HARNACK'S INEQUALITY FOR DEGENERATE SCHRÖDINGER OPERATORS

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Abstract. We prove a Harnack inequality for nonnegative weak solutions of certain Schrödinger equations of the form $L u - V u = 0$ where $L$ is a second order degenerate elliptic operator in divergence form and $V$ is a potential in certain class.

1. Introduction

The purpose of this paper is to establish a Harnack inequality for nonnegative weak solutions of certain degenerate Schrödinger equations of the form

$$L u - V u = \sum_{i,j=1}^{n} D_{x_{i}}(a_{ij}(x)D_{x_{j}}u) - Vu = 0,$$

$x \in \mathbb{R}^{n}$. The coefficients $a_{ij}$ are measurable real-valued functions, the coefficient matrix $a = (a_{ij})$ is symmetric and

$$\lambda^{-1} w(x)|\xi|^{2} \leq \langle a(x)\xi, \xi \rangle \leq \lambda w(x)|\xi|^{2},$$

where $\lambda > 0$, $\xi = (\xi_{1}, \ldots, \xi_{n})$, $\langle , \rangle$ is the Euclidean inner product and $w$ is a weight satisfying either

(i) $w \in A_{2}$, that is

$$\sup_{B} \left( \frac{1}{|A|} \int_{B} w(x) \, dx \right) \left( \frac{1}{|A|} \int_{B} w(x)^{-1} \, dx \right) = c_{0} < \infty,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^{n}$ and $\int_{B} w(x) \, dx$ denotes the average of $w$ over $B$. The constant $c_{0}$ is referred to as the $A_{2}$ constant of $w$; or

(ii) $w(x) = |f'(x)|^{2-2/n}$, where $f : \mathbb{R}^{n} \to \mathbb{R}^{n}$ is a quasiconformal mapping and $|f'(x)|$ denotes the absolute value of the Jacobian determinant of $f$. Quasiconformal means $f = (f_{1}, \ldots, f_{n})$ is one-to-one, the distributional derivatives of $f_{i}$ belong to $L^{n}_{\text{loc}}(\mathbb{R}^{n})$ and there is a constant $C_{0} > 0$, called the

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dilation constant of $f$, such that a.e.

$$
\left[ \sum_{i,j=1}^{n} (D_{ij}f_i)^2 \right]^{1/2} \leq C_0 |f'(x)|^{1/n}.
$$

We shall assume that the potential $V$ satisfies the following condition

(1.2) \quad \lim_{\delta \to 0} \sup_{B_{\delta/4}(x_0)} \int_{|x-y|<\delta} |V(y)| \int_{|x-y|}^{R} \frac{s^2}{w(B_s(x))} \frac{ds}{s} dy = 0,

for every $x_0 \in \mathbb{R}^n$ and $R > 0$, $B_s(x)$ denotes the ball centered at $x$ with radius $s$.

Let $\Omega$ be an open, bounded and connected set in $\mathbb{R}^n$. We say that the function $u$ is a weak solution of $Lu - Vu = 0$ in $\Omega$ if $u \in H_{\text{loc}}^1(\Omega, w)$ and

$$
-\int \langle a(x) \nabla u(x), \nabla \psi(x) \rangle dx = \int V(x)u(x)\psi(x) dx
$$

for every $\psi \in H_0^1(\Omega, w)$ (see definitions in §2).

Given $\Omega$ an open bounded subset of $\mathbb{R}^n$ let $B_R$ be the smallest ball containing $\Omega$. If $\eta$ is a nondecreasing function defined for $r > 0$ and such that $\lim_{r \to 0} \eta(r) = 0$ then we set

$$
K_\eta = \left\{ V: \sup_{x \in B_R} \int_{|x-y| \leq r} |V(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B_s(x))} \frac{ds}{s} dy \leq \eta(r), \ r > 0 \right\}.
$$

By $c_0$ we denote either the $A_2$ constant of $w$ or the dilation constant of $f$ if $w$ satisfies (ii). The main result is the following:

**Theorem.** Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$, $w$ is a weight satisfying (i) or (ii) and $V \in K_\eta$. Then there exist positive constants $r_0$ and $\gamma$ only depending on $\lambda, n, c_0, \Omega$ and $\eta$ such that if $u$ is any nonnegative solution of (1.1) in $\Omega$ then for any ball $B_r$ with $B_{8r} \subset \Omega$ and $0 < r \leq r_0$ we have

$$
\sup_{B_{r/2}} u \leq \gamma \inf_{B_{r/2}} u.
$$

The theorem in the nondegenerate case, i.e. $w \equiv 1$, was obtained by Chiarenza, Fabes and Garofalo in [1]. In this case (1.2) means that $V$ belongs to the Kato-Stummel class. In the degenerate case (1.2) is suggested by the following approximate formula for the Green's function $G_L(x,y)$ for $L$ in $B_R(x_0)$ valid when $w$ satisfies (i) or (ii),

(1.3) \quad G_L(x,y) \simeq \int_{|x-y|}^{R} \frac{s^2}{w(B_s(x))} \frac{ds}{s},

for $x, y \in B_{R/4}(x_0)$ (see [4] for a proof of this formula). It is easy to see that if $V/w \in L^p_w$ locally for $p > (n/2)\mu$ then $V$ satisfies (1.2). Here $\mu$ means the doubling order of $w$, i.e. $w \in D_\mu$ (see §2 for definitions).
The proof of our Theorem is based on the method developed in [1] and [6] which basically consists of estimating powers of the solution $u$. One of the ingredients used in the proof is a weighted interpolation inequality (Lemma (3.3)) having some independent interest.

As in the nondegenerate case our result implies the continuity of solutions.

The paper is organized as follows: in §2 we state some preliminary definitions and results, in §3 we show an $L^\infty$-estimate for solutions and in §4 we establish some properties of the Green's function for $L - V$ and the infimum estimate.

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2. Preliminaries

$\Omega$ denotes a bounded, open and connected subset of $\mathbb{R}^n$. $L^p(\Omega, w)$ denotes the class of functions $f$ such that

$$
\|f\|_p^p = \int_\Omega |f(x)|^p w(x) \, dx < \infty.
$$

$Lip(\overline{\Omega})$ denotes the class of Lipschitz functions in $\overline{\Omega}$. We say $\psi \in Lip_0(\Omega)$ if $\psi \in Lip(\overline{\Omega})$ and $\psi$ has compact support contained in $\Omega$. For $\psi \in Lip(\overline{\Omega})$ we define the norm

$$(2.1) \quad \int_\Omega |\psi(x)|^2 w(x) \, dx + \int_\Omega |\nabla \psi(x)|^2 w(x) \, dx.$$  

$H^1(\Omega, w)$ denotes the closure of $Lip(\overline{\Omega})$ under the norm (2.1). $H^1_0(\Omega, w)$ denotes the closure of $Lip_0(\Omega)$ under the norm (2.1). $H^{-1}(\Omega, w)$ denotes the dual space of $H^1_0(\Omega, w)$. When $w$ satisfies (i) or (ii) and $u \in H^1(\Omega, w)$ the gradient of $u$ is uniquely defined (see [5, §2]). It can be shown (see [4, p. 154]) that

$$H^{-1}(\Omega, w) = \{ f_0 - \text{div} \vec{f} : \vec{f} = (f_1, \ldots, f_n), f_i, f_i/w \in L^2(\Omega, w) \},$$

where $i = 0, 1, \ldots, n$.

We say $u \in H^1_{\text{loc}}(\Omega, w)$ if $u \in H^1(\Omega', w)$ for every $\Omega'$ with closure contained in $\Omega$. Let $u \in H^1(\Omega, w)$, $E \subset \overline{\Omega}$, then $u \geq 0$ on $E$ in the sense of $H^1(\Omega, w)$ if there exists a sequence $u_n \in Lip(\overline{\Omega})$ such that $u_n(x) \geq 0$ for $x \in E$ and $u_n \to u$ in $H^1(\Omega, w)$. If $w$ satisfies (i) or (ii) then Poincaré’s inequality holds, i.e. there exist constants $C$ and $\tau > 1$ depending only on $c_0$ such that

$$(2.2) \quad \left( \int_B |u - u_B|^{2\tau} w(x) \, dx \right)^{1/2\tau} \leq C |B|^{1/n} \left( \int_B |\nabla u|^2 w(x) \, dx \right)^{1/2}$$

for all $u \in H^1(\Omega, w)$, $u_B = \frac{1}{w(B)} \int_B u \, dx$. Also, if $u \in H^1_0(\Omega, w)$ we have Sobolev’s inequality

$$(2.3) \quad \left( \int_B |u|^2 w(x) \, dx \right)^{1/2} \leq C |B|^{1/n} \left( \int_B |\nabla u|^2 w \, dx \right)^{1/2}.$$
For a proof of (2.2) and (2.3) see [5]. We say that the weight \( w \) satisfies a doubling condition of order \( \mu \) if there exists a constant \( C > 0 \) such that
\[
   w(B_{t r}(x_0)) \leq C t^{n \mu} w(B_r(x_0))
\]
for every \( x_0 \in \mathbb{R}^n, \ r > 0 \) and \( t \geq 1 \). In this case we write \( w \in D_\mu \). It is well known that if \( w \) satisfies (i) or (ii) in §1 then \( w \in D_\mu \) for some \( \mu \geq 1 \).

3. THE L^\infty-ESTIMATE

In this section we will show the following

Theorem (3.1). Given \( p > 0 \) there exist positive constants \( r_0 \) and \( C \) only depending on \( p, \lambda, n, \eta \) and \( \Omega \) such that if \( u \) is any solution of \( Lu - Vu = 0 \) in \( \Omega \) and \( B_r \) is any ball with \( r \leq r_0 \) and \( B_{2r} \subset \Omega \) then we have
\[
   \sup_{B_{r/2}} |u| \leq C \left( \int_{B_r} |u|^p w(x) \, dx \right)^{1/p}.
\]

The proof will be a consequence of the following lemmas.

Lemma (3.2). Let \( u \) be a solution of \( Lu - Vu = 0 \) in \( \Omega \). Then there exists a positive constant \( C = C(\lambda, n, c_0, \eta, \Omega) \) such that if \( 0 < s < t \) and \( B_t \subset \Omega \) then we have
\[
   \int_{B_t} |\nabla u(x)|^2 w(x) \, dx \leq C \left( \frac{1}{(t-s)^2} \int_{B_s} u(x)^2 w(x) \, dx \right).
\]

Proof. Take \( \phi \in C^\infty_0(B_t) \) such that \( 0 \leq \phi \leq 1 \), \( \phi \equiv 1 \) on \( B_s \) and \( \|
abla \phi\|_\infty \leq \frac{C}{t-s} \). We have
\[
   \int |\nabla u(x)|^2 \phi(x)^2 w(x) \, dx \leq \lambda \int \langle a(\nabla u), \nabla \phi \rangle \phi(x)^2 \, dx
   = \lambda \int \langle a(\nabla u), \nabla (u\phi^2) \rangle \, dx - 2\lambda \int \langle a(\nabla u), \nabla \phi \phi(x) u(x) \rangle \, dx
   = -\lambda \int u(x)^2 \phi(x)^2 V(x) \, dx - 2\lambda \int \langle a(\nabla u), \nabla \phi \phi(x) u(x) \rangle \, dx.
\]
To estimate the second term we use the fact that for every \( \epsilon > 0 \) we have
\[
   |\langle a(\phi \nabla u), u \nabla \phi \rangle| \leq \frac{\epsilon}{2} \langle a(\phi \nabla u), \phi \nabla u \rangle + \frac{1}{2\epsilon} \langle a(\nabla \phi), u \nabla \phi \rangle.
\]
Hence by taking \( \epsilon \lambda^2 = 1/2 \) we obtain
\[
   \int |\nabla u|^2 \phi^2 w \, dx \leq -2\lambda \int u^2 \phi^2 V \, dx + 4\lambda^4 \int u^2 |\nabla \phi|^2 w \, dx.
\]
To estimate the second integral in the last inequality we use the following embedding lemma. (For a proof of this lemma in the unweighted case see [9, p. 138].)
Lemma (3.3). Let \( \Omega \) be an open, bounded and connected set in \( \mathbb{R}^n \) and let \( V \) be a potential satisfying (1.2). Then given \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon = C(\varepsilon, \Omega, w, n, V, \lambda) \) such that for any \( u \in H^0_0(\Omega, w) \) we have

\[
\int u^2(x)|V(x)| \, dx \leq \varepsilon \int |\nabla u|^2 w(x) \, dx + C_\varepsilon \int u^2 w(x) \, dx.
\]

Proof. Given \( D \) an open and bounded set in \( \mathbb{R}^n \) let \( B_R \) be a ball such that \( D \subseteq B_{R/4} \) and let \( G(x, y) \) be the Green's function for \( L \) in \( B_R \). We define

\[
\eta_D(s) = \sup_{x \in D} \int_{|x-y|<s} |V(y)|G(x, y) \, dy.
\]

By (1.3) we have that (1.2) is equivalent to

\[
\lim_{s \to 0} \eta_D(s) = 0
\]

for every bounded and open set \( D \subseteq \mathbb{R}^n \). It is enough to prove the lemma for \( u \in \textrm{Lip}_0(\Omega) \), then the desired result follows by passing to the limit. Let us suppose first that \( u \) has support contained in a ball \( B_r \subseteq \Omega \), then we claim that for every \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) independent of \( u \) and \( V \) such that

\[
\int u^2 V \, dx \leq \varepsilon \int |\nabla u|^2 w \, dx + C_\varepsilon \left[ \eta_{B_r}(3r) \int u^2 V \, dx + \eta_{B_r}(3r)^2 \int |\nabla u|^2 w \, dx \right].
\]

To prove the claim assume \( V \) has support contained in \( B_r \), \( V \geq 0 \), and set

\[
F(x) = \int V(y)G(x, y) \, dy
\]

where \( G \) is the Green's function of a ball \( B \) such that \( \frac{1}{4}B \supset \Omega \). Then \( F \) is the solution of \( LF = -V \) in \( B \) and \( F \in H^1_0(B, w) \). This is because since \( V \) satisfies (1.2) then

\[
\int_B \int_B G(x, y)V(x)V(y) \, dx \, dy < \infty,
\]

and therefore as in the proof of Theorem 4.8 of [4] we can conclude that \( \chi_B V \in H^{-1}(B, w), (\chi_B \textrm{ denotes the characteristic function of } B) \). Therefore

\[
\int u^2 V \, dx = \int \langle a(\nabla F), \nabla u^2 \rangle \, dx = 2 \int \langle a(u\nabla F), \nabla u \rangle \, dx
\]

\[
\leq \varepsilon \int \langle a(\nabla u), \nabla u \rangle \, dx + \frac{1}{\varepsilon} \int \langle a(u\nabla F), u\nabla F \rangle \, dx
\]

\[
\leq \lambda \varepsilon \int |\nabla u|^2 w(x) \, dx + \frac{\lambda}{\varepsilon} \int |\nabla F|^2 u^2 w \, dx.
\]
Now observe that
\[\int |\nabla F|^2 u^2 w \, dx \leq \lambda \int \langle a(\nabla F), \nabla F \rangle u^2 \, dx = \lambda \int \langle (\nabla F), (\nabla (Fu^2)) \rangle \, dx - 2\lambda \int \langle a(\nabla F), \nabla u \rangle uF \, dx \leq \lambda \int u^2 F(x) V(x) \, dx + \lambda \delta \int |\nabla F|^2 u^2 w \, dx + \frac{\lambda}{\delta} \int F^2 |\nabla u|^2 w \, dx\]
for every \(\delta > 0\). Note that for \(x \in B_r\), \(F(x) \leq \eta_{B_r}(3r)\) and then by taking \(\delta = (2\lambda)^{-1}\) we get
\[\int |\nabla F|^2 u^2 w \, dx \leq 2\eta_{B_r}(3r)\lambda \int u^2 V \, dx + 4\lambda^2 \eta_{B_r}(3r)^2 \int |\nabla u|^2 w \, dx.\]
The claim follows with \(C_\varepsilon = 4\lambda^4 / \varepsilon\).

To complete the proof of the lemma, given \(0 < \delta < 1\), let \(\{\psi_j\}_{j=1}^N\) be a finite partition of unity of \(\Omega\) such that \(\text{supp} \psi_j \subseteq B_{r_j}(x_j)\) with \(x_j \in \Omega\) and \(0 < r_j \leq \delta\). Set \(\Omega = \{x: d(x, \Omega) \leq 1\}\). Therefore
\[\int (u \psi_j)^2 V \, dx \leq \frac{\varepsilon}{2} |\nabla (u \psi_j)|^2 w \, dx + C_\varepsilon \left[\eta_{B_{r_j}}(3r_j) \int (\psi_j u)^2 V \, dx + \eta_{B_{r_j}}(3r_j)^2 \int |\nabla (u \psi_j)|^2 w \, dx\right] \leq \frac{\varepsilon}{2} \int |\nabla u|^2 \psi_j^2 w \, dx + \frac{\varepsilon}{2} \int u^2 |\nabla \psi_j|^2 w \, dx + C_\varepsilon \left[\eta_{\Omega}(3\delta) \int (\psi_j u)^2 V \, dx + \eta_{\Omega}(3\delta)^2 \int |\nabla \psi_j|^2 u^2 w \, dx\right].\]
We now choose \(\delta = \delta(\varepsilon) < 1\) such that
\[C_\varepsilon \eta_{\Omega}(3\delta) < \frac{1}{2} \quad \text{and} \quad C_\varepsilon \eta_{\Omega}(3\delta)^2 < \frac{\varepsilon}{2}.
\]
Hence
\[\frac{1}{2} \int (u \psi_j)^2 V \, dx \leq \varepsilon \int |\nabla u|^2 \psi_j^2 w \, dx + \varepsilon \int u^2 |\nabla \psi_j|^2 w \, dx.\]
By summing in \(j\) it follows that
\[\frac{1}{2} \int u^2 V \, dx \leq \varepsilon \int |\nabla u|^2 w \, dx + \varepsilon \frac{N(\varepsilon)}{\delta(\varepsilon)^2} \int u^2 w \, dx.\]

**Remarks.** (1) The constant \(C_\varepsilon\) only depends on \(\varepsilon, \Omega, \eta, n, c_0\) and \(\lambda\).
(2) By Sobolev’s inequality (2.3) Lemma (3.3) implies the following two-weights Sobolev inequality
\[\int u^2(x) V(x) \, dx \leq C \int |\nabla u|^2 w(x) \, dx.\]
Lemma (3.4). There exists a constant $C = C(n, \lambda, \eta, \Omega, c_0)$ such that if $u$ is a solution of $Lu - Vu = 0$ in $\Omega$ and $B_{2r}(x_0) \subset \Omega$ then

$$
\left( \int_{B_{2r}(x_0)} u^2 w \, dx \right)^{1/2} \leq C \int_{B_r(x_0)} |u| w \, dx.
$$

Proof. We claim that it is enough to prove the lemma when $r = 1$ and $x_0 = 0$. In fact, if $u_{x_0}(x) = u(x - x_0)$ is defined in $\Omega + x_0$ and if $a_{x_0}(x) = a(x - x_0)$, $V_{x_0}(x) = V(x - x_0)$ and $w_{x_0}(x) = w(x - x_0)$ then $u_{x_0}$ is a solution of $\text{div}(a_{x_0}(x) \nabla) - V_{x_0} = 0$ in $\Omega + x_0$. Note that the constant $c_0$ of $w_{x_0}$ does not change and $V_{x_0}$ is in $K_\eta$ defined with $w_{x_0}$. Therefore by translations we can assume $x_0 = 0$. Set $u_r(x) = u(rx)$, then $u_r$ is defined in $\frac{1}{r}\Omega$ (in particular in $B_2$) and if we set $a_r(x) = a(rx)$, $V_r(x) = r^2 V(rx)$ and $w_r(x) = w(rx)$ then $u_r$ is a solution of $\text{div}(a_r(x) \nabla) - V_r = 0$ in $\frac{1}{r}\Omega$. Note again that the constant $c_0$ of $w_r$ does not change. Also by changing variables is easy to see that

$$
\sup_{x \in 1/r\Omega} \int_{|x-y|<\delta} |V_r(y)| \int_{|x-y| \leq s} \frac{s}{w_r(B_{4r}(x))} \, ds \, dy,
$$

which if $r \leq 1$ implies that $V_r$ belongs to the class $K_\eta$ defined with $w_r$. Let us assume $\int_{B_1} |u|w \, dx = 1$ and for $\frac{1}{2} < s < 1$ consider

$$
I(s) = \left( \frac{1}{w(B_{1/2})} \int_{B_1} u^2 w \, dx \right)^{1/2}.
$$

If $I(\frac{1}{2}) \leq 1$ then there is nothing to show, so suppose $I(\frac{1}{2}) > 1$. We want to show $I(\frac{1}{2}) \leq C$, $C$ only depends on $n, \lambda, \eta, \Omega$ and $c_0$. Let $\tau$ be the exponent in the Poincaré inequality and choose $0 < \theta < 1$ such that $(2-\theta)/(1-\theta) = 2\tau$. By doubling and Poincaré we have

$$
I(s) = \left( \frac{1}{w(B_{1/2})} \int_{B_1} |u|^{2-\theta} |u|^\theta w \, dx \right)^{1/2} \leq C \left( \int_{B_1} |u|^{(2-\theta)/(1-\theta)} w \, dx \right)^{(1-\theta)/2} \leq Cs^{(1-\theta)\tau} \left( \int_{B_1} |\nabla u|^2 w \, dx \right)^{(1-\theta)/2\tau} + C \left( \int_{B_1} |u|^2 w \, dx \right)^{(1-\theta)/2\tau}.
$$

If $\frac{1}{2} \leq s < t \leq 1$ then by Lemma (3.2) and doubling we obtain $I(s) \leq C[(t-s)^{-1} I(t)]^{(1-\theta)\tau}$ which implies $I(\frac{1}{2}) \leq C$. (See Lemma 1.2 of [1].)

Lemma (3.5). Let $\Omega = B_2(0)$, $w \in D_\mu$ and $p > (n/2)\mu$. There exist constants $\delta_0 = \delta_0(\lambda, n, \eta, c_0)$ and $C = C(\lambda, n, \eta, c_0)$ such that if

$$
\sup_{B_t} \int_{B_{2t}} |V(y)| \int_{|x-y| \leq s} \frac{s}{w(B_{4t}(x))} \, ds \, dy < \delta_0
$$

then...
then given \( f/w \in L^p_w(\Omega) \) there exists a unique \( u \in H^1_0(\Omega, w) \) such that \( Lu - Vu = f \) in \( \Omega \) and
\[
\|u\|_{L^\infty(\Omega)} \leq \frac{c\|f/w\|_{L^p_w(\Omega)}}{w(B_1)}\frac{1}{1/p}. 
\]

**Proof.** The bilinear form
\[
\alpha(u,v) = \int_\Omega (a(\nabla u), \nabla v) \, dx + \int_\Omega uv \, V \, dx
\]
is continuous and coercive in \( H^1_0(\Omega, w) \) provided \( \delta_0 \) is small enough. This follows by the claim made in the proof of Lemma (3.3). Now if \( p \geq 2 \) and \( f \in L^{p-\delta}_w(\Omega) \) implies \( f \in H^{-1}(\Omega, w) \) and consequently the existence and uniqueness of \( u \) is a consequence of the Lax-Milgram theorem, (see [7]).

Let \( u_0 \) be the solution of the problem \( Lu = f \) in \( \Omega \), \( u/\partial\Omega = 0 \) (i.e. \( u \in H^1_0(\Omega, w) \)), and for \( j \geq 1 \), let \( u_j \) be the solution of \( Lu - Vu_{j-1} = f \) in \( \Omega \), \( u/\partial\Omega = 0 \). Then we have
\[
u_0(x) = \int_\Omega G_L(x, y)f(y) \, dy,
\]
where \( G_L(x, y) \) is the Green's function of \( L \) in \( \Omega \). By the maximum principle
\[
G_L(x, y) \leq C \int_{|x-y|}^8 \frac{s}{w(B_1(x))} \, ds.
\]
It is easy to see that
\[
\left( \int_\Omega G_L(x, y)w(y) \, dy \right)^{1/p'} = \frac{C}{w(B_1(0))^{1/p}} = C_1 < \infty, \quad \text{for } p > \frac{n}{2} \mu.
\]
Then
\[
|u_0(x)| \leq C_1 \|f/w\|_{L^p_w(\Omega)}, \quad x \in \Omega.
\]
We also have
\[
u_1(x) = \int_\Omega G_L(x, y)f(y) \, dy + \int_\Omega G_L(x, y)V(y)u_0(y) \, dy
\]
which implies
\[
|u_1(x)| \leq C_1 \|f\|_{L^p_w(\Omega)} + C_1 \delta \|f\|_{L^{p-\delta}_w(\Omega)}.
\]
provided \( \sup_\Omega \int G_L(x, y)|V(y)| \, dy < \delta \). Continuing in this manner we obtain
(3.6) \( u_j(x) = \int_\Omega G_L(x, y)f(y) \, dy + \int_\Omega G_L(x, y)V(y)u_{j-1}(y) \, dy \)
and
\[
|u_j(x)| \leq C_1 \cdot C_\delta \|f/w\|_{L^p_w(\Omega)}, \quad \text{for } j = 2, 3, \ldots.
\]

We claim that \( u_j \) is a Cauchy sequence in \( H^1_0(\Omega, w) \) and \( u_j \to u \) in \( H^1_0(\Omega, w) \). By (3.6) we have
\[
u_{j+1}(x) - u_j(x) = \int_\Omega G_L(x, y)V(y)[u_j(y) - u_{j-1}(y)] \, dy
\]
and therefore
\[ \| u_{j+1} - u_j \|_{L^\infty(\Omega)} \leq \delta \| u_j - u_{j-1} \|_{L^\infty(\Omega)}. \]
Consequently for \( m > n \)
\[ \| u_m - u_n \|_{L^\infty(\Omega)} \leq \sum_{j=n}^{m-1} \| u_{j+1} - u_j \|_{L^\infty(\Omega)} \]
\[ \leq \| u_0 - u_1 \|_{L^\infty(\Omega)} \sum_{j=n}^{m-1} \delta^j. \]
Therefore for \( \delta < 1 \) \( \{ u_j \} \) is a Cauchy sequence in \( L^\infty(\Omega) \) and therefore in \( L^2(\Omega, w) \). Also
\[ \int_{\Omega} |\nabla (u_m - u_n)|^2 w(x) \, dx \leq \lambda \int_{\Omega} (a(\nabla (u_m - u_n)) \cdot \nabla (u_m - u_n)) \, dx \]
\[ = - \int (u_m - u_n)(u_{m-1} - u_{n-1}) V(x) \, dx \]
\[ \leq \left( \int |u_m - u_n|^2 V(x) \, dx \right)^{1/2} \left( \int |u_{m-1} - u_{n-1}|^2 V(x) \, dx \right)^{1/2}, \]
and since \( V \in L^1(\Omega) \) we have \( \| \nabla (u_m - u_n) \|_{L^2(\Omega, w)} \) tends to 0 as \( m, n \to \infty \).
Consequently \( u_j \to \bar{u} \) in \( H^1_0(\Omega, w) \) and by (3.6) \( \bar{u} = u \).

Remark (3.7). Lemma (3.5) implies the existence and integrability of the Green's function \( G(x, y) \) of \( L - V \). In fact, if \( p > (n/2)\mu \) then
\[ \int_{\Omega} G(x, y) w(y)^{(p-1)/p} f(y) w(y)^{(1-p)/p} \, dy \leq \frac{C}{w(B_1)^{(1-p)/p}} \| w^{(1-p)/p} \|_{L^q(\Omega)}, \]
which implies
\[ \left( \int_{\Omega} G(x, y) w(y) \, dy \right)^{1/q} = \frac{C}{w(B_1)^{1/p}} < \infty \quad \text{for} \ 1 < q < \frac{n\mu}{n\mu - 2} \]
and a.e. \( x \in \Omega \ (\frac{1}{p} + \frac{1}{q} = 1) \).

Theorem 3.8. There exist \( r_0 = r_0(\lambda, n, c_0, \eta) \) and for each \( p > 0 \) a constant \( C = C(p, \lambda, n, c_0, \eta) \) such that if \( u \) is a solution of \( Lu - Vu = 0 \) in \( \Omega \) and \( B_{2r} \subset \Omega \) with \( r \leq r_0 \) then
\[ \sup_{B_{r/2}} |u| \leq C \left( \frac{1}{J_{B_r}} |u(x)|^p w(x) \, dx \right)^{1/p}. \]

Proof. By translation we may assume \( B_r \) is centered at 0. As before \( u_r(x) = u(rx) \) is a solution in \( B_2 = B_2(0) \) of \( L_r u_r - V_r u_r = 0 \) where \( L_r \) and \( V_r \) are defined in Lemma (3.4). Also observe that
\[ \int_{|x-y|<\delta} |V_r(y)| \int_{|x-y|}^8 \frac{s}{w_r(B_s(x))} \, ds \, dy \]
\[ = \int_{|x-z|<\delta} \int_{|x-z|}^{8r} \frac{s}{w_r(B_s(z))} \, ds \, dy \leq \eta(\delta r). \]
Therefore
\[ \sup_{x \in B_2} \int_{|x-y|<\delta} |V_r(y)| \int_{|x-y|}^{\delta} \frac{s}{w_r(B_s(x))} \, ds \, dy \leq \eta(\delta r). \]

Since \( \eta(\delta r) \to 0 \) as \( \delta \to 0 \) then it is enough to show that if \( \delta_0 \) is the number specified in Lemma (3.5) and
\[ \sup_{B_2} \int_{B_6} |V(y)| \int_{|x-y|}^{8} \frac{s}{w(B_s(x))} \, ds \, dy < \delta_0 \]
then we have
\[ \sup_{B_1} |u| \leq C \left( \int_{B_1} |u|^p w(y) \, dy \right)^{1/p} \]
with \( C = C(\lambda, \eta, \rho, \nu, \psi_0) \).

Let \( G(x, y) \) be the Green's function of \( L - V \) in \( B_2 \). Given \( \frac{1}{4} \leq s < t \leq 1 \), let \( \psi \) be in \( C_0^\infty(B_{t-(t-s)/4}(0)) \) such that \( 0 \leq \psi \leq 1 \), \( \psi \equiv 1 \) on \( B_{t+(t-s)/2}(0) \) and \( |\nabla \psi| \leq C/(t-s) \). We have
\[ u(x)\psi(x) = \int_{B_2} (a(\nabla_y G(x, y)), \nabla \psi(y)) u(y) \, dy \]
\[ - \int_{B_2} (a(\nabla u), \nabla \psi) G(x, y) \, dy = J_1 - J_2. \]

\[ J_1 = \int_{B_{t-(t-s)/4}\setminus B_{t+(t-s)/2}} (a(\nabla G(x, \cdot)), \nabla \psi) u(y) \, dy \]
\[ \leq \left( \int_{B_{t-(t-s)/4}\setminus B_{t+(t-s)/2}} |\nabla_y G(x, y)|^2 w(y) \, dy \right)^{1/2} \]
\[ \times \left( \int_{B_{t-(t-s)/4}\setminus B_{t+(t-s)/2}} u^2 |\nabla \psi|^2 w(y) \, dy \right)^{1/2} \]
\[ \leq \frac{C}{t-s} \left( \int_{B_{t-(t-s)/4}\setminus B_{t+(t-s)/2}} |\nabla_y G(x, y)|^2 w(y) \, dy \right)^{1/2} \left( \int_{B_{t-(t-s)/4}} u^2 w(y) \, dy \right)^{1/2}. \]

Analogously
\[ J_2 \leq \frac{C}{t-s} \left( \int_{B_{t-(t-s)/4}\setminus B_{t+(t-s)/2}} |G(x, y)|^2 w(y) \, dy \right)^{1/2} \]
\[ \times \left( \int_{B_{t-(t-s)/4}\setminus B_{t+(t-s)/2}} |\nabla u|^2 w(y) \, dy \right)^{1/2}. \]

Now by Lemma (3.2)
\[ \int_{B_{t-(t-s)/4}\setminus B_{t+(t-s)/2}} |\nabla u|^2 w(y) \, dy \leq \int_{B_{t-(t-s)/4}} |\nabla u|^2 w(y) \, dy \]
\[ \leq \frac{C}{(t-s)^2} \int_{B_t} u^2 w(y) \, dy. \]
We cover the annulus $B_{t-(t-s)/4} \setminus B_{(t+s)/2}$ by a union of $N$ balls $B_{(t-s)/4}(z_i)$, with $|z_i| = (t+s)/2 + (t-s)/8$ (observe that the annulus has width $(t-s)/4$). Therefore if $x \in B_s$ then $x \notin B_{(t-s)/2}(z_i)$ and then for $x \in B_s$ we have

$$
\int_{B_{t-(t-s)/4} \setminus B_{(t+s)/2}} |\nabla_y G(x, y)|^2 w(y) \, dy \leq \sum_{i=1}^N \int_{B_{(t-s)/4}(z_i)} |\nabla G(x, y)|^2 w(y) \, dy
$$

$$
\leq \sum_{i=1}^N \frac{C}{(t-s)^2} \int_{B_{(t-s)/4}(z_i)} |G(x, y)|^2 \, dy
$$

$$
= \frac{C}{(t-s)^2} \sum_{i=1}^N w(B_{(t-s)/4}(z_i)) \int_{B_{(t-s)/2}(z_i)} |G(x, y)|^2 w(y) \, dy
$$

which by Lemma (3.4) is less than

$$
\frac{C}{(t-s)^2} \sum_{i=1}^N w(B_{(t-s)/4}(z_i)) \left( \int_{B_{(t-s)/2}(z_i)} G(x, y) w(y) \, dy \right)^2.
$$

Now by doubling we have $w(B_1(0)) (t-s)^n \mu \leq c \cdot w(B_{t-s}(z_i))$ and then by Remark (3.7) we obtain

$$
\left( \int_{B_{t-(t-s)/4} \setminus B_{(t+s)/2}} |\nabla G(x, y)|^2 w(y) \, dy \right)^{1/2} \leq \frac{C}{(t-s)^{1+(n/2)\mu}} \frac{1}{w(B_1)^{1/2}}.
$$

Analogously we have

$$
\int_{B_{t-(t-s)/4} \setminus B_{(t+s)/2}} G(x, y) y^2 w(y) \, dy
$$

$$
\leq C \sum_{i=1}^N w(B_{t-s}(z_i)) \left( \int_{B_{(t-s)/2}(z_i)} G(x, y) w(y) \, dy \right)^2.
$$

Collecting estimates we obtain

$$
\|u\|_{L^\infty(B_s)} \leq \frac{C}{(t-s)^{2+\mu}} \left( \frac{1}{w(B_1)} \int_{B_s} u^2 w(y) \, dy \right)^{1/2}.
$$

For $\frac{1}{2} \leq s \leq 1$ we set $I(s) = (1/w(B_1)) \int_{B_1} u^2 w \, dx)^{1/2}$. Let $p > 0$ and assume $\int_{B_1} u^p w \, dy = 1$, then if $p < 2$ we have

$$
I(s) \leq \left( \sup_{B_s} |u| \right)^\theta, \quad \theta = 1 - \frac{p}{2},
$$

and therefore

$$
I(s) \leq \frac{C}{(t-s)^{(2+(n/2)\mu)\theta}} I(t)^\theta.
$$

By the argument in [6, p. 1004] we obtain the theorem.
4. THE INFIMUM ESTIMATE

We begin with the following

**Lemma (4.1).** Let \( u \geq 0 \) be a solution of \( Lu - Vu = 0 \) in \( \Omega \). Then there exists a constant \( C = C(\lambda, \eta, c_0) \) independent of \( u \) such that if \( B_{4r} \subset \Omega \) then for every \( \varepsilon > 0 \) we have

\[
\frac{1}{w(B_r)} \int_{B_r} \left| \log(u + \varepsilon) - \log(u + \varepsilon) \right| w(y) \, dy \leq C.
\]

**Proof.** Let \( \psi \in C_0^\infty(B_{3r/2}) \), \( \psi \equiv 1 \) on \( B_r \) and \( 0 \leq \psi \leq 1 \), \( |\nabla \psi| \leq c/r \), \( B_r = B_r(x) \) and set \( u_\varepsilon = u + \varepsilon \). Then

\[
\int |\nabla \log u_\varepsilon|^2 w \, dy = \int \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} \psi^2 w \, dy
\]

\[
\leq \lambda \int \left( a(\nabla u_\varepsilon), \nabla u_\varepsilon \right) \frac{\psi^2}{u_\varepsilon^2} \, dy
\]

\[
= 2\lambda \int \left( a(\nabla u_\varepsilon), \nabla \psi \right) \frac{\psi}{u_\varepsilon} \, dy - \lambda \int \left( a(\nabla u_\varepsilon), \nabla \left( \frac{\psi^2}{u_\varepsilon} \right) \right) \, dy
\]

\[
= 2\lambda \int \left( a(\nabla u_\varepsilon), \nabla \psi \right) \frac{\psi}{u_\varepsilon} \, dy + \lambda \int V(y) \frac{u_\varepsilon}{u_\varepsilon} \psi^2 \, dy
\]

\[
\leq 2\lambda \int \left( a(\nabla u_\varepsilon), \nabla \psi \right) \frac{\psi}{u_\varepsilon} \, dy + \lambda \int |V(y)| \psi^2 \, dy.
\]

We have

\[
\int V(y) \psi^2(y) \, dy
\]

\[
\leq \int_{B_{2r}(x)} |V(y)| \left( \int_{|x-y|}^{4r} \frac{s}{w(B_s(x))} \, ds \right) \left( \int_{|x-y|}^{4r} \frac{s}{w(B_s(x))} \, ds \right)^{-1} \, dy.
\]

If \( y \in B_{2r} \) then by doubling we have

\[
\int_{|x-y|}^{4r} \frac{s}{w(B_s(x))} \, ds \geq C \cdot \frac{r^2}{w(B_r(x))}.
\]

Consequently by the assumption on \( V \) we have

\[
\int V(y) \psi^2(y) \, dy \leq C \frac{w(B_r(x))}{r^2}.
\]

Also as in the proof of Lemma (3.2) we have for every \( \delta > 0 \)

\[
\int \left( a(\nabla u_\varepsilon), \nabla \psi \right) \frac{\psi}{u_\varepsilon} \, dy \leq \frac{\lambda}{\delta} \int |\nabla \log u_\varepsilon|^2 \psi^2 w(y) \, dy + \delta \cdot C \frac{w(B_r(x))}{r^2}.
\]

Hence if \( \delta \) is large we have

\[
\int |\nabla \log u_\varepsilon|^2 w \, dy \leq C \frac{w(B_r(x))}{r^2}.
\]
Now since $\log u_\varepsilon \in H^1(B_{2r}, w)$ then by Poincaré the lemma follows.

**Remark (4.2).** Note that if $w \in A^\infty$ then by Theorem 5 of [8], Lemma (4.1) implies that $\log(u + \varepsilon) \in \text{BMO}, \varepsilon > 0$.

**Lemma (4.3).** Let $u > 0$ be a solution of $Lu - Vu = 0$ in $\Omega$ and let $r_0$ be the number in Theorem (3.8) then there exists a constant $C = C(\lambda, \eta, n, c_0)$ such that

$$\int_{B_r} u(x)w(x)dx \leq C \int_{B_r} u(x)w(x)dx$$

for $0 < r \leq r_0$ and $B_{8r} \subset \Omega$.

**Proof.** By Lemma (4.1) and Theorem 5 of [8] there exist $\delta > 0$ and $C > 0$ such that

$$\left(\frac{1}{w(B_{r/2})} \int_{B_{r/2}} u_\varepsilon^\delta w dx\right) \leq C,$$

for $B_{4r} \subset \Omega$, i.e. $u_\varepsilon^\delta \in A_2(w)$, for every $\varepsilon > 0$. Since $w$ is doubling this implies

$$\int_{B_{2r}} u_\varepsilon^\delta w dx \leq C \int_{B_r} u_\varepsilon^\delta w dx.$$

Hence by Theorem (3.8) and $\delta < 1$ we have

$$\frac{1}{w(B_{2r})} \int_{B_{2r}} uw dx \leq C \left(\frac{1}{w(B_{4r})} \int_{B_{4r}} u_\varepsilon^\delta w dx\right)^{1/\delta} \leq C \left(\frac{1}{w(B_{r/2})} \int_{B_{r/2}} u_\varepsilon^\delta w dx\right)^{1/\delta} \leq \frac{1}{w(B_r)} \int_{B_r} u_\varepsilon w dx,$$

by letting $\varepsilon \to 0$ this implies the lemma.

**Lemma (4.4).** Set $V^+ = \max(V, 0)$ and let $G(x, y)$ denote the Green's function of $L - V^+ = 0$ in $\Omega$. Then for any $1 < q < n\mu/(n\mu - 2)$ ($w \in D_\mu$) there exists a constant $C = C(q, \lambda, \eta, c_0)$ such that

$$\left(\frac{1}{w(B_r)} \int_{B_r} G(x, y)^q w(y) dy\right)^{1/q} \leq C \frac{1}{w(B_r)} \int_{B_r} G(x, y) w(y) dy,$$

for $B_{8r} \subset \Omega$ and $0 < r \leq r_0$ ($r_0$ is as in Theorem (3.8)).

**Proof.** First observe that $G(x, \cdot)$ is a solution of $L - V^+ = 0$ for $y \neq x$, $x \in \Omega$. Suppose first that $x \not\in B_{8r}$, then by Theorem (3.8) we have for every $q > 0$ that

$$\left(\int_{B_r} G(x, y)^q w(y) dy\right)^{1/q} \leq \sup_{y \in B_r} G(x, y) \leq C \int_{B_r} G(x, y) w(y) dy$$

which by Lemma (4.3) is less than

$$\int_{B_r} G(x, y) w(y) dy.$$
Now assume \( x \in B_{8r} \) and let \( G_r(x, y) = G_{L - V^+ . B_8}(x, y) \). By the maximum principle \( G_r(x, y) \leq G(x, y) \) for \( x, y \in B_{8r} \). Then

\[
\int_{B_r} G(x, y)^q w(y) \, dy \leq 2^q \left\{ \int_{B_r} [G(x, y) - G_r(x, y)]^q w(y) \, dy + \int_{B_r} G_r(x, y)^q w(y) \, dy \right\}.
\]

Since \( G(x, .) - G_r(x, .) \) is a nonnegative solution of \( L - V^+ = 0 \) in \( B_{8r} \), then arguing as before we obtain

\[
\int_{B_r} [G(x, y) - G_r(x, y)]^q w(y) \, dy < C \left\{ \int_{B_r} [G(x, y) - G_r(x, y)] w(y) \, dy \right\}^q.
\]

By translation we can assume that \( B_r \) is centered at 0. Let \( L_r \) and \( V^+_r \) be defined as in Lemma (3.4) and let \( \tilde{G}(x, y) \) be the Green’s function to \( L_r - V^+_r \) in \( B_8 \). Then for \( x \in B_{8r} \) and \( z \in B_8 \) we have

\[
G_r(x, rz) = r^{2-n} \tilde{G}(x, z).
\]

Therefore

\[
\int_{B_r} G_r(x, y)^q w(y) \, dy = \frac{r^n}{w(B_r)} \left( \int_{|z| \leq 1} \tilde{G}(x, z)^q w(rz) \, dz \right) r^{2-n/q}
\]

and

\[
\int_{B_r} G_r(x, y) w(y) \, dy = \frac{r^n}{w(B_r)} \left( \int_{B_1} \tilde{G}(x, z) w(rz) \, dz \right) r^{2-n}.
\]

Then if we set

\[
u(x) = \int_{B_1} \tilde{G}(x, z) w(rz) \, dz\]

then \( (L_r - V^+_r)u = -\chi_{B_1} w_r \) \((w_r(z) = w(rz))\). Since \( u/\partial B_8 = 0 \) then we have

\[
u(x) = \int_{B_1} G_{L_r . B_8}(x, y) w_r(y) \, dy
- \int_{B_1} G_{L_r . B_8}(x, y) V^+_r(y) u(y) \, dy,
\]
where $G_{L_r, B_8}$ is the Green's function to $L_r$ in $B_8$. By the estimates (1.3) for $G_{L_r, B_8}$ we have

$$
\int_{B_1} G_{L_r, B_8}(x, y) w_r(y) dy \geq C \int_{B_1} \left( \int_{|x-y|}^8 \frac{s}{w_r(B_s(x))} ds \right) w_r(y) dy
$$

$$
= C \int_{B_1} \int_0^8 \chi_{(|x-y|, 8)}(s) \frac{s}{w_r(B_s(x))} ds w_r(y) dy
$$

$$
= C \int_0^8 \frac{s}{w_r(B_s(x))} \int_{B_1} \chi_{(|x-y|, 8)}(s) w_r(y) dy ds
$$

$$
= C \int_0^8 \frac{s}{w_r(B_s(x))} w_r(B_1 \cap B_s(x)) ds
$$

$$
\geq C \int_0^8 \frac{s}{w_r(B_s(x))} w_r(B_1 \cap B_s(x)) ds.
$$

If $x \in B_3$ and $s > 4$ then $B_1 \subset B_s(x)$ and by doubling we have

$$
\inf_{x \in B_3} \int_{B_1} G_{L_r, B_8}(x, y) w_r(y) dy \geq C
$$

$C$ independent of $r$. Also by (1.2) we have

$$
\int_{B_1} G_{L_r, B_8}(x, y) V^+(y) u(y) dy \leq \|u\|_{L^\infty(B_1)}.
$$

Now by Lemma (3.5) (with $f = \chi_{B_1} w_r$) we have $\|u\|_{L^\infty(B_1)} \leq C$ with $C = C(\lambda, n, \eta, c_0)$. This implies that there exist $\delta, C > 0$ depending only on the parameters such that if

$$
\sup_{B_4} \int_{B_4} [V^+(y)] \int_{|x-y|}^8 \frac{s}{w(B_s(x))} ds dy \leq \delta
$$

then we have

$$
\inf_{B_4} u \geq C_1.
$$

Consequently for $1 < q < n\mu/(n\mu - 2)$ we have

$$
\int_{B_r} G_r(x, y)^q w(y) dy = \frac{r^n}{w(B_r)} \left( \int_{|z| \leq 1} \tilde{G}(x, z)^q w_r(z) dz \right) \left( \int_{B_r} G_r(x, y) w(y) dy \right)^q
$$

$$
\leq C \left[ \frac{r^n}{w(B_r)} \right]^{1-q} \left( \int_{B_r} G_r(x, y) w(y) dy \right)^q \times \left( \int_{|z| \leq 1} \tilde{G}(x, z)^q w_r(z) dz \right).
$$

By Remark (3.7) we have

$$
\int_{|z| \leq 1} \tilde{G}(x, z)^q w_r(z) dz \leq \frac{c}{w_r(B_1)^{q-1}},
$$

and since $w_r(B_1) = r^{-n} w(B_r)$ then the lemma follows.
We are now in a position to show

**Theorem (4.5).** Let \( u \) be a nonnegative solution of \( Lu - Vu = 0 \) in \( \Omega \). There exist positive constants \( r_0 = r_0(\lambda, \eta, n, c_0) \), \( p_0 = p_0(\lambda, c_0) \) and \( C = C(\lambda, \eta, n, c_0) \) all independent of \( u \) such that

\[
\left( \int_{B_r} u^{p_0} w(y) \, dy \right)^{1/p_0} \leq C \inf_{B_{r/2}} u
\]

for \( B_{3r} \subset \Omega \) and \( r \leq r_0 \).

**Proof.** By translation and dilation we can assume \( \Omega = B_2(0) \) and as in the proof of Theorem (3.8) we can assume \( r = 1 \) and all balls are centered at 0. We show that if \( u \geq 1 \) on a closed set \( \Gamma \subset B_1 \) in the sense of \( H^1(B_2) \) then we have

\[
\inf_{B_{1/2}} u \geq C \left[ \frac{w(\Gamma)}{w(B_1)} \right]^M,
\]

where \( C \) and \( M \) only depend on \( \lambda, \eta \) and \( c_0 \). Set

\[
z(x) = \int_{\Gamma} G_{L-V^+,B_2}(x,y)w(y) \, dy,
\]

then \( (L - V^+)z = -\chi_\Gamma w \) in \( B_2 \). Also by Remark (3.7) we have

\[
z(x) \leq \left( \int_{B_2} G_{L-V^+,B_2}(x,y)w(y) \, dy \right)^{1/q} w(\Gamma)^{1/q'} \leq C_1, \quad \text{a.e. in } B_2,
\]

here \( C_1 \) only depends on \( \lambda, n, \eta \) and \( c_0 \). Consequently \( (1/C_1)z(x) \leq 1 \) in the \( H^1 \) sense in \( B_2 \) and therefore \( (1/C_1)z(x) \leq 1 \) in the \( H^1 \) sense in \( \Gamma \). Then \( u(x) \geq (1/C_1)z(x) \) in \( H^1 \) sense in \( \Gamma \). Since \( z/\partial B_2 = 0 \) and \( u \geq 0 \) in \( B_2 \) a.e. then we have \( u \geq (1/C_1)z \) in the \( H^1 \) sense in \( \partial B_2 \). Also \((L - V^+)(u - (1/C_1)z) = -V^+u - (1/C_1)\chi_\Gamma w \leq 0\), in \( B_2 \setminus \Gamma \), then by the maximum principle we have \( u(x) \geq (1/C_1)z(x) \) in \( H^1 \) sense in \( B_2 \) and consequently a.e. in \( B_2 \). Now Lemma (4.4) implies that there exists \( C > 0 \) and \( M > 0 \) such that

\[
\int_{\Gamma} G_{L-V^+,B_2}(x,y)w(y) \, dy \geq C \left[ \frac{w(\Gamma)}{w(B_1)} \right]^M \int_{B_{1/4}} G_{L-V^+,B_2}(x,y)w(y) \, dy.
\]

By the argument used to prove Lemma (4.4) and (1.2) we obtain

\[
\inf_{x \in B_{1/2}} \int_{B_{1/4}} G_{L-V^+,B_2}(x,y)w(y) \, dy \geq C,
\]

where \( C = C(\lambda, n, \eta, c_0) \). This implies (4.6). We claim that

\[
\inf_{B_{1/4}} u \geq C \left[ \frac{w\{x \in B_1: u(x) \geq 2 \text{ a.e.}\}}{w(B_1)} \right]^M.
\]

In fact, since \( u \in H^1(B_{3/2}) \) there is a sequence \( u_n \in \text{Lip}(B_{3/2}) \) such that \( u_n \to u \) in the \( H^1(B_{3/2}) \) sense and a.e. Let \( F = \{x \in B_1: u(x) \geq 2 \text{ a.e.}\} \),
then by Egorov’s theorem given $\varepsilon > 0$ there exist a closed set $F_\varepsilon \subset F$ such that $w(F - F_\varepsilon) < \varepsilon$ and $u_n \to u$ uniformly in $F_\varepsilon$. Therefore $u \geq 1$ in the $H^1(B_{3/2})$ sense in $F_\varepsilon$ and then

$$\inf_{B_{1/2}} u \geq C \left[ \frac{w(T_{F_\varepsilon})}{w(B_1)} \right]^M \geq C \left[ \frac{w(F) - \varepsilon}{w(B_1)} \right]^M.$$ 

Now letting $\varepsilon \to 0$ the claim follows. By taking $u/t$ we obtain that

$$\inf_{B_{1/2}} u \geq Ct \left[ \frac{w\left\{ x \in B_1 : u(x) \geq 2t \text{ a.e.} \right\}}{w(B_1)} \right]^M,$$

and consequently the theorem follows for $0 < p_0 < 1/M$.

References


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