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RIGIDITY FOR COMPLETE WEINGARTEN HYPERSURFACES

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ABSTRACT. We classify, locally and globally, the ruled Weingarten hypersurfaces of the Euclidean space. As a consequence of the local classification and a rigidity theorem of Dajczer and Gromoll, it follows that a complete Weingarten hypersurface which does not contain an open subset of the form $L^3 \times \mathbf{R}^{n-3}$, where L^3 is unbounded and $n \geq 3$, is rigid.

INTRODUCTION

Recently Dajczer and Gromoll $[DG]_2$ showed that a complete hypersurface M^n, $n \geq 4$, of the euclidean space $\mathbf{R}^{n+1}$ is rigid, unless it contains an open subset U such that either $U = L^3 \times \mathbf{R}^{n-3}$ with L^3 unbounded or U is completely ruled. We recall that a *completely ruled* submanifold is a ruled submanifold with complete rulings. It is not known if there exists a nowhere ruled three-dimensional irreducible hypersurface which is not rigid (see $[DG_2]$).

We observe that there is an abundance of hypersurfaces of the euclidean space which admit local isometric deformations. A classification of such hypersurfaces was obtained by Sbrana [S] and Cartan [C]. A special case is given by the minimal hypersurfaces of rank two discussed in $[DG]_1$.

In this paper we consider the rigidity question for complete hypersurfaces M^n which satisfy the additional condition of being Weingarten, i.e. there exists a differentiable function relating the mean curvature and the scalar curvature of M. Our main result is the following.

Theorem A. *Let M^n, $n \geq 4$, be a complete Weingarten immersed hypersurface of $\mathbf{R}^{n+1}$, which does not contain an open subset $U = L^3 \times \mathbf{R}^{n-3}$ with L^3 unbounded. Then M is rigid.*

The above result is an immediate consequence of the rigidity theorem of Dajczer and Gromoll and the following local classification of ruled Weingarten hypersurfaces.

Theorem B. *Let M^n, $n \geq 3$, be a connected ruled Weingarten hypersurface of $\mathbf{R}^{n+1}$. Then M^n is either*

(i) *flat*;

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or it is an open subset of one of the following:

(ii) $Q^3 \times \mathbf{R}^{n-3}$, *where* $Q^3 \subset \mathbf{R}^4$ *is a cone over a product of circles in* S^3, *or over a minimal ruled surface in* S^3;

(iii) $Q^2 \times \mathbf{R}^{n-2}$, *where* $Q^2 \subset \mathbf{R}^3$ *is a ruled helicoidal surface or a hyperboloid of revolution.*

The classification for $n = 2$ was obtained in 1865 by Beltrami [B] and Dini [D], see (2.29). We observe that the classfication of Theorem B is complete since the minimal ruled surfaces in S^3 are given in [L], see (2.16).

Now if we assume M to be complete, we have

Corollary C. *Let* M^n, $n \geq 3$, *be a complete connected ruled Weingarten hypersurface in* $\mathbf{R}^{n+1}$. *Then,* M *is either*

(i) *a product* $Q^2 \times \mathbf{R}^{n-2}$, *where* Q^2 *is a complete ruled helicoidal surface of a hyperboloid of revolution*; *or*

(ii) *a cylinder over a complete curve.*

1. Preliminares

Let $M^n \subset \mathbf{R}^{n+1}$ be a connected orientable immersed hypersurface endowed with the induced metric. The *relative nullity* of the immersion at a point $p \in M$, is $\ker A(p)$, where A denotes the second fundamental form of the hypersurface. Suppose that the relative nullity has constant dimension $\overline{\nu} = n - k$. Then the Gauss map $\phi: M^n \to S^n \subset \mathbf{R}^{n+1}$ is parallel along each leaf of the relative nullity foliation, and provides (locally) a *Gauss parametrization* of M as it was defined in $[\mathrm{DG}]_1$. More precisely, there exists an isometric immersion $g: L^k \to S^n$, which is a local parametrization of the image of the Gauss map ϕ, and a differentiable function $\gamma: L^k \to R$ (support function) such that

$$\begin{aligned} &X: U \subset \Lambda \to M^n \subset \mathbf{R}^{n+1}, \\ &(x, v) \mapsto X(x, v) = \gamma(x) g(x) + \operatorname{grad} \gamma(x) + v \end{aligned} \tag{1.1}$$

is a local parametrization of M^n, where Λ is the normal bundle of the immersion g. X is the so-called Gauss parametrization of M.

For each $(x, v) \in U \subset \Lambda$, let Hess $\gamma(x)$ denote the hessian of γ and B_v the second fundamental form of the immersion g at $x \in L^k$, relative to the normal vector v. Then the selfadjoint operator defined on the tangent space of L^k at x,

$$P_{(x,v)} = \gamma(x) I + \operatorname{Hess} \gamma(x) - B_v \tag{1.2}$$

is nonsingular. Moreover, the second fundamental form $A_{(x,v)}$ of X at (x, v) is given by $-P^{-1}$, when restricted to the orthogonal complement of the relative nullity distribution. We refer to $[\mathrm{DG}]_1$ for the above results.

For each vector field $e: L^k \to \mathbf{R}^{n+1}$, we may consider an associated vector field $\overline{e}: U \subset \Lambda \to \mathbf{R}^{n+1}$ defined by

$$\overline{e}(x, v) = e(x), \quad \forall (x, v) \in U,$$

i.e. $\overline{e}$ is the euclidean parallel transport of $e(x)$ along the leaves of the relative nullity foliation of M. Therefore, if e is a vector field normal (resp. tangent) to the immersion g, then the associated vector field $\overline{e}$ belongs (resp. is orthogonal) to the relative nullity distribution.

In what follows we consider hypersurfaces $M^n \subset \mathbf{R}^{n+1}$ with constant index of relative nullity $\overline{\nu} = n-2$, locally parametrized as in (1.1). Moreover, we choose orthonormal vector fields $e_1, \dots, e_n$, locally defined on L^2, such that $e_1(x)$, $e_2(x)$ are tangent to the immersion g at x and $e_3(x), \dots, e_n(x)$ generate the normal space of the immersion in S^n. Let $\overline{e}_i(x, v) = e_i(x)$, $1 \leq i \leq n$, $(x, v) \in U \subset \Lambda$, be the associated vector fields on M. With respect to this frame the second fundamental form of X at (x, v) is given by

$$A = \begin{pmatrix} -P^{-1}_{(x,v)} & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.3}$$

where P is defined by (1.2).

It follows that the mean curvature $\overline{H}$ and the scalar curvature $\overline{S}$ of M at (x, v) are given respectively by

$$\overline{H}(x, v) = -\operatorname{tr} A = \frac{\operatorname{tr} P}{\det P}, \tag{1.4}$$

$$\overline{S}(x, v) = \frac{1}{\det P}. \tag{1.5}$$

Lemma 1.6. *Let $M^n \subset \mathbf{R}^{n-1}$ be a ruled immersed hypersurface with constant index of relative nullity $\overline{\nu} = n-2$. Then the immersion g is a ruled surface in S^n.*

Proof. Let

$$X(s, \lambda, \mu_j) = c(s) + \lambda\xi(s) + \sum_{j=1}^{n-2} \mu_j \eta_j(s)$$

be a local parameterization of M, where $c(s)$ is a curve orthogonal to the ruling, η_j, $1 \leq j \leq n-2$, generate the relative nullity and $\{\xi, \eta_j\}$ generate the ruling of M^n. Then the Gauss map depends only on the parameters s, λ, since η_j generate the relative nullity distribution. Moreover, for $s = s_0$, the Gauss map describes a curve which is orthogonal to the subspace generated by $\xi(s_0)$, $\eta_j(s_0)$, $1 \leq j \leq n-2$. Therefore it is contained in a great circle of S^n. Q.E.D.

Fact 1.7. It follows from the above lemma that if M is a ruled hypersurface then the frame considered earlier may be chosen such that $e_1(x)$ is tangent to the ruling of the immersion g. Thus the second fundamental form θ of g with values in the normal bundle satisfies $\theta(e_1, e_1) = 0$. Therefore, the associated frame tangent to M, $\overline{e}_i(x, v) = e_i(x)$, is such that $\overline{e}_i$, $3 \le i \le n$, generate the relative nullity, $\overline{e}_i$, $2 \le i \le n$, generate the ruling and $\langle A\overline{e}_2, \overline{e}_2\rangle = 0$.

For such a frame, the second fundamental form of the immersion g, with respect to e_i, $3 \le i \le n$, will be denoted by

$$B_i(x) = \begin{pmatrix} 0 & \beta_i \\ \beta_i & \lambda_i \end{pmatrix}, \qquad 3 \le i \le n, \tag{1.8}$$

and the operator $\gamma(x)I + \operatorname{Hess}\gamma(x)$ will be denoted by

$$\gamma(x)I + \operatorname{Hess}\gamma(x) = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix}. \tag{1.9}$$

Now we assume that the submanifold $M^n \subset \mathbf{R}^{n+1}$ is Weingarten, i.e. there exists a differentiable function $F(\overline{H}, \overline{S}) = 0$. Taking exterior derivatives we obtain

$$\frac{\partial F}{\partial \overline{H}} d\overline{H} + \frac{\partial F}{\partial \overline{S}} d\overline{S} = 0.$$

Therefore, applying to vector fields tangent to M, we conclude that

$$d\overline{H} \wedge d\overline{S} = 0, \tag{1.10}$$

since the partial derivatives of F are not simultaneously zero.

Fact 1.11. Let $M^n \subset \mathbf{R}^{n+1}$ be a ruled Weingarten hypersurface with constant index of relative nullity $\overline{\nu} = n - 2$. Then it follows from (1.4) to (1.10) that

$$d\left(\alpha(x) - \sum_{i=3}^{n} t_i \lambda_i(x)\right) \wedge d\left(h(x) - \sum_{j=3}^{n} t_j \beta_j(x)\right) = 0 \tag{1.12}$$

for $t_i \in \mathbf{R}$.

2. Proofs of the theorems

For the proof of Theorem B we will need the following three propositions.

Proposition 2.1. *Let $M^n \subset \mathbf{R}^{n+1}$ be a connected ruled Weingarten hypersurface without flat points. Suppose that the dimension of the first normal space of g is constant equal to 1. Then, there exists a totally geodesic submanifold $S^3 \subset S^n$ such that $g(L^2) \subset S^3$ is a ruled Weingarten surface which satisfies*

$$H^2 + c^2(K - 1) = 0,$$

where H and K are the mean and Gaussian curvature and c is a constant. Moreover, M^n is contained in a euclidean product $Q^3 \times \mathbf{R}^{n-3}$, where $Q^3 \subset \mathbf{R}^4$ is a ruled Weingarten surface with index of relative nullity $\nu = 1$.

Proposition 2.2. *Let $g: L^2 \to S^3$ be a connected ruled surface in S^3 such that*

$$H^2 + c^2(K - 1) = 0.$$

Then either $H = 0$ or $H = c \neq 0$ and $K = 0$. In the latter case the immersed surface is contained in the product of two circles.

Proposition 2.3. *Let $M^3 \subset \mathbf{R}^4$ be a connected ruled Weingarten hypersurface, with index of relative nullity $\bar{\nu} = 1$. Suppose that the image of the Gauss map $g(L^2)$ is either*

(i) *a minimal surface in S^3; or*
(ii) *it is contained in the product of two circles.*

Then M^3 is an open subset of a cone over $g(L^2)$.

We need the following result. Recall that the first normal space of an immersion is the subspace generated by the second fundamental form.

Lemma 2.4. *Let M^n be a ruled Weingarten hypersurface without flat points. Then, for each $x \in L^2$, the dimension of the first normal space N_1 of g is less than or equal to 1.*

Proof. The ruled hypersurface M has no flat points if and only if the index of relative nullity is constant $\bar{\nu} = n - 2$.

Since M^n is a ruled hypersurface, it follows from Lemma 1.6 that $g(L^2)$ is a ruled surface. Let $e_1(x)$, $e_2(x)$ be a locally defined tangent frame to the immersion g such that $e_1(x)$ is tangent to the ruling. Let $N_1(x)$ be the first normal space of g at x. Since N_1 is generated by $\theta(e_1, e_2)$, $\theta(e_2, e_2)$, it follows that $\dim N_1 \leq 2$.

Suppose $\dim N_1(x) = 2$. We choose $e_3(x)$, $e_4(x)$ generating N_1 such that e_4 is orthogonal to $\theta(e_1, e_2)$. Then the second fundamental form with respect to e_3 and e_4 in the tangent basis e_1, e_2 is given respectively by

$$B_3 = \begin{pmatrix} 0 & \beta_3 \\ \beta_3 & \lambda_3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_4 \end{pmatrix}.$$

Since M^n is a Weingarten hypersurface it follows from (1.13) that $\lambda_4 \beta_3 = 0$. If $\lambda_4 = 0$, then $B_4 = 0$. If $\beta_3 = 0$, then $\theta(e_1, e_2) = 0$. In both cases we have a contradiction, since we assumed that $\dim N_1 = 2$. Q.E.D.

Proof of Proposition 2.1. Since M is a ruled hypersurface without flat points, the index of relative nullity is constant $\bar{\nu} = n - 2$. Let $e_1(x), \dots, e_n(x)$ be a orthonormal frame defined locally on L^2 as in Fact 1.7. Moreover, we can choose $e_3(x)$ to generate the first normal space N_1 of g. For such a frame, the second fundamental form of g (1.8) reduces to

$$B_3 = \begin{pmatrix} 0 & \beta \\ \beta & \lambda \end{pmatrix}, \qquad B_i = 0, \ 4 \leq i \leq n. \tag{2.5}$$

Since M is Weingarten, it follows from (1.12) that

$$d(\alpha - t\lambda) \wedge d(h - t\beta) = 0, \qquad t \in \mathbf{R}. \tag{2.6}$$

Applying (2.6) to the pair $(e_i, \partial/\partial t)$, $i = 1, 2$, we obtain

$$-\beta d\alpha + \lambda dh = 0, \tag{2.7}$$

$$\beta d\lambda - \lambda d\beta = 0. \tag{2.8}$$

It follows from (2.8) that there exist constants c_1, c_2, not simultaneously zero, such that

$$c_1\beta + c_2\lambda = 0. \tag{2.9}$$

Observe that $c_2 \neq 0$. In fact if $c_2 = 0$, then from (2.9) we have $\beta = 0$. Now, $\dim N_1 = 1$ implies that $\lambda \neq 0$ and (2.7) implies that h is a constant. Therefore, it follows from (1.5) that $\overline{S} = -1/h^2$ is constant. However, Theorem 3.4 in $[\mathrm{DG}]_1$ implies that $\dim N_1 = 0$, which is a contradiction.

Therefore, we have

$$\lambda = c\beta \tag{2.10}$$

and $\beta \neq 0$ in L^2. Moreover, it follows from (2.7) that $\alpha = ch + \overline{c}$, where c and $\overline{c}$ are constants.

Now we prove that the first normal space of the immersion $g: L^2 \to S^n$ is parallel. In fact, let η be any vector field generated by $e_4, \dots, e_n$. Then it follows from the Codazzi equation that

$$B_{\nabla^{\perp}_{e_1}\eta}e_2 = B_{\nabla^{\perp}_{e_2}\eta}e_1,$$

where $\nabla^{\perp}$ is the connection in the normal bundle. Hence

$$\langle\nabla^{\perp}_{e_1}\eta, e_3\rangle B_3 e_2 = \langle\nabla^{\perp}_{e_2}\eta, e_3\rangle B_3 e_1.$$

Using (2.5) we get

$$\langle\nabla^{\perp}_{e_1}\eta, e_3\rangle \beta e_1 + [\langle\nabla^{\perp}_{e_1}\eta, e_3\rangle\lambda - \langle\nabla^{\perp}_{e_2}\eta, e_3\rangle\beta]e_2 = 0.$$

Since $\beta \neq 0$ we conclude that

$$\langle\nabla^{\perp}_{e_1}\eta, e_3\rangle = \langle\nabla^{\perp}_{e_2}\eta, e_3\rangle = 0.$$

Hence the first normal space of the immersion g is parallel. It follows that there exists a totally geodesic submanifold $S^3 \subset S^n$ which contains the image of g. Therefore, the normal bundle Λ of g splits into $\Lambda = \Lambda_1 + \Lambda_{n-3}$, where Λ_1 is the normal bundle in S^3 and the orthogonal complement Λ_{n-3} is parallel in $\mathbf{R}^{n+1}$. Hence, M^n splits as a consequence of the Gauss parametrization.

Finally, from (2.5) we obtain that the mean curvature H and the Gaussian curvature K satisfies $H = \lambda$ and $K - 1 = \beta^2$. Therefore, it follows from (2.10) that $H^2 + c^2(K-1) = 0$. Q.E.D.

Fact 2.11. It follows from the preceding proof that if M^n satisfies the hypothesis of Proposition 1 then there is a frame locally defined on L^2 for which

$$B_3 = \begin{pmatrix} 0 & \beta \\ \beta & \lambda \end{pmatrix}, \qquad B_i = 0,\ 4 \leq i \leq n,$$

$$\gamma I + \operatorname{Hess}\gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix},$$

where $\lambda = c\beta$, $\alpha = ch + \overline{c}$, and c, $\overline{c}$ are constants.

Proof of Proposition 2.2. If the constant c is zero, then $H = 0$. Otherwise, we will show that $K = 0$ and hence $H = c$.

Let $g: L^2 \to S^3$ be a parametrized ruled surface in S^3. We may consider

$$g(s, t) = \cos t\sigma(s) + \sin te(s),$$

where $\sigma(s)$ and $e(s)$ are vectors in $\mathbf{R}^4$ such that

$$|\sigma| = 1 = |e|, \quad \langle e, \sigma\rangle = 0, \quad \langle e', \sigma'\rangle = 0.$$

Moreover, we may choose the parameter s such that $|e'| = 1$. We introduce the following notation

$$\begin{aligned} p(s) &= \langle e', \sigma' \times e \times \sigma\rangle = (e'\sigma' e\sigma), \\ A(s) &= |\sigma'|^2 - \langle \sigma', e\rangle^2, \\ B(s) &= 1 - \langle \sigma', e\rangle^2, \\ G(s, t) &= A\cos^2 t + B\sin^2 t. \end{aligned} \tag{2.12}$$

We observe that $p = \sqrt{AB}$. Moreover, it follows by a straightforward computation that the mean and Gaussian curvature of the surface are given by

$$H = \frac{l - 2p\langle \sigma', e\rangle}{2G^{3/2}}, \quad K - 1 = -\frac{p^2}{G^2}, \tag{2.13}$$

where

$$\begin{aligned} l(s, t) = &\cos^2 t(\sigma'' \sigma' e\sigma) + \sin^2 t(e'' e' e\sigma) \\ &+ \sin t\cos t[(\sigma'' e' e\sigma)_+(e''\sigma' e\sigma)]. \end{aligned} \tag{2.14}$$

By hypothesis $H^2 + c^2(K - 1) = 0$, therefore, without loss of generality, we have

$$l - 2p\langle\sigma', e\rangle - 2cpG^{1/2} = 0.$$

Taking a derivative with respect to t we get

$$\partial l/\partial t - cpG^{-1/2}\partial G/\partial t = 0. \tag{2.15}$$

In particular for $t = 0$, it follows from (2.14) and (2.15) that

$$(\sigma'' e' e\sigma) + (e''\sigma' e\sigma) = 0.$$

Hence (2.15) reduces to

$$2\sin t\cos t[-(\sigma''\sigma' e\sigma) + (e'' e' e\sigma)] - cpG^{-1/2}\partial G/\partial t = 0,$$

which is equivalent to

$$\left[\frac{1}{AB}\left(\frac{d}{ds}(\sigma' \times e \times \sigma)e' e\sigma\right) - cG^{1/2}\right]\frac{\partial G}{\partial t} = 0, \quad \forall s, t.$$

Since c is a nonzero constant, it follows that

$$\partial G/\partial t = 0, \quad \forall s, t.$$

Therefore, using (2.12) we get $A = B$, $p = A = G$. From (2.13) we get $K = 0$ and hence $H = c$.

In order to show that the surface is contained in a product of two circles, we consider a local orthonormal frame field such that the second fundamental form is given by

$$B = \begin{pmatrix} 0 & \beta \\ \beta & \lambda \end{pmatrix}.$$

From $K = 0$ and $H = c$, we have $\det B = -1$ and $\lambda = 2c$, so that

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 2c \end{pmatrix}.$$

We conclude the proof by using the uniqueness part of the fundamental theorem for surfaces in the sphere, see [S]. Q.E.D.

Proof of Proposition 2.3. *Part* (i). Since $\nu = 1$, there exists an immersion

$$g: L^2 \to G \subset S^3$$

and a local Gauss parametrization of M^3 given by

$$X: \Lambda \to M^3 \subset R^4, \quad (x, v) \mapsto \gamma(x)g(x) + \operatorname{grad}\gamma(x) + v,$$

where Λ is the normal bundle of the immersion g and $\gamma: L^2 \to R$ is a differentiable function.

M^3 is a ruled hypersurface, therefore it follows from Lemma 1.6 that $g(L^2)$ is a ruled surface in S^3. Since $g(L^2)$ is also minimal, we have g locally given by

$$\begin{aligned} g(x_1, x_2) = {} & \cos x_1(\cos kx_2, \sin kx_2, 0, 0) \\ & + \sin x_1(0, 0, \cos x_2, \sin x_2), \end{aligned} \tag{2.16}$$

where k is a positive constant, see [L or BDJ]. Let us consider the orthonormal tangent frame

$$e_1 = \frac{\partial g}{\partial x_1}, \quad e_2 = \frac{1}{\sqrt{E}}\frac{\partial g}{\partial x_2}, \tag{2.17}$$

where $E = k^2\cos^2 x_1 + \sin^2 x_1$. Let e_3 be a unitary normal vector field for the immersion g. Then the second fundamental form with respect to this frame is given by

$$B(x) = \begin{pmatrix} 0 & -k/E \\ -k/E & 0 \end{pmatrix}, \qquad x = (x_1, x_2). \tag{2.18}$$

Let $\bar{e}_i$ be the associated frame defined on M^3, i.e.

$$\bar{e}_i(x, v) = e_i(x), \qquad 1 \le i \le 3, \ (x, v) \in \Lambda.$$

Then the second fundamental form for $M^3 \subset R^4$, with respect to this frame, is given by

$$A(x, v) = \begin{pmatrix} -P^{-1}_{(x,v)} & 0 \\ 0 & 0 \end{pmatrix}, \qquad (x, v) \in \Lambda,$$

where

$$P(x, v) = \gamma(x) + \operatorname{Hess}\gamma(x) - \langle v, e_3\rangle B(x).$$

Moreover, it follows from Fact 2.11 that $\gamma I + \operatorname{Hess}\gamma$ is of the form

$$\gamma I + \operatorname{Hess}\gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix},$$

where

$$0 = -ck/E, \quad \alpha = ch + \bar{c}.$$

Hence, $c = 0$ and $\alpha = \bar{c}$.

Now, we want to determine $\gamma: L^2 \to R$ such that

$$\gamma I + \operatorname{Hess}\gamma = \begin{pmatrix} 0 & * \\ * & \bar{c} \end{pmatrix}. \tag{2.19}$$

It follows from (2.17) and (2.19) that γ must satisfy

$$\gamma + \frac{2}{x_1^2} = 0, \quad \gamma + \frac{1}{E}\left(\frac{\partial^2\gamma}{\partial x_2^2} + \sin x_1 \cos x_1 (1 - k^2)\frac{\partial\gamma}{\partial x_1}\right) = \bar{c}. \tag{2.20}$$

From the first equation we get

$$\gamma = f(x_2)\cos x_1 + h(x_2)\sin x_1.$$

Substituting into (2.20) we get $\bar{c} = 0$. Therefore, the trace of P and hence the trace of A is zero, i.e. M^3 is a minimal surface in R^4.

To conclude the proof in this case we use [BDJ].

Part (ii). By hypothesis $g(L^2)$ is contained in the product of two circles, therefore the immersion g is locally given by

$$g(x_1, x_2) = r_1\left(\sin\frac{x_1}{r_1}, \cos\frac{x_1}{r_1}, 0, 0\right) + r_2\left(0, 0, \sin\frac{x_2}{r_2}, \cos\frac{x_2}{r_2}\right) \tag{2.21}$$

where $r_1^2 + r_2^2 = 1$.

Let us consider the orthonormal frame field defined by

$$e_1 = r_1\frac{\partial g}{\partial x_1} - r_2\frac{\partial g}{\partial x_2}, \quad e_2 = r_2\frac{\partial g}{\partial x_1} + r_1\frac{\partial g}{\partial x_2}. \tag{2.22}$$

Then the second fundamental form of the immersion g with respect to e_1, e_2 is given by

$$B = \begin{pmatrix} 0 & 1 \\ 1 & \frac{r_2^2 - r_1^2}{r_1 r_2} \end{pmatrix}.$$

Let e_3 be a unitary normal vector field for the immersion g and

$$\bar{e}_i(x, v) = e_i(x), \qquad 1 \le i \le 3, \ (x, v) \in \Lambda,$$

the associated frame defined on M^3. Then the second fundamental form for $M^3 \subset R^4$, with respect to this frame, is given by

$$A(x, v) = \begin{pmatrix} -P_{(x,v)}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

where $P(x, v) = \gamma(x) + \operatorname{Hess}\gamma(x) - \langle v, e_3\rangle B$.

Moreover, it follows from Fact 2.11 that with respect to this frame

$$\gamma I + \operatorname{Hess}\gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix},$$

where

$$c = \frac{r_2^2 - r_1^2}{r_1 r_2}, \quad \alpha = ch + \bar{c}.$$

We want to determine $\gamma\colon L^2 \to \mathbf{R}$ which satisfies the above conditions. It follows from (2.22) that γ must satisfy

$$\begin{aligned} &\gamma + r_1^2\frac{\partial^2\gamma}{\partial x_1^2} - 2r_1r_2\frac{\partial^2\gamma}{\partial x_1\partial x_2} + r_2^2\frac{\partial^2\gamma}{\partial x_2^2} = 0, \\ &\gamma + r_1^2\frac{\partial^2\gamma}{\partial x_1^2} + \frac{r_2^4 + r_1^4}{r_1r_2}\frac{\partial^2\gamma}{\partial x_1\partial x_2} + r_2^2\frac{\partial^2\gamma}{\partial x_2^2} = \bar{c}. \end{aligned} \tag{2.23}$$

Subtracting the above equation we get

$$\frac{1}{r_1r_2}\frac{\partial^2\gamma}{\partial x_1\partial x_2} = \bar{c}.$$

Therefore

$$\gamma(x_1, x_2) = \bar{c}r_1r_2x_1x_2 + \gamma_1(x_1) + \gamma_2(x_2), \tag{2.24}$$

where γ_1 and γ_2 are functions which depend only on x_1 and x_2 respectively. Substituting (2.24) into (2.23), we obtain

$$\gamma_1 + r_1^2\frac{d^2\gamma_1}{dx_1^2} + \gamma_2 + r_2^2\frac{d^2\gamma_2}{dx_2^2} + \bar{c}r_1r_2x_1x_2 = 2r_1^2r_2^2\bar{c}. \tag{2.25}$$

Taking derivatives with respect to x_1, and then with respect to x_2 we conclude that $\bar{c} = 0$. Therefore, (2.24) reduces to

$$\gamma(x_1, x_2) = \gamma_1(x_1) + \gamma_2(x_2), \tag{2.26}$$

where γ_1 and γ_2 satisfy the following equations

$$\gamma_1 + r_1^2\frac{\partial^2\gamma_1}{\partial x_1^2} = a, \quad \gamma_2 + r_2\frac{\partial^2\gamma_2}{\partial x_2^2} = a, \tag{2.27}$$

where a is a constant.

Now we want to show that the Gauss parametrization of M^3 describes a cone over G. In fact

$$X(x_1, x_2, s) = \gamma(r_1u_1 + r_2v_1) + \frac{d\gamma_1}{\partial x_1}v_1 + \frac{d\gamma_2}{\partial x_2}v_2 + s(-r_2u_1 + r_1u_2),$$

where

$$u_1 = \left(\sin\frac{x_1}{r_1}, \cos\frac{x_1}{r_1}, 0, 0\right),$$

$$u_2 = \left(0, 0, \sin\frac{x_2}{r_2}, \cos\frac{x_2}{r_2}\right),$$

$$v_1 = \partial X/\partial x_1, \quad v_2 = \partial X/\partial x_2.$$

It follows from (2.26) and (2.27) that $X(x_1, x_2, s(x_1, x_2))$ is constant for

$$s(x_1, x_2) = \frac{a}{r_1 r_2} - \frac{r_2}{r_1}\gamma_1 + \frac{r_1}{r_2}\gamma_2,$$

which concludes the proof of case (ii). Q.E.D.

Finally, we prove Theorem B using the preceding results.

Proof of Theorem B. Let $\overline{M} = \{p \in M;\ \overline{S}(p) \neq 0\}$. Since M is a ruled hypersurface, the sectional curvature $\overline{K}$ at points of $\overline{M}$ is not identically zero. It follows from Lemma 2.4 applied to $\overline{M}$ that at each point of the image of the Gauss map the first normal space N_1 has dimension ≤ 1. We have $\overline{M} = \overline{M}_0 \cup \overline{M}_1$, where at $\overline{M}_0$ the Gauss map is totally geodesic in S^n and $\overline{M}_1$ is the open subset of points where N_1 has dimension 1.

Let V_1 be a connected component of $\overline{M}_1$, let $X: U \subset \Lambda \to V_1 \subset R^{n+1}$ be a Gauss parametrization and let $g: L^2 \to S^n$ be the associated local parametrization of the Gauss map of V_1. It follows from Proposition 2.1 that there exists a totally geodesic submanifold $S^3 \subset S^n$ such that $g(L^2) \subset S^3$ is a ruled Weingarten surface which satisfies $H^2 + c^2(K-1) = 0$. Moreover, V_1 is contained in a euclidean product $Q^3 \times \mathbf{R}^{n-3}$, where $Q^3 \subset \mathbf{R}^4$ is a ruled Weingarten surface with constant index of relative nullity $\nu = 1$.

Using Proposition 2.2, we obtain that either g is a minimal immersion in S^3 or $K = 0$, $H = c$, and the image of g is contained in the product of two circles of S^3. It follows from Proposition 2.3, that Q^3 is an open subset of a cone over the image of g, i.e. V_1 satisfies (ii).

Let V_0 be a connected open subset of $\overline{M}_0$. We have a Gauss parametrization for V_0 and g the associated local parametrization of the image of the Gauss map of V_0. Since g is totally geodesic in S^n, the normal bundle Λ of the immersion g is parallel in $\mathbf{R}^{n+1}$. Hence, using the Gauss parametrization we obtain that V_0 is an open subset of $Q^2 \times \mathbf{R}^{n-2}$, where $Q^2 \subset \mathbf{R}^3$ is a ruled Weingarten surface. It follows from the classical result of Beltrami [B] and Dini [D] that Q^2 is a ruled helicoidal surface or a hyperboloid of revolution, i.e. V_0 satisfies (iii).

We now observe that the boundary of V_0 does not intersect the boundary of V_1, since the determinant of the second fundamental form of the image of the Gauss map of V_1 in S^3 is bounded away from zero. Moreover, the boundaries of V_0 and of V_1 do not contain points where the scalar curvature $\overline{S}$ is zero. Since M is connected, this concludes the proof of the theorem. Q.E.D.

Remark 2.28. The ruled Weingarten surfaces $Q^2 \subset \mathbf{R}^3$ classified by Beltrami and Dini are given by

(2.29) $$X(s, t) = (a\cos s + ct\sin s,\ a\sin s - ct\cos s,\ bs + \sqrt{1-c^2}t)$$

where a, b, c are constants.

Proof of Corollary C. We use Theorem B. If M is complete, then it cannot be a cone. If M splits as in (iii), then $M = Q^2 \times \mathbf{R}^2$, where Q^2 is a complete ruled helicoidal surface or a hyperboloid of revolution. If M is flat, it follows from [HN] that M is a cylinder over a complete curve. Q.E.D.

Proof of Theorem A. If $M^n \subset \mathbf{R}^{n+1}$, $n \geq 4$, is a complete hypersurface, it follows from $[\mathrm{DG}]_2$ that M is rigid, unless it contains an open subset U which is completely ruled.

We will show that the existence of such a subset U contradicts the hypothesis of Theorem A. In fact, if we apply Theorem B to each connected component U_0 of U, we conclude that U_0 is completely ruled and flat. We consider a connected component of U_0 where the nullity is $n-1$. Then the ruling coincides with the nullity and therefore the nullity is complete. The argument used in [HN] implies that this component of U_0 is a cylinder over a curve (not necessarily complete). Moreover, each connected component of U_0 where the nullity is n is totally geodesic. Hence, in both cases we obtain open subsets of type $L^3 \times \mathbf{R}^{n-3}$, with L^3 unbounded, which is a contradiction. Therefore, M is rigid. Q.E.D.

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