr{B}_1$ a basis for G_1. We say that the pair $\mathscr{B} = (\mathscr{B}_0, \mathscr{B}_1)$ is a *basis for G* if both are closed under finite meets and

(i) if $B_0 \in \mathscr{B}_0$ then $d_0^{-1}(B_0), d_1^{-1}(B_0) \in \mathscr{B}_1$,

(ii) if $B_1 \in \mathscr{B}_1$ then $s^{-1}(B_1) \in \mathscr{B}_0$,

(iii) $\mathscr{B}_1$ is closed under "closure for composition", i.e. if $B \in \mathscr{B}_1$ then ${}^mB \in \mathscr{B}_1$ (here for $U \in \mathscr{O}(G_1)$, $U^+ = m(U \times_{G_0} U)$—recall that $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1$ is open, cf. 5.1—and $U^{(0)} = U$, $U^{(n+1)} = U^{(n)+}$, ${}^mU = \bigcup_n U^{(n)}$). Then clearly, given any object $[G,U,N]$ of $\mathbf{S}_G$, there is a canonical map $[G,B_0,B_1] \to [G,U,N]$ whenever $B_0 \subset U$ and $B_1 \subset N$ (the map induced by the section $B_0 \to U \xrightarrow{[s]} [G,U,N]$), and these form a cover in $\mathbf{S}_G$, $\{[G,B_0,B_1] \to [G,U,N]\}_{B_0B_1}$, indexed by all $B_0 \in \mathscr{B}_0$ with $B_0 \subset U$ and $B_1 \in \mathscr{B}_1$ with $B_1 \subset N$.

Consequently by the comparison lemma (**SGA 4**, III.4), the full subcategory of $\mathbf{S}_G$ consisting of only those objects $[G,B_0,B_1]$, $B_0 \in \mathscr{B}_0$, $B_1 \in \mathscr{B}_1$, for some basis $(\mathscr{B}_0, \mathscr{B}_1)$ for G, equipped with the Grothendieck topology induced from $\mathbf{S}_G$, still form a site for BG.

It is clear that if $\mathscr{B} = (\mathscr{B}_0, \mathscr{B}_1)$ is a basis for G, then for any geometric morphism $\mathscr{E} \xrightarrow{p} \mathscr{S}$, the bases (presentations, cf. 1.1) $p^*(\mathscr{B}_0)$ for $p^\#(G_0)$ and $p^*(\mathscr{B}_1)$ for $p^\#(G_1)$ still satisfy conditions (i)–(iii), and thus define a basis $p^*(\mathscr{B})$ for $p^\#(G)$.

6.6. LEMMA. (a) *Let $E \xrightarrow{p} X$ and $F \xrightarrow{q} Y$ be étale maps of (generalized) spaces and let $U \xrightarrow{t} E \times F$ be a section of $p \times q$ over an open $U \subset X \times Y$. Then there exists a cover $U = \bigvee_i (V_i \times W_i)$ such that $t|V_i \times W_i$ is of the form $r_i \times s_i$ for sections $V_i \xrightarrow{r_i} E$, $W_i \xrightarrow{s_i} F$.*

(b) *In particular, taking $q =$ identity, sections of $E \times Y \xrightarrow{p \times Y} X \times Y$ locally do not depend on the Y-variable.*

PROOF. Obvious.

6.7. STABILITY THEOREM. *Let $\mathscr{F} \xrightarrow{p} \mathscr{E}$ be a geometric morphism, and let G be an open continuous groupoid in $\mathscr{E}$. Then the canonical geometric morphism $B(\mathscr{F}, p^\#(G)) \xrightarrow{\sim} \mathscr{F} \times_{\mathscr{E}} B(\mathscr{E}, G)$ is an equivalence of toposes.*

PROOF. By arguing constructively in $\mathscr{E}$, it is enough to consider the case $\mathscr{E} =$ Sets. We may also assume that $\mathscr{F}$ is sheaves on a space Y, because for any $\mathscr{F}$ there is an open surjection $\mathrm{Sh}(Y) \xrightarrow{q} \mathscr{F}$, and if we prove the theorem for the composite $p \circ q$, then it will follow for p since equivalences of toposes are reflected down open surjections; i.e. if

$$\begin{array}{ccc} \mathscr{Z} & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathscr{Y} & \longrightarrow & \mathscr{S} \end{array}$$

is a pullback, $\mathscr{Y} \to \mathscr{S}$ is an open surjection and $\mathscr{Z} \to \mathscr{Y}$ is an equivalence, then $\mathscr{X} \to \mathscr{S}$ is also an equivalence (see Moerdijk (to appear), lemma in 2.4).

Let $\mathscr{B} = (\mathscr{B}_0, \mathscr{B}_1)$ be the maximal basis for $\mathscr{G}$, $\mathscr{B}_i = \mathscr{O}(G_i)$, with corresponding site $\mathbf{S}_G$ for $BG = B(\mathscr{E}, G)$. By applying p^* to the category $\mathbf{S}_G$ and to the covers of $\mathbf{S}_G$ we obtain a site $\mathbf{S} = p^{\#}(\mathbf{S}_G)$ in $\mathscr{F}$ (or rather, a category with a stable generating system for the topology) for $\mathscr{F} \times_{\mathscr{E}} B(\mathscr{E}, G)$.

On the other hand, $p^*(\mathscr{B}_0)$, $p^*(\mathscr{B}_1)$ give a basis for $p^{\#}(G)$ as noted in 6.5, with a corresponding site $\mathbf{T} = \mathbf{S}_{p^*(\mathscr{B})} \subset \mathbf{S}_{p^{\#}(G)}$ for $B(\mathscr{F}, p^{\#}G)$ in $\mathscr{F}$.

We will compare these two sites $\mathbf{S}$ and $\mathbf{T}$ in $\mathscr{F}$ by an obvious functor $\mathbf{S} \xrightarrow{P} \mathbf{T}$ induced by p^*.

(i) The objects of $\mathbf{S} \in \mathscr{F}$ are (at least locally) the same as those of $\mathbf{S}_G \in \mathscr{E}$ (i.e. $\mathbf{S}_0 = p^*(\mathbf{S}_{G0})$, the constant sheaf), and P sends an object $G_1 \cap d_1^{-1}(V)/N$ of $\mathbf{S}_G$ to the corresponding object $p^{\#}(G)_1 \cap d_1^{-1}(p^{\#}(V))/p^{\#}(N)$. But observe that the quotient $G_1 \cap d_1^{-1}(V)/N$ is stable, i.e. we have

$$p^{\#}(G)_1 \cap d_1^{-1}(p^{\#}(V))/p^{\#}(N) \cong p^{\#}(G_1 \cap d_1^{-1}(V)/N),$$

so $\mathbf{S}$ and $\mathbf{T}$ have essentially the same objects.

(ii) A map $G_1 \cap d_1^{-1}(V)/N \to G_1 \cap d_1^{-1}(W)/M$ in $\mathbf{S}$ comes (locally) from a section $V \to G_1 \cap d_1^{-1}(W)/M$ (cf. 6.3), and this gives a section

$$p^{\#}(V) \to p^{\#}(G_1 \cap d_1^{-1}(W)/M) \cong p^{\#}(G)_1 \cap d_1^{-1} p^{\#}(W)/p^{\#}(M),$$

and hence a map

$$p^{\#}(G)_1 \cap d_1^{-1}(p^{\#}V)/p^{\#}(M) \to p^{\#}(G)_1 \cap d_1^{-1} p^{\#}(W)/p^{\#}(M)$$

(this describes P on maps). There are many more maps $p^{\#}(G_1 \cap d_1^{-1}(V)/N) \xrightarrow{u} p^{\#}(G_1 \cap d_1^{-1}(W)/M)$ in $\mathbf{T}$ than there are maps $G_1 \cap d_1^{-1}(V)/M \to G_1 \cap d_1^{-1}(W)/M$ in $\mathbf{S}$ (i.e. essentially, in $\mathbf{S}_G$). However, if u is such a map in $\mathbf{T}$ corresponding to a section $p^{\#}(V) \xrightarrow{t} p^{\#}(G_1 \cap d_1^{-1}(W))/p^{\#}(M)$ of $p^{\#}(d_0)$, then it is true internally (in $\mathscr{F}$) that there exists an open cover $\{U_i\}_{i\in I}$ of $p^{\#}(V)$ such that $t|U_i$ is of the form $p^{\#}(r_i)$ for a section r_i of $G_1 \cap d_1^{-1}(W)/M \xrightarrow{d_0} G_0$ in $\mathscr{E}$—this follows from Lemma 6.6(b). Moreover $p^*(\mathscr{B}_0)$ gives a presentation of the locale $p^{\#}(G_0)$, so the covers of $p^{\#}(V)$ are generated by families of the form $\{p^{\#}(V_j)\}_{j\in p^*(J)}$, where $\{V_j\}_{j\in J}$ is a cover of V by elements of $\mathscr{B}_0$ in $\mathscr{E}$. But any such cover $V = \bigvee V_j$ in G_0 gives a cover of $G_1 \cap d_1^{-1}(V)/M$ in the site $\mathbf{S}_G$ in $\mathscr{E}$: $\{G_1 \cap d_1^{-1}(V_i)/(M|V_j) \xrightarrow{\eta_j} G_1 \cap d_1^{-1}(V)/M\}_j$ (cf. 6.4). So what we conclude is that the composite $t \circ p^{\#}(\eta_j)$ is of the form $p^{\#}(r_j)$. In other words, $\mathbf{S} \xrightarrow{P} \mathbf{T}$ is *locally full* in the sense that for any map $PS \xrightarrow{u} PS'$ in $\mathbf{T}$ there is a cover $\{S_j \xrightarrow{\alpha_j} S\}_j$ in $\mathbf{S}$ such that $u \circ P(\alpha_j) = P(v_j)$ for some $v_j \colon S_j \to S'$ in $\mathbf{S}$.

(iii) It remains to compare the covers in $\mathbf{S}$ and in $\mathbf{T}$. Clearly P preserves covers (cf. 6.4). P also reflects covers, since every cover $\{PS_i \xrightarrow{v_i} PS\}_{i\in I}$ in $\mathbf{T}$ has a refinement of the form $\{PS_j \xrightarrow{Pw_j} PS\}_{j\in J}$ where $\{w_j\}$ is a cover in $\mathbf{S}$. To see this, we may first assume that $v_i = P(f_i)$, by (ii). Now factor $f_i = u_i \circ \pi_i$ as in 6.4, and construct the pullback 6.4(1). Clearly π_i is a cover in $\mathbf{S}$. Moreover the u_i give a family $\{V_i \to V\}_{i\in I}$ such that $\{p^{\#}(V_i)\}_i$ is a cover of $p^{\#}(V)$ in $p^{\#}(G_0)$. This does not necessarily mean that $\{V_i\}$ is a cover of V (in $\mathscr{E}$), but it certainly implies that the induced family $\{[p^{\#}G_1, p^{\#}(V_i), p^{\#}(N|V_i)] \to [p^{\#}(G_1), p^{\#}(V), p^{\#}(N)]\}$ is a cover in $\mathbf{S}$, since the cover $\{p^{\#}(V_i)\}$ of $p^{\#}(V)$ must at least be *generated* from open covers in G_0 (in $\mathscr{E}$), essentially by definition of the space $p^{\#}(G_0)$ (cf. 1.5).

It now follows by the Comparison Lemma (**SGA 4**, III) that $\mathbf{S}$ and $\mathbf{T}$ give equivalent toposes of sheaves, and the proof of Theorem 6.7 is complete.

6.8. THEOREM. *Let $\{G^i\}_i$ be a family of open continuous groupoids in a topos $\mathscr{E}$. Then there is an equivalence of $\mathscr{E}$-toposes $\prod_{\mathscr{E}} B(\mathscr{E}, G^i) \xrightarrow{\sim} B(\mathscr{E}, \prod G^i)$ (on the left, $\prod_{\mathscr{E}}$ is the (possibly infinite) fibered product over $\mathscr{E}$; on the right, $\prod$ is the obvious direct product of continuous groupoids).*

PROOF. The proof is similar to that of 5.7, in fact easier. I will only say a few words about the case of a binary product $G^1 \times G^2$. If $\mathscr{B}^i = (\mathscr{B}_0^i, \mathscr{B}_1^i)$ is a basis for G^i $(i = 1, 2)$, then $\mathscr{B} = (\mathscr{B}_0^1 \times \mathscr{B}_0^2, \mathscr{B}_1^1 \times \mathscr{B}_1^2)$ is a basis for $G^1 \times G^2$ (where $\mathscr{B}_i^1 \times \mathscr{B}_i^2 = \{B^1 \times B^2 | B^j \in \mathscr{B}_i^j\}$), and there is a comparison functor $\mathbf{S}_{\mathscr{B}^1} \times \mathbf{S}_{\mathscr{B}^2} \to \mathbf{S}_{\mathscr{B}}$ which is essentially the identity on objects, locally full by 6.6(a), and preserves and reflects covers because any cover of a product of spaces $B^1 \times B^2$ is generated from covers in each coordinate separately (by the very definition of the product of spaces).

As a corollary of 6.7 and 6.8, we obtain some stable coequalizers of spaces (see also 5.6).

6.9. COROLLARY. (1) *Let G be an open groupoid over a space T, and let*

$$G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0 \xrightarrow{p} \pi G$$

be a coequalizer of spaces (over T). Then $Y \times_T G_1 \rightrightarrows Y \times_T G_0 \to Y \times_T \pi(G)$ is again a coequalizer, for any space Y over T.

(2) *Let G and H be continuous groupoids over a space T, and let*

$$G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0 \xrightarrow{p} \pi G \quad \textit{and} \quad H_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} H_0 \xrightarrow{q} \pi H$$

be coequalizers of spaces over T. Then

$$G_1 \times_T H_1 \rightrightarrows G_0 \times_T H_0 \to \pi(G) \times_T \pi(H)$$

is again a coequalizer.

PROOF. G is a continuous groupoid in the topos $\mathrm{Sh}(T)$, and $\pi(G)$ is the reflection of $BG \to \mathrm{Sh}(T)$ into spaces over T (i.e. spaces in $\mathrm{Sh}(T)$). So (1) follows from 6.7 and the fact that the spatial reflection is preserved by pullback (cf. 1.6). (2) follows similarly from 6.8 and the fact that the spatial reflection commutes with products (1.6).

Just like 6.8, 6.9(2) holds for arbitrary (small) products, and not just for binary ones.

7. Toposes as a localization of continuous groupoids. The aim of this section is to obtain the category of toposes (over a given base topos $\mathscr{S}$) as a category of fractions from the category of continuous groupoids. The category of toposes here *could* mean the 2-category with geometric morphisms as 1-cells and natural isomorphisms as 2-cells. This 2-categorical version requires a calculus of fractions for 2-categories. Here, however, one looses oneself in an orgy of coherence conditions involved in pseudofunctors, pseudo-natural transformations, etc. Therefore, I will only present a version for ordinary categories, namely the category [toposes] of toposes and isomorphism classes of geometric morphisms (under natural isomorphism of inverse image functors), and a category of continuous groupoids and isomorphism classes of continuous homomorphisms, as in the following definition.

7.1. DEFINITION. Let $G \underset{\psi}{\overset{\phi}{\rightrightarrows}} H$ be maps of continuous groupoids. A map (2-cell) $\alpha\colon \phi \Rightarrow \psi$ is a continuous map of spaces $G_0 \xrightarrow{\alpha} H_1$ such that $d_0\alpha = \phi_0$, $d_1\alpha = \psi_0$, and the diagram (1) commutes:

$$(1) \qquad G_1 \underset{(\alpha d_1,\phi)}{\overset{(\psi,\alpha d_0)}{\rightrightarrows}} H_1 \times_{H_0} H_1 \xrightarrow{m} H_1.$$

Any 2-cell is an isomorphism; ϕ and $\psi\colon G \to H$ are isomorphic if there exists a 2-cell $\phi \Rightarrow \psi$. $\underline{\mathrm{Hom}}(G, H)$ denotes the category of maps $G \to H$ and 2-cells between them.

7.2. *Étale-complete groupoids.* A continuous groupoid G is étale-complete if

$$(1) \qquad \begin{tikzcd} \mathrm{Sh}(G_1) \arrow[r, "d_1"] \arrow[d, "d_0"'] & \mathrm{Sh}(G_0) \arrow[d, "p"] \arrow[dl, phantom, "\swarrow \mu_G"] \\ \mathrm{Sh}(G_0) \arrow[r, "p"'] & BG \end{tikzcd}$$

is a pullback of toposes. Here p is the canonical map (p^* is the forgetful functor $(E \to G, \cdot) \mapsto E$), and $\mu_G\colon d_1^*p^* \Rightarrow d_0^*p^*$ is the natural transformation corresponding to the action of G on étale-spaces.

If x is a point of G_0, let $\mathrm{ev}_x\colon BG \to \mathcal{S}$ denote the functor "evaluate at x", which takes the fiber at x: $\mathrm{ev}_x(E) = E_x$. Using the Stability Theorem (6.7), one can describe étale-completeness as follows: G is étale-complete iff for any two points $x, y \in G_0$, any natural isomorphism $\alpha\colon \mathrm{ev}_x \overset{\sim}{\Rightarrow} \mathrm{ev}_y$ is induced by a unique point $y \xrightarrow{g} x$ of G_1 (here *point* means: point in any base extension).

If G is an arbitrary continuous groupoid, there is an étale-completion $\hat{G}$ associated to it, defined by $\hat{G}_0 = G_0$ and $\hat{G}_1$ is given by the pullback $\mathrm{Sh}(G_1) = \mathrm{Sh}(G_0) \times_{BG} \mathrm{Sh}(G_0)$. (So in point-set language, for $x, y \in G_0$ the maps $x \to y$ in $\hat{G}_1$ are the natural isomorphisms $\mathrm{ev}_y \Rightarrow \mathrm{ev}_x\colon BG \to \mathcal{S}$.) G is étale-complete iff G is isomorphic to $\hat{G}$. Note that $\hat{G}$ is open if G is (cf. 5.1).

There is a canonical map $G \xrightarrow{\eta} \hat{G}$ of continuous groupoids, which is universal in the sense that for any étale-complete groupoid H, any $G \xrightarrow{\phi} H$ has a unique extension $\hat{\phi}$ to $\hat{G}$

$$\begin{tikzcd} G \arrow[rr, "\eta"] \arrow[dr, "\phi"'] & & \hat{G} \arrow[dl, "\hat{\phi}"] \\ & H & \end{tikzcd}$$

Moreover, if $G \xrightarrow{\psi} H$ is another map with unique extension $\hat{\psi}$, there is a natural 1-1 correspondence between 2-cells $\phi \Rightarrow \psi$ and 2-cells $\hat{\phi} \Rightarrow \hat{\psi}$. (More precisely, if $\alpha\colon G_0 \to H_1$ is a 2-cell from ϕ to ψ, i.e. the diagram 7.1(1) commutes, then the same α also makes (1) commute with G_1 replaced by $\hat{G}_1$). Thus, η induces an isomorphism of categories $\underline{\mathrm{Hom}}(\hat{G}, H) \to \underline{\mathrm{Hom}}(G, H)$.

7.3. LEMMA. *Let $\mathcal{F} \xrightarrow{p} \mathcal{E}$ be a geometric morphism, and let G be a continuous groupoid in $\mathcal{E}$. If G is étale-complete, then so is $p^{\#}(G)$.*

PROOF. Obvious from the Definition and the Stability Theorem (6.7).

7.4. *Étale-complete groups.* At least in the case of a continuous *group* G (i.e. $G_0 = 1$, $G_1 = G$), the étale completion is easy to describe explicitly. Let $\mathbf{S}$ be the system of open subgroups of G, partially ordered by inclusion. If $U \subset V$ in $\mathbf{S}$, there is a canonical projection map of discrete spaces (of right G-sets) $G/U \to G/V$. Here G/U is defined as the coequalizer $R_U \rightrightarrows G \xrightarrow{q_U} G/U$, where $R_U \subset G \times G$ is given by the pullback

$$\begin{array}{ccc} R_U & \longrightarrow & G \times G \\ \downarrow & & \downarrow{\scriptstyle xy^{-1}} \\ U & \longrightarrow & G \end{array}$$

(so this construction of G/U is a special case of the construction of $G_1 \cap d_1^{-1}(V)/N$ from §6). Taking the inverse limit of this system of projections, one obtains a prodiscrete space

$$M_G = \varprojlim_{U \in \mathbf{S}} G/U.$$

We claim that M_G is a *monoid* object in the category of spaces. The neutral element $1 \xrightarrow{e} M_G$ is defined by $\pi_U \circ e = 1 \xrightarrow{e} G \xrightarrow{q_U} G/U$, where $M_G \xrightarrow{\pi_U} G/U$ is the projection. Multiplication $\mu\colon M_G \times M_G \to M_G$ (also denoted by $\cdot$), is defined as follows: Write $G/U \times M_G = \coprod_{K \in G/U} \{K\} \times M_G$ (G/U is discrete), and let $m^U\colon G/U \times M_G \to G/U$ be the map with $m^U|\{K\} \times M_G$ equal to the composite $\{K\} \times M_G \xrightarrow{\pi_{\tilde{K}}} G/\tilde{K} \xrightarrow{l} G/U$, where $\tilde{K} = q_U^{-1}(K)^{-1} \cdot q_U(K) \subset G$, and l is the function (of discrete sets!) defined by the commutative diagram

$$\begin{array}{ccc} q_U^{-1}(K) \times G & \xrightarrow{\ \cdot\ } & G \\ \downarrow{\scriptstyle q_{\tilde{K}} \circ \pi_2} & & \downarrow \\ G/\tilde{K} & \xrightarrow{\ l\ } & G/U. \end{array}$$

A more intelligible definition of μ_G can be given in point-set language: points of M_G "are" systems $\bar{x} = (Ux_U)_{U \in \mathbf{S}}$ of cosets, and μ is simply defined by

$$\mu(\bar{x}, \bar{y})_U = (\bar{x} \cdot \bar{y})_U = Ux_U y_{(x_U^{-1} U x_U)}. \tag{1}$$

REMARK. It is important to note that (1) may actually be taken as a definition, by the usual techniques of change of base: We have to define a natural (in X) function $\mu_X\colon \mathrm{Cts}(X, M_G \times M_G) \to \mathrm{Cts}(X, M_G)$. By working in $\mathrm{Sh}(X)$, it is enough to give an explicit definition for the case $X = 1$. So take two points $\bar{x}$ and $\bar{y}$ of M_G. These are sequences of elements of G/U, $U \in \mathbf{S}$. By changing the base along an open surjection (pulling back along $G \to G/U$), every point $1 \to G/U$ can be represented as an actual coset Ux_U for a point x_U of G. (Going to a base extension does not affect M_G, since (i) the quotient G/U is stable, and (ii) it is enough to consider a cofinal system of open subgroups.) By taking the filtered inverse limit over $\mathbf{S}$ of all these base extensions, we obtain another open surjection $\mathrm{Sh}(A) \to \mathscr{S}$ (Moerdijk **(1986)**, Theorem 5.1(ii)), such that in $\mathrm{Sh}(A)$, $\bar{x}$ is given as a sequence of cosets $(Ux_U)_{U \in \mathbf{S}}$ for points $x_U \in G$; and similarly we may choose $\bar{y}$ to be represented as $(Uy_U)_{U \in \mathbf{S}}$. Then $(\bar{x} \cdot \bar{y})_U = Ux_U y_{(x_U^{-1} U x_U)}$ as in (1) defines a point of M_G inside $\mathrm{Sh}(A)$, i.e. a map $A \to M_G$. We have to show that it factors through $\mathrm{Sh}(A)$, i.e. a map $A \to \mathcal{S}$ and gives an actual point $1 \to M_G$ in $\mathcal{S}$. But since $\mathrm{Sh}(A) \to \mathcal{S}$ is an open surjection, and open surjections are coequalizers of their kernel pairs, this precisely means that the definition (1) is independent of the points x_U, y_U chosen to represent the cosets Ux_U, Uy_U, which is obvious.

Using such change of base techniques, it is easy to check that M_G is a well-defined monoid, and that the map $G \xrightarrow{\pi} M_G$ given by $\pi_U \circ \pi = q_U : G \to G/U$, is a continuous homomorphism. Now let $\mathcal{I}(M_G)$ be the subspace of invertible elements of M_G. Then this is precisely the étale completion of G: $\hat{G} \cong \mathcal{I}(M_G)$.

7.5. *Essential equivalences.* Call a map $G \xrightarrow{\phi} H$ an essential equivalence if ϕ is open, essentially surjective, and fully faithful (cf. 5.5; note that if ϕ is fully faithful, $\phi_1 : G_1 \to H_1$ is open when $\phi_0 : G_0 \to H_0$ is). Let $\underline{E}$ denote the class of essential equivalences.

Clearly $\underline{E}$ is closed under composition. Moreover, if

$$\begin{tikzcd} P \arrow[r, "\mu"] \arrow[d, "\nu"'] & H \arrow[d, "\phi"] \arrow[dl, phantom, "\swarrow"] \\ K \arrow[r, "\psi"'] & G \end{tikzcd} \tag{1}$$

is a lax pullback-square of continuous groupoids and ϕ is an essential equivalence, then so is ν. The laxpullback (1) is the groupoid P defined by $P_0 = H_0 \times_{G_0} G_1 \times_{G_0} K_0$ (i.e. "the space of triples (y, g, z), $y \in H_0$, $z \in K_0$, $g : \phi(y) \to \psi(z)$ in G_1"), and P_1 is the equalizer

$$P_1 \rightarrowtail H_1 \times_{(H_0 \times H_0)} (P_0 \times P_0) \times_{(K_0 \times K_0)} K_1 \underset{m(\pi_3, \phi\pi_1)}{\overset{m(\psi\pi_4, \pi_2)}{\rightrightarrows}} G_1$$

(i.e. "the maps $(y, g, z) \to (y', g', z')$ in P are pairs $y \xrightarrow{h} y'$, $z \xrightarrow{k} z'$ such that $\psi(k) \circ g = g' \circ \phi(h)$"). Finally, observe that if $G \underset{\psi}{\overset{\phi}{\rightrightarrows}} H$ are maps of continuous groupoids, and $H \xrightarrow{\varepsilon} K$ is an essential equivalence, then any 2-cell $\alpha : \varepsilon\phi \Rightarrow \varepsilon\psi$ (cf. 7.2) factors through ε—i.e. there is a 2-cell $\beta : \phi \Rightarrow \psi$ with $\varepsilon \cdot \beta = \alpha$.

Let CG be the category of open (5.1) continuous groupoids and isomorphism classes of maps (cf. 7.2). Let $\underline{E}$ also stand for the family of morphisms in CG which come from essential equivalences. The properties of $\underline{E}$ as just pointed out show that $\underline{E} \subset CG$ admits a calculus of right fractions (see Gabriel and Zisman **(1967)**).

Let ECG be the full subcategory of CG whose objects are étale-complete continuous groupoids. Clearly, $\underline{E}$ also admits a calculus of right fractions on ECG (by 7.3, the inclusion $ECG \hookrightarrow CG$ has a left adjoint).

7.6. LEMMA. *Let* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ *be a functor, and* $\Sigma \subset \mathbf{C}$ *a class of morphisms admitting a right calculus of fractions. Suppose*

(i) *F is surjective on objects, and faithful,*

(ii) *F sends morphisms from Σ to isomorphisms,*

(iii) *for any map* $FC \xrightarrow{g} FC'$ *there is a commutative diagram*

$$\begin{tikzcd} FC \arrow[r, "g"] & FC' \\ FC_0 \arrow[u, "F\sigma"] \arrow[ur, "Ff"'] & \end{tikzcd}$$

with $\sigma \in \Sigma$.

Then the functor $\mathbf{C}[\Sigma^{-1}] \to \mathbf{D}$ *induced by* F *(by* (ii)*) is an equivalence of categories.*

PROOF. Since Σ admits a calculus of right fractions, we can explicitly construct $\mathbf{C}[\Sigma^{-1}]$ as in Gabriel and Zisman (**1967**). F then induces a functor $F'\colon \mathbf{C}[\Sigma^{-1}] \to \mathbf{D}$ and the conditions of the lemma simply state that F' is surjective on objects, and fully faithful.

Note that "F is faithful" can be replaced by the weaker condition that whenever f, g are parallel arrows in $\mathbf{C}$ with $Ff = Fg$, then there is a $\sigma \in \Sigma$ such that $f\sigma = g\sigma$.

7.7. THEOREM. *The functor* (*continuous groupoids*) $\xrightarrow{B}$ (*toposes*) *induces an equivalence of categories* $ECG[\underline{E}^{-1}] \xrightarrow{\sim} [\textit{toposes}]$.

PROOF. The descent theorem of Joyal and Tierney (Joyal-Tierney (**1984**); see also Moerdijk (**1985**)) implies that the restriction of B to étale complete groupoids is essentially surjective. Moreover, B restricts to a faithful functor $ECG \to$ [toposes]. In fact, if $G \underset{\psi}{\overset{\phi}{\rightrightarrows}} H$ are maps of continuous groupoids and H is étale complete, then any 2-isomorphism $(B\phi)^* \overset{\alpha}{\underset{\sim}{\Rightarrow}} (B\psi)^*\colon BH \to BG$ comes from a 2-cell $\beta\colon \psi \Rightarrow \phi$ as in 7.2, as is immediate from étale completeness. By 7.6 it therefore suffices to show that for any geometric morphism $BG \xrightarrow{f} BH$ there exists a diagram $G \xleftarrow{\varepsilon} K \xrightarrow{\phi} H$ of continuous groupoids such that ε is an essential equivalence, and

$$\begin{tikzcd} BG \arrow[rr, "f"] & & BH \\ & BK \arrow[ul, "B\varepsilon"] \arrow[ur, "B\phi"'] & \end{tikzcd} \tag{1}$$

commutes (up to natural isomorphism). Construct the pullback square

$$\begin{tikzcd} \mathscr{F} \arrow[rr, "\phi_0"] \arrow[d, "\varepsilon_0"] & & \mathrm{Sh}(H_0) \arrow[d, "q"] \arrow[dl, phantom, "\Swarrow\alpha"] \\ \mathrm{Sh}(G_0) \arrow[r, "p"] & BG \arrow[r, "f"] & BH \end{tikzcd} \tag{2}$$

of toposes. $\mathscr{F}$ must be of the form $\mathrm{Sh}(K_0)$ for a unique space K_0. The points of K_0 can be thought of as triples (x, α, y) where $x \in G_0$, $y \in H_0$ are points, and $\alpha\colon \mathrm{ev}_y \xrightarrow{\sim} \mathrm{ev}_x \circ f^*$ is a natural isomorphism. Notice that $\varepsilon_0\colon K_0 \to G_0$ is an open surjection since $\mathrm{Sh}(H_0) \to BH$ is.

Write $\alpha\colon q\phi_0 \overset{\sim}{\Rightarrow} fp\varepsilon_0$ for the 2-isomorphism up to which (2) commutes.

The space K_1 is similarly defined as the pullback of toposes

(3)
$$\begin{tikzcd} \mathrm{Sh}(K_1) \arrow[rr, "\phi_1"] \arrow[d, "\varepsilon_1"] & & \mathrm{Sh}(H_1) \arrow[d, "qd_0"] \\ \mathrm{Sh}(G_1) \arrow[r, "pd_0"] & BG \arrow[r, "f"] & BH \end{tikzcd}$$

with a natural isomorphism $\beta\colon qd_0\pi_H \overset{\sim}{\Rightarrow} fpd_0\pi_G$. So the points of K_1 can be thought of as triples (g,h,β), where $x \xrightarrow{g} x'$ is a point of G_1, $y \xrightarrow{h} y'$ a point of H_1, and $\beta\colon \mathrm{ev}_y \to \mathrm{ev}_x f^*$ is a natural isomorphism.

There are canonical maps $K_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} K_0$ resulting from the universal properties of (2) and (3). In fact, d_0 and d_1 are most easily described by using test spaces and stability (6.7): Given a map $T \xrightarrow{t} K_1$, this can be thought of (inside $\mathrm{Sh}(T)$) as a triple $t = (g,h,\beta)$ of points as above, and $d_0 t = (x,y,\beta)$ defines a point of K_0 in $\mathrm{Sh}(T)$, i.e. a map $T \xrightarrow{d_0 t} K_0$. Similarly, $d_1(g,h,\beta) = (x', y', g^{-1} \circ \beta \circ h)$. In other words, if we allow change of base, then given two points (x,y,α), (x',y',α') of K_0, the maps $(x,y,\alpha) \to (x',y',\alpha')$ in K_1 are pairs (g,h) such that $x \xrightarrow{g} x' \in G$, $y \xrightarrow{h} y' \in H$, and the diagram of natural transformations

$$\begin{tikzcd} \mathrm{ev}_y \arrow[r, "\alpha"] & \mathrm{ev}_x f^* \\ \mathrm{ev}_{y'} \arrow[r, "\alpha'"] \arrow[u, "\cdot h"] & \mathrm{ev}_{x'} f^* \arrow[u, "\cdot g"] \end{tikzcd}$$

commutes. K is a groupoid in the obvious way, and ε_1, ϕ_1 give maps of continuous groupoids $K \xrightarrow{\varepsilon} G$, $K \xrightarrow{\phi} H$.

If G is étale complete, it easily follows from the construction that

$$\begin{tikzcd} K_1 \arrow[r, "(d_0,d_1)"] \arrow[d, "\varepsilon_1"] & K_0 \times K_0 \arrow[d, "\varepsilon_0 \times \varepsilon_0"] \\ G_1 \arrow[r, "(d_0,d_1)"] & G_0 \times G_0 \end{tikzcd}$$

is a pullback. Thus $K \xrightarrow{\varepsilon} G$ is an essential equivalence.

Notice that K is indeed an object of ECG if G and H are. First of all, the pullback (3) can be constructed stepwise

(4)
$$\begin{tikzcd} K_1 = G_1 \times_{G_0} K_0 \times_{H_0} H_1 \arrow[r] \arrow[d] & K_0 \times_{H_0} H_1 \arrow[rr] \arrow[d] & & H_1 \arrow[d] \\ G_1 \times_{G_0} K_0 \arrow[r] \arrow[d] & K_0 \arrow[rr] \arrow[d] & & H_0 \arrow[d] \\ G_1 \arrow[r] & G_0 \arrow[r] & BG \arrow[r] & BH \end{tikzcd}$$

and all the vertical maps are open surjections. It follows that d_0 and $d_1 : K_1 \rightrightarrows K_0$ are open maps. Moreover, since $K \xrightarrow{\varepsilon} G$ is fully faithful and G is étale complete, so is K.

Finally, it remains to show that the diagram

$$\begin{tikzcd} & BK \arrow[dl, "B\varepsilon"'] \arrow[dr, "B\phi"] & \\ BG \arrow[rr, "f"] & & BH \end{tikzcd}$$

commutes, up to isomorphism. Write $\mathrm{Sh}(K_0) \xrightarrow{t} BK$ for the canonical geometric morphism. Then $\alpha\colon q\phi_0 \Rightarrow fp\varepsilon_0$ gives a natural isomorphism $\nu\colon B\phi \circ t \Rightarrow f \circ B\varepsilon \circ t$. For $E \in BH$, the component ν_E is a priori just a map of spaces $\phi^*(E) \to \varepsilon^* f^*(E)$ over K_0. However, it easily follows from the construction of K_1 (from the naturality of β) that it is in fact a map of K-spaces, i.e. $\nu_H \cong t^*\mu_H$ for some (unique) μ_H.

This completes the proof of 7.7.

There are several variants, such as the following analogue of 7.7 for open $\mathscr{S}$-toposes and open maps.

7.8. *Open toposes.* Let ($\mathscr{O}$-toposes) denote the category of open $\mathscr{S}$-toposes and open geometric morphisms (over $\mathscr{S}$), and [$\mathscr{O}$-toposes] the corresponding category with isomorphism classes of open geometric morphisms as maps. Let $\mathscr{O}$-ECG be the subcategory of ECG given by open maps of continuous groupoids (5.5). It can be shown that $B\colon \mathscr{O}\text{-}ECG \to [\mathscr{O}\text{-toposes}]$ induces an equivalence $\mathscr{O}\text{-}ECG[\underline{E}^{-1}] \xrightarrow{\sim} [\mathscr{O}\text{-toposes}]$. The proof is completely analogous to that of 7.7.

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