HARNACK'S INEQUALITY FOR DEGENERATE SCHröDINGER OPERATORS

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ABSTRACT. We prove a Harnack inequality for nonnegative weak solutions of certain Schrödinger equations of the form $Lu - Vu = 0$ where $L$ is a second order degenerate elliptic operator in divergence form and $V$ is a potential in certain class.

1. Introduction

The purpose of this paper is to establish a Harnack inequality for nonnegative weak solutions of certain degenerate Schrödinger equations of the form

$$Lu - Vu = \sum_{i,j=1}^{n} D_{x_i} (a_{ij}(x) D_{x_j} u) - Vu = 0,$$

$x \in \mathbb{R}^n$. The coefficients $a_{ij}$ are measurable real-valued functions, the coefficient matrix $a = (a_{ij})$ is symmetric and

$$\lambda^{-1} w(x) |\xi|^2 \leq \langle a(x) \xi, \xi \rangle \leq \lambda w(x) |\xi|^2,$$

where $\lambda > 0$, $\xi = (\xi_1, \ldots, \xi_n)$, $\langle , \rangle$ is the Euclidean inner product and $w$ is a weight satisfying either

(i) $w \in A_2$, that is

$$\sup_B \left( \frac{1}{B} \int_B w(x) \, dx \right) \left( \frac{1}{B} \int_B w(x)^{-1} \, dx \right) = c_0 < \infty,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$ and $\frac{1}{B} \int_B w(x) \, dx$ denotes the average of $w$ over $B$. The constant $c_0$ is referred to as the $A_2$ constant of $w$; or

(ii) $w(x) = |f'(x)|^{-2/n}$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal mapping and $|f'(x)|$ denotes the absolute value of the Jacobian determinant of $f$. Quasiconformal means $f = (f_1, \ldots, f_n)$ is one-to-one, the distributional derivatives of $f_i$ belong to $L^1_{\text{loc}}(\mathbb{R}^n)$ and there is a constant $C_0 > 0$, called the...
dilation constant of $f$, such that a.e.

$$\left[ \sum_{i,j=1}^{n} (D_{x_i} f_{j})^2 \right]^{1/2} \leq C_0 |f'(x)|^{1/n}.$$ 

We shall assume that the potential $V$ satisfies the following condition

$$(1.2) \quad \lim_{\delta \to 0} \sup_{B_R(x_\delta)} \int_{|x-y|<\delta} |V(y)| \int_{|x-y|}^{R} \frac{s^2}{w(B_s(x))} \frac{ds}{s} dy = 0,$$

for every $x_\delta \in \mathbb{R}^n$ and $R > 0$, $B_s(x)$ denotes the ball centered at $x$ with radius $s$.

Let $\Omega$ be an open, bounded and connected set in $\mathbb{R}^n$. We say that the function $u$ is a weak solution of $Lu - Vu = 0$ in $\Omega$ if $u \in H^1_{\text{loc}}(\Omega, w)$ and

$$- \int \langle a(x) \nabla u(x), \nabla \psi(x) \rangle dx = \int V(x) u(x) \psi(x) \, dx$$

for every $\psi \in H^1_0(\Omega, w)$ (see definitions in §2).

Given $\Omega$ an open bounded subset of $\mathbb{R}^n$ let $B_R$ be the smallest ball containing $\Omega$. If $\eta$ is a nondecreasing function defined for $r > 0$ and such that $\lim_{r \to 0} \eta(r) = 0$ then we set

$$K_\eta = \left\{ V : \sup_{x \in B_R} \int_{|x-y| \leq r} |V(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B_s(x))} \frac{ds}{s} dy \leq \eta(r), \ r > 0 \right\}.$$

By $c_0$ we denote either the $A_2$ constant of $w$ or the dilation constant of $f$ if $w$ satisfies (ii). The main result is the following:

**Theorem.** Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$, $w$ is a weight satisfying (i) or (ii) and $V \in K_\eta$. Then there exist positive constants $\gamma_0$ and $\gamma$ only depending on $\lambda, n, c_0, \Omega$ and $\eta$ such that if $u$ is any nonnegative solution of (1.1) in $\Omega$ then for any ball $B_r$ with $B_{cr} \subset \Omega$ and $0 < r \leq r_0$ we have

$$\sup_{B_{\gamma r}} u \leq \gamma \inf_{B_r} u.$$ 

The theorem in the nondegenerate case, i.e. $w \equiv 1$, was obtained by Chiarenza, Fabes and Garofalo in [1]. In this case (1.2) means that $V$ belongs to the Kato-Stummel class. In the degenerate case (1.2) is suggested by the following approximate formula for the Green's function $G_L(x, y)$ for $L$ in $B_R(x_0)$ valid when $w$ satisfies (i) or (ii),

$$(1.3) \quad G_L(x, y) \simeq \int_{|x-y|}^{R} \frac{s^2}{w(B_s(x))} \frac{ds}{s},$$

for $x, y \in B_{R/4}(x_0)$ (see [4] for a proof of this formula). It is easy to see that if $V/w \in L^p_w$ locally for $p > (n/2)\mu$ then $V$ satisfies (1.2). Here $\mu$ means the doubling order of $w$, i.e. $w \in D_\mu$ (see §2 for definitions).
The proof of our Theorem is based on the method developed in [1] and [6] which basically consists of estimating powers of the solution $u$. One of the ingredients used in the proof is a weighted interpolation inequality (Lemma (3.3)) having some independent interest.

As in the nondegenerate case our result implies the continuity of solutions.

The paper is organized as follows: in §2 we state some preliminary definitions and results, in §3 we show an $L^\infty$-estimate for solutions and in §4 we establish some properties of the Green's function for $L - V$ and the infimum estimate.

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2. Preliminaries

$\Omega$ denotes a bounded, open and connected subset of $R^n$. $L^p(\Omega, w)$ denotes the class of functions $f$ such that

$$\|f\|_{p,w}^p = \int_\Omega |f(x)|^p w(x) \, dx < \infty.$$ 

$\text{Lip}(\Omega)$ denotes the class of Lipschitz functions in $\Omega$. We say $\psi \in \text{Lip}_0(\Omega)$ if $\psi \in \text{Lip}(\Omega)$ and $\psi$ has compact support contained in $\Omega$. For $\psi \in \text{Lip}(\Omega)$ we define the norm

$$(2.1) \quad \int_\Omega |\psi(x)|^2 w(x) \, dx + \int_\Omega |\nabla \psi(x)|^2 w(x) \, dx.$$ 

$H^1(\Omega, w)$ denotes the closure of $\text{Lip}(\Omega)$ under the norm (2.1). $H^1_0(\Omega, w)$ denotes the closure of $\text{Lip}_0(\Omega)$ under the norm (2.1). $H^{-1}(\Omega, w)$ denotes the dual space of $H^1_0(\Omega, w)$. When $w$ satisfies (i) or (ii) and $u \in H^1(\Omega, w)$ the gradient of $u$ is uniquely defined (see [5, §2]). It can be shown (see [4, p. 154]) that

$$H^{-1}(\Omega, w) = \{ f_0 - \text{div} f : f = (f_1, \ldots, f_n), f_i w \in L^2(\Omega, w) \}.$$ 

We say $u \in H^1_{\text{loc}}(\Omega, w)$ if $u \in H^1(\Omega', w)$ for every $\Omega'$ with closure contained in $\Omega$. Let $u \in H^1(\Omega, w)$, $E \subset \overline{\Omega}$, then $u \geq 0$ on $E$ in the sense of $H^1(\Omega, w)$ if there exists a sequence $u_n \in \text{Lip}(\Omega)$ such that $u_n(x) \geq 0$ for $x \in E$ and $u_n \to u$ in $H^1(\Omega, w)$. If $w$ satisfies (i) or (ii) then Poincaré's inequality holds, i.e. there exist constants $C$ and $\tau > 1$ depending only on $c_0$ such that

$$(2.2) \quad \left( \int_B |u - u_B|^{2\tau} w(x) \, dx \right)^{1/2\tau} \leq C |B|^{1/\tau} \left( \int_B |\nabla u|^2 w(x) \, dx \right)^{1/2}\nabla$$

for all $u \in H^1(\Omega, w)$, $u_B = \frac{1}{M_B} \int_B u w \, dx$. Also, if $u \in H^1_0(\Omega, w)$ we have Sobolev's inequality

$$(2.3) \quad \left( \int_B |u|^{2\tau} w(x) \, dx \right)^{1/2\tau} \leq C |B|^{1/\tau} \left( \int_B |\nabla u|^2 w \, dx \right)^{1/2}.$$
For a proof of (2.2) and (2.3) see [5]. We say that the weight \( w \) satisfies a doubling condition of order \( \mu \) if there exists a constant \( C > 0 \) such that

\[
w(B_{r}(x_0)) \leq Cr^\mu w(B_r(x_0))
\]

for every \( x_0 \in \mathbb{R}^n \), \( r > 0 \) and \( t \geq 1 \). In this case we write \( w \in D_\mu \). It is well known that if \( w \) satisfies (i) or (ii) in §1 then \( w \in D_\mu \) for some \( \mu \geq 1 \).

3. The \( L^\infty \)-estimate

In this section we will show the following

**Theorem (3.1).** Given \( p > 0 \) there exist positive constants \( r_0 \) and \( C \) only depending on \( p, \lambda, n, \eta \) and \( \Omega \) such that if \( u \) is any solution of \( Lu - Vu = 0 \) in \( \Omega \) and \( B_r \) is any ball with \( r \leq r_0 \) and \( B_{2r} \subseteq \Omega \) then we have

\[
\sup_{B_r} |u| \leq C \left( \int_{B_r} |u|^p w(x) \, dx \right)^{1/p}.
\]

The proof will be a consequence of the following lemmas.

**Lemma (3.2).** Let \( u \) be a solution of \( Lu - Vu = 0 \) in \( \Omega \). Then there exists a positive constant \( C = C(\lambda, n, c_0, \eta, \Omega) \) such that if \( 0 < s < t \) and \( B_s \subseteq \Omega \) then we have

\[
\int_{B_s} |\nabla u(x)|^2 w(x) \, dx \leq C \frac{1}{(t-s)^2} \int_{B_t} u(x)^2 w(x) \, dx.
\]

**Proof.** Take \( \phi \in C^\infty_0(B_t) \) such that \( 0 \leq \phi \leq 1 \), \( \phi \equiv 1 \) on \( B_s \) and \( \|\nabla \phi\|_\infty \leq \frac{C}{t-s} \). We have

\[
\int |\nabla u(x)|^2 \phi(x)^2 w(x) \, dx \leq \lambda \int \langle a(\nabla u), \nabla u \rangle \phi(x)^2 \, dx
\]

\[
= \lambda \int \langle a(\nabla u), \nabla (u\phi^2) \rangle \, dx - 2\lambda \int \langle a(\nabla u), \nabla \phi \rangle \phi(x) u(x) \, dx
\]

\[
= -\lambda \int u(x)^2 \phi(x)^2 V(x) \, dx - 2\lambda \int \langle a(\nabla u), \nabla \phi \rangle \phi(x) u(x) \, dx.
\]

To estimate the second term we use the fact that for every \( \varepsilon > 0 \) we have

\[
|\langle a(\phi \nabla u), u \nabla \phi \rangle| \leq \frac{\varepsilon}{2} \langle a(\phi \nabla u), \phi \nabla u \rangle + \frac{1}{2\varepsilon} \langle a(u \nabla \phi), u \nabla \phi \rangle.
\]

Hence by taking \( \varepsilon \lambda^2 = 1/2 \) we obtain

\[
\int |\nabla u|^2 \phi^2 w \, dx \leq -2\lambda \int u^2 \phi^2 V \, dx + 4\lambda^4 \int u^2 |\nabla \phi|^2 w \, dx.
\]

To estimate the second integral in the last inequality we use the following embedding lemma. (For a proof of this lemma in the unweighted case see [9, p. 138].)
**Lemma (3.3).** Let \( \Omega \) be an open, bounded and connected set in \( \mathbb{R}^n \) and let \( V \) be a potential satisfying (1.2). Then given \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon = C(\varepsilon, \Omega, \omega, n, V, \lambda) \) such that for any \( u \in H^1_0(\Omega, \omega) \) we have

\[
\int u^2(x)|V(x)| dx \leq \varepsilon \int |\nabla u|^2 w(x) dx + C_\varepsilon \int u^2 w(x) dx.
\]

**Proof.** Given \( D \) an open and bounded set in \( \mathbb{R}^n \) let \( B_R \) be a ball such that \( D \subset B_{R/4} \) and let \( G(x, y) \) be the Green's function for \( L \) in \( B_R \). We define

\[
\eta_D(s) = \sup_{x \in D} \int_{|x-y|<s} |V(x)|G(x, y) dy.
\]

By (1.3) we have that (1.2) is equivalent to

\[
limit_{s \to 0} \eta_D(s) = 0\]

for every bounded and open set \( D \subset \mathbb{R}^n \). It is enough to prove the lemma for \( u \in \text{Lip}_0(\Omega) \), then the desired result follows by passing to the limit. Let us suppose first that \( u \) has support contained in a ball \( B_r \subset \Omega \), then we claim that for every \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) independent of \( u \) and \( V \) such that

\[
\int u^2 V dx \leq \varepsilon \int |\nabla u|^2 w dx + C_\varepsilon \left[ \eta_{B_r}(3r) \right] \int u^2 V dx + \frac{\varepsilon}{2} \int |\nabla V|^2 u^2 dx.
\]

To prove the claim assume \( V \) has support contained in \( B_r, V \geq 0 \), and set

\[
F(x) = \int V(y)G(x, y) dy
\]

where \( G \) is the Green's function of a ball \( B \) such that \( \frac{1}{4}B \supset \Omega \). Then \( F \) is the solution of \( LF = -V \) in \( B \) and \( F \in H^1_0(B, \omega) \). This is because since \( V \) satisfies (1.2) then

\[
\int_B \int_B G(x, y)V(x)V(y) dx dy < \infty,
\]

and therefore as in the proof of Theorem 4.8 of [4] we can conclude that \( \chi_B V \in H^{-1}(B, \omega) \), (\( \chi_B \) denotes the characteristic function of \( B \)). Therefore

\[
\int u^2 V dx = \int \langle a(\nabla F), \nabla u^2 \rangle dx = 2 \int \langle a(u\nabla F), \nabla u \rangle dx
\]

\[
\leq \varepsilon \int \langle a(\nabla u), \nabla u \rangle dx + \frac{1}{\varepsilon} \int \langle a(u\nabla F), u\nabla F \rangle dx
\]

\[
\leq \lambda \varepsilon \int |\nabla u|^2 w(x) dx + \frac{\lambda}{\varepsilon} \int |\nabla F|^2 u^2 w dx.
\]
Now observe that
\[
\int |
abla F|^2 u^2 w \, dx \leq \lambda \int \langle a(\nabla F), \nabla F \rangle u^2 \, dx \\
= \lambda \int \langle a(\nabla F), \nabla (Fu^2) \rangle \, dx - 2\lambda \int \langle a(\nabla F), \nabla u \rangle uF \, dx \\
= \lambda \int u^2 F' \, dx - 2\lambda \int \langle a(u\nabla F), F \nabla u \rangle \, dx \\
\leq \lambda \int u^2 F(x)V(x) \, dx + \lambda \delta \int |\nabla F|^2 u^2 w \, dx + \frac{\lambda}{\delta} \int F^2 |\nabla u|^2 w \, dx
\]
for every \( \delta > 0 \). Note that for \( x \in B_r, \ F(x) \leq \eta_{B_r}(3r) \) and then by taking \( \delta = (2\lambda)^{-1} \) we get
\[
\int |\nabla F|^2 u^2 w \, dx \leq 2\eta_{B_r}(3r)\lambda \int u^2 V' \, dx + 4\lambda^2 \eta_{B_r}(3r)^2 \int |\nabla u|^2 w \, dx.
\]
The claim follows with \( C_{\epsilon} = 4\lambda^2 /\epsilon \).

To complete the proof of the lemma, given \( 0 < \delta < 1 \), let \( \{\psi_j\}_{1}^{N} \) be a finite partition of unity of \( \Omega \) such that \( \text{supp}\ psi_j \subseteq B_{\delta r}(x_j) \) with \( x_j \in \Omega \) and \( 0 < r_j \leq \delta \). Set \( \Omega = \{x: d(x, \Omega) \leq 1\} \). Therefore
\[
\int (u\psi_j)^2 V' \, dx \leq \frac{\epsilon}{2} |\nabla (u\psi_j)|^2 w \, dx \\
+ C_{\epsilon} \left[ \eta_{B_{\delta r}}(3r_j) \int (\psi_j u)^2 V' \, dx + \eta_{B_{\delta r}}(3r_j)^2 \int |\nabla (u\psi_j)|^2 w \, dx \right] \\
\leq \frac{\epsilon}{2} \int |\nabla u|^2 \psi_j^2 w \, dx + \frac{\epsilon}{2} \int u^2 |\nabla \psi_j|^2 w \, dx \\
+ C_{\epsilon} \left[ \eta_{\Omega}(3\delta) \int (\psi_j u)^2 V \, dx \\
+ \eta_{\Omega}(3\delta)^2 \int |\nabla u|^2 \psi_j^2 w \, dx + \eta_{\Omega}(3\delta)^2 \int |\nabla \psi_j|^2 u^2 w \, dx \right].
\]
We now choose \( \delta = \delta(\epsilon) < 1 \) such that
\[
C_{\epsilon} \eta_{\Omega}(3\delta) < \frac{1}{2} \quad \text{and} \quad C_{\epsilon} \eta_{\Omega}(3\delta)^2 < \frac{\epsilon}{2}.
\]
Hence
\[
\frac{1}{2} \int (u\psi_j)^2 V' \, dx \leq \epsilon \int |\nabla u|^2 \psi_j^2 w \, dx + \epsilon \int u^2 |\nabla \psi_j|^2 w \, dx.
\]
By summing in \( j \) it follows that
\[
\frac{1}{2} \int u^2 V' \, dx \leq \epsilon \int |\nabla u|^2 w \, dx + \epsilon \frac{N(\epsilon)}{\delta(\epsilon)^2} \int u^2 w \, dx.
\]
Remarks. (1) The constant \( C_{\epsilon} \) only depends on \( \epsilon, \Omega, \eta, n, c_0 \) and \( \lambda \).
(2) By Sobolev's inequality (2.3) Lemma (3.3) implies the following two-weights Sobolev inequality
\[
\int u^2(x)V(x) \, dx \leq C \int |\nabla u|^2 w(x) \, dx.
\]
Lemma (3.4). There exists a constant $C = C(n, \lambda, \eta, \Omega, c_0)$ such that if $u$ is a solution of $Lu - Vu = 0$ in $\Omega$ and $B_{2r}(x_0) \subseteq \Omega$ then

$$\left( \int_{B_{2r}(x_0)} u^2 w \, dx \right)^{1/2} \leq C \int_{B_r(x_0)} |u| w \, dx.$$ 

Proof. We claim that it is enough to prove the lemma when $r = 1$ and $x_0 = 0$. In fact, $u_{x_0}(x) = u(x - x_0)$ is defined in $\Omega + x_0$ and if $a_{x_0}(x) = a(x - x_0)$, $V_{x_0}(x) = V(x - x_0)$ and $w_{x_0}(x) = w(x - x_0)$ then $u_{x_0}$ is a solution of $\text{div}(a_{x_0}(x) \nabla) - V_{x_0} = 0$ in $\Omega + x_0$. Note that the constant $c_0$ of $w_{x_0}$ does not change and $V_{x_0}$ is in $K_\eta$ defined with $w_{x_0}$. Therefore by translations we can assume $x_0 = 0$. Set $u_r(x) = u(rx)$, then $u_r$ is defined in $\frac{1}{r} \Omega$ (in particular in $B_2$) and if we set $a_r(x) = a(rx)$, $V_r(x) = r^2 V(rx)$ and $w_r(x) = w(rx)$ then $u_r$ is a solution of $\text{div}(a_r(x) \nabla) - V_r = 0$ in $\frac{1}{r} \Omega$. Note again that the constant $c_0$ of $w_r$ does not change. Also by changing variables it is easy to see that

$$\sup_{x \in \frac{1}{r} \Omega} \int_{|x - y| < \delta} |V_r(y)| \int_{|x - y| \frac{s}{w_r(B_s(x))}}^{4R/r} ds \, dy$$

$$= \sup_{x \in \Omega} \int_{|x - y| < \delta} |V(y)| \int_{|x - y| \frac{s}{w(B_s(x))}}^{4R} ds \, dy,$$

which if $r \leq 1$ implies that $V_r$ belongs to the class $K_\eta$ defined with $w_r$. Let us assume $\int_{B_1} |u| w \, dx = 1$ and for $\frac{1}{2} < s < 1$ consider

$$I(s) = \left( \frac{1}{w(B_{1/2})} \int_{B_{1/2}} u^2 w \, dx \right)^{1/2}.$$ 

If $I(\frac{1}{2}) \leq 1$ then there is nothing to show, so suppose $I(\frac{1}{2}) > 1$. We want to show $I(\frac{1}{2}) \leq C$, $C$ only depends on $n, \lambda, \eta, \Omega$ and $c_0$. Let $\tau$ be the exponent in the Poincaré inequality and choose $0 < \theta < 1$ such that $(2 - \theta)/(1 - \theta) = 2\tau$. By doubling and Poincaré we have

$$I(s) = \left( \frac{1}{w(B_{1/2})} \int_{B_{1/2}} |u|^{2-\theta} |u|^{\theta} w \, dx \right)^{1/2} \leq C \left( \int_{B_{1/2}} |u|^{(2-\theta)/(1-\theta)} w \, dx \right)^{(1-\theta)/2}$$

$$\leq Cs^{(1-\theta)\tau} \left( \int_{B_{1/2}} |\nabla u|^{2\theta} w \, dx \right)^{(1-\theta)/2\tau} + C \left( \int_{B_{1/2}} |u|^{2\theta} w \, dx \right)^{(1-\theta)/2\tau}.$$ 

If $\frac{1}{2} \leq s < t \leq 1$ then by Lemma (3.2) and doubling we obtain $I(s) \leq C[\tau - s]^{-1} I(t)^{(1-\theta)/\tau}$ which implies $I(\frac{1}{2}) \leq C$. (See Lemma 1.2 of [1].)

Lemma (3.5). Let $\Omega = B_2(0)$, $w \in D_\mu$ and $p > (n/2)\mu$. There exist constants $\delta_0 = \delta_0(\lambda, n, \eta, c_0)$ and $C = C(\lambda, n, \eta, c_0)$ such that if

$$\sup_{B_6} \int_{B_8} |V(y)| \int_{|x - y| \frac{s}{w(B_s(x))}}^{8} ds \, dy < \delta_0$$
then given \( f/w \in L_w^p(\Omega) \) there exists a unique \( u \in H_0^1(\Omega, w) \) such that \( Lu - Vu = f \) in \( \Omega \) and
\[
\|u\|_{L^\infty(\Omega)} \leq \frac{c}{w(B_1)^{1/p}} \|f/w\|_{L_w^p(\Omega)}.
\]

**Proof.** The bilinear form
\[
\alpha(u, v) = \int_\Omega (a(\nabla u), \nabla v) dx + \int_\Omega uv V dx
\]
is continuous and coercive in \( H_0^1(\Omega, w) \) provided \( \delta_0 \) is small enough. This follows by the claim made in the proof of Lemma (3.3). Now if \( p \geq 2 \) and \( f \in L_w^{p-1}(\Omega) \) implies \( f \in H^{-1}(\Omega, w) \) and consequently the existence and uniqueness of \( u \) is a consequence of the Lax-Milgram theorem, (see [7]).

Let \( u_0 \) be the solution of the problem \( Lu = f \) in \( \Omega \), \( u/\partial \Omega = 0 \) (i.e. \( u \in H_0^1(\Omega, w) \)), and for \( j \geq 1 \), let \( u_j \) be the solution of \( Lu - Vu_{j-1} = f \) in \( \Omega \), \( u/\partial \Omega = 0 \). Then we have
\[
u_0(x) = \int_\Omega G_L(x, y)f(y) dy,
\]
where \( G_L(x, y) \) is the Green's function of \( L \) in \( \Omega \). By the maximum principle
\[
G_L(x, y) \leq C \int_{|x-y|}^8 \frac{s}{w(B_\frac{1}{2})} ds.
\]
It is easy to see that
\[
\left( \int_\Omega G_L(x, y)^p w(y) dy \right)^{1/p} = \frac{C}{w(B_1(0))^{1/p}} = C_1 < \infty, \quad \text{for} \quad p > \frac{n}{2} m.
\]
Then
\[
|u_0(x)| \leq C_1 \|f/w\|_{L_w^p(\Omega)}, \quad x \in \Omega.
\]
We also have
\[
u_1(x) = \int_\Omega G_L(x, y)f(y) dy + \int_\Omega G_L(x, y)V(y)u_0(y) dy
\]
which implies
\[
|u_1(x)| \leq C_1 \|f\|_{L^p_w} + C_1 \|f\|_{L_w^{p-1}},
\]
provided \( \sup_\Omega \int G_L(x, y)|V(y)| dy < \delta \). Continuing in this manner we obtain
\[
u_j(x) = \int_\Omega G_L(x, y)f(y) dy + \int_\Omega G_L(x, y)V(y)u_{j-1}(y) dy
\]
and
\[
|u_j(x)| \leq C_1 \cdot C_2 \|f/w\|_{L_w^p(\Omega)} \quad \text{for} \quad j = 2, 3, \ldots.
\]

We claim that \( u_j \) is a Cauchy sequence in \( H_0^1(\Omega, w) \) and \( u_j \to u \) in \( H_0^1(\Omega, w) \). By (3.6) we have
\[
u_{j+1}(x) - u_j(x) = \int_\Omega G_L(x, y)V(y)[u_j(y) - u_{j-1}(y)] dy
\]
and therefore

\[ \|u_{j+1} - u_j\|_{L^\infty(\Omega)} \leq \delta \|u_j - u_{j-1}\|_{L^\infty(\Omega)}. \]

Consequently for \( m > n \)

\[ \|u_m - u_n\|_{L^\infty(\Omega)} \leq \sum_{j=n}^{m-1} \|u_{j+1} - u_j\|_{L^\infty(\Omega)} \]

\[ \leq \|u_0 - u_1\|_{L^\infty(\Omega)} \sum_{j=n}^{m-1} \delta^j. \]

Therefore for \( \delta < 1 \) \( \{u_j\} \) is a Cauchy sequence in \( L^\infty(\Omega) \) and therefore in \( L^2(\Omega, w) \). Also

\[ \int_{\Omega} |\nabla (u_m - u_n)|^2 w(x) \, dx \leq \lambda \int_{\Omega} \langle a(\nabla (u_m - u_n)), \nabla (u_m - u_n) \rangle \, dx \]

\[ = - \int (u_m - u_n)(u_{m-1} - u_{n-1}) V(x) \, dx \]

\[ \leq \left( \int |u_m - u_n|^2 V(x) \, dx \right)^{1/2} \left( \int |u_{m-1} - u_{n-1}|^2 V(x) \, dx \right)^{1/2}, \]

and since \( V \in L^1(\Omega) \) we have \( \|\nabla (u_m - u_n)\|_{L^2(\Omega, w)} \) tends to 0 as \( m, n \to \infty \). Consequently \( u_j \to \tilde{u} \) in \( H^1_0(\Omega, w) \) and by (3.6) \( \tilde{u} = u \).

**Remark (3.7).** Lemma (3.5) implies the existence and integrability of the Green’s function \( G(x, y) \) of \( L - V \). In fact, if \( p > (n/2)\mu \) then

\[ \int_{\Omega} G(x, y) w(y)^{(p-1)/p} f(y) w(y)^{(1-p)/p} \, dy \leq \frac{C}{w(B_1)^{1-p}} \|f w^{(1-p)/p}\|_{L^2(\Omega)}, \]

which implies

\[ \left( \int_{\Omega} G(x, y)^q w(y) \, dy \right)^{1/q} = \frac{C}{w(B_1)^{1/p}} < \infty \quad \text{for} \quad 1 < q < \frac{n\mu}{n\mu - 2} \]

and a.e. \( x \in \Omega \) \( (\frac{1}{p} + \frac{1}{q} = 1) \).

**Theorem 3.8.** There exist \( r_0 = r_0(\lambda, n, c_0, \eta) \) and for each \( p > 0 \) a constant \( C = C(p, \lambda, n, c_0, \eta) \) such that if \( u \) is a solution of \( Lu - V u = 0 \) in \( \Omega \) and \( B_{2r} \subset \Omega \) with \( r \leq r_0 \) then

\[ \sup_{B_{r/2}} |u| \leq C \left( \int_{B_r} |u(x)|^p w(x) \, dx \right)^{1/p}. \]

**Proof.** By translation we may assume \( B_r \) is centered at 0. As before \( u_r(x) = u(rx) \) is a solution in \( B_2 = B_2(0) \) of \( L_r u_r - V_r u_r = 0 \) where \( L_r \) and \( V_r \) are defined in Lemma (3.4). Also observe that

\[ \int_{|x-y|<\delta} |V_r(y)| \int_{|x-y|}^{s} \frac{w_r(B_s(x))}{w_r(B_s(y))} \, ds \, dy \]

\[ = \int_{|rx-z|<\delta} \int_{|rx-z|}^{8r} \frac{w_r(B_s(x))}{w_r(B_s(z))} \, ds \, dy \leq \eta(\delta r). \]
Therefore

\[
\sup_{x \in B_2} \int_{|x-y| < \delta} |V_r(y)| \int_{|x-y|}^8 \frac{s}{w_r(B_s(x))} \, ds \, dy \leq \eta(\delta r).
\]

Since \(\eta(\delta r) \to 0\) as \(\delta \to 0\) then it is enough to show that if \(\delta_0\) is the number specified in Lemma (3.5) and

\[
\sup_{B_2} \int_{B_0} |V(y)| \int_{|x-y|}^8 \frac{s}{w(B_s(x))} \, ds \, dy < \delta_0
\]

then we have

\[
\sup_{B_{1/2}} |u| \leq C \left( \int_{B_1} |u|^p w(y) \, dy \right)^{1/p},
\]

with \(C = C(\lambda, \eta, p, n, c_0)\).

Let \(G(x, y)\) be the Green's function of \(L - V\) in \(B_2\). Given \(\frac{1}{2} \leq s < t \leq 1\), let \(\psi\) be in \(C_0^\infty(B_{t-(t-s)/4}(0))\) such that \(0 \leq \psi \leq 1\), \(\psi \equiv 1\) on \(B_{(t+s)/2}(0)\) and \(|\nabla \psi| \leq C/(t-s)\). We have

\[
u(x)\psi(x) = \int_{B_2} (a(\nabla G(x, y)), \nabla \psi(y)) u(y) \, dy
\]

\[-\int_{B_2} (a(\nabla u), \nabla \psi) G(x, y) \, dy = J_1 - J_2.\]

\[
J_1 = \int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} (a(\nabla G(x, y)), \nabla \psi) u(y) \, dy
\]

\[
\leq \left( \int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\nabla G(x, y)|^2 w(y) \, dy \right)^{1/2}
\times \left( \int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} u^2 |\nabla \psi|^2 w(y) \, dy \right)^{1/2}
\leq \frac{C}{t-s} \left( \int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\nabla G(x, y)|^2 w(y) \, dy \right)^{1/2} \left( \int_{B_{t-s}} u^2 w(y) \, dy \right)^{1/2}.
\]

Analogously

\[
J_2 \leq \frac{C}{t-s} \left( \int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |G(x, y)|^2 w(y) \, dy \right)^{1/2}
\times \left( \int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\nabla u|^2 w(y) \, dy \right)^{1/2}.
\]

Now by Lemma (3.2)

\[
\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\nabla u|^2 w(y) \, dy \leq \int_{B_{(t-s)/4}} |\nabla u|^2 w(y) \, dy
\leq \frac{C}{(t-s)^2} \int_{B_{t-s}} u^2 w(y) \, dy.
\]
We cover the annulus \( B_{t-(t-s)/4} \setminus B_{t+s/2} \) by a union of \( N \) balls \( B_{(t-s)/4}(z_i) \), with \( |z_i| = (t + s)/2 + (t - s)/8 \) (observe that the annulus has width \( (t - s)/4 \)). Therefore if \( x \in B_s \) then \( x \not\in B_{(t-s)/2}(z_i) \) and then for \( x \in B_s \) we have

\[
\int_{B_{t-(t-s)/4} \setminus B_{t+s/2}} |\nabla_y G(x, y)|^2 w(y) \, dy \leq \sum_{i=1}^{N} \int_{B_{(t-s)/4}(z_i)} |\nabla G(x, y)|^2 w(y) \, dy
\]

\[
\leq \sum_{i=1}^{N} \frac{C}{(t-s)^2} \int_{B_{(t-s)/2}(z_i)} |G(x, y)|^2 \, dy
\]

\[
= \frac{C}{(t-s)^2} \sum_{i=1}^{N} w(B_{(t-s)/2}(z_i)) \int_{B_{(t-s)/2}(z_i)} |G(x, y)|^2 w(y) \, dy
\]

which by Lemma (3.4) is less than

\[
\frac{C}{(t-s)^2} \sum_{i=1}^{N} w(B_{t-s}(z_i))^{-1} \left( \int_{B_{t-s}(z_i)} G(x, y) w(y) \, dy \right)^2.
\]

Now by doubling we have \( w(B_1(0))(t - s)^{n\mu} \leq c \cdot w(B_{t-s}(z_i)) \) and then by Remark (3.7) we obtain

\[
\left( \int_{B_{t-(t-s)/4} \setminus B_{t+s/2}} |\nabla G(x, y)|^2 w(y) \, dy \right)^{1/2} \leq \frac{C}{(t-s)^{1+(n/2)\mu}} \frac{1}{w(B_1)^{1/2}}.
\]

Analogously we have

\[
\int_{B_{t-(t-s)/4} \setminus B_{t+s/2}} G(x, y)^2 w(y) \, dy
\]

\[
\leq C \sum_{i=1}^{N} w(B_{t-s}(z_i)) \left( \int_{B_{t-s}(z_i)} G(x, y) w(y) \, dy \right)^2.
\]

Collecting estimates we obtain

\[
\|u\|_{L^\infty(B_s)} ^2 \leq \frac{C}{(t-s)^{2+(n/2)\mu}} \left( \frac{1}{w(B_1)} \int_{B_1} u^2 w(y) \, dy \right)^{1/2}.
\]

For \( \frac{1}{2} \leq s \leq 1 \) we set \( I(s) = (1/w(B_1)) \int_{B_1} u^2 w \, dx \)^{1/2}. Let \( p > 0 \) and assume \( \int_{B_t} u^p w \, dy = 1 \), then if \( p < 2 \) we have

\[
I(s) \leq \left( \sup_{B_t} |u| \right)^\theta, \quad \theta = 1 - \frac{p}{2},
\]

and therefore

\[
I(s) \leq \frac{C}{(t-s)^{(2+(n/2)\mu)\theta}} I(t)^\theta.
\]

By the argument in [6, p. 1004] we obtain the theorem.
4. The Infimum Estimate

We begin with the following

**Lemma (4.1).** Let \( u \geq 0 \) be a solution of \( Lu - Vu = 0 \) in \( \Omega \). Then there exists a constant \( C = C(\lambda, \eta, c_0) \) independent of \( u \) such that if \( B_{4r} \subset \Omega \) then for every \( \varepsilon > 0 \) we have

\[
\frac{1}{w(B_r)} \int_{B_r} \left| \log(u + \varepsilon) - \int_{B_r} \log(u + \varepsilon) w(y) \, dy \right|^2 w(y) \, dy \leq C.
\]

**Proof.** Let \( \psi \in C_0^\infty(B_{3r/2}) \), \( \psi \equiv 1 \) on \( B_r \) and \( 0 \leq \psi \leq 1 \), \( |\nabla\psi| \leq c/r \), \( B_r = B_r(x) \) and set \( u_\varepsilon = u + \varepsilon \). Then

\[
\int |\nabla \log u_\varepsilon|^2 w \, dy = \int \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \psi^2 w \, dy
\]

\[
\leq \lambda \int \left( a(\nabla u_\varepsilon) , \nabla u_\varepsilon \right) \frac{\psi^2}{u_\varepsilon} \, dy
\]

\[
= 2\lambda \int \left( a(\nabla u_\varepsilon) , \nabla \psi \right) \frac{\psi}{u_\varepsilon} \, dy - \lambda \int \left( a(\nabla u_\varepsilon) , \nabla \left( \frac{\psi^2}{u_\varepsilon} \right) \right) \, dy
\]

\[
= 2\lambda \int \left( a(\nabla u_\varepsilon) , \nabla \psi \right) \frac{\psi}{u_\varepsilon} \, dy + \lambda \int V(y) \frac{u}{u_\varepsilon} \psi^2 \, dy
\]

\[
\leq 2\lambda \int \left( a(\nabla u_\varepsilon) , \nabla \psi \right) \frac{\psi}{u_\varepsilon} \, dy + \lambda \int |V(y)| \psi^2 \, dy.
\]

We have

\[
\int V(y) \psi^2(y) \, dy
\]

\[
\leq \int_{B_{3r}(x)} |V(y)| \left( \int_{|x-y|}^{4r} s \frac{s}{w(B_s(x))} \, ds \right) \left( \int_{|x-y|}^{4r} s \frac{s}{w(B_s(x))} \, ds \right)^{-1} \, dy.
\]

If \( y \in B_{2r} \) then by doubling we have

\[
\int_{|x-y|}^{4r} \frac{s}{w(B_s(x))} \, ds \geq C \cdot \frac{r^2}{w(B_r(x))}.
\]

Consequently by the assumption on \( V \) we have

\[
\int V(y) \psi^2(y) \, dy \leq C \frac{w(B_r(x))}{r^2}.
\]

Also as in the proof of Lemma (3.2) we have for every \( \delta > 0 \)

\[
\int \left( a(\nabla u_\varepsilon) , \nabla \psi \right) \frac{\psi}{u_\varepsilon} \, dy \leq \frac{\lambda}{\delta} \int |\nabla \log u_\varepsilon|^2 \psi^2 w(y) \, dy + \delta \cdot C \frac{w(B_r(x))}{r^2}.
\]

Hence if \( \delta \) is large we have

\[
\int |\nabla \log u_\varepsilon|^2 w \, dy \leq C \frac{w(B_r(x))}{r^2}.
\]
Now since \( \log u_\varepsilon \in H^1(B_{2r}, w) \) then by Poincaré the lemma follows.

**Remark (4.2).** Note that if \( w \in A_\infty \) then by Theorem 5 of [8], Lemma (4.1) implies that \( \log(u + \varepsilon) \in BMO, \varepsilon > 0 \).

**Lemma (4.3).** Let \( u > 0 \) be a solution of \( Lu - Vu = 0 \) in \( \Omega \) and let \( r_0 \) be the number in Theorem (3.8) then there exists a constant \( C = C(\lambda, \eta, n, c_0) \) such that

\[
\int_{B_{2r}} u(x)w(x)dx \leq C \int_{B_r} u(x)w(x)dx
\]

for \( 0 < r \leq r_0 \) and \( B_{8r} \subset \Omega \).

**Proof.** By Lemma (4.1) and Theorem 5 of [8] there exist \( S > 0 \) and \( C > 0 \) such that

\[
\left( \frac{1}{w(B_r)} \int_{B_r} u_\varepsilon^\delta w dx \right) \left( \frac{1}{w(B_r)} \int_{B_r} u_{-\varepsilon}^\delta w dx \right) \leq C,
\]

for \( B_{4r} \subset \Omega \), i.e. \( u_\varepsilon^\delta \in A_2(w) \), for every \( \varepsilon > 0 \). Since \( w \) is doubling this implies

\[
\int_{B_{2r}} u_\varepsilon^\delta w dx \leq C \int_{B_r} u_\varepsilon^\delta w dx.
\]

Hence by Theorem (3.8) and \( \delta < 1 \) we have

\[
\frac{1}{w(B_{2r})} \int_{B_{2r}} uw dx \leq C \left( \frac{1}{w(B_{4r})} \int_{B_{4r}} u_\varepsilon^\delta w dx \right)^{1/\delta}
\leq C \left( \frac{1}{w(B_{r/2})} \int_{B_{r/2}} u_\varepsilon^\delta w dx \right)^{1/\delta} \leq \frac{1}{w(B_r)} \int_{B_r} u_\varepsilon w dx,
\]

by letting \( \varepsilon \to 0 \) this implies the lemma.

**Lemma (4.4).** Set \( V^+ = \max(V, 0) \) and let \( G(x, y) \) denote the Green's function of \( L - V^+ = 0 \) in \( \Omega \). Then for any \( 1 < q < n\mu/(n\mu - 2) \) \( (w \in D_\mu) \) there exists a constant \( C = C(q, \lambda, \eta, c_0) \) such that

\[
\left( \frac{1}{w(B_r)} \int_{B_r} G(x, y)^q w(y) dy \right)^{1/q} \leq C \frac{1}{w(B_r)} \int_{B_r} G(x, y)w(y) dy,
\]

for \( B_{8r} \subset \Omega \) and \( 0 < r \leq r_0 \) (\( r_0 \) is as in Theorem (3.8)).

**Proof.** First observe that \( G(x, \cdot) \) is a solution of \( L - V^+ = 0 \) for \( y \neq x, x \in \Omega \). Suppose first that \( x \notin B_{8r} \), then by Theorem (3.8) we have for every \( q > 0 \) that

\[
\left( \int_{B_r} G(x, y)^q w(y) dy \right)^{1/q} \leq \sup_{y \in B_r} G(x, y) \leq C \int_{B_r} G(x, y)w(y) dy
\]

which by Lemma (4.3) is less than

\[
\int_{B_r} G(x, y)w(y) dy.
\]
Now assume \( x \in B_{8r} \) and let \( G_r(x, y) = G_{L-V^+, B_8}(x, y) \). By the maximum principle \( G_r(x, y) \leq G(x, y) \) for \( x, y \in B_{8r} \). Then

\[
\int_{B_r} G(x, y)^q w(y) \, dy \leq 2^q \left\{ \int_{B_r} [G(x, y) - G_r(x, y)]^q w(y) \, dy + \int_{B_r} G_r(x, y)^q w(y) \, dy \right\}.
\]

Since \( G(x, \cdot) - G_r(x, \cdot) \) is a nonnegative solution of \( L - V^+ = 0 \) in \( B_{8r} \), then arguing as before we obtain

\[
\int_{B_r} [G(x, y) - G_r(x, y)]^q w(y) \, dy < C \left\{ \int_{B_r} [G(x, y) - G_r(x, y)] w(y) \, dy \right\}^q.
\]

By translation we can assume that \( B_r \) is centered at \( 0 \). Let \( L_r \) and \( V_r^+ \) be defined as in Lemma (3.4) and let \( \tilde{G}(x, y) \) be the Green’s function to \( L_r - V_r^+ \) in \( B_8 \). Then for \( x \in B_{8r} \) and \( z \in B_8 \) we have

\[
G_r(x, rz) = r^{2-n} \tilde{G}(x, z).
\]

Therefore

\[
\int_{B_r} G_r(x, y)^q w(y) \, dy = \frac{r^n}{w(B_r)} \left( \int_{|z| \leq 1} \tilde{G}(x, z)^q w(rz) \, dz \right) r^{(2-n)q}
\]

and

\[
\int_{B_r} G_r(x, y) w(y) \, dy = \frac{r^n}{w(B_r)} \left( \int_{B_1} \tilde{G}(x, z) w(rz) \, dz \right) r^{2-n}.
\]

Then if we set

\[
u(x) = \int_{B_1} \tilde{G}(x, z) w(rz) \, dz
\]

then \( (L_r - V_r^+)u = -\chi_{B_1} w_r \) \((w_r(z) = w(rz))\). Since \( u/\partial B_8 = 0 \) then we have

\[
u(x) = \int_{B_1} G_{L_r, B_8}(x, y) w_r(y) \, dy
\]

\[- \int_{B_1} G_{L_r, B_8}(x, y) V_r^+(y) u(y) \, dy ,
\]
where \( G_{Lr,B_8} \) is the Green's function to \( L_r \) in \( B_8 \). By the estimates (1.3) for \( G_{Lr,B_8} \) we have

\[
\int_{B_1} G_{Lr,B_8}(x,y)w_r(y)\,dy \geq C \int_{B_1} \left( \int_{|x-y|}^8 \frac{s}{w_r(B_s(x))} \,ds \right) w_r(y)\,dy \\
= C \int_{B_1} \int_0^8 \chi_{[|x-y|,8]}(s) \frac{s}{w_r(B_s(x))} \,ds w_r(y)\,dy \\
= C \int_0^8 \frac{s}{w_r(B_s(x))} \int_{B_1} \chi_{[|x-y|,8]}(s)w_r(y)\,dy \,ds \\
= C \int_0^8 \frac{s}{w_r(B_s(x))} w_r(B_1 \cap B_s(x)) \,ds \\
\geq C \int_0^8 \frac{s}{w_r(B_s(x))} w_r(B_1 \cap B_s(x)) \,ds.
\]

If \( x \in B_3 \) and \( s > 4 \) then \( B_1 \subset B_s(x) \) and by doubling we have

\[
\inf_{x \in B_3} \int_{B_1} G_{Lr,B_8}(x,y)w_r(y)\,dy \geq C_{\independent \text{of } r}.
\]

Also by (1.2) we have

\[
\int_{B_1} G_{Lr,B_8}(x,y)V^+(y)u(y)\,dy \leq \|u\|_{L^\infty(B_1)}.
\]

Now by Lemma (3.5) (with \( f = \chi_{B_1} w_r \)) we have \( \|u\|_{L^\infty(B_1)} \leq C \) with \( C = C(\lambda, n, \eta, c_0) \). This implies that there exist \( \delta, C > 0 \) depending only on the parameters such that if

\[
\sup_{B_4} \int_{B_4} |V_r(y)| \int_{|x-y|}^8 \frac{s}{w(B_s(x))} \,ds \,dy \leq \delta
\]

then we have

\[
\inf_{B_3} u \geq C_1.
\]

Consequently for \( 1 < q < n\mu/(n\mu - 2) \) we have

\[
\int_{B_r} G_r(x,y)^q w(y)\,dy = \frac{r^n}{w(B_r)} \left( \int_{|z| \leq 1} \tilde{G}(x,z)^q w_r(z) \,dz \right) r^{(2-n)q} \frac{u(x)^q}{u(x)^q} \\
\leq C \left[ \frac{r^n}{w(B_r)} \right]^{1-q} \left( \int_{B_r} G_r(x,y)w(y)\,dy \right)^q \\
\times \left( \int_{|z| \leq 1} \tilde{G}(x,z)^q w_r(z) \,dz \right).
\]

By Remark (3.7) we have

\[
\int_{|z| \leq 1} \tilde{G}(x,z)^q w_r(z) \,dz \leq \frac{c}{w_r(B_1)^{q-1}}, \]

and since \( w_r(B_1) = r^{-n}w(B_r) \) then the lemma follows.
We are now in a position to show

**Theorem (4.5).** Let $u$ be a nonnegative solution of $Lu - Vu = 0$ in $\Omega$. There exist positive constants $r_0 = r_0(\lambda, \eta, n, c_0)$, $p_0 = p_0(\lambda, c_0)$ and $C = C(\lambda, \eta, n, c_0)$ all independent of $u$ such that

$$\left( \int_{B_r} u^{p_0} w(y) \, dy \right)^{1/p_0} \leq C \inf_{B_{r/2}} u$$

for $B_{8r} \subset \Omega$ and $r \leq r_0$.

**Proof.** By translation and dilation we can assume $\Omega = B_2(0)$ and as in the proof of Theorem (3.8) we can assume $r = 1$ and all balls are centered at 0. We show that if $u \geq 1$ on a closed set $\Gamma \subset B_1$ in the sense of $H^1(B_2)$ then we have

$$\inf_{B_{1/2}} u \geq C \left[ \frac{w(\Gamma)}{w(B_1)} \right]^M ,$$

where $C$ and $M$ only depend on $\lambda, \eta$ and $c_0$. Set

$$z(x) = \int_{\Gamma} G_{L-V^+, B_2}(x, y) w(y) \, dy ,$$

then $(L - V^+)z = -\chi_{\Gamma}w$ in $B_2$. Also by Remark (3.7) we have

$$z(x) \leq \left( \int_{B_2} G_{L-V^+, B_2}(x, y)^q w(y) \, dy \right)^{1/q} w(\Gamma)^{1/q} \leq C_1 , \quad \text{a.e. in } B_2 ,$$

here $C_1$ only depends on $\lambda, n, \eta$ and $c_0$. Consequently $(1/C_1)z(x) \leq 1$ in the $H^1$ sense in $B_2$ and therefore $(1/C_1)z(x) \leq 1$ in the $H^1$ sense in $\Gamma$. Then $u(x) \geq (1/C_1)z(x)$ in $H^1$ sense in $\Gamma$. Since $z/\partial B_2 = 0$ and $u \geq 0$ in $B_2$ a.e. then we have $u \geq (1/C_1)z$ in the $H^1$ sense in $\partial B_2$. Also $(L - V^+)(u - (1/C_1)z) = -\chi_{\Gamma}w \leq 0$, in $B_2 \setminus \Gamma$, then by the maximum principle we have $u(x) \geq (1/C_1)z(x)$ in $H^1$ sense in $B_2$ and consequently a.e. in $B_2$. Now Lemma (4.4) implies that there exists $C > 0$ and $M > 0$ such that

$$\int_{\Gamma} G_{L-V^+, B_2}(x, y) w(y) \, dy \geq C \left[ \frac{w(\Gamma)}{w(B_1)} \right]^M \int_{B_{1/4}} G_{L-V^+, B_2}(x, y) w(y) \, dy .$$

By the argument used to prove Lemma (4.4) and (1.2) we obtain

$$\inf_{x \in B_{1/2}} \int_{B_{1/4}} G_{L-V^+, B_2}(x, y) w(y) \, dy \geq C ,$$

where $C = C(\lambda, n, \eta, c_0)$. This implies (4.6). We claim that

$$\inf_{B_{1/4}} u \geq C \left[ \frac{w\{x \in B_1 : u(x) \geq 2 \text{ a.e.}\}}{w(B_1)} \right]^M .$$

In fact, since $u \in H^1(B_{3/2})$ there is a sequence $u_n \in \text{Lip}(B_{3/2})$ such that $u_n \to u$ in the $H^1(B_{3/2})$ sense and a.e. Let $F = \{x \in B_1 : u(x) \geq 2 \text{ a.e.}\}$,
then by Egorov’s theorem given $\varepsilon > 0$ there exist a closed set $F_\varepsilon \subset F$ such that $w(F - F_\varepsilon) < \varepsilon$ and $u_n \to u$ uniformly in $F_\varepsilon$. Therefore $u \geq 1$ in the $H^1(B_{3/2})$ sense in $F_\varepsilon$ and then

$$\inf_{B_{1/2}} u \geq C \left[ \frac{w(F_\varepsilon)}{w(B_1)} \right]^M \geq C \left[ \frac{w(F) - \varepsilon}{w(B_1)} \right]^M.$$ 

Now letting $\varepsilon \to 0$ the claim follows. By taking $u/t$ we obtain that

$$\inf_{B_{1/2}} u \geq C t \left[ \frac{w\{x \in B_1 : u(x) \geq 2t \text{ a.e.}\}}{w(B_1)} \right]^M,$$

and consequently the theorem follows for $0 < p_0 < 1/M$.

**References**


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