THE SPECTRUM OF THE SCHRÖDINGER OPERATOR

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ABSTRACT. We describe the negative spectrum of the Schrödinger operator with a singular potential. We determine the exact value of the bottom of the spectrum and estimate it from above and below. We describe the dependence of a crucial constant on the eigenvalue parameter and discuss some of its properties. We show how recent results of others are simple consequences of a theorem proved by the author in 1972.

1. INTRODUCTION

For $V(x) \geq 0$ in $L^{\text{loc}}(\mathbb{R}^n)$, the smallest constant $C_{\lambda}(V)$ which satisfies

$$
(1.1) \quad (Vu, u) \leq C_{\lambda}(V)(\|\nabla u\|^2 + \lambda^2\|u\|^2), \quad u \in C_0^\infty,
$$

is of importance in the study of the spectrum of the Schrödinger operator

$$
(1.2) \quad H = -\Delta - V
$$

We shall show that $-\lambda_0^2$ is the smallest point of the spectrum of $H$ if and only if, $\lambda_0$ is the smallest value of $\lambda \geq 0$ such that $C_{\lambda}(V) \leq 1$ (if $C_{\lambda}(V) > 1$ for all $\lambda \geq 0$, then the operator $H$ is not bounded from below; the smallest point in the spectrum is $-\infty$). In 1972 the author obtained an expression determining the exact value of $C_{\lambda}(V)$ (cf. [1, p. 498]). It is given by

$$
(1.3) \quad C_{\lambda}(V) = \inf_{\psi > 0} \sup_{x \in \mathbb{R}^n} \psi(x)^{-1} \int_{\mathbb{R}^n} V(y)\psi(y)G_{2,\lambda}(x-y)\,dy
$$

where $G_{2,\lambda}(x)$ is the Bessel potential of order 2. It is the kernel of the operator

$$
(1.4) \quad G_{2,\lambda}f = (\lambda^2 - \Delta)^{-1}f, \quad I_2 = G_{2,0}.
$$

In (1.3) one obtains an upper bound for $C_{\lambda}(V)$ by picking a particular function $\psi(x) > 0$, e.g., $\psi(x) \equiv 1$. One can improve the estimate by varying $\psi$.

The cases $\lambda = 0$ and $\lambda = 1$ have received much attention. In 1962 Mazya [2] showed that for $n > 2$, $C_0(V) \leq 1$ if

$$
(1.5) \quad \int_\epsilon V(x)\,dx \leq \frac{n-2}{4} \omega \text{cap}(\epsilon)
$$

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holds for all compact sets $e \subset \mathbb{R}^n$. Here $\omega$ is the surface area of the unit ball in $\mathbb{R}^n$ and $\text{cap}(e)$ is the Green capacity of $e$. More recently Adams [3] showed that
\begin{equation}
\int \left( \int_e |x-y|^{2-n} V(y) \, dy \right)^2 \, dx \leq C \int_e V(y) \, dy, \quad e \subset \mathbb{R}^n,
\end{equation}
implies a bound for $C_0(V)$. In [4], Fefferman and Phong show that
\begin{equation}
C_0(V) \leq C_p \sup_{\delta, x} \left( \delta^{2-p-n} \int_{|x-y|<\delta} V(y)^p \, dy \right)^{1/p}
\end{equation}
if $p > 1$. The proof of (1.7) given in [4] is rather long and involved. In the next section we shall show that it is a simple consequence of (1.3). In fact, we shall give a direct easy proof of (1.7) without involving the ideas of [4]. In [1] Kerman and Sawyer show that
\begin{equation}
C_\lambda(V) \sim \sup_{|Q| \leq 2^{-n}} \frac{\int_Q G_{1,\lambda}(x-y)V(y)\, dy}{\int_Q V(y)\, dy}
\end{equation}
where the supremum is taken over all dyadic cubes $Q \subset \mathbb{R}^n$. Previous to [1], sufficient conditions for (1.1) to hold for various values of $\lambda$ were obtained by Kato, Rollnik, Schechter, Simon (cf. [14, 15] for references). Other sets of sufficient conditions were recently obtained in [12 and 16]. These authors were apparently unaware of the results of [1] where a condition which is both necessary and sufficient is obtained.

In §3 we show that there is a constant $C_p$ depending only on $n$ and $p$ such that
\begin{equation}
C_\lambda(V) \leq C_p \|M_{2p,1/\lambda}[V^p]\|_\infty^{1/p}, \quad \lambda \geq 0,
\end{equation}
where
\begin{equation}
M_{\alpha,\delta}[V](x) = \sup_{r \leq \delta} r^{-\alpha-n} \int_{|y-x|<r} V(y) \, dy.
\end{equation}
This allows us to show that the lowest point $-\mu^2$ of the spectrum of the operator (1.2) satisfies
\[\mu^2 \leq \sup_{x, \delta} \left( 2C_p \left( \delta^{-n} \int_{|y-x|<\delta} V(y)^p \, dy \right)^{1/p} - \delta^{-2} \right)\]
which is another estimate of Fefferman-Phong [4]. In our estimate only one constant appears (the one from (1.9)) and can be readily estimated. In proving (1.9) we show that there is a constant $C_{s,q}$ depending only on $s, n$ and $q$ such that
\begin{equation}
\|I_{s,\delta}f\|_q \leq C_{s,q} \|M_{s,\delta}f\|_q
\end{equation}
where
\[I_{s,\delta}f(x) = \int_{|y-x|<\delta} |y-x|^{s-n} f(y) \, dy.\]
The estimate (1.10) is of interest in its own right. Our proof extends a method of Muckenhoupt-Wheeden [5]. As a consequence of (1.10) we obtain

\[(1.11) \|G_{s,\lambda}f\|_q \leq C'_{s,\lambda}\|M_{s,1/\lambda}f\|_q\]

where

\[G_{s,\lambda}f = (\lambda^2 - \Delta)^{-s/2}f.\]

In §4 we show that the constant \(C_\lambda(V)\) is continuous in \(\lambda\) in the interval \([0, \infty)\). Moreover

\[\mu^2 = \inf_{C_i(V) \leq 1} \lambda^2 C_\lambda(V) = \inf_{C_i(V) \leq 1} \lambda^2\]

\[= \sup_{C_i(V) > 1} \lambda^2 = \sup_{C_i(V) > 1} \lambda^2 C_\lambda(V).\]

From this it follows easily that

\[\sup_\lambda \lambda^2 [C_\lambda(V) - 1] \leq \mu^2 \leq \sup_\lambda \lambda^2 [2C_\lambda(V) - 1].\]

Next we show that if \(V\) is in the Muckenhoupt-Wheeden class \(A_\infty\) (cf. [5]), then

\[(1.12) C_\lambda(V) \leq N_\mu \|M_{2,1/\lambda}\|_\infty.\]

In §5 we show that the essential spectrum of \(H\) is the same as that of \(-\Delta\), i.e.,

\[(1.13) \sigma_e(H) = [0, \infty)\]

provided

(a) \(C_\lambda(V) \to 0\) as \(\lambda \to \infty\);

(b) \(C_{\lambda_0}(V^R) \to 0\) as \(R \to \infty\)

for some \(\lambda_0 \geq 0\), where

\[V^R(x) = \begin{cases} 0, & |x| \leq R, \\ V(x), & |x| > R. \end{cases}\]

2. A simple proof of the Fefferman-Phong estimate

We now show that (1.7) is a simple consequence of (1.3). Let

\[(2.1) M_{n}[V](x) = \sup_r r^{n-n} \int_{|y-x|<r} V(y) \, dy, \quad M = M_0,\]

denote the maximal function. The right-hand side of (1.7) is equivalent to

\[K_p\|M_{2p}[V^p]\|_\infty^{1/p}.\]

By Hölder's inequality

\[(2.2) M_1[V^{1/2}u] \leq M_q[V^{q/2}]^{1/q} M[|u|^{q'}]^{1/q'}.\]
holds for any \( q \geq 1 \), where \( 1/q + 1/q' = 1 \). If we take \( q = 2p > 2 \), we have

\[
\|M[V^{1/2}u]\|_2 \leq K_p^{1/2} \|M[u^{q'}]^{1/q'}\|_2 = K_p^{1/2} \|M[u^{q'}]\|^{1/q'}_2,
\]

since \( q' < 2 \). By a theorem of Muckenhoupt and Wheeden [5], this implies

\[
\|I_1[V^{1/2}u]\|_2 \leq C''K_p^{1/2} \|u\|_2,
\]

where \( I_1 = G_{s,0} \). Inequality (2.4) is equivalent to

\[
\|V^{1/2}I_2[V^{1/2}u]\|_2 \leq C''K_p \|u\|_2.
\]

If \( C > C''K_p \) and \( h > 0 \) is in \( L^2 \), then there is a \( \phi > 0 \) in \( L^2 \) such that

\[
\phi = h + C^{-1}V^{1/2}I_2[V^{1/2}\phi].
\]

This shows that the right-hand side of (1.3) is bounded by a constant times \( K_p \). Hence, (1.7) holds.

Another approach is to note that (2.4) is equivalent to

\[
\|V^{1/2}I_2 \nu\|_2 \leq C''K_p^{1/2} \|\nu\|_2
\]

which in turn is equivalent to

\[
(Vu, u) \leq C''K_p \|\nabla u\|^2
\]

which shows that (1.7) holds.

3. Estimates for arbitrary \( \lambda \)

For \( \mu \) a locally finite Borel measure, we define

\[
I_{s,\delta} \mu(y) = \int_{|x-y|<\delta} |x-y|^{s-n} \, d\mu(x), \quad 0 < s \leq n,
\]

and

\[
M_{s,\delta} \mu(y) = \sup_{r \leq \delta} \int_{|x-y|<r} |x-y|^{s-n} \, d\mu(x), \quad 0 \leq s \leq n,
\]

For \( 1 \leq q < \infty \) we let

\[
\|\mu\|_q = \left( \int_{\mathbb{R}^n} |\mu(x)|^q \, dx \right)^{1/q}
\]

by the norm in \( L^q(\mathbb{R}^n) \). Our first result is

**Theorem 3.1.** There is a constant \( C_{s,q} \) depending only on \( s, n \) and \( q \) such that

\[
\|I_{s,\delta} \mu\|_q \leq C_{s,q} \|M_{s,\delta} \mu\|_q.
\]

Moreover

\[
C_{s,q} \leq 2^{n-s+1} + (\omega/s)5^{n-s}n^{n/2}2^{(n+2-s)q+2s+2}.
\]

Before proving Theorem 3.1 we state some consequences.
Theorem 3.2. For each $p > 1$ there is a constant $C_p$ depending only on $n$ and $p$ such that

$$C_\lambda(V) \leq C_p \sup_x (M_{2p,1/\lambda} V^p)^{1/p}, \quad \lambda \geq 0.$$  

Moreover, there is a constant $C_1$ depending only on $n$ such that

$$C_\lambda(V) \geq C_1 M_{2,1/\lambda} V.$$  

Corollary 3.3. If $-\mu^2$ is the lowest point of the spectrum of the operator (1.2), then

$$\mu^2 \leq \sup_{\delta > 0} \left( 2C_p \delta^{-2} \sup_x (M_{2p,\delta} V^p)^{1/p} - \delta^{-2} \right)$$

$$= \sup_{x,\delta} \left( 2C_p \left( \delta^{-n} \int_{|y-x|<\delta} V(y)^p \, dy \right)^{1/p} - \delta^{-2} \right)$$

and

$$\mu^2 \geq \sup_{\delta} \left( C_1 \delta^{-n} \sup_x M_{2,\delta} V - \delta^{-2} \right)$$

$$= \sup_{x,\delta} \left( C_1 \delta^{-n} \int_{|y-x|<\delta} V(y) \, dy - \delta^{-2} \right).$$

Corollary 3.4. If $C_p M_{2p} V^p \leq 1$, then $\mu = 0$.

Corollaries 3.3 and 3.4 are proved by Fefferman and Phong [4]. Their proof is rather long and involved. They require two constants in (3.6) and do not provide a way of estimating them. Our proof is much shorter. They were unaware of the authors results in [1].

Proof of Theorem 3.1. Let

$$S_t = \{ x \in \mathbb{R}^n | I_{s,\delta} \, d\mu(x) < t \}$$

for each $t > 0$. If $S_t \neq \mathbb{R}^n$, then

$$S_t = \bigcup_{j=1}^{\infty} Q_j,$$

where the cubes $Q_j$ have sides parallel to the coordinate axes, have disjoint interiors and satisfy

$$d(Q_j, S_t) \leq 3\sqrt{n} l(Q_j),$$

where $M^c$ is the complement of $M$ in $\mathbb{R}^n$ and $l(Q)$ is the edge length of $Q$ (cf. [6, p. 10]). By subdividing $Q_j$ if necessary, we may require that

$$\rho_j = 4\sqrt{n} l(Q_j) \leq \delta.$$
If (3.10) is achieved by subdivision, we lose (3.9). But in this case we can require

\begin{equation}
\delta \leq 2\rho_j.
\end{equation}

Thus we can make each \( Q_j \) satisfy (3.10). If it does not satisfy (3.11) as well, then it will satisfy (3.9).

Let \( b, d \) be positive numbers to be determined later. Define

\begin{equation}
E_j = \{ x \in Q_j \mid I_{s, \delta/2} d\mu(x) > tb, M_{s, \delta} d\mu(x) \leq td \}.
\end{equation}

Let \( Q \) be one of the cubes \( Q_j \), and let \( E \subset Q \) be the set given by (3.12). Assume first that \( Q \) satisfies (3.10) and (3.11). Then we have

\[
|E| \leq \int_Q I_{s, \delta/2} d\mu(x) dx
\]

where \( \omega \) is the surface area of the unit sphere in \( \mathbb{R}^n \) and \( Q + \delta \) is the cube having the same center as \( Q \) but edge length equal to \( l(Q) + \delta \). Assume that \( E \) is not empty, and let \( x_0 \) be any point in \( E \). The cube \( Q + \delta \) is contained in the ball with center \( x_0 \) and radius \( \sqrt{n}l(Q) + (\delta/2) \leq (\rho/4) + (\delta/2) \leq 3\delta/4 \) by (3.10). Hence by (3.11)

\[
tb|E| \leq (\omega/s)(\delta/2)^{\omega/2} (\rho/4 + \delta/2)^{n-s} M_{s, \delta} d\mu(x_0)
\]

where

\[
\leq (\omega/s)\rho^s (\delta/4)^{n-s} td
\]

Consequently,

\begin{equation}
|E| \leq (\omega/s)4^s 5^{n-s} n^{n/2} |Q|.
\end{equation}

Note that (3.13) holds if \( E \) is empty. Next assume that (3.9) and (3.10) hold. Then there is a point \( x_1 \) not in \( S \), that

\[
d(x_1, Q) \leq 3\sqrt{n}l(Q).
\]

If \( x \) is in \( Q \), then

\begin{equation}
|x - x_1| < \rho.
\end{equation}

Consequently, if \( y \) is any point such that

\begin{equation}
|y - x| > \rho.
\end{equation}
then
\[(3.16) \quad |y - x_1| \leq |y - x| + |x - x_1| < 2|y - x|.
\]

Hence we have
\[
I_{s, \delta/2} d\mu(x) = \int_{|y-x|<\delta} + \int_{\rho<|y-x|<\delta/2} |y - x|^{s-n} d\mu(y)
\leq I_{s, \rho} d\mu(x) + 2^n \int_{|y-x|<\delta} |y - x_1|^{s-n} d\mu(y)
\leq I_{s, \rho} d\mu(x) + 2^n I_{s, \delta} d\mu(x_1)
\leq I_{s, \rho} d\mu(x) + 2^n \delta t
\]
since \(x_1\) is not in \(S_t\). We now take \(b = 2^{n+1-s}\). This implies that if \(x \in E\), we have
\[
tb \leq I_{s, \rho} d\mu(x) + tb/2
\]
and consequently
\[
tb/2 \leq I_{s, \rho} d\mu(x).
\]

Thus \(E\) is contained in the set
\[
\{x \in Q | I_{s, \rho} d\mu(x) > tb/2, M_s \delta d\mu(x) \leq td\}.
\]
Hence, if \(x \in E\)
\[
tb|E|/2 \leq \int_Q I_{s, \rho} d\mu(x) dx
\]
\[
= \int \int_{|x-y|<\rho \cap x \in E} |x - y|^{s-n} dx d\mu(y)
\leq (\omega/s) \rho^s \int_{Q+2\rho} d\mu.
\]

Since \(2\rho \leq \delta\) and the cube \(Q + 2\rho\) is contained in a ball of radius \(5\rho/4 < \delta\) about any point in \(Q\), we see that
\[
tb|E|/2 \leq (\omega/s) \rho^s (5\rho/4)^{n-s} M_s \delta d\mu(x_0)
\leq (\omega/s) \rho^s 5^{n-s} n^{n/2} td|Q|
\]
or
\[(3.17) \quad |E| \leq 2^{2s+1} (\omega/s) 5^{n-s} n^{n/2} (d/b)|Q|
\]
if we take \(x_0 \in E\). If \(E\) is empty, (3.17) holds as well. Thus we see that (3.17) holds in all cases. If we sum over all the cubes \(Q_j\), we see that
\[
|\{|I_{s, \delta/2} d\mu(x) \geq tb, M_s \delta d\mu(x) \leq td\}| \leq C_{n,s} d|S_t|
\]
where
\[
C_{n,s} = \omega 5^{n-s} n^{n/2} 2^{3s-n}/s.
\]
Hence
\[ |\{I_{s, \delta/2} d\mu(x) > tb\}| \leq C_{n,s} d|S| + |\{M_{s, \delta} d\mu(x) > td\}|.\]
This means that
\[ \int_0^N |\{I_{s, \delta/2} d\mu(x) > tb\}| dt^q \]
\[ \leq C_{n,s} d \int_0^N |S| dt^q + \int_0^N |\{M_{s, \delta} d\mu(x) > td\}| dt^q \]
or
\[ b^{-q} \int_0^{Nb} |\{I_{s, \delta/2} d\mu(x) > \tau\}| d\tau^q \]
\[ \leq C_{n,s} d \int_0^N |S| dt^q + d^{-q} \int_0^{Nd} |\{M_{s, \delta} d\mu(x) > \tau\}| d\tau^q. \]
Letting \( N \to \infty \), we have
\[ \|I_{s, \delta/2} d\mu\|^q \leq C_{n,s} d b^q \|I_{s, \delta} d\mu\|^q + (b/d)^q \|M_{s, \delta} d\mu\|^q \]
and consequently
\[ \|I_{s, \delta/2} d\mu\|^q \leq C_{n,s}^1 d^{1/q} b \|I_{s, \delta} d\mu\|^q + (b/d)^q \|M_{s, \delta} d\mu\|^q. \]
Now
\[ I_{s, \delta} d\mu(x) = I_{s, \delta/2} d\mu(x) + \int_{\delta/2 < |y-x| < \delta} |x-y|^{n-s} d\mu(y) \]
\[ \leq I_{s, \delta/2} d\mu + 2^{n-s} M_{s, \delta} d\mu. \]
Hence
\[ \|I_{s, \delta} d\mu\|^q \leq C_{n,s}^1 d^{1/q} b \|I_{s, \delta} d\mu\|^q + (bd^{-1} + 2^{n-s}) \|M_{s, \delta} d\mu\|^q. \]
Take \( 1/d = C_{n,s} 2^q b^q \) . Then
\[ \|I_{s, \delta} d\mu\|^q \leq b(2d^{-1} + 1) \|M_{s, \delta} d\mu\|^q \]
\[ = (2^{n-s+1} + (\omega/s) 2^{n/2} 2^{(n+2-s)q+2s+2}) \|M_{s, \delta} d\mu\|^q. \]
This gives the theorem.
Next we shall prove

**Theorem 3.5.** Under the same hypothesis,
\[ \|G_{s, \lambda} d\mu\|^q \leq C_{s,q} \|M_{s, 1/\lambda} d\mu\|^q \]
where the constant depends only on \( s, n \) and \( q \). Here
\[ G_{s, \lambda} d\mu(x) = \int G_{s, \lambda}(x - y) d\mu(y). \]
\[ (\lambda^2 - \Delta)^{-s/2} f(x) = \int G_{s, \lambda}(x - y)f(y) dy. \]
Proof. For each $s > 0$, the function $G_{s,\lambda}(x)$ has been studied extensively by Aronszajn-Smith [7]. In particular, it satisfies

$$G_{s,\lambda}(x) \leq \begin{cases} c_0|x|^{s-n}, & \lambda |x| \leq 1, \\ c_1\lambda^{n-s}e^{-\lambda |x|}, & \lambda |x| > 1, \end{cases}$$

where $\gamma = (n-s-1)/2$ and the $c_j$ do not depend on $\lambda$. Let

$$\tilde{G}_{s,\lambda}(x) = \begin{cases} 0, & \lambda |x| \leq 1, \\ G_{s,\lambda}(x), & \lambda |x| > 1. \end{cases}$$

It suffices to show that

$$\|\tilde{G}_{s,\lambda} d\mu\|_q \leq C\|M_{s,1/\lambda} d\mu\|_q.$$  

For by Theorem 3.1 and (3.20)

$$\|G_{s,\lambda} - \tilde{G}_{s,\lambda} d\mu\|_q \leq c_0\|I_{s,1/\lambda} d\mu\|_q \leq c_0 C_{s,q}\|M_{s,1/\lambda} d\mu\|_q.$$  

Now by (3.20) and (3.21)

$$\tilde{G}_{s,\lambda} d\mu(y) \leq c_1 \int_{|x-y| > 1} \lambda^{n-s}e^{-\lambda (x-y)} d\mu(x) \leq c_1\lambda^{n-s} \sum_{k=1}^{N(k)} \int_{k-1 < \lambda |x-y| < k+1} (k+1)^{\gamma} e^{-k} d\mu(x).$$

The set $k < |x| < k+1$ can be covered by $N(k)$ balls of radius 1 and centers $z(1), \ldots, z^{N(k)}$ with $N(k) \leq c_2 k^{n-1}$. Thus the set $k < \lambda |x| < k+1$ can be covered by $N(k)$ balls with centers $z^{(1)}/\lambda, \ldots, z^{N(k)}/\lambda$ having radius $1/\lambda$. Hence

$$\tilde{G}_{s,\lambda} d\mu(y) \leq c_1\lambda^{n-s} \sum_{k=1}^{N(k)} \sum_{j=1}^{N(k)} (k+1)^{\gamma} e^{-k} \int_{|x-y-z^{(j)}/\lambda| < 1/\lambda} d\mu(x) \leq c_1\lambda^{n-s} \sum_{k=1}^{N(k)} \sum_{j=1}^{N(k)} (k+1)^{\gamma} e^{-k} M_{s,1/\lambda} d\mu(y + z^{(j)}/\lambda).$$

Consequently

$$\|\tilde{G}_{s,\lambda} d\mu\|_q \leq c_1 \sum_{k=1}^{N(k)} \sum_{j=1}^{N(k)} (k+1)^{\gamma} e^{-k} \|M_{s,1/\lambda} d\mu\|_q.$$  

This gives (3.22).

We can now give the

Proof of Theorem 3.2. Let $\delta = 1/\lambda$ and put

$$K_p = \sup_x (M_{2p,\delta} V^p)^{1/p}.$$  

If $q = 2p > 2$, then Hölder's inequality gives

$$M_{1,\delta}[V^{1/2} u] \leq M_{q,\delta}(V^{1/2})^{1/q} M_{0,\delta}(|u|^q)^{1/q'} \leq K_p^{1/2} (M |u|^q)^{1/q'}.$$
Hence
\[ \|M_{\lambda} [V^{1/2} u]\|_2 \leq K_p^{1/2} \|(M |u|^{q'})^{1/q'}\|_2. \]

Since \( q' < 2 \), this is bounded by
\[ K_p^{1/2} \|(M |u|^{q'})^{1/q'}\|_2 \leq c' K_p^{1/2} \|u|^{q'}\|_2^{1/q'} = c' K_p^{1/2} \|u\|_2. \]

By Theorem 3.5, this implies
\[ \|G_{1, \lambda} [V^{1/2} u]\|_2 \leq c' C_{1, \lambda}^{1/2} K_p^{1/2} \|u\|_2. \]

This implies by duality
\[ \|V^{1/2} G_{1, \lambda} \nu\|_2 \leq c' C_{1, \lambda}^{1/2} K_p^{1/2} \|\nu\|_2 \]
which is equivalent to
\[ (Vu, u) \leq c' C_{1, \lambda}^{1/2} \lambda^{2-n} (\|\nabla u\|^2 + \lambda^2 \|u\|^2). \]

Thus
\[ C_\lambda(V) \leq c' C_{1, \lambda}^{1/2} \lambda^{2-n} \lambda K_p \]
which is precisely (3.4). To prove (3.5), let \( \phi(x) \) be a test function which equals 1 for \( |x| < 1 \) and 0 for \( |x| > 2 \). Put \( \phi_\lambda(x) = \phi(\lambda(x - z)) \), where \( z \in \mathbb{R}^n \) is fixed. Then
\[ (V\phi_\lambda, \phi_\lambda) \leq C_\lambda(V) (\|\nabla \phi_\lambda\|^2 + \lambda^2 \|\phi_\lambda\|^2) \]
\[ = C_\lambda(V) \lambda^{2-n} (\|\nabla \phi\|^2 + \|\phi\|^2) = C_\lambda^{2-n} C_\lambda(V). \]

Hence
\[ \lambda^{n-2} \int_{|x-z|<1} V(x) \, dx \leq C C_\lambda(V) \]
and consequently
\[ M_{2,1/\lambda} V(z) \leq C C_\lambda(V). \]

**Remark 3.6.** The constant \( C_p \) in (3.4) can be estimated readily from the proofs of Theorems 3.1, 3.2 and 3.5.

### 4. Properties of \( C_\lambda(V) \)

In this section we shall derive some properties of the constant \( C_\lambda(V) \).

**Theorem 4.1.** \( C_\lambda(V) \) is continuous in \( \lambda \) in the interval \([0, \infty)\).

**Proof.** Suppose
\[ C_\nu(V) \leq A, \quad \nu > \lambda. \]

Then \( C_\lambda(V) \leq A \). For we have
\[ (Vu, u) \leq A (\|\nabla u\|^2 + \nu^2 \|u\|^2), \quad u \in C_0^\infty. \]

Let \( \nu \to \lambda \). Then
\[ (Vu, u) \leq A (\|\nabla u\|^2 + \lambda^2 \|u\|^2), \quad u \in C_0^\infty. \]
Thus \( C_\lambda(V) \leq A \). Next, suppose \( \lambda > 0 \) and \( C_\nu(V) \geq A \), \( \nu < \lambda \).

Then \( C_\lambda(V) \geq A \). For if \( C_\lambda(V) \leq A - \epsilon \), we can find for each \( \nu < \lambda \) a function \( u_\nu \in C_0^{\infty} \) such that

\[
(4.1) \quad \|\nabla u_\nu\|^2 + \nu^2 \|u_\nu\|^2 = 1
\]

and

\[
C_\nu(V) - \epsilon/2 \leq (Vu_\nu, u_\nu) \leq C_\lambda(V)(\|\nabla u_\nu\|^2 + \lambda^2 \|u_\nu\|^2).
\]

Thus

\[
A - \epsilon/2 \leq C_\lambda(V)(1 + (\lambda^2 - \nu^2)\|u_\nu\|^2) \leq C_\lambda(V)\lambda^2/\nu^2
\]

in view of (4.1). Let \( \nu \to \lambda \). We have

\[
A - \epsilon/2 < C_\lambda(V) < A - \epsilon
\]

providing a contradiction. Since \( C_\lambda(V) \) is a decreasing function of \( \lambda \), it must be continuous.

**Theorem 4.2.** \(-\mu^2\) is the lowest point of the spectrum of \(-\Delta - V\), then

\[
\mu^2 = \inf_{C_\lambda(V) \leq 1} \lambda^2 = \sup_{C_\lambda(V) > 1} \lambda^2
\]

\[
= \inf_{C_\lambda(V) \leq 1} \lambda^2 C_\lambda(V) = \sup_{C_\lambda(V) > 1} \lambda^2 C_\lambda(V).
\]

If the set \( C_\lambda(V) \leq 1 \) is empty, then \( \mu = \infty \). If the set \( C_\lambda(V) > 1 \) is empty, then \( \mu = 0 \).

**Proof.** Let \( H \) be the operator (1.2). If \( C_\lambda(V) \leq 1 \), then (1.1) implies

\[
-C_\lambda(V)\lambda^2 \|u\|^2 \leq (Hu, u).
\]

Thus

\[
(4.2) \quad \mu^2 \leq C_\lambda(V)\lambda^2 \leq \lambda^2, \quad C_\lambda(V) \leq 1.
\]

If \( C_\lambda(V) > 1 \), then for every \( \epsilon > 0 \) there is a \( u \in C_0^{\infty} \) such that

\[
(Vu, u) \geq (C_\lambda(V) - \epsilon)(\|\nabla u\|^2 + \lambda^2 \|u\|^2).
\]

Thus

\[
(Hu, u) + \lambda^2(C_\lambda(V) - \epsilon)\|u\|^2 \leq (1 + \epsilon - C_\lambda(V))\|\nabla u\|^2.
\]

For \( \epsilon \) sufficiently small, this is \( \leq 0 \). Thus

\[
-\mu^2 \leq -\lambda^2(C_\lambda(V) - \epsilon) \quad \text{or} \quad \mu^2 \geq \lambda^2(C_\lambda(V) - \epsilon).
\]

Letting \( \epsilon \to 0 \), we have

\[
(4.3) \quad \mu^2 \geq \lambda^2 C_\lambda(V) \geq \lambda^2, \quad C_\lambda(V) > 1.
\]

In particular we see from this that \( C_\mu(V) \leq 1 \).
If \( \mu \neq 0 \), we see by (4.2) that

\begin{equation}
C_\mu(V) = 1.
\end{equation}

By (4.2),

\begin{equation}
\mu^2 \leq \inf_{\lambda^2 C_\lambda(V) \leq 1} \lambda^2 C_\lambda(V) \leq \inf_{\lambda^2 C_\lambda(V) \leq 1} \lambda^2.
\end{equation}

But by (4.4) we see that equality holds. Similarly, by (4.2) we see that

\begin{equation}
\mu^2 \geq \sup_{\lambda^2 C_\lambda(V) \leq 1} \lambda^2 C_\lambda(V) \geq \sup_{\lambda^2 C_\lambda(V) \leq 1} \lambda^2.
\end{equation}

But there cannot be a positive \( \epsilon \) such that \( \mu^2 \geq \epsilon + \lambda^2 \) holds for all \( \lambda \) satisfying \( C_\lambda(V) > 1 \). For that would imply the existence of a \( \nu < \mu \) such that \( C_\nu(V) \leq 1 \), contradicting (4.5). Thus, equality holds throughout (4.6) as well.

**Corollary 4.3.**

\begin{equation}
\mu^2 \leq \sup_{\lambda^2} \lambda^2 [2 C_\lambda(V) - 1].
\end{equation}

\begin{equation}
\mu^2 \geq \sup_{\lambda^2} \lambda^2 [C_\lambda(V) - 1].
\end{equation}

**Proof.** If \( C_\lambda(V) > 1 \), then

\begin{equation}
\lambda^2 \leq \lambda^2 [2 C_\lambda(V) - 1].
\end{equation}

Thus \( \sup \lambda^2 \) over the set \( C_\lambda(V) > 1 \) is bounded by the right-hand side of (4.7). Similarly, if \( C_\lambda(V) > 1 \), then

\begin{equation}
\lambda^2 C_\lambda(V) \geq \lambda^2 [C_\lambda(V) - 1].
\end{equation}

On the other hand, the right-hand side of (4.9) is negative if \( C_\lambda(V) < 1 \). Thus \( \sup \lambda^2 C_\lambda(V) \) over the set \( C_\lambda(V) > 1 \) is \( \geq \) the right-hand side of (4.8).

Now we turn to the

**Proof of Corollary 3.3.** By (4.7) and (3.4)

\begin{equation}
\mu^2 \leq \sup_{\lambda^2} \lambda^2 [2 C_\rho \sup_{x} (M_{2p} x, y(V)^p V^p)^{1/p} - 1]
\end{equation}

\begin{equation}
= \sup_{x, \delta} [2 C_\rho \delta^{-2} (M_{2p} \delta V^p)^{1/p} - \delta^{-2}].
\end{equation}

This equals the last expression in (3.6). For let \( L \) be the latter expression. Then

\[ \left( \delta^{-n} \int_{|y-x|<\delta} V(y)^p dy \right)^{1/p} \leq \left( L + \delta^{-2} \right)/2C_\rho, \quad \delta > 0. \]

This implies

\[ (M_{2p} \delta V^p)^{1/p} \leq (\delta^2 L + 1)/2C_\rho. \]
If we substitute this into (4.10), we obtain\[
\mu^2 \leq \sup_{x, \delta} [\delta^{-2}(\delta^2 L + 1) - \delta^{-2}] = L.
\]
The same reasoning works in reverse. The second estimate in Corollary 3.3 is proved in the same way using inequality (3.5).

Corollary 3.4 is an immediate consequence of (3.4) taking $\lambda = 0$.

A function $V(x)$ is said to satisfy the $A_\infty$ condition if there is $p > 1$ such that
\[
\left(\frac{1}{|Q|} \int_Q (V(x))^p \, dx\right)^{1/p} \leq L_p |Q|^{-1} \int_Q V(x) \, dx
\]
holds for all cubes $Q$, where $|Q|$ is the volume of $Q$ (cf. [8]). We have

Corollary 4.4. If $V(x)$ satisfies the $A_\infty$ condition, then
\[
C_\lambda(V) \leq N_p \|M_{2,1/\lambda} V\|_{\infty}.
\]

Proof. From the definition we see that there is a constant $L_p'$ such that
\[
(M^{2p,s} V^p)_{L_p'} \leq L'_p M_{2p,\delta} V.
\]

5. Invariance of the essential spectrum

For a closed operator $A$ on a Banach space we define the essential spectrum of $A$ as
\[
\sigma_e(A) = \bigcap_K \sigma(A + K)
\]
where the intersection is taken over all compact operators $K$. We give sufficient conditions for $H$ to have the same essential spectrum as $-\Delta$.

Theorem 5.1. Assume that
(a) $C_\lambda(V) \to 0$ as $\lambda \to \infty$.
(b) For some $\lambda_0 \geq 0$,
\[
C_{\lambda_0}(V^R) \to 0 \text{ as } R \to \infty
\]
where
\[
V^R(x) = \begin{cases} 0, & |x| \leq R, \\ V(x), & |x| > R. \end{cases}
\]
Then
\[
\sigma_e(H) = \sigma_e(-\Delta) = [0, \infty).
\]

Proof. By (a) and (1.1), for each $\epsilon > 0$ there is a constant $C_\epsilon$ such that
\[
(Vu, u) \leq \epsilon \|\nabla u\|^2 + C_\epsilon \|u\|^2.
\]
Moreover, if $\phi(x) \in C_0^\infty$ is the function used in the proof of Theorem 3.2, then

$$C_{\lambda_0}((V(1-\phi_R)) \to 0 \text{ as } R \to \infty$$

by (b). These two conditions are necessary and sufficient for $V^{1/2}$ to be compact from $H^{1,2}$ to $L^2$ (cf. [14, p. 172]). This in turn is sufficient for $H$ to have a $1/2$ extension satisfying (5.1) (cf. [14, p. 149]).

**Bibliography**


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