TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 311, Number 1, January 1989

REALIZING ROTATION VECTORS FOR TORUS HOMEOMORPHISMS

BY JOHN FRANKS

ABSTRACT. We consider the rotation set $\rho(F)$ for a lift F of a homeomorphism $f: T^2 \to T^2$, which is homotopic to the identity. Our main result is that if a vector v lies in the interior of $\rho(F)$ and has both coordinates rational, then there is a periodic point $x \in T^2$ with the property that

$$\frac{F^q(x_0) - x_0}{q} = v$$

where $x_0 \in R^2$ is any lift of x and q is the least period of x.

In this article we consider the rotation set $\rho(F)$ as defined in [MZ], for a lift F of a homeomorphism $f: T^2 \to T^2$, which is homotopic to the identity. Our main result is that if a vector v lies in the interior of $\rho(F)$ and has both coordinates rational, then there is a periodic point $x \in T^2$ with the property that

$$\frac{F^q(x_0) - x_0}{q} = v$$

where $x_0 \in R^2$ is any lift of x and q is the least period of x. This should be compared with the well-known fact that if a homeomorphism of the circle has rational rotation number p/q then it has a periodic point (with rotation number p/q).

R. MacKay and J. Llibre [ML] have proved a similar result using the ideas our Proposition (2.4) below. They require the stronger hypothesis that v is in the interior of the convex hull of vectors in $\rho(F)$ which represent periodic orbits of f.

1. BACKGROUND AND DEFINITIONS

Suppose $f: T^2 \to T^2$ is a homeomorphism homotopic to the identity map, and let $F: R^2 \to R^2$ be a lift.

(1.1) Definition. Let $\rho(F)$ denote the set of accumulation points of the subset of R^2

$$\left\{ \frac{F^n(x) - x}{n} \;\middle|\; x \in R^2, n \in Z^+ \right\},$$

Received by the editors August 23, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F22.

0002-9947/89 $1.00 + $.25 per page

thus $\nu \in \rho(F)$ if there are sequences $x_i \in R^2$ and $n_i \in Z^+$ with $\lim n_i = \infty$ such that

$$\lim_{i\to\infty} \frac{F^{n_i}(x_i) - x_i}{n_i} = \nu.$$

In [MZ] the rotation set is defined for a map homotopic to the identity (rather than a homeomorphism) $f: T^n \to T^n$. However, we shall be concerned only with homeomorphisms of T^2. In [MZ] it is shown that for homeomorphisms of T^2, $\rho(F)$ is convex.

We now briefly review the elementary theory of attractor-repeller pairs and chain recurrence developed by Charles Conley in [C]. In the following $f: X \to X$ will denote a homeomorphism of a compact metric space X.

(1.2) Definition. An *ε-chain* for f is a sequence $x_1, x_2, \dots, x_n$ of points in X such that

$$d(f(x_i), x_{i+1}) < \varepsilon \qquad \text{for } 1 \le i \le n-1.$$

If $x_1 = x_n$ it is called a periodic ε-chain.

A point $x \in X$ is called *chain recurrent* if for every $\varepsilon > 0$ there is an n (depending on ε) and an ε-chain $x_1, x_2, \dots, x_n$ with $x_1 = x_n = x$. The set $\mathbf{R}$ of chain recurrent points is called the *chain recurrent set* of f.

It is easily seen that $\mathbf{R}$ is compact and invariant under f.

If $A \subset X$ is a compact subset and there is an open neighborhood U of A such that $f(\mathrm{cl}(U)) \subset U$ and $\bigcap_{n\geq 0} f^n(\mathrm{cl}(U)) = A$, then A is called an *attractor* and U is its isolating neighborhood. It is easy to see that if $V = X - \mathrm{cl}(U)$ and $A^* = \bigcap_{n\geq 0} f^{-n}(\mathrm{cl}(V))$, then A^* is an attractor for f^{-1} with isolating neighborhood V. The set A^* is called the *repeller* dual to A. It is clear that A^* is independent of the choice of isolating neighborhood U for A. Obviously $f(A) = A$ and $f(A^*) = A^*$.

If we define a relation $\sim$ on $\mathbf{R}$ by $x \sim y$ if for every $\varepsilon > 0$ there is an $\varepsilon > 0$ there is an ε-chain from x to y and another from y to x, then it is clear that $\sim$ is an equivalence relation.

The equivalence classes in $\mathbf{R}(f)$ for the equivalence relation $\sim$ above are called the *chain transitive components* of $\mathbf{R}(f)$.

(1.3) Definition. A complete Lyapounov function for $f: X \to X$ is a continuous function $g: X \to R$ satisfying:

(1) If $x \notin \mathbf{R}(f)$, then $g(f(x)) < g(x)$.

(2) If $x, y \in \mathbf{R}(f)$, then $g(x) = g(y)$ iff $x \sim y$ (i.e., x and y are in the same chain transitive component.

(3) $g(\mathbf{R}(f))$ is a compact nowhere dense subset of R.

By analogy with the smooth setting, elements of $g(\mathbf{R}(f))$ are called *critical values* of g.

A theorem of C. Conley [C] asserts that a complete Lyapounov function exists for any flow or homeomorphism of a compact space. The proof in [C] is given for flows; for an exposition in the case of homeomorphisms see [F2].

In general the number of chain transitive components for a homeomorphism can be infinite (even uncountable). However, if we specify a fixed $\delta > 0$ and work with δ-chains we can decompose $R(f)$ into a finite number of pieces.

(1.4) Definition. For a fixed $\delta > 0$ we say that $x, y \in \mathbf{R}(f)$ are *δ-equivalent* if there is a δ-chain from x to y and one from y to x. This is an equivalence relation and the equivalence classes will be called *δ-transitive components* of $R(f)$. We will say a compact f-invariant set $\Lambda \subset \mathbf{R}(f)$ is *δ-transitive* if for every $x, y \in \Lambda$, x is δ-equivalent to y.

(1.5) Lemma. *Given $\delta > 0$ and a homeomorphism $f: X \to X$ of a compact space, then there are finitely many δ-transitive components.*

Proof. A δ-transitive component is a union of chain transitive components. Two chain transitive components which are in different δ-transitive components must be at least distance δ apart. Hence if there were infinitely many δ-transitive components, there would be infinitely many subsets each at least distance δ from the others. This is impossible since X is compact. □

(1.6) Theorem. *Given $\delta > 0$ and a homeomorphism of a compact space $f: X \to X$, there is a complete Lyapounov function $g: X \to R$ for f, and regular values for g, $c_0 < c_1 < c_2 < \cdots < c_n$ such that if $\Lambda_i = \mathbf{R}(f) \cap g^{-1}([c_{i-1}, c_i])$, then $\{\Lambda_i\}$, $1 \le i \le n$, are the δ-transitive components of f.*

Proof. Let $\Lambda_1, \dots, \Lambda_n$ be the δ-transitive components for f. We order them in such a way that if $i < j$ there is no δ-chain from Λ_i to Λ_j. This is possible since there can be no "cycle" of Λ_i's with each one having a δ-chain to the next and the last having a δ-chain to the first.

Let U_i denote the set of all $z \in X$ such that there is a δ-chain from Λ_i to z. U_i is an open set. Moreover, $f(\mathrm{cl}(U_i)) \subset U_i$, because if $z \in \mathrm{cl}(U_i)$, there is $z_0 \in U_i$ such that $d(f(z), f(z_0)) < \delta$ and consequently a δ-chain from x to z_0 gives a δ-chain $x = x_1, x_2, \dots, x_k, z_0, f(z)$ from x to $f(z)$.

Thus if $A_i = \bigcap_{n\ge 0} f^n(\mathrm{cl}\, U_i)$ and $A_i^* = \bigcap_{n \ge 0} f^{-n}(X - U_i)$, then A_i, A_i^* are an attractor repeller pair and $\Lambda_i \subset A_i$. A result of Conley (see Lemma (1.7) of [F2] for a proof) asserts there is a continuous function $g_i : X \to [0, 1]$ such that $A_i = g_i^{-1}(0)$, $A_i^* = g_i^{-1}(1)$ and $g_i(f(x)) < g_i(x)$ for all $x \in X - (A_i \cup A_i^*)$. If $i < j$, then $\Lambda_j \subset A_i^*$ so $g_i(\Lambda_j) = \{1\}$.

Let $h(x) = \sum_{i=1}^n 2^i g_i(x)$ and note that $h(f(x)) \le h(x)$ for all $x \in X$. For $x \in \mathbf{R}(f) = \bigcup \Lambda_i$, $h(x)$ is an even integer between 0 and 2^{n+1}. Also note if $x, y \in \mathbf{R}(f)$, then $h(x) = h(y)$ if and only if $g_i(x) = g_i(y)$ for all i. Hence if $x \in \Lambda_i$, $y \in \Lambda_j$, $i < j$, then $h(x) \ne h(y)$ since $g_i(x) \ne g_i(y)$. Now if $g_0 : X \to [0, 1]$ is a complete Lyapounov function, then $g(x) = g_0(x) + h(x)$ is the desired function. □

2. The δ-transitive case

We begin with a sequence of results leading to our main theorem. Assume throughout that $f: T^2 \to T^2$ is a homeomorphism homotopic to the identity and $F: R^2 \to R^2$ is a lift, i.e., if $\pi: R^2 \to T^2$ is the covering projection then $\pi \circ F = f \circ \pi$.

(2.1) Lemma. *If F has no fixed points, then there is an $\varepsilon > 0$ such that no periodic ε-chain for F exists.*

Proof. This result and its proof are quite similar to (2.1) of [F1] and (2.2) of [F2]. Let

$$\delta = \min_{x \in R^2} |F(x) - x|.$$

Note this minimum is assumed since it suffices to consider only x in a compact fundamental domain for π. Hence $\delta > 0$.

A result of Oxtoby [Ox] says that there is a $\gamma > 0$ such that for any finite set of pairs $\{(x_i, y_i)\}$ of elements in R^2 with $\|x_i - y_i\| < \gamma$ there is a pairwise disjoint set of piecewise linear arcs α_i from x_i to y_i with the diameter of each $< \delta$.

Let $\varepsilon = \gamma$; we will show there is no periodic ε-chain for F. Suppose to the contrary that $z_1 = z, z_2, z_3, \dots, z_n = z$ is a periodic ε-chain. Letting $y_i = z_i$, $x_i = F(z_{i-1})$, we see that there are pairwise disjoint arcs α_i from $F(z_{i-1})$ to z_i, with diameter $< \delta$. By isotoping in a neighborhood of these arcs we can produce a perturbation G of F satisfying

(1) $\|F(x) - G(x)\| < \delta$ for all $x \in R^2$, and

(2) $G(z_{i-1}) = z_i$.

Now G has a periodic point, namely z. Hence by results of [Br or Fa] G has a fixed point p. Thus $\|F(p) - p\| \leq \|F(p) - G(p)\| + \|G(p) - p\| < \delta$ which is a contradiction. □

(2.2) Lemma. *Suppose Λ is a δ-transitive compact invariant subset of $\mathbf{R}(f)$ for a homeomorphism $f: T^2 \to T^2$ and F is a lift of f. There is a constant $K > 0$, such that for any $x_0, y_0 \in \Lambda$, $x \in \pi^{-1}(x_0)$ there is a δ-chain for F from x to a point $y \in \pi^{-1}(y_0)$ with $\|y - x\| < K$.*

Proof. Fix $\omega \in \pi^{-1}(\Lambda)$ and let Q_n denote the set of $z \in \Lambda$ such that there is a δ-chain for f from $\pi(\omega)$ to z of length less than n. Q_n is open by definition and $\Lambda = \bigcup_{n \geq 1} Q_n$ so compactness of Λ implies $Q_N = \Lambda$ for some $N > 0$. Hence given $y_0 \in \Lambda$ there is a δ-chain from $\pi(\omega)$ to y_0 of length less than N. Lifting this to R^2, starting at w, we obtain a δ-chain from w to some $y' \in \pi^{-1}(y_0)$. If $P = \sup \|F(\nu) - \nu\|$, then since this δ-chain from w to y' has length less than N, it follows that $\|y' - w\| < C_1 = N(P + \delta)$.

A similar argument shows that given $x_0 \in \Lambda$ there is an $x' \in \pi^{-1}(x_0)$ with a δ-chain from x' to w and $\|x' - w\| < C_2$ for some constant C_2 independent of

x_0. Piecing these together we obtain a δ-chain from x' to y' with $\|y'-x'\| < K = C_1 + C_2$. Now given any $x \in \pi^{-1}(x_0)$ translate this δ-chain by the integer vector $x - x'$ to obtain a δ-chain from x to y, where $y = y' + (x - x')$ satisfies $\pi(y) = y_0$ and $\|y - x\| = \|y' - x'\| < K$. □

(2.3) Definition. If $\Lambda \subset T^2$ is a compact invariant set for $f: T^2 \to T^2$, and F is a lift of f, we denote by $\rho(f, \Lambda)$, the accumulation points of the set

$$\left\{ \frac{F^n(x) - x}{n} \mid \pi(x) \in \Lambda \text{ and } n > 0 \right\}.$$

(2.4) Proposition. *Suppose $\Lambda \subset T^2$ is a compact invariant subset of $\mathbf{R}(f)$ for $f: T^2 \to T^2$ and for some $\delta > 0$, Λ is δ-transitive. If 0 is in the interior of the convex hull of $\rho(F, \Lambda)$, then there is a periodic δ-chain for F.*

Proof. The hypothesis guarantees that there are vectors $\nu_1, \nu_2, \nu_3, \nu_4 \in \rho(F, \Lambda)$ such that 0 is in the interior of their convex hull (see Steinitz's theorem in [HDK]). Choose neighborhoods U_i of ν_i in R^2 so small that whenever $\nu_i' \in U_i$, 0 is also in the interior of the convex hull of ν_1', ν_2', ν_3' and ν_4'. Fix $z_0 \in \Lambda$ and $z \in \pi^{-1}(z_0)$. Now by (2.2) and the fact that $\nu_1 \in \rho(F, \Lambda)$ we can find $x_i \in R^2$ and $n_i > i$ such that

(1) $\lim_{i\to\infty} \frac{F^{n_i}(x_i)-x_i}{n_i} = \nu_1$.
(2) There is a δ-chain from z to x_i and $\|x_i - z\| < K$.
(3) There is a δ-chain from $F^{n_i}(x_i)$ to $z_i' \in \pi^{-1}(z_0)$ and $\|F^{n_i}(x_i) - z_i'\| < K$.

Notice that piecing together the δ-chain from z to x_i, the orbit segment from x_i to $F^{n_i}(x_i)$ and the δ-chain from $F^{n_i}(x_i)$ to z_i' we obtain a δ-chain from z to z_i'. Also (1), (2), and (3) imply

$$\lim_{n\to\infty} \frac{z_i' - z}{n_i} = \nu_1.$$

Choose i sufficiently large that

$$\frac{z_i' - z}{n_i} \in U_1$$

and set $w_1 = z_i' - z$, $m_1 = n_i$ so that there is a δ-chain from z to $z + w_1$ and $w_1/m_1 \in U_1$. Note that $\pi(z_i') = \pi(z) = z_0$ implies w_1 is an integer vector.

Now in a similar fashion construct w_2, m_2, w_3, m_3, and w_4, m_4, with the analogous properties.

Since 0 is in the convex hull of w_1/m_1, w_2/m_2, w_3/m_3, w_4/m_4 and the vectors w_1, w_2, w_3, w_4 are integers, it is possible to solve

$$Aw_1 + Bw_2 + Cw_3 + Dw_4 = 0$$

for positive integers A, B, C, D. Any translate of a δ-chain by an integer vector is another δ-chain. Hence piecing together A translates of the δ-chain from z to $z+w$, with B translates of the δ-chain from z to $z+w_2$, C translates of the δ-chain from z to $z+w_3$, etc., we obtain a δ-chain from z to $z+Aw_1+Bw_2+Cw_3+Dw_4=z$ as desired. □

3. The general case

As before we assume $f: T^2 \to T^2$ is a homeomorphism and $F: R^2 \to R^2$ is a lift.

(3.1) Proposition. *Suppose ν_1, ν_2, ν_3 and ν_4 are extreme points of the convex set $\rho(F)$ and 0 is in the interior of their convex hull. Then F possesses a fixed point.*

Proof. In [MZ] it is shown that since ν_i is an extreme point of $\rho(F)$ there is an ergodic Borel measure realizing ν_i and hence a nonwandering point $x_i \in T^2$ such that if $x \in \pi^{-1}(x_i)$

$$\lim_{n\to\infty} \frac{F^n(x)-x}{n} = \nu_i.$$

We will need only the fact that such an x_i exists with $x_i \in \mathbf{R}(f)$.

To show that F has a fixed point it suffices by (2.1) to show that for every $\delta > 0$ there is a periodic δ-chain for F. Given $\delta > 0$, let $\mathbf{R}(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_m$ be a decomposition of the chain recurrent set into δ-transitive pieces as given in (1.6) and let $g: T^2 \to R$ be a complete Lyapounov function compatible with this decomposition. We will show that there exists a piece Λ_j of this decomposition and points $y_i \in \Lambda_j$, $i = 1, 2, 3, 4$, such that whenever $y \in \pi^{-1}(y_i)$,

$$\nu_i = \lim_{n\to\infty} \frac{F^n(y)-y}{n}.$$

It then follows by (2.4) that F has a δ-chain. Since this holds for all $\delta > 0$ we conclude by (2.1) that F has a fixed point.

Choose a smooth approximation $g_0: T^2 \to R$ to g and regular values $c_1, c_2, \ldots, c_m$ such that the manifolds with boundary $M_i = g_0^{-1}((-\infty, c_i])$ satisfy

(1) $f(M_i) \subset \operatorname{int} M_i$, and
(2) $\Lambda_i \subset M_i - M_{i-1}$.

Let N_i be the manifold $\operatorname{cl}(M_i - M_{i-1})$, so $T^2 = \bigcup N_i$ and $N_i \cap N_k$ consists of a finite set of circles if $k = i \pm 1$ and otherwise is empty if $i \neq k$.

These circles are the components of $g_0^{-1}(\{c_1, c_2, \ldots, c_m\})$. We first observe that none of these circles is essential in T^2. If there were such a circle, say

γ, then it would be in the boundary of M_j for some j and M_j would have to have another boundary component which is isotopic to γ. (There might also be some inessential circles in the boundary of M_j.) It follows that M_j is an essential annulus (perhaps with some disks removed) in T^2. Let $\tilde{M}_j$ be a component of $\pi^{-1}(M_j)$ and choose a lift F_0 of f so that $F_0(\tilde{M}_j) \subset \tilde{M}_j$. Now $\tilde{M}_j$ is an infinite strip (perhaps with holes) which has a rational slope. It follows since $F_0(\tilde{M}_j) \subset \tilde{M}_j$ that for any $x \in R^2$, if $\lim_{n\to\infty} \frac{F^n(x)-x}{n}$ exists, then it must lie on a line with this slope, since $F_0^n(x)$ is constrained between parallel translates of $\tilde{M}_j$. From this and the fact that $F(x) = F_0(x) + w$ for some integer vector w, it follows that the convex hull of the vectors ν_i given in our hypothesis is a line segment. This contradicts the assumption that 0 is in the interior of the convex hull; so none of the boundary components of the N_i can be essential in T^2.

Since each of these boundary circles is inessential, each of them bounds a unique smooth disk in T^2. The complement of the union of these disks consists of the interior of a single one of the N_i's, say N_j. The complement of $\operatorname{int}(N_j)$ in T^2 consists of a finite set of disks, say $D_1, D_2, \dots, D_r$. Number these disks so that

$$D_i \subset M_j \qquad \text{for } 1 \le i \le s$$

and

$$D_i \subset \operatorname{cl}(T^2 - M_j) \qquad \text{for } s < i \le r.$$

Then

$$f(D_i) \subset \bigcup_{k=1}^{s} D_k \qquad \text{for } 1 \le i \le s$$

and

$$f^{-1}(D_i) \subset \bigcup_{k=s+1}^{r} D_k \qquad \text{if } s < i \le r.$$

Consider now a point $x \in \pi^{-1}(x_1)$ such that

$$\nu_1 = \lim_{n\to\infty} \frac{F^n(x) - x}{n}.$$

We will show that if x_1 is not in Λ_j, there is another point $y_1 \in \Lambda_j$ so that whenever $y \in \pi^{-1}(y_1)$,

$$\nu_1 = \lim_{n\to\infty} \frac{F^n(y) - y}{n}.$$

Since the same is true for ν_2, ν_3, and ν_4, we will have completed the proof by the remarks above.

Suppose now that $x_1 \in D_p$ for $1 \le p \le s$. There exists $q > 0$ such that $f^q(D_p) \subset D_p$ (recall that x_1 is recurrent). Hence if $D \subset R^2$ is the lift of

D_p containing x, then $F^q(D) \subset D + w$ for some integer vector w. If we set $G(z) = F^q(z) - w$, then $G(D) \subset D$ so there is a fixed point z_0 for G. Clearly

$$\nu_1 = \lim_{n\to\infty} \frac{F^n(x) - x}{n} = \lim_{n\to\infty} \frac{F^n(z_0) - z_0}{n} = \frac{w}{q}.$$

If $x_1 \in D_p$ and $s < p$, then a similar argument applied to f^{-1} leads to a fixed point z_0 of G with the same properties.

We want to find a fixed point for G which is in $\pi^{-1}(N_j)$. To do this we consider fixed points of f^q on T^2. We will use the fact that f^q is homotopic to a map with no fixed points so the index sum of the set of fixed points in any Nielsen class for f^q is zero (see [B, Theorem 3, p. 94]). Recall that two fixed points p_1 and p_2 are in the same Nielsen class for f^q provided any lift of f^q to R^2 which pointwise fixes $\pi^{-1}(p_1)$ also pointwise fixes $\pi^{-1}(p_2)$.

We will consider points in the Nielsen class of the point $\pi(z_0)$ where z_0 is the fixed point of G mentioned above. Any such points which are not in N_j will lie in a D_i with a lift $\tilde{D}_i$ for which $G(\tilde{D}_i) \subset \tilde{D}_i$ or with $G^{-1}(\tilde{D}_i) \subset \tilde{D}_i$. Hence the contribution to the index of the points in D_i will be $+1$. Thus the index of the set of fixed points in the Nielsen class of $\pi(z_0)$ which are not in N_j is positive (the disk D_p contributes at least one $+1$). It follows there must be a fixed point $y_1 \in N_j$ of f^q in the Nielsen class of $\pi(z_0)$. Since y_1 is in the Nielsen class of $\pi(z_0)$, if $y \in \pi^{-1}(y_1)$, then $G(y) = y$. Hence

$$\nu_1 = \lim_{n\to\infty} \frac{F^n(y) - y}{n}.$$

Also y_1 is a periodic point of f in N_j so $y_1 \in \Lambda_j$. The same argument implies the existence of $y_2, y_3, y_4 \in \Lambda_j$, so this completes the proof. □

(3.2) Theorem. *Suppose $f\colon T^2 \to T^2$ is a homeomorphism homotopic to the identity and $F\colon R^2 \to R^2$ is a lift. If ν is a vector with rational coordinates in the interior of $\rho(F)$, then there is a point $p \in R^2$ such that $\pi(p) \in T^2$ is a periodic point for f and*

$$\nu = \lim_{n\to\infty} \frac{F^n(p) - p}{n}.$$

Proof. Suppose $\nu = (r/q, s/q)$ with the greatest common divisor of r, s, and q equal to 1. If $G(x) = F^q(x) - (r, s)$, then a fixed point p of G will satisfy $F^q(p) = p + (r, s)$ and hence be the desired point.

It is easy to check (see [MZ]) that $\rho(G) = q\rho(F) - (r, s)$. Thus since $(r/q, s/q)$ is in the interior of $\rho(F)$, it follows that 0 is in the interior of $\rho(G)$. Since $\rho(G)$ is closed and convex there exist extreme points $\nu_1, \nu_2, \nu_3, \nu_4 \in \rho(G)$ such that 0 is in their convex hull (see Steinitz's theorem in [HDK]). It now follows from (3.1) that G possesses a fixed point p. □

REFERENCES

[B] R. F. Brown, *The Lefschetz fixed point theorem*, Scott Foresman and Co., Glenview, Ill., 1978.

[Br] M. Brown, *A new proof of Brouwer's lemma on translation arcs*, Houston J. Math. **10** (1984), 35–41.

[C] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conf. Ser. in Math., no. 38, Amer. Math. Soc., Providence, R. I., 1978.

[Fa] A. Fathi, *An orbit closing proof of Brouwer's lemma on translation arcs*, Enseign. Math. **33** (1987), 315–322.

[F] J. Franks, *Recurrence and fixed points of surface homeomorphisms*, Ergodic Theory Dynamical Systems **8** * (1988), 99–107.

[F2] ——, *A variation on the Poincaré-Birkhoff theorem*, Hamiltonian Dynamics, Contemp. Math., Amer. Math. Soc., Providence, R. I. (to appear).

[HDK] H. Hadwiger, H. Debrunner, and V. Klee, *Combinatorial geometry in the plane*, Holt Rinehart and Winston, New York, 1964.

[ML] R. MacKay and J. Llibre, *Rotation vectors and entropy for homeomorphisms homotopic to the identity*, preprint.

[MZ] M. Misiurewicz and K. Ziemian, *Rotation sets of toral maps* (to appear).

[Ox] J. Oxtoby, *Diameters of arcs and the gerrymandering problem*, Amer. Math. Monthly **84** (1977), 155–162.

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201