RIGIDITY FOR COMPLETE WEINGARTEN HYPERSURFACES

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Abstract. We classify, locally and globally, the ruled Weingarten hypersurfaces of the Euclidean space. As a consequence of the local classification and a rigidity theorem of Dajczer and Gromoll, it follows that a complete Weingarten hypersurface which does not contain an open subset of the form $L^3 \times \mathbb{R}^{n-3}$, where $L^3$ is unbounded and $n \geq 3$, is rigid.

Introduction

Recently Dajczer and Gromoll [DG] showed that a complete hypersurface $M^n$, $n \geq 4$, of the Euclidean space $\mathbb{R}^{n+1}$ is rigid, unless it contains an open subset $U$ such that either $U = L^3 \times \mathbb{R}^{n-3}$ with $L^3$ unbounded or $U$ is completely ruled. We recall that a completely ruled submanifold is a ruled submanifold with complete rulings. It is not known if there exists a nowhere ruled three-dimensional irreducible hypersurface which is not rigid (see [DG2]).

We observe that there is an abundance of hypersurfaces of the Euclidean space which admit local isometric deformations. A classification of such hypersurfaces was obtained by Sbrana [S] and Cartan [C]. A special case is given by the minimal hypersurfaces of rank two discussed in [DG].

In this paper we consider the rigidity question for complete hypersurfaces $M^n$ which satisfy the additional condition of being Weingarten, i.e. there exists a differentiable function relating the mean curvature and the scalar curvature of $M$. Our main result is the following.

Theorem A. Let $M^n$, $n \geq 4$, be a complete Weingarten immersed hypersurface of $\mathbb{R}^{n+1}$, which does not contain an open subset $U = L^3 \times \mathbb{R}^{n-3}$ with $L^3$ unbounded. Then $M$ is rigid.

The above result is an immediate consequence of the rigidity theorem of Dajczer and Gromoll and the following local classification of ruled Weingarten hypersurfaces.

Theorem B. Let $M^n$, $n \geq 3$, be a connected ruled Weingarten hypersurface of $\mathbb{R}^{n+1}$. Then $M^n$ is either

(i) flat.
or it is an open subset of one of the following:

(ii) $Q^3 \times \mathbb{R}^{n-3}$, where $Q^3 \subseteq \mathbb{R}^4$ is a cone over a product of circles in $S^3$, or over a minimal ruled surface in $S^3$;

(iii) $Q^2 \times \mathbb{R}^{n-2}$, where $Q^2 \subseteq \mathbb{R}^3$ is a ruled helicoidal surface or a hyperboloid of revolution.

The classification for $n = 2$ was obtained in 1865 by Beltrami [B] and Dini [D], see (2.29). We observe that the classification of Theorem B is complete since the minimal ruled surfaces in $S^3$ are given in [L], see (2.16).

Now if we assume $M$ to be complete, we have

Corollary C. Let $M^n$, $n \geq 3$, be a complete connected ruled Weingarten hypersurface in $\mathbb{R}^{n+1}$. Then, $M$ is either

(i) a product $Q^2 \times \mathbb{R}^{n-2}$, where $Q^2$ is a complete ruled helicoidal surface of a hyperboloid of revolution; or

(ii) a cylinder over a complete curve.

1. Preliminares

Let $M^n \subseteq \mathbb{R}^{n+1}$ be a connected orientable immersed hypersurface endowed with the induced metric. The relative nullity of the immersion at a point $p \in M$, is $\ker A(p)$, where $A$ denotes the second fundamental form of the hypersurface. Suppose that the relative nullity has constant dimension $\nu = n - k$. Then the Gauss map $\phi: M^n \to S^n \subseteq \mathbb{R}^{n+1}$ is parallel along each leaf of the relative nullity foliation, and provides (locally) a Gauss parametrization of $M$ as it was defined in [DG]. More precisely, there exists an isometric immersion $g: L^k \to S^n$, which is a local parametrization of the image of the Gauss map $\phi$, and a differentiable function $\gamma: L^k \to \mathbb{R}$ (support function) such that

$$X: U \subseteq \Lambda \to M^n \subseteq \mathbb{R}^{n+1},$$

$$(x, v) \mapsto X(x, v) = \gamma(x)g(x) + \text{grad}\, \gamma(x) + v$$

is a local parametrization of $M^n$, where $\Lambda$ is the normal bundle of the immersion $g$. $X$ is the so-called Gauss parametrization of $M$.

For each $(x, v) \in U \subseteq \Lambda$, let $\text{Hess}\, \gamma(x)$ denote the hessian of $\gamma$ and $B_v$ the second fundamental form of the immersion $g$ at $x \in L^k$, relative to the normal vector $v$. Then the selfadjoint operator defined on the tangent space of $L^k$ at $x$,

$$P_{(x, v)} = \gamma(x)I + \text{Hess}\, \gamma(x) - B_v,$$

is nonsingular. Moreover, the second fundamental form $A_{(x, v)}$ of $X$ at $(x, v)$ is given by $-P^{-1}$, when restricted to the orthogonal complement of the relative nullity distribution. We refer to [DG] for the above results.
For each vector field \( e: L^k \to \mathbb{R}^{n+1} \), we may consider an associated vector field \( \overline{e}: U \subset \Lambda \to \mathbb{R}^{n+1} \) defined by

\[
\overline{e}(x , v) = e(x), \quad \forall (x , v) \in U,
\]

i.e. \( \overline{e} \) is the Euclidean parallel transport of \( e(x) \) along the leaves of the relative nullity foliation of \( M \). Therefore, if \( e \) is a vector field normal (resp. tangent) to the immersion \( g \), then the associated vector field \( \overline{e} \) belongs (resp. is orthogonal) to the relative nullity distribution.

In what follows we consider hypersurfaces \( M^n \subset \mathbb{R}^{n+1} \) with constant index of relative nullity \( \overline{\nu} = n - 2 \), locally parametrized as in (1.1). Moreover, we choose orthonormal vector fields \( e_1 , \ldots , e_n \), locally defined on \( L^2 \), such that \( e_1(x) \), \( e_2(x) \) are tangent to the immersion \( g \) at \( x \) and \( e_3(x) , \ldots , e_n(x) \) generate the normal space of the immersion in \( S^n \). Let \( \overline{e}_i(x , v) = e_i(x) \), \( 1 \leq i \leq n \), \( (x , v) \in U \subset \Lambda \), be the associated vector fields on \( M \). With respect to this frame the second fundamental form of \( X \) at \( (x , v) \) is given by

\[
A = \begin{pmatrix}
-P_{(x,v)}^{-1} & 0 \\
0 & 0
\end{pmatrix},
\]

where \( P \) is defined by (1.2).

It follows that the mean curvature \( \overline{H} \) and the scalar curvature \( \overline{S} \) of \( M \) at \( (x , v) \) are given respectively by

\[
\overline{H}(x , v) = -\text{tr} A = \frac{\text{tr} P}{\det P},
\]

\[
\overline{S}(x , v) = \frac{1}{\det P}.
\]

**Lemma 1.6.** Let \( M^n \subset \mathbb{R}^{n-1} \) be a ruled immersed hypersurface with constant index of relative nullity \( \overline{\nu} = n - 2 \). Then the immersion \( g \) is a ruled surface in \( S^n \).

**Proof.** Let

\[
X(s , \lambda , \mu) = c(s) + \lambda \xi(s) + \sum_{j=1}^{n-2} \mu_j \eta_j(s)
\]

be a local parameterization of \( M \), where \( c(s) \) is a curve orthogonal to the ruling, \( \eta_j \), \( 1 \leq j \leq n - 2 \), generate the relative nullity and \( \{ \xi , \eta_j \} \) generate the ruling of \( M^n \). Then the Gauss map depends only on the parameters \( s , \lambda \), since \( \eta_j \) generate the relative nullity distribution. Moreover, for \( s = s_0 \), the Gauss map describes a curve which is orthogonal to the subspace generated by \( \xi(s_0) , \eta_j(s_0) \), \( 1 \leq j \leq n - 2 \). Therefore it is contained in a great circle of \( S^n \). Q.E.D.

**Fact 1.7.** It follows from the above lemma that if \( M \) is a ruled hypersurface, then the frame considered earlier may be chosen such that \( e_j(x) \) is tangent to the ruling of the immersion \( g \). Thus the second fundamental form \( \theta \) of \( g \) with
values in the normal bundle satisfies $\theta(e_i, e_i) = 0$. Therefore, the associated frame tangent to $M$, $\bar{e}_i(x, v) = e_i(x)$, is such that $\bar{e}_i$, $3 \leq i \leq n$, generate the relative nullity, $\bar{e}_i$, $2 \leq i \leq n$, generate the ruling and $\langle A\bar{e}_2, \bar{e}_2 \rangle = 0$.

For such a frame, the second fundamental form of the immersion $g$, with respect to $e_i$, $3 \leq i \leq n$, will be denoted by
\[ B_i(x) = \begin{pmatrix} 0 & \beta_i \\ \beta_i & \lambda_i \end{pmatrix}, \quad 3 \leq i \leq n, \]
and the operator $\gamma(x)I + \text{Hess} \gamma(x)$ will be denoted by
\[ \gamma(x)I + \text{Hess} \gamma(x) = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix}. \]

Now we assume that the submanifold $M^n \subset \mathbb{R}^{n+1}$ is Weingarten, i.e. there exists a differentiable function $F(H, S) = 0$. Taking exterior derivatives we obtain
\[ \partial F \partial H dH + \partial F \partial S dS = 0. \]
Therefore, applying to vector fields tangent to $M$, we conclude that
\[ dH \wedge dS = 0, \]
since the partial derivatives of $F$ are not simultaneously zero.

**Fact 1.11.** Let $M^n \subset \mathbb{R}^{n+1}$ be a ruled Weingarten hypersurface with constant index of relative nullity $\nu = n - 2$. Then it follows from (1.4) to (1.10) that
\[ d\left( \alpha(x) - \sum_{i=3}^{n} t_i \lambda_i(x) \right) \wedge d\left( h(x) - \sum_{j=3}^{n} t_j \beta_j(x) \right) = 0 \]
for $t_i \in \mathbb{R}$.

2. Proofs of the theorems

For the proof of Theorem B we will need the following three propositions.

**Proposition 2.1.** Let $M^n \subset \mathbb{R}^{n+1}$ be a connected ruled Weingarten hypersurface without flat points. Suppose that the dimension of the first normal space of $g$ is constant equal to 1. Then, there exists a totally geodesic submanifold $S^3 \subset S^n$ such that $g(L^2) \subset S^3$ is a ruled Weingarten surface which satisfies
\[ H^2 + c^2(K - 1) = 0, \]
where $H$ and $K$ are the mean and Gaussian curvature and $c$ is a constant. Moreover, $M^n$ is contained in a euclidean product $Q^3 \times \mathbb{R}^{n-3}$, where $Q^3 \subset \mathbb{R}^4$ is a ruled Weingarten surface with index of relative nullity $\nu = 1$.

**Proposition 2.2.** Let $g: L^2 \rightarrow S^3$ be a connected ruled surface in $S^3$ such that
\[ H^2 + c^2(K - 1) = 0. \]
Then either $H = 0$ or $H = c = 0$ and $K = 0$. In the latter case the immersed surface is contained in the product of two circles.

**Proposition 2.3.** Let $M^3 \subset \mathbb{R}^4$ be a connected ruled Weingarten hypersurface, with index of relative nullity $\tilde{\nu} = 1$. Suppose that the image of the Gauss map $g(L^2)$ is either

(i) a minimal surface in $S^3$; or

(ii) it is contained in the product of two circles.

Then $M^3$ is an open subset of a cone over $g(L^2)$.

We need the following result. Recall that the first normal space of an immersion is the subspace generated by the second fundamental form.

**Lemma 2.4.** Let $M^n$ be a ruled Weingarten hypersurface without flat points. Then, for each $x \in L^2$, the dimension of the first normal space $N_1$ of $g$ is less than or equal to 1.

**Proof.** The ruled hypersurface $M$ has no flat points if and only if the index of relative nullity is constant $\tilde{\nu} = n - 2$.

Since $M^n$ is a ruled hypersurface, it follows from Lemma 1.6 that $g(L^2)$ is a ruled surface. Let $e_1(x), e_2(x)$ be a locally defined tangent frame to the immersion $g$ such that $e_1(x)$ is tangent to the ruling. Let $N_1(x)$ be the first normal space of $g$ at $x$. Since $N_1$ is generated by $\theta(e_1, e_2), \theta(e_2, e_2)$, it follows that $\dim N_1 \leq 2$.

Suppose $\dim N_1(x) = 2$. We choose $e_3(x), e_4(x)$ generating $N_1$ such that $e_4$ is orthogonal to $\theta(e_1, e_2)$. Then the second fundamental form with respect to $e_3$ and $e_4$ in the tangent basis $e_1, e_2$ is given respectively by

$$B_3 = \begin{pmatrix} 0 & \beta_3 \\ \beta_3 & \lambda_3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_4 \end{pmatrix}.$$ 

Since $M^n$ is a Weingarten hypersurface it follows from (1.13) that $\lambda_4 \beta_3 = 0$. If $\lambda_4 = 0$, then $B_4 = 0$. If $\beta_3 = 0$, then $\theta(e_1, e_2) = 0$. In both cases we have a contradiction, since we assumed that $\dim N_1 = 2$. Q.E.D.

**Proof of Proposition 2.1.** Since $M$ is a ruled hypersurface without flat points, the index of relative nullity is constant $\tilde{\nu} = n - 2$. Let $e_1(x), \ldots, e_n(x)$ be an orthonormal frame defined locally on $L^2$ as in Fact 1.7. Moreover, we can choose $e_3(x)$ to generate the first normal space $N_1$ of $g$. For such a frame, the second fundamental form of $g$ (1.8) reduces to

$$B_3 = \begin{pmatrix} 0 & \beta_3 \\ \beta_3 & \lambda_3 \end{pmatrix}, \quad B_i = 0, \quad 4 \leq i \leq n.$$ 

Since $M$ is Weingarten, it follows from (1.12) that

$$d(\alpha - t\lambda) \wedge d(h - t\beta) = 0,$$

$t \in \mathbb{R}$.

Applying (2.6) to the pair $(e_i, \partial/\partial t)$, $i = 1, 2$, we obtain

$$-\beta d\alpha + \lambda dh = 0.$$
It follows from (2.8) that there exist constants $c_1, c_2$, not simultaneously zero, such that

$$c_1 \beta + c_2 \lambda = 0. \quad (2.9)$$

Observe that $c_2 \neq 0$. In fact if $c_2 = 0$, then from (2.9) we have $\beta = 0$. Now, dim $N_1 = 1$ implies that $\lambda \neq 0$ and (2.7) implies that $h$ is a constant. Therefore, it follows from (1.5) that $\overline{S} = -1/h^2$ is constant. However, Theorem 3.4 in [DG] implies that dim $N_1 = 0$, which is a contradiction.

Therefore, we have

$$\lambda = c\beta \quad (2.10)$$

and $\beta \neq 0$ in $L^2$. Moreover, it follows from (2.7) that $\alpha = ch + \bar{c}$, where $c$ and $\bar{c}$ are constants.

Now we prove that the first normal space of the immersion $g: L^2 \to S^n$ is parallel. In fact, let $\eta$ be any vector field generated by $e_4, \ldots, e_n$. Then it follows from the Codazzi equation that

$$B e_4 \eta e_3 = B e_2 \eta e_1,$$

where $\nabla^\perp$ is the connection in the normal bundle. Hence

$$\langle \nabla_{e_1}^\perp \eta, e_3 \rangle B e_2 = \langle \nabla_{e_2}^\perp \eta, e_3 \rangle B e_1.$$ 

Using (2.5) we get

$$\langle \nabla_{e_1}^\perp \eta, e_3 \rangle \beta e_1 + [\langle \nabla_{e_1}^\perp \eta, e_3 \rangle \lambda - \langle \nabla_{e_2}^\perp \eta, e_3 \rangle \beta] e_2 = 0.$$ 

Since $\beta \neq 0$ we conclude that

$$\langle \nabla_{e_1}^\perp \eta, e_3 \rangle = \langle \nabla_{e_2}^\perp \eta, e_3 \rangle = 0.$$ 

Hence the first normal space of the immersion $g$ is parallel. It follows that there exists a totally geodesic submanifold $S^3 \subset S^n$ which contains the image of $g$. Therefore, the normal bundle $\Lambda$ of $g$ splits into $\Lambda = \Lambda_1 + \Lambda_{n-3}$, where $\Lambda_1$ is the normal bundle in $S^3$ and the orthogonal complement $\Lambda_{n-3}$ is parallel in $R^{n+1}$. Hence, $M^n$ splits as a consequence of the Gauss parametrization.

Finally, from (2.5) we obtain that the mean curvature $H$ and the Gaussian curvature $K$ satisfies $H = \lambda$ and $K - 1 = \beta^2$. Therefore, it follows from (2.10) that $H^2 + c^2(K - 1) = 0$. Q.E.D.

Fact 2.11. It follows from the preceding proof that if $M^n$ satisfies the hypothesis of Proposition 1 then there is a frame locally defined on $L^2$ for which

$$B_3 = \begin{pmatrix} 0 & \beta \\ \beta & \lambda \end{pmatrix}, \quad B_i = 0, \ 4 \leq i \leq n,$$

$$\gamma I + \text{Hess} \gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix},$$

where $\lambda = c\beta$, $\alpha = ch + \bar{c}$, and $c$, $\bar{c}$ are constants.
Proof of Proposition 2.2. If the constant \( c \) is zero, then \( H = 0 \). Otherwise, we will show that \( K = 0 \) and hence \( H = c \).

Let \( g: \mathbb{R}^2 \to S^3 \) be a parametrized ruled surface in \( S^3 \). We may consider
\[
g(s, t) = \cos t \sigma + \sin t e(s),
\]
where \( \sigma(s) \) and \( e(s) \) are vectors in \( \mathbb{R}^4 \) such that
\[
|\sigma| = 1 = |e|, \quad \langle e, \sigma \rangle = 0, \quad \langle e', \sigma' \rangle = 0.
\]
Moreover, we may choose the parameter \( s \) such that \( |e'| = 1 \). We introduce the following notation
\[
p(s) = \langle e', \sigma \times e \times \sigma \rangle = \langle e', e' \sigma \rangle,
\]
\[
A(s) = |\sigma|^2 - \langle \sigma', e \rangle^2,
\]
\[
B(s) = 1 - \langle \sigma', e \rangle^2,
\]
\[
G(s, t) = A \cos^2 t + B \sin^2 t.
\]
We observe that \( p = \sqrt{AB} \). Moreover, it follows by a straightforward computation that the mean and Gaussian curvature of the surface are given by
\[
H = \frac{l - 2p(\sigma', e)}{2G^{3/2}}, \quad K - 1 = \frac{-p^2}{G^2},
\]
where
\[
l(s, t) = \cos^2 t (\sigma'' \sigma' e \sigma) + \sin^2 t (e'' e' e \sigma)
\]
\[
\quad + \sin t \cos t [((\sigma'' e' e \sigma) + (e'' e' \sigma'))].
\]
By hypothesis \( H^2 + c^2 (K - 1) = 0 \), therefore, without loss of generality, we have
\[
l - 2p(\sigma', e) - 2cG^{1/2} = 0.
\]
Taking a derivative with respect to \( t \) we get
\[
\frac{\partial l}{\partial t} - cG^{-1/2} \frac{\partial G}{\partial t} = 0.
\]
In particular for \( t = 0 \), it follows from (2.14) and (2.15) that
\[
(\sigma'' e' e \sigma) + (e'' e' \sigma') = 0.
\]
Hence (2.15) reduces to
\[
2 \sin t \cos t [-(\sigma'' \sigma' e \sigma) + (e'' e' e \sigma)] - cG^{-1/2} \frac{\partial G}{\partial t} = 0,
\]
which is equivalent to
\[
\left[ \frac{1}{AB} \left( \frac{d}{ds} (\sigma' \times e \times \sigma) e' e \sigma \right) - cG^{1/2} \right] \frac{\partial G}{\partial t} = 0, \quad \forall s, t.
\]
Since \( c \) is a nonzero constant, it follows that
\[
\frac{\partial G}{\partial t} = 0, \quad \forall s, t.
\]
Therefore, using (2.12) we get \( A = B \), \( p = A = G \). From (2.13) we get \( K = 0 \) and hence \( H = c \).

In order to show that the surface is contained in a product of two circles, we consider a local orthonormal frame field such that the second fundamental form is given by

\[
B = \begin{pmatrix} 0 & \beta \\ \beta & \lambda \end{pmatrix}.
\]

From \( K = 0 \) and \( H = c \), we have \( \det B = -1 \) and \( \lambda = 2c \), so that

\[
B = \begin{pmatrix} 0 & 1 \\ 1 & 2c \end{pmatrix}.
\]

We conclude the proof by using the uniqueness part of the fundamental theorem for surfaces in the sphere, see [S]. Q.E.D.

**Proof of Proposition 2.3.** Part (i). Since \( \nu = 1 \), there exists an immersion

\[
g: L^2 \to G \subset S^3
\]

and a local Gauss parametrization of \( M^3 \) given by

\[
X: \Lambda \to M^3 \subset R^4, \quad (x, v) \mapsto \gamma(x)g(x) + \text{grad} \gamma(x) + v,
\]

where \( \Lambda \) is the normal bundle of the immersion \( g \) and \( \gamma: L^2 \to R \) is a differentiable function.

\( M^3 \) is a ruled hypersurface, therefore it follows from Lemma 1.6 that \( g(L^2) \) is a ruled surface in \( S^3 \). Since \( g(L^2) \) is also minimal, we have \( g \) locally given by

\[
g(x_1, x_2) = \cos x_1(\cos kx_2, \sin kx_2, 0, 0) + \sin x_1(0, 0, \cos x_2, \sin x_2),
\]

where \( k \) is a positive constant, see [L or BDJ]. Let us consider the orthonormal tangent frame

\[
e_1 = \frac{\partial g}{\partial x_1}, \quad e_2 = \frac{1}{\sqrt{E}} \frac{\partial g}{\partial x_2},
\]

where \( E = k^2 \cos^2 x_1 + \sin^2 x_1 \). Let \( e_3 \) be a unitary normal vector field for the immersion \( g \). Then the second fundamental form with respect to this frame is given by

\[
B(x) = \begin{pmatrix} 0 & -k/E \\ -k/E & 0 \end{pmatrix}, \quad x = (x_1, x_2).
\]

Let \( \bar{e}_i \) be the associated frame defined on \( M^3 \), i.e.

\[
\bar{e}_i(x, v) = e_i(x), \quad 1 \leq i \leq 3, \quad (x, v) \in \Lambda.
\]

Then the second fundamental form for \( M^3 \subset R^4 \), with respect to this frame, is given by

\[
A(x, v) = \begin{pmatrix} -P_{(x,v)}^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad (x, v) \in \Lambda,
\]
where

\[ P(x, v) = \gamma(x) + \text{Hess} \gamma(x) - \langle v, e_3 \rangle B(x). \]

Moreover, it follows from Fact 2.11 that \( \gamma I + \text{Hess} \gamma \) is of the form

\[ \gamma I + \text{Hess} \gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix}, \]

where

\[ 0 = -ck/E, \quad \alpha = ch + \bar{c}. \]

Hence, \( c = 0 \) and \( \alpha = \bar{c}. \)

Now, we want to determine \( \gamma: L^2 \to R \) such that

\[ (2.19) \quad \gamma I + \text{Hess} \gamma = \begin{pmatrix} 0 & * \\ * & \bar{c} \end{pmatrix}. \]

It follows from (2.17) and (2.19) that \( \gamma \) must satisfy

\[ (2.20) \quad \gamma + \frac{2}{x_1^2} = 0, \quad \gamma + \frac{1}{E} \left( \frac{\partial^2 \gamma}{\partial x_2^2} + \sin x_1 \cos x_1 (1 - k^2) \frac{\partial \gamma}{\partial x_1} \right) \gamma = \bar{c}. \]

From the first equation we get

\[ \gamma = f(x_2) \cos x_1 + h(x_2) \sin x_1. \]

Substituting into (2.20) we get \( \bar{c} = 0 \). Therefore, the trace of \( P \) and hence the trace of \( A \) is zero, i.e. \( M^3 \) is a minimal surface in \( R^4 \).

To conclude the proof in this case we use [BDJ].

**Part (ii).** By hypothesis \( g(L^2) \) is contained in the product of two circles, therefore the immersion \( g \) is locally given by

\[ (2.21) \quad g(x_1, x_2) = r_1 \left( \sin \frac{x_1}{r_1}, \cos \frac{x_1}{r_1}, 0, 0 \right) + r_2 \left( 0, 0, \sin \frac{x_2}{r_2}, \cos \frac{x_2}{r_2} \right) \]

where \( r_1^2 + r_2^2 = 1 \).

Let us consider the orthonormal frame field defined by

\[ (2.22) \quad e_1 = r_1 \frac{\partial g}{\partial x_1}, \quad e_2 = r_2 \frac{\partial g}{\partial x_2}. \]

Then the second fundamental form of the immersion \( g \) with respect to \( e_1, e_2 \) is given by

\[ B = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}. \]

Let \( e_3 \) be a unitary normal vector field for the immersion \( g \) and

\[ \bar{e}_i(x, v) = e_i(x), \quad 1 \leq i \leq 3, \quad (x, v) \in \Lambda, \]

the associated frame defined on \( M^3 \). Then the second fundamental form for \( M^3 \subset R^4 \), with respect to this frame, is given by

\[ A(x, v) = \begin{pmatrix} -P^{-1}(x, v) & 0 \\ 0 & 0 \end{pmatrix} \]

where \( P(x, v) = \gamma(x) + \text{Hess} \gamma(x) - \langle v, e_3 \rangle B \).
Moreover, it follows from Fact 2.11 that with respect to this frame
\[ \gamma I + \text{Hess} \gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix}, \]
where
\[ c = \frac{r_2^2 - r_1^2}{r_1 r_2}, \quad \alpha = ch + \bar{c}. \]

We want to determine \( \gamma: L^2 \to \mathbb{R} \) which satisfies the above conditions. It follows from (2.22) that \( \gamma \) must satisfy
\[
\begin{align*}
\gamma + r_1^2 \frac{\partial^2 \gamma}{\partial x_1^2} - 2r_1 r_2 \frac{\partial^2 \gamma}{\partial x_1 \partial x_2} + r_2^2 \frac{\partial^2 \gamma}{\partial x_2^2} &= 0, \\
\gamma + r_1^2 \frac{\partial^2 \gamma}{\partial x_1^2} + r_2^4 + r_1^4 + r_2^2 \frac{\partial^2 \gamma}{\partial x_1 \partial x_2} + r_2^2 \frac{\partial^2 \gamma}{\partial x_2^2} &= \bar{c}.
\end{align*}
\]
(2.23)

Subtracting the above equation we get
\[
\frac{1}{r_1 r_2} \frac{\partial^2 \gamma}{\partial x_1 \partial x_2} = \bar{c}.
\]

Therefore
\[
\gamma(x_1, x_2) = \bar{c} r_1 r_2 x_1 x_2 + \gamma_1(x_1) + \gamma_2(x_2),
\]
where \( \gamma_1 \) and \( \gamma_2 \) are functions which depend only on \( x_1 \) and \( x_2 \) respectively. Substituting (2.24) into (2.23), we obtain
\[
\begin{align*}
\gamma_1 + r_1^2 \frac{d^2 \gamma_1}{dx_1^2} + \gamma_2 + r_2^2 \frac{d^2 \gamma_2}{dx_2^2} + \bar{c} r_1 r_2 x_1 x_2 &= 2r_1^2 r_2^2 \bar{c}.
\end{align*}
\]
(2.25)

Taking derivatives with respect to \( x_1 \), and then with respect to \( x_2 \) we conclude that \( \bar{c} = 0 \). Therefore, (2.24) reduces to
\[
\gamma(x_1, x_2) = \gamma_1(x_1) + \gamma_2(x_2),
\]
where \( \gamma_1 \) and \( \gamma_2 \) satisfy the following equations
\[
\begin{align*}
\gamma_1 + r_1^2 \frac{\partial^2 \gamma_1}{\partial x_1^2} &= a, \\
\gamma_2 + r_2^2 \frac{\partial^2 \gamma_2}{\partial x_2^2} &= a,
\end{align*}
\]
(2.27)

where \( a \) is a constant.

Now we want to show that the Gauss parametrization of \( M^3 \) describes a cone over \( G \). In fact
\[
X(x_1, x_2, s) = \gamma(r_1 u_1 + r_2 u_2) + \frac{dy_1}{\partial x_1} v_1 + \frac{dy_2}{\partial x_2} v_2 + s(-r_2 u_1 + r_1 u_2),
\]
where
\[
\begin{align*}
u_1 &= \left( \sin \frac{x_1}{r_1}, \cos \frac{x_1}{r_1}, 0, 0 \right), \\
u_2 &= \left( 0, 0, \sin \frac{x_2}{r_2}, \cos \frac{x_2}{r_2} \right), \\
v_1 &= \partial X / \partial x_1, \\
v_2 &= \partial X / \partial x_2.
\end{align*}
\]
It follows from (2.26) and (2.27) that \( X(x_1, x_2, s(x_1, x_2)) \) is constant for
\[
s(x_1, x_2) = \frac{a}{r_1 r_2} - \frac{r_2}{r_1} \gamma_1 + \frac{1}{r_2} \gamma_2,
\]
which concludes the proof of case (ii). Q.E.D.

Finally, we prove Theorem B using the preceding results.

**Proof of Theorem B.** Let \( \overline{M} = \{ p \in M; \ S(p) \neq 0 \} \). Since \( M \) is a ruled hypersurface, the sectional curvature \( \overline{K} \) at points of \( \overline{M} \) is not identically zero. It follows from Lemma 2.4 applied to \( \overline{M} \) that at each point of the image of the Gauss map the first normal space \( N_1 \) has dimension \( \leq 1 \). We have \( \overline{M} = \overline{M}_0 \cup \overline{M}_1 \), where at \( \overline{M}_0 \) the Gauss map is totally geodesic in \( S^n \) and \( \overline{M}_1 \) is the open subset of points where \( N_1 \) has dimension 1.

Let \( V_1 \) be a connected component of \( \overline{M}_1 \), let \( X: U \subset \Lambda \to V_1 \subset \mathbb{R}^{n+1} \) be a Gauss parametrization and let \( g: L^2 \to S^n \) be the associated local parametrization of the Gauss map of \( V_1 \). It follows from Proposition 2.1 that there exists a totally geodesic submanifold \( S^3 \subset S^n \) such that \( g(L^2) \subset S^3 \) is a ruled Weingarten surface which satisfies \( H^2 + c^2(K - 1) = 0 \). Moreover, \( V_1 \) is contained in a euclidean product \( Q^3 \times \mathbb{R}^{n-3} \), where \( Q^3 \subset \mathbb{R}^4 \) is a ruled Weingarten surface with constant index of relative nullity \( \nu = 1 \).

Using Proposition 2.2, we obtain that either \( g \) is a minimal immersion in \( S^3 \) or \( K = 0 \), \( H = c \), and the image of \( g \) is contained in the product of two circles of \( S^3 \). It follows from Proposition 2.3, that \( Q^3 \) is an open subset of a cone over the image of \( g \), i.e. \( V_1 \) satisfies (ii).

Let \( V_0 \) be a connected open subset of \( \overline{M}_0 \). We have a Gauss parametrization for \( V_0 \) and \( g \) the associated local parametrization of the image of the Gauss map of \( V_0 \). Since \( g \) is totally geodesic in \( S^n \), the normal bundle \( \Lambda \) of the immersion \( g \) is parallel in \( \mathbb{R}^{n+1} \). Hence, using the Gauss parametrization we obtain that \( V_0 \) is an open subset of \( Q^2 \times \mathbb{R}^{n-2} \), where \( Q^2 \subset \mathbb{R}^3 \) is a ruled Weingarten surface. It follows from the classical result of Beltrami [B] and Dini [D] that \( Q^2 \) is a ruled helicoidal surface or a hyperboloid of revolution, i.e. \( V_0 \) satisfies (iii).

We now observe that the boundary of \( V_0 \) does not intersect the boundary of \( V_1 \), since the determinant of the second fundamental form of the image of the Gauss map of \( V_1 \) in \( S^3 \) is bounded away from zero. Moreover, the boundaries of \( V_0 \) and of \( V_1 \) do not contain points where the scalar curvature \( S \) is zero. Since \( M \) is connected, this concludes the proof of the theorem. Q.E.D.

**Remark 2.28.** The ruled Weingarten surfaces \( Q^2 \subset \mathbb{R}^3 \) classified by Beltrami and Dini are given by
\[
(2.29) \quad X(s, t) = (a \cos s + ct \sin s, \ a \sin s - ct \cos s, \ bs + \sqrt{1 - c^2 t})
\]
where \( a, b, c \) are constants.
Proof of Corollary C. We use Theorem B. If $M$ is complete, then it cannot be a cone. If $M$ splits as in (iii), then $M = Q^2 \times \mathbb{R}^2$, where $Q^2$ is a complete ruled helicoidal surface or a hyperboloid of revolution. If $M$ is flat, it follows from [HN] that $M$ is a cylinder over a complete curve. Q.E.D.

Proof of Theorem A. If $M^n \subset \mathbb{R}^{n+1}$, $n \geq 4$, is a complete hypersurface, it follows from [DG] that $M$ is rigid, unless it contains an open subset $U$ which is completely ruled.

We will show that the existence of such a subset $U$ contradicts the hypothesis of Theorem A. In fact, if we apply Theorem B to each connected component $U_0$ of $U$, we conclude that $U_0$ is completely ruled and flat. We consider a connected component of $U_0$ where the nullity is $n - 1$. Then the ruling coincides with the nullity and therefore the nullity is complete. The argument used in [HN] implies that this component of $U_0$ is a cylinder over a curve (not necessarily complete). Moreover, each connected component of $U_0$ where the nullity is $n$ is totally geodesic. Hence, in both cases we obtain open subsets of type $L^3 \times \mathbb{R}^{n-3}$, with $L^3$ unbounded, which is a contradiction. Therefore, $M$ is rigid. Q.E.D.

References


