

## UNIFORM ANALYTICITY OF ORTHOGONAL PROJECTIONS

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**ABSTRACT.** Let  $X$  denote the circle  $T$  or the interval  $[-1, 1]$ , and let  $d\mu$  denote a nonnegative, absolutely continuous measure on  $X$ . Under what conditions does the Gram-Schmidt procedure in the weighted space  $L^2(X, \omega^2 d\mu)$  depend analytically on the logarithm of the weight function  $\omega$ ? In this paper, we show that, in numerous examples of interest,  $\log \omega \in BMO$  is a sufficient (often necessary!) condition for analyticity of the Gram-Schmidt procedure. These results are then applied to establish the local analyticity of certain infinite-dimensional Toda flows.

### 1. INTRODUCTION

Let  $X$  denote the circle  $T$  or the interval  $[-1, 1]$ , let  $d\mu$  be a nonnegative measure on  $X$  which is absolutely continuous with respect to Lebesgue measure, and let  $L(0)$  denote the complex Hilbert space  $L^2(X, d\mu)$ . Let  $\omega$  be a nonnegative  $d\mu$ -measurable function on  $X$  such that  $\omega^2 + \omega^{-2} \in L^1(X, d\mu)$  and let  $\beta = \log \omega$ ; clearly also  $\beta \in L^1(X, d\mu)$ . Let  $L(\beta)$  denote the complex weighted Hilbert space  $L^2(X, \omega^2 d\mu)$  and, for each nonnegative integer  $n$ , let  $H_n(\beta)$  denote the closure of the polynomials (in the case  $X = T$ , trigonometric polynomials) of degree at most  $n$  in  $L(\beta)$ . Let  $S_n(\beta)$  denote the orthogonal projection of  $L(\beta)$  onto  $H_n(\beta)$ . We wish to study the dependence of the family of operators  $\langle S_n(\beta) : n \in \mathbf{N} \rangle$  upon the functional parameter  $\beta$ . Each  $S_n(\beta)$  is a bounded operator on  $L(\beta)$ , which varies with  $\beta$ , so to facilitate our study, we "lift" each operator  $S_n(\beta)$  to  $L(0)$  by means of the operator  $M_\omega$  of pointwise multiplication by  $\omega$ , which is an isometry from  $L(\beta)$  to  $L(0)$ . If we define, for each nonnegative integer  $n$ , the operator  $Q_n(\beta) = M_\omega S_n(\beta) M_\omega^{-1}$ , then we see that the  $L(0)$ -boundedness of  $Q_n(\beta)$  is equivalent to the  $L(\beta)$ -boundedness of  $S_n(\beta)$ , and the operator norms are equal. In fact,  $Q_n(\beta)$  is easily seen to be the self-adjoint projection of  $L(0)$  onto  $M_\omega H_n(\beta) \subseteq L(0)$ .

We would like to determine conditions on  $\beta$  under which the family of operators  $\langle Q_n(\beta) \rangle$  depends analytically (in a sense to be made precise) upon the functional parameter  $\beta$ . In the specific examples which we consider, it is

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difficult if not impossible to write down the operators  $S_n(\beta)$  and  $Q_n(\beta)$  explicitly. On the other hand, the "base projection"  $S_n(0)$  is an integral operator whose kernel is comparatively easy to write down. Rather than study the operators  $\langle Q_n(\beta) \rangle$  directly, it is much more convenient to work with the family of operators  $\langle P_n(\beta) \rangle$  defined by

$$(1.1) \quad P_n(\beta) = M_\omega S_n(0) M_\omega^{-1}.$$

For each nonnegative integer  $n$ ,  $P_n(\beta)$  is an oblique (i.e., non-self-adjoint) projection from  $L(0)$  onto  $M_\omega H_n(\beta) \subseteq L(0)$ , and its adjoint,  $P_n(\beta)^*$ , is simply  $P_n(-\beta)$ .

A remarkable formula due to Kerzman and Stein ([8]) shows that, in fact,  $\langle Q_n(\beta) \rangle$  depends analytically on  $\beta$  whenever  $\langle P_n(\beta) \rangle$  depends analytically on  $\beta$ . Moreover, the analytic dependence of  $\langle P_n(\beta) \rangle$  upon  $\beta$  is essentially equivalent to a uniform weighted norm inequality of the form

$$(1.2) \quad \int_X |S_n(0)f(x)|^2 \omega^2(x) d\mu(x) \leq C \int_X |f(x)|^2 \omega^2(x) d\mu(x)$$

where  $C$  is a constant independent of  $n$  and  $f$ .

In addition, we would like to determine the "space of uniform holomorphy" for the family  $\langle Q_n(\beta) \rangle$ , i.e. the largest Banach function space on which the family  $\langle Q_n \rangle$  is analytic at the origin. In practice, this amounts to determining necessary and sufficient conditions on the function  $\beta$  such that

$$(1.3) \quad \int_X |[M_\beta, S_n(0)]f(x)|^2 d\mu(x) \leq C \int_X |f(x)|^2 d\mu(x)$$

where  $M_\beta$  is the operator of pointwise multiplication by  $\beta$ , and  $C$  is a constant independent of  $n$  and  $f$ .

Our work in this paper has been inspired in part by the related work of Coifman and Rochberg in [2], and by questions arising from the study of Toda flows in infinite dimensions (see, for example, [4]).

In this paper, we consider a number of examples. In the case  $X = \mathbf{T}$ ,  $d\mu = d\theta$ , the uniform weighted norm inequality (1.2), and the uniform commutator estimate (1.3), are equivalent to the same inequalities with  $S_n(0)$  replaced by the conjugate function. In this simplest example, the space of uniform holomorphy for  $\langle Q_n(\beta) \rangle$  is easily seen to be the space of functions of bounded mean oscillation on  $\mathbf{T}$ . In the case  $X = [-1, 1]$ ,  $d\mu =$  Lebesgue measure weighted by a Jacobi weight, the uniform weighted norm inequality (1.2) follows from a weighted norm inequality for the Hilbert transform. In this case, we prove that  $\langle Q_n(\beta) \rangle$  depends analytically on  $\beta$  when  $\beta$  is in a neighborhood of 0 in the space  $\mathbf{BMO}([-1, 1])$ . As an application of this result, we show that the Toda flow corresponding to the measure  $\omega(x)^{2t} d\mu(x)$  on  $[-1, 1]$  is analytic in  $t$  in a neighborhood of the origin provided that  $\beta \in \mathbf{BMO}([-1, 1])$ .

We conjecture that  $\mathbf{BMO}([-1, 1])$  is the space of uniform holomorphy for  $\langle Q_n \rangle$  in the case where  $d\mu =$  Lebesgue measure weighted by a Jacobi weight.

We consider the example  $X = [0, \pi]$ ,  $d\mu = d\theta$ ,  $S_n =$   $n$ th partial sum operator for cosine series, and show that, in this example,  $\mathbf{BMO}([0, \pi])$  is the space of uniform holomorphy for  $\langle Q_n \rangle$ . From this result it is immediate that the conjecture is true for  $d\mu(x) = (1 - x^2)^{-1/2} dx$ . Classical equiconvergence results for Jacobi series and cosine series (see [10]) suggest that the conjecture is probably true in general.

2. UNIFORM ANALYTICITY OF PROJECTIONS: GENERAL SETTING

Let  $(X, d\mu)$  be a  $\sigma$ -finite measure space; set  $L(0) = L^2(X, d\mu)$ . Suppose that  $\omega$  is a nonnegative real-valued function such that  $\omega^2 + \omega^{-2} \in L^1_{loc}(X, d\mu)$ . We write  $\beta = \log \omega$  and observe that  $\beta \in L^1_{loc}(X, d\mu)$ . Let  $\mathbf{N}$  denote the set of nonnegative integers, and suppose that for each  $n \in \mathbf{N}$ ,  $H_n(0)$  is a closed subspace of  $L(0)$ . We assume that

- $L(0, \beta) = L^2(X, (\omega^2 + \omega^{-2})d\mu)$  is dense in  $L(0)$  and
- $H_n(0, \beta) = L(0, \beta) \cap H_n(0)$  is dense in  $H_n(0)$ , for each  $n \in \mathbf{N}$ .

We define the spaces:

- $L(\beta) = L^2(X, \omega^2 d\mu)$ ,
- $\mathcal{L}(\beta) =$  bounded linear operators on  $L(\beta)$ ,
- $H_n(\beta) =$  closure of  $H_n(0, \beta)$  in  $L(\beta)$ , for each  $n \in \mathbf{N}$ .

We assume that the foregoing are complex Hilbert spaces.

For each  $n \in \mathbf{N}$ , let  $S_n(\beta) \in \mathcal{L}(\beta)$  be the selfadjoint projection of  $L(\beta)$  onto  $H_n(\beta)$ . We wish to study the dependence of the operators  $\langle S_n(\beta): n \in \mathbf{N} \rangle$  upon the functional parameter  $\beta$ . To facilitate our study, we "lift" each operator  $S_n(\beta)$  back to  $\mathcal{L}(0)$  by means of the operator  $M_\omega$  of pointwise multiplication by  $\omega$ , to wit: for each  $n \in \mathbf{N}$ , define  $Q_n(\beta) = M_\omega S_n(\beta) M_\omega^{-1}$ . Then  $Q_n(\beta)$  is the self-adjoint projection of  $L(0)$  onto  $M_\omega H_n(\beta) \subseteq L(0)$ , and  $\|Q_n(\beta)\|_{\mathcal{L}(0)} = \|S_n(\beta)\|_{\mathcal{L}(\beta)}$ .

We would like to formulate a clear conception of the "analytic dependence" of the family of operators  $\langle Q_n(\beta): n \in \mathbf{N} \rangle$  upon the functional parameter  $\beta$ . To this end, we make the following definitions.

**Definitions.** Let  $B$  be a real Banach space,  $\mathcal{L}(H)$  the space of bounded linear operators on a complex Hilbert space  $H$ . For  $n \in \mathbf{N}$ , let  $T_n: B \rightarrow \mathcal{L}(H)$  be an operator-valued function on  $B$ .

(a)  $\langle T_n: n \in \mathbf{N} \rangle$  is said to be *uniformly (real-) analytic* in a neighborhood of 0 in  $B$  if and only if there is a constant  $C > 0$  such that, whenever  $b \in B$  with  $\|b\|_B \leq C$  and whenever  $f \in H$ , we have

$$(2.1) \quad T_n(b)f = \sum_{k=0}^{\infty} \Lambda_{n,k}(b, \dots, b, f), \quad \text{for all } n \in \mathbf{N},$$

where  $\Lambda_{n,k}$  is a bounded,  $(k + 1)$ -multilinear operator from  $B^k \times H \rightarrow H$  which satisfies an estimate of form

$$(2.2) \quad \|\Lambda_{n,k}(b, \dots, b, f)\|_H \leq C_0^{k+1} \|b\|_B^k \|f\|_H$$

where  $C_0$  is independent of  $b, f, n$ , and  $k$ .

(b) Let  $\mathbf{B}$  denote the complexification of  $B$ .  $\langle T_n; n \in \mathbf{N} \rangle$  is said to be *uniformly holomorphic* in a neighborhood of 0 in  $\mathbf{B}$  if and only if there is a neighborhood of 0 in  $\mathbf{B}$  to which each  $T_n$  can be extended, and there is a constant  $C > 0$  such that, whenever  $b \in \mathbf{B}$  with  $\|b\|_{\mathbf{B}} \leq C$  and whenever  $f \in H$ , we have (2.1) and (2.2) with ' $\mathbf{B}$ ' in place of ' $B$ '.

(c)  $\mathbf{B}$  is called the *space of uniform holomorphy* at 0 for the family  $\langle T_n; n \in \mathbf{N} \rangle$  if and only if  $\langle T_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in  $\mathbf{B}$  and a necessary and sufficient condition for

$$(2.3) \quad \sup_n \|\Lambda_{n,1}(b, \cdot)\|_{\mathcal{L}(H)}$$

to be finite is that  $b \in \mathbf{B}$ .

We pause to observe that the operator  $\Lambda_{n,k}(b, \dots, b, \cdot)$  occurring in (2.1) is just the  $k$ th Gâteaux (or Fréchet) differential of  $T_n$  at 0 in the direction  $b$  (see, for example, [1, Chapter 2]). We give one equivalent formulation of the notion of uniform holomorphy in terms of Gâteaux differentiability, which will be useful in practice.

**Proposition 2.1.** *Let  $\mathbf{B}$  be a complex Banach space,  $\mathcal{L}(H)$  the space of bounded linear operators on a complex Hilbert space  $H$ . For  $n \in \mathbf{N}$ , let  $T_n: \mathbf{B} \rightarrow \mathcal{L}(H)$  be an operator-valued function on  $\mathbf{B}$ . Then  $\langle T_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in  $\mathbf{B}$  if and only if there exists a neighborhood  $U$  of 0 in  $\mathbf{B}$  on which each  $T_n$  is Gâteaux differentiable and there exists a constant  $C$  such that for all  $n \in \mathbf{N}$  and for all  $\beta \in U$ ,  $\|T_n(\beta)\|_{\mathcal{L}(H)} \leq C$ .*

*Proof.* The proof is an easy modification of the proof of Theorem 2.3.3 of [1] and will be omitted.  $\square$

We can now formulate our general problem precisely, as follows: we wish to identify the space of uniform holomorphy at 0 for the family  $\langle Q_n; n \in \mathbf{N} \rangle$ . In practice it is often difficult to characterize the projections  $S_n(\beta)$  and  $Q_n(\beta)$ ; it is much more convenient to work with the operators  $\langle P_n(\beta); n \in \mathbf{N} \rangle$  defined by

$$(2.4) \quad P_n(\beta) = M_\omega S_n(0) M_\omega^{-1}.$$

For  $n \in \mathbf{N}$ ,  $P_n(\beta)$  is an oblique projection from  $L(0)$  onto  $M_\omega H_n(\beta)$  with adjoint  $P_n(\beta)^* = P_n(-\beta)$ . The formula of Kerzman and Stein (see [8, §3.4]) makes it possible to deduce the uniform holomorphy of  $\langle Q_n \rangle$  from that of  $\langle P_n \rangle$ , and greatly simplifies the computation of the Gâteaux differentials of the operators  $\langle Q_n \rangle$ :

**Proposition 2.2 (Kerzman-Stein Formula).** *Let  $H$  be a complex Hilbert space,  $K$  a closed subspace. Let  $Q$  be the self-adjoint projection of  $H$  onto  $K$ , and let  $P$  be a bounded oblique projection from  $H$  onto  $K$ . Then:*

- (a)  $I + (P - P^*)$  is invertible.
- (b)  $Q = P[I + (P - P^*)]^{-1}$ ;

(c) whenever  $c_0$  and  $M$  are positive constants with  $\|P - P^*\| \leq c_0$  and  $M > \frac{1}{2}(c_0^2 - 1)$ , the series

$$(2.5) \quad P \left\{ \frac{1}{M+1} \sum_{k=0}^{\infty} \left[ \frac{MI - (P - P^*)}{M+1} \right]^k \right\}$$

converges in the operator norm topology to  $Q$ .

*Proof.* The operator  $P - P^*$  is skew-adjoint, so its spectrum is purely imaginary. In particular,  $-1$  is not in the spectrum of  $P - P^*$ , from which (a) follows.

Clearly  $QP = P$  and  $PQ = Q$ . Now let  $h \in H$  and let  $(\cdot | \cdot)$  denote the inner product on  $H$ . We have

$$(2.6) \quad \begin{aligned} (QP^*h|h) &= (P^*h|Qh) \quad \text{since } Q^* = Q \\ &= (h|PQh) \\ &= (h|Qh) \quad \text{since } PQ = Q \\ &= (Qh|h) \quad \text{since } Q^* = Q. \end{aligned}$$

Hence  $QP^* = Q$ ; consequently

$$(2.7) \quad P = Q + (P - Q) = Q + (QP - QP^*) = Q[I + (P - P^*)]$$

whereupon we obtain (b).

It is tempting to expand  $[I + (P - P^*)]^{-1}$  in a Neumann series, but we do not know that  $\|P - P^*\| < 1$ . Instead we proceed as follows. For any constant  $M > 0$ , we have

$$(2.8) \quad \begin{aligned} I + (P - P^*) &= I + MI - [MI - (P - P^*)] \\ &= (1 + M) \left[ I - \frac{MI - (P - P^*)}{1 + M} \right]. \end{aligned}$$

Recall that, if  $S^* = S$ ,  $T^* = -T$ , and  $ST = TS$ , then

$$\|S + T\|^2 \leq \|S\|^2 + \|T\|^2.$$

Thus

$$(2.9) \quad \left\| \frac{MI - (P - P^*)}{1 + M} \right\|^2 = \frac{M^2 + \|P - P^*\|^2}{M^2 + 2M + 1}$$

which is less than 1 provided  $\|P - P^*\|^2 < 2M + 1$ , i.e.,  $M > \frac{1}{2}(\|P - P^*\|^2 - 1)$ . In particular, if  $M > \frac{1}{2}(c_0^2 - 1)$ , we see that

$$(2.10) \quad Q = P \left\{ \frac{1}{M+1} \left[ I - \frac{MI - (P - P^*)}{1 + M} \right]^{-1} \right\}$$

may be expanded in a Neumann series to give (2.5).  $\square$

We obtain, as an immediate consequence, the following:

**Corollary 2.2.1.** *With  $\langle P_n \rangle$  and  $\langle Q_n \rangle$  as in the foregoing discussion, let  $B$  be a real Banach function space on  $(X, d\mu)$  such that  $\langle P_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in  $\mathbf{B}$ . Then  $\langle Q_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in  $\mathbf{B}$ .  $\square$*

The mappings  $P_n$  may be extended to complex-valued functions in a straightforward way: if  $\beta = a + ib$  is complex valued, we define, for  $n \in \mathbf{N}$ ,

$$(2.11) \quad P_n(\beta) = P_n(a + ib) = M_{e^{ib}}P_n(a)M_{e^{-ib}}.$$

If  $P_n(\beta)$  is a bounded operator on  $L(0)$  for all  $\beta$  in a neighborhood of 0 in  $\mathbf{B}$ , then the series (2.5) may be used to extend  $Q_n$  to complex-valued functions in a natural way, such that (by Proposition 2.2(b))

$$(2.12) \quad Q_n(\beta) = P_n(\beta)[I + (P_n(\beta) - P_n(-\beta))]^{-1}.$$

Now let us compute the first Gâteaux differential at 0, in the direction  $\beta \in \mathbf{B}$ , of  $P_n$  and  $Q_n$ . We have

$$(2.13) \quad \begin{aligned} \left. \frac{d}{ds} P_n(s\beta) \right|_{s=0} &= \left. \frac{d}{ds} \{ M_{e^{s\beta}} S_n(0) M_{e^{-s\beta}} \} \right|_{s=0} \\ &= \left. \left\{ M_\beta M_{e^{s\beta}} S_n(0) M_{e^{-s\beta}} - M_{e^{s\beta}} S_n(0) M_\beta M_{e^{-s\beta}} \right\} \right|_{s=0} \\ &= [M_\beta, S_n(0)]. \end{aligned}$$

Thus, by (2.12), we have

$$(2.14) \quad \begin{aligned} \left. \frac{d}{ds} Q_n(s\beta) \right|_{s=0} &= \left. \frac{d}{ds} P_n(s\beta) \right|_{s=0} - P_n(0) \left. \frac{d}{ds} \{ P_n(s\beta) - P_n(-s\beta) \} \right|_{s=0} \\ &= [M_\beta, S_n(0)] - S_n \{ [M_\beta, S_n(0)] + [M_\beta, S_n(0)] \} \\ &= \{ I - 2S_n(0) \} [M_\beta, S_n(0)]. \end{aligned}$$

In light of these calculations we obtain

**Corollary 2.2.2.** *Let  $\mathbf{B}$  be the complexification of a real Banach function space  $B$  on  $(X, d\mu)$ , and suppose that  $\langle P_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in  $\mathbf{B}$ . Then:*

(a)  $\mathbf{B}$  is the space of uniform holomorphy at 0 for  $\langle P_n \rangle$  if and only if  $\beta \in \mathbf{B}$  is a necessary and sufficient condition for

$$(2.15) \quad \sup\{ \| [M_\beta, S_n(0)] \|_{\mathcal{L}(0)} : n \in \mathbf{N} \} < \infty.$$

(b)  $\mathbf{B}$  is the space of uniform holomorphy at 0 for  $\langle Q_n \rangle$  if and only if  $\beta \in \mathbf{B}$  is a necessary and sufficient condition for

$$(2.16) \quad \sup\{ \| \{ I - 2S_n(0) \} [M_\beta, S_n(0)] \|_{\mathcal{L}(0)} : n \in \mathbf{N} \} < \infty. \quad \square$$

In practice it is frequently the case that the base projections  $\langle S_n(0) \rangle$  are given by integration against a kernel. In this case, the uniform holomorphy of  $\langle P_n \rangle$

in a neighborhood of 0 may be reduced to the problem of obtaining a uniform weighted norm inequality for the base projections. This is a consequence of the following general result:

**Proposition 2.3.** *Let  $B$  be a real Banach function space on  $(X, d\mu)$ . Let  $\langle K_n(0): n \in \mathbf{N} \rangle$  be a family of integral operators in  $\mathcal{L}(0)$ , and suppose that, for all  $n \in \mathbf{N}$ , there exists a kernel  $D_n(x, y)$  such that, for  $f \in L(0)$  and  $x \in X$ ,*

$$(2.17) \quad \{K_n(0)f\}(x) = \int_X D_n(x, y)f(y)d\mu(y).$$

For each  $\beta \in \mathbf{B}$ , define  $K_n(\beta) = M_{e^\beta}K_n(0)M_{e^{-\beta}}$ . Then the following are equivalent:

- (a)  $\langle K_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in  $\mathbf{B}$ .
- (b) There exist constants  $\delta_0, C_0 > 0$  such that, for every  $n \in \mathbf{N}$  and for all  $\beta \in \mathbf{B}$  with  $\|\beta\|_B < \delta_0$ ,

$$(2.18) \quad \|K_n(\beta)\|_{\mathcal{L}(0)} \leq C_0.$$

*Proof.* That (a) implies (b) is evident from Proposition 2.1. Now suppose (b) is true. By virtue of the fact that, for all  $\alpha \in B$ , the operator of multiplication by  $e^{i\alpha}$  is an isometry of  $L(0)$ , it is clear that (b) continues to hold with ‘ $B$ ’ replaced everywhere by ‘ $\mathbf{B}$ ’. Thus, by Proposition 2.1, it suffices to show that each  $K_n$  is Gâteaux differentiable in a common neighborhood of 0 in  $\mathbf{B}$ . Our proof follows an idea of Coifman, Rochberg, and Weiss (see [3, §2]; see also [5, Chapter 4, Note 7.12]).

For  $\beta \in \mathbf{B}$ ,  $f \in L(0)$ , and  $x \in X$  we have

$$(2.19) \quad \{K_n(\beta)f\}(x) = \int_X \exp(\beta(x) - \beta(y))D_n(x, y)f(y)d\mu(y).$$

If  $\alpha \in \mathbf{B}$ , then the first Gâteaux differential of  $K_n$  at  $\alpha$  in the direction  $\beta$  is given by

$$(2.20) \quad \left\{ \frac{d}{dz}K_n(\alpha + z\beta)f \right\}(x) = \int_X (\beta(x) - \beta(y)) \exp\{\alpha(x) - \alpha(y) + z(\beta(x) - \beta(y))\}D_n(x, y)f(y)d\mu(y).$$

Now let  $\alpha, \beta \in \mathbf{B}$  with  $\|\alpha\|_B < \delta_0/2$  and  $\|\beta\|_B < (\delta_0/2) - \|\alpha\|_B$ . For  $\theta \in [0, 2\pi]$ , define the operator

$$(2.21) \quad K_{n,\theta} = K_n(\alpha + (z + e^{i\theta})\beta).$$

Now we have

$$(2.22) \quad \begin{aligned} \|\alpha + (z + e^{i\theta})\beta\|_B &\leq \|\alpha\|_B + (1 + |z|)\|\beta\|_B \\ &< \|\alpha\|_B + (1 + |z|)\{(\delta_0/2) - \|\alpha\|_B\} \\ &= (1 + |z|)(\delta_0/2) - |z|\|\alpha\|_B \end{aligned}$$

which is less than  $\delta_0$  provided  $|z| < 1$ . Consequently, for  $|z| < 1$ , we have  $\|K_{n,\theta}\|_{\mathcal{L}(0)} \leq C_0$ .

Now we claim that, for  $|z| < 1$ ,

$$(2.23) \quad \frac{d}{dz}K_n(\alpha + z\beta) = \frac{1}{2\pi} \int_0^{2\pi} K_{n,\theta} e^{-i\theta} d\theta.$$

In view of (2.19)–(2.21), we see that, to establish (2.23), it suffices to show that

$$(2.24) \quad \frac{1}{2\pi} \int_0^{2\pi} \exp\{e^{i\theta}(\beta(x) - \beta(y))\} e^{-i\theta} d\theta = \beta(x) - \beta(y).$$

But note that, if  $A$  is a complex constant,

$$(2.25) \quad \frac{1}{2\pi} \int_0^{2\pi} \exp(Ae^{i\theta}) e^{-i\theta} d\theta = \frac{1}{2\pi} \sum_{k=0}^{\infty} \int_0^{2\pi} \frac{(Ae^{i\theta})^k}{k!} e^{-i\theta} d\theta = A.$$

Letting  $A = \beta(x) - \beta(y)$  in (2.25), we obtain (2.24).

From (2.23), we see that, for  $|z| < 1$ ,

$$(2.26) \quad \left\| \frac{d}{dz}K_n(\alpha + z\beta) \right\|_{\mathcal{L}(0)} \leq \frac{1}{2\pi} \int_0^{2\pi} \|K_{n,\theta}\|_{\mathcal{L}(0)} d\theta \leq C_0.$$

From this we conclude that each  $K_n$  is Gâteaux differentiable on the open ball of radius  $\delta_0/2$  in  $\mathbf{B}$ .  $\square$

With an additional assumption regarding the strong convergence of the operators  $\langle K_n(\beta) \rangle$ , we obtain the following useful result:

**Proposition 2.4.** *Under the hypotheses of Proposition 2.3, let us make the additional assumption that there is an operator  $K_\infty(0) \in \mathcal{L}(0)$  such that, for all  $\beta$  in a neighborhood of 0 in  $B$ ,  $K_n(\beta) - K_\infty(\beta)$  converges to 0 in the strong operator topology on  $\mathcal{L}(0)$  as  $n \rightarrow \infty$ , where  $K_\infty(\beta) = M_{e^\beta} K_\infty(0) M_{e^{-\beta}}$ . Then the following are equivalent:*

(a) *There exist constants  $\delta_0, C_0 > 0$  such that, for all  $n \in \mathbf{N}$  and for all  $\beta \in B$  with  $\|\beta\|_B < \delta_0$ , inequality (2.18) holds.*

(b) *There exist constants  $\delta_1, C_1 > 0$  such that, for all  $\beta \in B$  with  $\|\beta\|_B < \delta_1$ ,*

$$(2.27) \quad \|K_\infty(\beta)\|_{\mathcal{L}(0)} \leq C_1.$$

(c)  *$\langle K_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in  $\mathbf{B}$ .*

*Proof.* The equivalence of (a) and (c) is simply Proposition 2.3. The equivalence of (a) and (b) is the essence of a remark made by Garnett [6, p. 109]. We give the details.

Suppose that (a) holds. Let  $\varepsilon > 0$  and  $\beta \in B$  with  $\|\beta\|_B < \delta_0$ . For every  $f \in L(0)$  we can find  $N > 0$  such that  $n \geq N$  implies

$$(2.28) \quad \|\{K_n(\beta) - K_\infty(\beta)\}f\|_{L(0)} < \varepsilon.$$

Then, for such  $n$ , we have

$$(2.29) \quad \|K_\infty(\beta)f\|_{L(0)} \leq \|\{K_n(\beta) - K_\infty(\beta)\}f\|_{L(0)} + \|K_n(\beta)f\|_{L(0)} < \varepsilon + C_0\|f\|_{L(0)}.$$

Since  $\varepsilon$  was arbitrary, we see that (b) is true with  $\delta_1 = \delta_0$  and  $C_1 = C_0$ .

Conversely, assume that (b) is true. Let  $\varepsilon > 0$  and  $\beta \in B$  with  $\|\beta\|_B < \delta_1$ . For every  $f \in L(0)$  we can find  $N > 0$  such that  $n \geq N$  implies (2.28). For such  $n$ , we have

$$(2.30) \quad \begin{aligned} \|K_n(\beta)f\|_{L(0)} &\leq \| \{K_n(\beta) - K_\infty(\beta)\}f \|_{L(0)} + \|K_\infty(\beta)f\|_{L(0)} \\ &< \varepsilon + C_1 \|f\|_{L(0)}. \end{aligned}$$

Thus the family of operators  $\{K_n(\beta): n \in \mathbf{N}, \beta \in B, \|\beta\|_B < \delta_1\}$  is “pointwise bounded” on  $L(0)$ . Then (a) follows from the principle of uniform boundedness.  $\square$

### 3. UNIFORM ANALYTICITY ON THE CIRCLE

In this section we apply our work in §2 to the case of trigonometric polynomials on the circle,  $\mathbf{T}$ . We parametrize  $\mathbf{T}$  by the interval  $[-\pi, \pi)$ , and let  $L(0) = L^2(\mathbf{T}) = L^2([-\pi, \pi), d\theta)$ , where  $d\theta$  is ordinary Lebesgue measure. Let  $\omega$  be a nonnegative weight function on  $\mathbf{T}$  such that  $\omega^2 + \omega^{-2} \in L^1(\mathbf{T})$ , and write  $\beta = \log \omega$ . For each integer  $k$ , define the function  $e_k$  by  $e_k(\theta) = e^{ik\theta}$ . We define, for  $n \in \mathbf{N}$ , the space

$$(3.1) \quad H_n(0) = \text{span}_{\mathbf{C}} \langle e_k : |k| \leq n \rangle$$

of trigonometric polynomials of degree at most  $n$ ; we define  $L(\beta)$ ,  $\mathcal{L}(\beta)$ ,  $H_n(\beta)$ , etc. as in §2. We note that the base projections  $\langle S_n(0): n \in \mathbf{N} \rangle$  are simply the partial sum operators for Fourier series, defined by

$$(3.2) \quad S_n(0)f = \sum_{k=-n}^n \hat{f}(k)e_k$$

where, for each integer  $k$ ,  $\hat{f}(k)$  is the  $k$ th Fourier coefficient of  $f$ , given by

$$(3.3) \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e_{-k}(\theta)d\theta$$

and thus

$$(3.4) \quad \sum_{k=-\infty}^{\infty} \hat{f}(k)e_k$$

is the Fourier series for  $f$ . The operator  $S_n(0)$  is an integral operator, given by

$$(3.5) \quad \{S_n(0)f\}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(\theta, \psi)f(\psi)d\psi$$

where  $D_n(\theta, \psi)$  is the Dirichlet kernel, given by

$$(3.6) \quad D_n(\theta, \psi) = \sum_{k=-n}^n e_k(\theta)e_{-k}(\psi) = \frac{\sin[(2n+1)\frac{\theta-\psi}{2}]}{\sin(\frac{\theta-\psi}{2})}$$

(see, for example, [10, p. 12]).

We shall show that, in this example, the space of functions of bounded mean oscillation on  $\mathbf{T}$  is the space of uniform holomorphy at 0 for the families  $\langle P_n \rangle$  and  $\langle Q_n \rangle$ .

We adopt the convention that  $I$  is a *subinterval* of  $\mathbf{T}$  if and only if it is a subinterval of  $[-\pi, \pi)$  in the usual sense, or it is the union of an interval of the form  $(c, \pi)$  or  $[c, \pi)$  with an interval of the form  $[-\pi, d)$  or  $[-\pi, d]$ , with  $-\pi < d < c < \pi$ . We let  $|I|$  denote the Lebesgue measure of  $I$ . If  $b \in L^1(\mathbf{T}) = L^1([-\pi, \pi), d\theta)$ , we define the mean of  $b$  on  $I$  to be

$$(3.7) \quad m_I(b) = |I|^{-1} \int_I b(\theta) d\theta.$$

The function  $b$  is said to have *bounded mean oscillation* on  $\mathbf{T}$  if and only if the quantity

$$(3.8) \quad \|b\|_* \equiv \sup_I |I|^{-1} \int_I |b(\theta) - m_I(b)| d\theta = \sup_I m_I(|b - m_I(b)|)$$

is finite, where the supremum is taken over all subintervals  $I$  of  $\mathbf{T}$ . The space  $BMO(\mathbf{T})$  of real-valued functions (modulo constants) having bounded mean oscillation on  $\mathbf{T}$  is a Banach space with  $\|\cdot\|_*$  as its norm. For ease of notation in this section we shall refer to  $BMO(\mathbf{T})$  as simply  $BMO$ ; its complexification will be denoted by  $\mathbf{BMO}$ .

To begin, we shall show that  $\langle P_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in  $\mathbf{BMO}$ , from which it follows that the same is true of  $\langle Q_n \rangle$ , by Corollary 2.2.1. By Proposition 2.3, the uniform holomorphy of  $\langle P_n \rangle$  in a neighborhood of 0 in  $\mathbf{BMO}$  is equivalent to a uniform estimate of the form

$$(3.9) \quad \|P_n(\beta)\|_{\mathcal{L}(0)} \leq C_0 \quad \text{for all } \beta \in BMO \quad \text{with } \|\beta\|_* < \delta_0,$$

where  $C_0, \delta_0$  are constants independent of  $n$ .

We can, in fact, characterize the weight functions  $\omega = e^\beta$  for which  $\langle \|P_n(\beta)\|_{\mathcal{L}(0)} \rangle$  is uniformly bounded in  $n$ . Recall that the weight  $\omega$  is said to belong to the class  $A_2$  if and only if

$$(3.10) \quad \sup_I m_I(\omega) m_I(\omega^{-1}) < \infty$$

where the supremum is taken over all subintervals  $I$  of  $\mathbf{T}$ . The quantity (3.10) is called the  $A_2$  constant of  $\omega$ . We have the following result:

**Proposition 3.1.** *The quantity  $\sup \langle \|P_n(\beta)\|_{\mathcal{L}(0)} : n \in \mathbf{N} \rangle$  is finite if and only if  $\omega^2 \in A_2$ .*

*Proof.* The proof is analogous to that of [5, Corollary 3.12, Chapter 4]. The idea is to exploit the relationship between  $S_n(0)$  and the orthogonal projection  $P_+$  of  $L(0)$  onto the Hardy space  $\mathcal{H}_+^2 = \{f \in L(0) : \hat{f}(k) = 0 \text{ for } k < 0\}$ .

Consider the operator  $T_n = e_n S_n(0) e_{-n}$ . A simple computation shows that, for  $f \in L(0)$ ,

$$(3.11) \quad T_n f = \sum_{k=0}^{2n} \hat{f}(k) e_k.$$

Moreover, as  $n \rightarrow \infty$ ,  $T_n \rightarrow P_+$  in the strong operator topology on  $\mathcal{L}(0)$ . By a slight modification of the proof of Proposition 2.4, it follows that  $\sup\{\|M_\omega T_n M_\omega^{-1}\|_{\mathcal{L}(0)}; n \in \mathbf{N}\}$  is finite if and only if  $M_\omega P_+ M_\omega^{-1} \in \mathcal{L}(0)$ . By virtue of the relationship between  $P_+$  and the conjugate operator (cf. [6, p. 108]), we see that  $M_\omega P_+ M_\omega^{-1} \in \mathcal{L}(0)$  if and only if  $\omega^2 \in A_2$  (see [7]). Now note that

$$(3.12) \quad P_n(\beta) = M_{e_{-n}} M_\omega T_n M_\omega^{-1} M_{e_n};$$

moreover,  $M_{e_k}$  is an isometry for each integer  $k$ . Consequently, for each integer  $n$ ,

$$(3.13) \quad \|P_n(\beta)\|_{\mathcal{L}(0)} = \|M_\omega T_n M_\omega^{-1}\|_{\mathcal{L}(0)},$$

from which the result follows.  $\square$

**Corollary 3.1.1.**  $\langle P_n \rangle$  and  $\langle Q_n \rangle$  are uniformly holomorphic in a neighborhood of 0 in **BMO**.

*Proof.* There exist constants  $\delta_0, C > 0$  such that, if  $\beta \in \mathbf{BMO}$  and  $\|\beta\|_* < \delta_0$ , then  $\omega^2 \in A_2$ , and the  $A_2$  constant of  $\omega^2$  is less than or equal to  $C$  (see [5, Chapter 2, Corollary 3.10 and Chapter 4, Corollary 2.18]). For  $\omega^2 \in A_2$ , the  $\mathcal{L}(0)$ -norm of  $M_\omega P_+ M_\omega^{-1}$  depends upon the  $A_2$  constant of  $\omega^2$ ; so by the proof of Proposition 3.1, we obtain (3.9), from which the corollary follows.  $\square$

Next, we would like to show that **BMO** is actually the space of uniform holomorphy at 0 for  $\langle P_n \rangle$  and  $\langle Q_n \rangle$ . A simple computation shows that  $I - 2S_n(0)$  is an isometry of  $L(0)$  for each  $n \in \mathbf{N}$ ; so, by Corollary 2.2.2, it suffices to show that  $\beta \in \mathbf{BMO}$  is a necessary and sufficient condition for boundedness of the set  $\{\| [M_\beta, S_n(0)] \|_{\mathcal{L}(0)}; n \in \mathbf{N}\}$ . We have the following:

**Proposition 3.2.** *There exist constants  $C_1, C_2 > 0$  such that for all  $\beta \in L^1(\mathbf{T})$ ,*

$$(3.14) \quad C_1 \|\beta\|_* \leq \sup_n \| [M_\beta, S_n(0)] \|_{\mathcal{L}(0)} \leq C_2 \|\beta\|_*.$$

*Proof.* The existence of  $C_2$  with the desired property follows from the fact that  $\langle P_n \rangle$  is uniformly holomorphic in a neighborhood of 0 in **BMO**. Therefore it suffices to prove that there exists a constant  $\mu > 0$  such that, for all  $\beta \in L^1(\mathbf{T})$ ,

$$(3.15) \quad \|\beta\|_* \leq \mu \sup_n \| [M_\beta, S_n(0)] \|_{\mathcal{L}(0)}.$$

Let  $I$  be a subinterval of  $\mathbf{T}$  and let  $\theta \in [-\pi, \pi)$ . Define the function  $f_\theta = f_{I, \theta, n}$  by setting, for  $\psi \in [-\pi, \pi)$ ,

$$(3.16) \quad f_\theta(\psi) = 2i \sin\left(\frac{\theta - \psi}{2}\right) \exp\left[i(2n + 1)\frac{\theta - \psi}{2}\right] \chi_I(\psi).$$

Observe that, by (3.6), when  $\psi \in I$ ,

$$\begin{aligned}
 (3.17) \quad D_n(\theta, \psi) f_\theta(\psi) &= 2i \sin \left[ (2n+1) \frac{\theta - \psi}{2} \right] \exp \left[ i(2n+1) \frac{\theta - \psi}{2} \right] \chi_I(\psi) \\
 &= 2i \cos \left[ (2n+1) \frac{\theta - \psi}{2} \right] \sin \left[ (2n+1) \frac{\theta - \psi}{2} \right] \chi_I(\psi) \\
 &\quad - 2 \sin^2 \left[ (2n+1) \frac{\theta - \psi}{2} \right] \chi_I(\psi).
 \end{aligned}$$

In view of the fact that

$$\begin{aligned}
 (3.18) \quad \exp[i(2n+1)(\theta - \psi)] \\
 &= 1 - 2 \sin^2 \left[ (2n+1) \frac{\theta - \psi}{2} \right] \\
 &\quad + 2i \cos \left[ (2n+1) \frac{\theta - \psi}{2} \right] \sin \left[ (2n+1) \frac{\theta - \psi}{2} \right],
 \end{aligned}$$

we obtain, for  $\psi \in I$ ,

$$(3.19) \quad D_n(\theta, \psi) f_\theta(\psi) = (\exp[i(2n+1)(\theta - \psi)] - 1) \chi_I(\psi).$$

Consequently, by (3.5) and (3.19),

$$\begin{aligned}
 (3.20) \quad \{[M_\beta, S_n(0)]f_\theta\}(\theta) \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\exp i(2n+1)(\theta - \psi) - 1\} (\beta(\theta) - \beta(\psi)) \chi_I(\psi) d\psi \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[i(2n+1)(\theta - \psi)] (\beta(\theta) - \beta(\psi)) \chi_I(\psi) d\psi \\
 &\quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} (\beta(\theta) - \beta(\psi)) \chi_I(\psi) d\psi \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[i(2n+1)(\theta - \psi)] (\beta(\theta) - \beta(\psi)) \chi_I(\psi) d\psi \\
 &\quad - \frac{1}{2\pi} |I| (\beta(\theta) - m_I(\beta)).
 \end{aligned}$$

Now note that

$$\begin{aligned}
 (3.21) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[i(2n+1)(\theta - \psi)] (\beta(\theta) - \beta(\psi)) \chi_I(\psi) d\psi \\
 &= \exp[i(2n+1)\theta] \left\{ \beta(\theta) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i(2n+1)\psi] \chi_I(\psi) d\psi \right. \\
 &\quad \left. - \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i(2n+1)\psi] \beta(\psi) \chi_I(\psi) d\psi \right\} \\
 &= \exp[i(2n+1)\theta] \{ \beta(\theta) \hat{\chi}_I(2n+1) - \widehat{\beta \chi}_I(2n+1) \}.
 \end{aligned}$$

Consequently, by (3.20) and (3.21),

$$\begin{aligned}
 (3.22) \quad \{[M_\beta, S_n(0)]f_\theta\}(\theta) &= \exp[i(2n+1)\theta] \{ \beta(\theta) \hat{\chi}_I(2n+1) - \widehat{\beta \chi}_I(2n+1) \} \\
 &\quad - \frac{1}{2\pi} |I| (\beta(\theta) - m_I(\beta)).
 \end{aligned}$$

Letting

$$(3.23) \quad g_n(\theta) = \chi_I(\theta) \exp[i(2n + 1)\theta] \{ \beta(\theta) \hat{\chi}_I(2n + 1) - \widehat{\beta\chi}_I(2n + 1) \},$$

$$(3.24) \quad g(\theta) = \frac{1}{2\pi} |I| (\beta(\theta) - m_I(\beta)),$$

we see that

$$(3.25) \quad \int_I | \{ [M_\beta, S_n(0)] f_\theta \}(\theta) | d\theta = \|g - g_n\|_1.$$

Now

$$(3.26) \quad \|g_n\|_1 \leq |\hat{\chi}_I(2n + 1)| \|\beta\|_1 + |\widehat{\beta\chi}_I(2n + 1)| |I|;$$

the right-hand side of (3.26) tends to 0 as  $n \rightarrow \infty$ , by the Riemann-Lebesgue lemma, so  $\lim_{n \rightarrow \infty} \|g_n\|_1 = 0$ . Consequently, by Fatou's lemma,

$$(3.27) \quad \lim_{n \rightarrow \infty} \int_I | \{ [M_\beta, S_n(0)] f_\theta \}(\theta) | d\theta = \|g\|_1 = \frac{1}{2\pi} |I| \int_I |\beta(\theta) - m_I(\beta)| d\theta.$$

Thus (3.15) follows, once we prove an estimate of the form

$$(3.28) \quad \overline{\lim}_{n \rightarrow \infty} \int_I | \{ [M_\beta, S_n(0)] f_\theta \}(\theta) | d\theta \leq \frac{1}{2\pi} \mu C |I|^2$$

where

$$(3.29) \quad C = \sup_n \| [M_\beta, S_n(0)] \|_{\mathcal{L}(0)}$$

and  $\mu$  is a constant independent of  $\beta$  and  $I$ .

Now we have

$$(3.30) \quad \begin{aligned} & \{ [M_\beta, S_n(0)] f_\theta \}(\theta) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(\theta, \psi) (\beta(\theta) - \beta(\psi)) f_\theta(\psi) d\psi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(\theta, \psi) (\beta(\theta) - \beta(\psi)) 2i \sin\left(\frac{\theta - \psi}{2}\right) \\ & \quad \times \exp\left[i(2n + 1)\frac{\theta - \psi}{2}\right] \chi_I(\psi) d\psi \\ &= 2i \exp\left[i(2n + 1)\frac{\theta}{2}\right] \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(\theta, \psi) (\beta(\theta) - \beta(\psi)) \sin\left(\frac{\theta - \psi}{2}\right) \\ & \quad \times \exp\left[-i(2n + 1)\frac{\psi}{2}\right] \chi_I(\psi) d\psi \end{aligned}$$

so that

$$(3.31) \quad \begin{aligned} & | \{ [M_\beta, S_n(0)] f_\theta \}(\theta) | \\ &= 2 \cdot \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(\theta, \psi) (\beta(\theta) - \beta(\psi)) \sin\left(\frac{\theta - \psi}{2}\right) \right. \\ & \quad \left. \times \exp\left[-i(2n + 1)\frac{\psi}{2}\right] \chi_I(\psi) d\psi \right|. \end{aligned}$$

Now suppose  $I$  is a subinterval of  $\mathbf{T}$  which is also a subinterval of  $[-\pi, \pi]$  in the ordinary sense, and let  $\psi_0$  denote the ordinary midpoint of  $I$ . (If  $I$  is a subinterval of  $\mathbf{T}$  comprising two disjoint subintervals of  $[-\pi, \pi]$ , we need to make minor adjustments to the argument.) Now, we write

$$(3.32) \quad \begin{aligned} \sin\left(\frac{\theta - \psi}{2}\right) &= \sin\left(\frac{\theta - \psi_0 + \psi_0 - \psi}{2}\right) \\ &= \sin\left(\frac{\theta - \psi_0}{2}\right) \cos\left(\frac{\psi_0 - \psi}{2}\right) + \cos\left(\frac{\theta - \psi_0}{2}\right) \sin\left(\frac{\psi_0 - \psi}{2}\right), \end{aligned}$$

so that

$$(3.33) \quad \begin{aligned} | \{[M_\beta, S_n(0)]f_\theta\}(\theta) | &\leq 2 \left| \sin\left(\frac{\theta - \psi_0}{2}\right) \{[M_\beta, S_n(0)]h_1\}(\theta) \right| \\ &\quad + 2 \left| \cos\left(\frac{\theta - \psi_0}{2}\right) \{[M_\beta, S_n(0)]h_2\}(\theta) \right| \end{aligned}$$

where

$$(3.34) \quad h_1(\psi) = \cos\left(\frac{\psi_0 - \psi}{2}\right) \exp\left[-i(2n+1)\frac{\psi}{2}\right] \chi_I(\psi),$$

$$(3.35) \quad h_2(\psi) = \sin\left(\frac{\psi_0 - \psi}{2}\right) \exp\left[-i(2n+1)\frac{\psi}{2}\right] \chi_I(\psi).$$

Now, for  $\alpha \in \mathbf{R}$ , we have  $|\sin \alpha| \leq |\alpha| \cosh \alpha$ , so that, for  $\psi, \theta \in I$ ,

$$(3.36) \quad |h_2(\psi)| \leq \left| \frac{\psi_0 - \psi}{2} \right| \cosh\left(\frac{\psi_0 - \psi}{2}\right) \leq \frac{1}{2}(\cosh \pi) |\psi_0 - \psi|,$$

$$(3.37) \quad \left| \sin\left(\frac{\theta - \psi_0}{2}\right) \right| \leq \left| \frac{\theta - \psi_0}{2} \right| \cosh\left(\frac{\theta - \psi_0}{2}\right) \leq \frac{1}{2}(\cosh \pi) |\theta - \psi_0|.$$

Thus, by Schwarz' inequality, (3.33) and (3.36),

$$(3.38) \quad \begin{aligned} &\frac{1}{2} \int_I | \{[M_\beta, S_n(0)]f_\theta\}(\theta) | d\theta \\ &\leq \left( \int_I \left| \sin\left(\frac{\theta - \psi_0}{2}\right) \right|^2 d\theta \right)^{1/2} \| [M_\beta, S_n(0)]h_1 \|_2 \\ &\quad + \left( \int_I \left| \cos\left(\frac{\theta - \psi_0}{2}\right) \right|^2 d\theta \right)^{1/2} \| [M_\beta, S_n(0)]h_2 \|_2 \\ &\leq \frac{1}{2}(\cosh \pi) \left( \int_I |\theta - \psi_0|^2 d\theta \right)^{1/2} C \|h_1\|_2 + |I|^{1/2} C \|h_2\|_2 \\ &= \frac{\sqrt{3}}{12}(\cosh \pi) |I|^{3/2} C \|h_1\|_2 + |I|^{1/2} C \|h_2\|_2. \end{aligned}$$

We have

$$(3.39) \quad \|h_1\|_2 = \left( \int_I \left| \cos\left(\frac{\psi - \psi_0}{2}\right) \right|^2 d\psi \right)^{1/2} \leq |I|^{1/2},$$

$$\begin{aligned}
 (3.40) \quad \|h_2\| &= \left( \int_I \left| \sin \left( \frac{\psi - \psi_0}{2} \right) \right|^2 d\psi \right)^{1/2} \\
 &\leq \frac{1}{2} (\cosh \pi) \left( \int_I |\psi_0 - \psi|^2 d\psi \right)^{1/2} \\
 &= \frac{\sqrt{3}}{12} (\cosh \pi) |I|^{3/2},
 \end{aligned}$$

so that

$$(3.41) \quad \int_I |\{[M_\beta, S_n(0)]f_\theta\}(\theta)| d\theta \leq \frac{\sqrt{3}}{3} (\cosh \pi) C |I|^2$$

whence

$$(3.42) \quad \overline{\lim}_{n \rightarrow \infty} \int_I |\{[M_\beta, S_n(0)]f_\theta\}(\theta)| d\theta \leq \frac{1}{2\pi} \left[ \frac{2\pi\sqrt{3} \cosh \pi}{3} \right] C |I|^2,$$

which is (3.28), with  $\mu = (2\pi\sqrt{3} \cosh \pi)/3$ . This completes the proof.  $\square$

The following corollary is immediate.

**Corollary 3.2.1.** *BMO is the space of uniform holomorphy at 0 for  $\langle P_n \rangle$  and  $\langle Q_n \rangle$ .*  $\square$

4. UNIFORM ANALYTICITY ON  $[-1, 1]$  IN A NEIGHBORHOOD OF A JACOBI WEIGHT

In this section we consider an example on the interval  $[-1, 1]$ . For  $\gamma, \delta > -1$ , we define the *Jacobi weight* with parameters  $\gamma, \delta$ , by

$$(4.1) \quad \omega_{\gamma, \delta}^2(x) = (1-x)^\gamma (1+x)^\delta.$$

Our basic Hilbert space is

$$(4.2) \quad L_{\gamma, \delta}(0) = L^2([-1, 1], \omega_{\gamma, \delta}^2(x) dx),$$

and we shall denote the orthogonal projection onto polynomials of degree  $\leq n$  in  $L_{\gamma, \delta}(0)$  by  $S_n^{\gamma, \delta}(0)$ . We shall make a small multiplicative perturbation of  $\omega_{\gamma, \delta}^2(x) dx$  by a nonnegative weight function  $\omega^2 = e^{2\beta}$ , where

$$(4.3) \quad \omega^2 + \omega^{-2} \in L^1([-1, 1], \omega_{\gamma, \delta}^2(x) dx), \quad \beta \in L^1([-1, 1], dx).$$

The corresponding Hilbert space is

$$(4.4) \quad L_{\gamma, \delta}(\beta) = L^2([-1, 1], \omega_{\gamma, \delta}^2(x) \omega^2(x) dx).$$

This orthogonal projection onto polynomials of degree  $\leq n$  in  $L_{\gamma, \delta}(\beta)$  will be denoted by  $S_n^{\gamma, \delta}(\beta)$ . Setting

$$(4.5) \quad Q_n^{\gamma, \delta}(\beta) = M_\omega S_n^{\gamma, \delta}(\beta) M_\omega^{-1},$$

$$(4.6) \quad P_n^{\gamma, \delta}(\beta) = M_\omega S_n^{\gamma, \delta}(0) M_\omega^{-1},$$

we conjecture that the space of functions of bounded mean oscillation on  $[-1, 1]$  is the space of uniform holomorphy at 0 for the families  $\langle Q_n^{\gamma, \delta} \rangle$  and  $\langle P_n^{\gamma, \delta} \rangle$  whenever  $\gamma, \delta > -1$ . In fact, we shall show that  $\langle Q_n^{\gamma, \delta} \rangle$  and  $\langle P_n^{\gamma, \delta} \rangle$  are uniformly holomorphic in a neighborhood of 0 in  $\mathbf{BMO}([-1, 1])$  whenever  $\gamma, \delta \geq -\frac{1}{2}$ .

We pause to remark that the operator  $S_n^{\gamma, \delta}(0)$  is the  $n$ th partial sum operator for Jacobi series with parameters  $\gamma, \delta$ ; it is an integral operator whose kernel may be expressed in terms of the Jacobi polynomials  $\langle P_n^{(\gamma, \delta)}(x) : n \in \mathbf{N} \rangle$ , which form an orthogonal polynomial system in  $L_{\gamma, \delta}(0)$  (see [10, Chapters 3 and 4]; see also [9]). Making use of Christoffel-Darboux formulas for  $S_n^{\gamma, \delta}(0)$ , it is possible to reduce the problem of obtaining uniform weighted norm inequalities for  $S_n^{\gamma, \delta}(0)$  to that of obtaining weighted norm inequalities for the Hilbert transform, as we shall see.

We begin with some terminology. A function  $b \in L^1([-1, 1], dx)$  is in  $\mathbf{BMO}([-1, 1], dx)$  if and only if

$$(4.7) \quad \|b\|_* \equiv \sup_I |I|^{-1} \int_I |b(\theta) - m_I(b)| d\theta$$

is finite, where the supremum is taken over all subintervals  $I$  of  $[-1, 1]$ .  $\mathbf{BMO}([-1, 1], dx)$  is a Banach space of functions (modulo constants). The space of real-valued functions in  $\mathbf{BMO}([-1, 1], dx)$  will be denoted by  $BMO([-1, 1], dx)$ ; we will use the abbreviations  $\mathbf{BMO}$  and  $BMO$ . A non-negative weight function  $\omega$  belongs to the class  $A_2$  if and only if

$$(4.8) \quad \sup_I m_I(\omega) m_I(\omega^{-1}) < \infty$$

where the supremum is taken over all subintervals  $I$  of  $[-1, 1]$ ; the quantity (4.8) is called the  $A_2$  constant of  $\omega$ .

By analogy to the notation of §2, we let  $\mathcal{L}_{\gamma, \delta}(0)$  denote the Hilbert space of bounded linear operators on  $L_{\gamma, \delta}(0)$ . We wish to obtain a uniform estimate of the form

$$(4.9) \quad \|P_n^{\gamma, \delta}(\beta)\|_{\mathcal{L}_{\gamma, \delta}(0)} \leq C$$

which is valid for all  $\beta$  in some neighborhood of the origin in  $BMO$ . Note that the estimate (4.9) is equivalent to an estimate of form

$$(4.10) \quad \left\| M_\omega M_{\omega, \delta} S_n^{\gamma, \delta}(0) M_{\omega, \delta}^{-1} M_\omega^{-1} \right\|_{\mathcal{L}(0)} \leq C,$$

where  $\mathcal{L}(0) = \mathcal{L}_{0,0}(0)$ , which is valid for all  $\beta$  in some neighborhood of the origin in  $BMO$ . In order to prove an estimate of this form, we need the following lemma.

**Lemma 4.1.** *Suppose  $w^2 \in A_2$ . Then there exist constants  $\delta_1, C > 0$ , depending only upon  $w$ , such that for all  $\beta \in BMO$  with  $\|\beta\|_* < \delta_1$ ,  $e^{2\beta} w^2$  is also in  $A_2$  with an  $A_2$  constant less than or equal to  $C$ .*

*Proof.* We make use of the following characterization of  $A_2$ : a function  $\varphi \in L^1([-1, 1], dx)$  is the logarithm of  $A_2$  weight if and only if the quantity

$$(4.11) \quad \sup_I |I|^{-1} \int_I \exp(|\varphi(x) - m_I(\varphi)|) dx$$

(where the supremum is taken over all subintervals  $I$  of  $[-1, 1]$ ) is finite; the quantity (4.11) is equivalent to the square root of the  $A_2$  constant of  $e^\varphi$  (see [5, Chapter 4, Theorem 2.17 and Corollary 2.18]).

Now suppose  $w^2 \in A_2$ ; let  $f = \log w$ . By [5, Theorem 2.7, Chapter 4] there is a constant  $\varepsilon > 0$  such that  $w^{2+\varepsilon} \in A_2$ . Consequently, if  $I$  is a subinterval of  $[-1, 1]$ , we have

$$(4.12) \quad \begin{aligned} & |I|^{-1} \int_I \exp(|(2\beta + 2f)(x) - m_I(2\beta + 2f)|) dx \\ &= |I|^{-1} \int_I \exp|2\beta(x) - m_I(2\beta)| \exp|2f(x) - m_I(2f)| dx \\ &\leq |I|^{-1} \left( \int_I [\exp|2\beta(x) - m_I(2\beta)|]^{(2+\varepsilon)/\varepsilon} dx \right)^{\varepsilon/(2+\varepsilon)} \\ &\quad \times \left( \int_I [\exp|2f(x) - m_I(2f)|]^{(2+\varepsilon)/2} dx \right)^{2/(2+\varepsilon)} \\ &= \left( |I|^{-1} \int_I \exp \left| \left( \frac{4 + 2\varepsilon}{\varepsilon} \right) (\beta(x) - m_I(\beta)) \right| dx \right)^{\varepsilon/(2+\varepsilon)} \\ &\quad \times \left( |I|^{-1} \int_I \exp|(2 + \varepsilon)(f(x) - m_I(f))| dx \right)^{2/(2+\varepsilon)} \end{aligned}$$

by Hölder's inequality. Now, there exist constants  $\delta_0, C > 0$  such that, if  $\varphi \in BMO$  and  $\|\varphi\|_* < \delta_0$ , then  $e^{2\varphi} \in A_2$ , with an  $A_2$  constant less than or equal to  $C$  (see [5, Chapter 2, Corollary 3.10 and Chapter 4, Corollary 2.18]). Taking  $\delta_1 = \varepsilon\delta_0/(2 + \varepsilon)$ , then  $\|((2 + \varepsilon)/\varepsilon)\beta\|_* < \delta_0$  when  $\|\beta\|_* < \delta_1$ . Consequently, if  $\|\beta\|_* < \delta_1$  then (4.12) will be dominated by a constant which depends only upon  $w$  and which is independent of  $I$ . The result follows.  $\square$

We can now prove:

**Proposition 4.2.** *Let  $\gamma, \delta \geq -\frac{1}{2}$ . Then there exist constants  $C, \delta_1 > 0$  such that for all  $n \in \mathbb{N}$  and for all  $\beta \in BMO$  with  $\|\beta\|_* < \delta_1$ ,*

$$(4.13) \quad \|M_\omega M_{\omega_{\gamma, \delta}} S_n^{\gamma, \delta}(0) M_{\gamma, \delta}^{-1} M_\omega^{-1}\|_{\mathcal{L}(0)} \leq C$$

where  $\omega = e^\beta$ .

*Proof.* We make use of Muckenhoupt's work in [9]. First of all, we note that, for  $f \in L_{\gamma, \delta}(0)$ ,

$$(4.14) \quad S_n^{\gamma, \delta}(0)f(x) = \int_{-1}^1 K_n^{\gamma, \delta}(x, y)f(y)\omega_{\gamma, \delta}^2(y)dy,$$

where

$$(4.15) \quad K_n^{\gamma, \delta}(x, y) = \sum_{j=0}^n P_j^{(\gamma, \delta)}(x) P_j^{(\gamma, \delta)}(y) \|P_j^{(\gamma, \delta)}\|_{L_{\gamma, \delta}(0)}^{-2}.$$

Thus we have, for  $f \in L(0)$ ,

$$(4.16) \quad M_\omega M_{\omega_{\gamma, \delta}} S_n^{\gamma, \delta}(0) M_{\omega_{\gamma, \delta}}^{-1} M_\omega^{-1} f(x) = \int_{-1}^1 K_n^{\gamma, \delta}(x, y) \omega_{\gamma, \delta}(x) \omega_{\gamma, \delta}(y) \frac{\omega(x)}{\omega(y)} f(y) dy.$$

Using estimates from [10], Muckenhoupt writes

$$(4.17) \quad K_n^{\gamma, \delta}(x, y) = A(n, \gamma, \delta) H_1^{\gamma, \delta}(n; x, y) \\ + B(n, \gamma, \delta) [H_2^{\gamma, \delta}(n; x, y) + H_2^{\gamma, \delta}(n; y, x)]$$

where  $|A(n, \gamma, \delta)|$ ,  $|B(n, \gamma, \delta)|$  are bounded above by a constant independent of  $n$ , and

$$(4.18) \quad H_1^{\gamma, \delta}(n; x, y) = (n+1) P_n^{(\gamma, \delta)}(x) P_n^{(\gamma, \delta)}(y),$$

$$(4.19) \quad H_2^{\gamma, \delta}(n; x, y) = \frac{n(1-y^2) P_n^{(\gamma, \delta)}(x) P_{n-1}^{(\gamma+1, \delta+1)}(y)}{x-y}.$$

The Jacobi polynomials satisfy

$$(4.20) \quad P_n^{(\gamma, \delta)}(x) = (-1)^n P_n^{(\gamma, \delta)}(-x), \quad x \in [-1, 1]$$

(see [10, p. 59, (4.13)]); moreover, there is a constant  $K(\gamma, \delta)$  such that, for  $n \in \mathbf{N}$ ,

$$(4.21) \quad |P_n^{(\gamma, \delta)}(x)| \leq K(\gamma, \delta) n^{-1/2} (1-x+n^{-2})^{-\gamma/2-1/4}, \quad x \in [0, 1]$$

(see [9, equation (2.2)] and [10, Theorem 7.32.2]). We shall use (4.20) and (4.21) to estimate

$$(4.22) \quad T_{j,n}^{\gamma, \delta} f(x) = \int_{-1}^1 H_j^{\gamma, \delta}(n; x, y) \omega_{\gamma, \delta}(x) \omega_{\gamma, \delta}(y) \frac{\omega(x)}{\omega(y)} f(y) dy$$

for  $j = 1, 2$ , and

$$(4.23) \quad S_{2,n}^{\gamma, \delta} f(x) = \int_{-1}^1 H_2^{\gamma, \delta}(n; y, x) \omega_{\gamma, \delta}(x) \omega_{\gamma, \delta}(y) \frac{\omega(x)}{\omega(y)} f(y) dy.$$

We shall begin by considering the operator  $T_{2,n}^{\gamma, \delta}$  in some detail.

Note first that, for  $n \in \mathbf{N}$  and  $x, y \in [-1, 1]$ , we have

$$(4.24) \quad H_2^{\gamma, \delta}(n; x, y) \omega_{\gamma, \delta}(x) \omega_{\gamma, \delta}(y) = b_1^{\gamma, \delta}(n; x) b_2^{\gamma, \delta}(n; y) \frac{\omega(x)}{\omega(y)} \frac{1}{x-y}$$

where

$$(4.25) \quad w(x) = (1-x^2)^{-1/4},$$

$$(4.26) \quad b_1^{\gamma, \delta}(n; x) = (1-x)^{\gamma/2+1/4} (1+x)^{\delta/2+1/4} n^{1/2} P_n^{(\gamma, \delta)}(x),$$

$$(4.27) \quad b_2^{\gamma, \delta}(n; y) = (1-y)^{\gamma/2+3/4} (1+y)^{\delta/2+3/4} n^{1/2} P_{n-1}^{(\gamma+1, \delta+1)}(y).$$

We shall restrict our attention to  $n \geq 2$  (for  $n = 0, 1$  we need only consider operators of the form  $T_{1,n}^{\gamma,\delta}$ ;  $T_{2,n}^{\gamma,\delta}$  and  $S_{2,n}^{\gamma,\delta}$  need not be considered).

For  $x \in [-1, 0]$ , (4.20) and (4.21) imply that

$$(4.28) \quad |b_1^{\gamma,\delta}(n; x)| \leq (1-x)^{\gamma/2+1/4}(1+x)^{\delta/2+1/4}K(\delta, \gamma)(1+x+n^{-2})^{-\delta/2-1/4};$$

since  $\delta \geq -\frac{1}{2}$ , we have  $\delta/2 + 1/4 \geq 0$  so that

$$(4.29) \quad |b_1^{\gamma,\delta}(n; x)| \leq (1-x)^{\gamma/2+1/4}K(\gamma, \delta) \leq 2^{\gamma/2+1/4}K(\delta, \gamma).$$

For  $x \in [0, 1]$ , (4.21) implies that

$$(4.30) \quad |b_1^{\gamma,\delta}(n; x)| \leq (1-x)^{\gamma/2+1/4}(1+x)^{\delta/2+1/4}K(\gamma, \delta)(1-x+n^{-2})^{-\gamma/2-1/4};$$

since  $\gamma \geq -\frac{1}{2}$ , we have  $\gamma/2 + 1/4 \geq 0$  so that

$$(4.31) \quad |b_1^{\gamma,\delta}(n; x)| \leq (1+x)^{\delta/2+1/4}K(\gamma, \delta) \leq 2^{\delta/2+1/4}K(\gamma, \delta).$$

Similarly, for  $y \in [-1, 0]$ , (4.20) and (4.21) imply that

$$(4.32) \quad |b_2^{\gamma,\delta}(n; y)| \leq (1-y)^{\gamma/2+3/4}(1+y)^{\delta/2+3/4}n^{1/2}(n-1)^{-1/2} \\ \times K(\delta+1, \gamma+1)[1+y+(n-1)^{-2}]^{-\delta/2-3/4} \\ \leq 2^{1/2}K(\delta+1, \gamma+1)(1-y)^{\gamma/2+3/4} \\ \times (1+y)^{\delta/2+3/4}[1+y+(n-1)^2]^{-\delta/2-3/4};$$

since  $\delta > -1$ , we have  $\delta/2 + 3/4 > 1/4 \geq 0$ , so that

$$(4.33) \quad |b_2^{\gamma,\delta}(n; y)| \leq 2^{1/2}K(\delta+1, \gamma+1)(1-y)^{\gamma/2+3/4} \leq 2^{\gamma/2+5/4}K(\delta+1, \gamma+1).$$

For  $y \in [0, 1]$ , (4.21) implies that

$$(4.34) \quad |b_2^{\gamma,\delta}(n; y)| \leq (1-y)^{\gamma/2+3/4}(1+y)^{\delta/2+3/4}n^{1/2}(n-1)^{-1/2} \\ \times K(\gamma+1, \delta+1)[1-y+(n-1)^{-2}]^{-\gamma/2-3/4} \\ \leq 2^{1/2}K(\gamma+1, \delta+1)(1+y)^{\delta/2+3/4}(1-y)^{\gamma/2+3/4} \\ \times [1-y+(n-1)^{-2}]^{-\gamma/2-3/4};$$

since  $\gamma > -1$ , we have  $\gamma/2 + 3/4 > 1/4 \geq 0$ , so that

$$(4.35) \quad |b_2^{\gamma,\delta}(n; y)| \leq 2^{1/2}K(\gamma+1, \delta+1)(1+y)^{\delta/2+3/4} \leq 2^{\delta/2+5/4}K(\gamma+1, \delta+1).$$

Consequently, the functions  $b_j^{\gamma,\delta}(n; \cdot)$ ,  $j = 1, 2$ , are uniformly bounded for  $n \geq 2$ . Letting  $M_{j,n}^{\gamma,\delta}$  denote the operator of multiplication by  $b_j^{\gamma,\delta}(n; \cdot)$ , we have

$$(4.36) \quad T_{2,n}^{\gamma,\delta}f = M_{1,n}^{\gamma,\delta}(M_{\omega w}HM_{\omega w}^{-1})M_{2,n}^{\gamma,\delta}\chi_{[-1,1]}f$$

where  $H$  denotes the Hilbert transform. Now it is easily seen that  $w^2$  is an  $A_2$  weight; by Lemma 4.1 there exist constants  $\delta_1, C > 0$  depending only upon

$w$  such that if  $\beta \in BMO$  with  $\|\beta\|_* < \delta_1$  then  $(\omega w)^2$  is also in  $A_2$  with an  $A_2$  constant depending only upon  $w$ . Consequently, for  $\|\beta\|_* < \delta_1$ ,

$$(4.37) \quad \|M_{\omega w} H M_{\omega w}^{-1}\|_{\mathcal{L}(0)} \leq C$$

where  $C$  is a constant depending only upon  $w$  (see [7]). From (4.36) and (4.37) it follows that  $\|T_{2,n}^{\gamma,\delta}\|_{\mathcal{L}(0)}$  is bounded by a constant independent of  $n$  and  $\beta$ .

The analysis of  $S_{2,n}^{\gamma,\delta}$  (for  $n \geq 2$ ) is similar. It can be seen without difficulty that

$$(4.38) \quad H_2^{\gamma,\delta}(n; y, x) \omega_{\gamma,\delta}(x) \omega_{\gamma,\delta}(y) = b_2^{\gamma,\delta}(n; x) b_1^{\gamma,\delta}(n; y) \frac{w^{-1}(x)}{w^{-1}(y)} \frac{1}{y-x}$$

so that

$$(4.39) \quad S_{2,n}^{\gamma,\delta} f = -M_{2,n}^{\gamma,\delta} (M_{\omega w^{-1}} H M_{\omega w^{-1}}^{-1}) M_{1,n}^{\gamma,\delta} \chi_{[-1,1]} f;$$

since  $w^{-2}$  is an  $A_2$  weight, applying Lemma 4.1 as before shows that for  $\|\beta\|_* < \delta_1$ ,  $\|S_{2,n}^{\gamma,\delta}\|_{\mathcal{L}(0)}$  is bounded by a constant independent of  $n, \beta$ .

The analysis of  $T_{1,n}^{\gamma,\delta}$  is somewhat easier. Since  $\gamma, \delta \geq -\frac{1}{2}$ , it is easy to see that for  $x, y \in [-1, 1]$ ,  $n \in \mathbb{N}$ ,

$$(4.40) \quad |H_1^{\gamma,\delta}(n; x, y) \omega_{\gamma,\delta}(x) \omega_{\gamma,\delta}(y)| \leq C(\gamma, \delta) w(x) w(y)$$

where  $C(\gamma, \delta)$  is independent of  $n$ . Then, by (4.22) and (4.40),

$$(4.41) \quad \int_{-1}^1 |T_{1,n}^{\gamma,\delta} f(x)|^2 dx \leq C(\gamma, \delta)^2 \int_{-1}^1 \left| \int_{-1}^1 w(x) w(y) \omega(x) \omega(y)^{-1} f(y) dy \right|^2 dx.$$

Letting  $I = [-1, 1]$ ,  $b = \log w$ , we may write

$$(4.42) \quad w(x) w(y) \omega(x) \omega(y)^{-1} = \exp[(b + \beta)(x) - m_I(b + \beta)] \times \exp[(b - \beta)(y) - m_I(b - \beta)] \exp[m_I(2b)].$$

Then, by (4.42) and Schwarz' inequality

$$(4.43) \quad \int_I \left| \int_I w(x) w(y) \omega(x) \omega(y)^{-1} f(y) dy \right|^2 dx \leq \exp[m_I(4b)] \cdot \left( \int_I \exp[2(b + \beta)(x) - m_I(2(b + \beta))] dx \right) \times \left( \int_I \exp[2(b - \beta)(y) - m_I(2(b - \beta))] dy \right) \|f\|_{L(0)}^2.$$

When  $\|\beta\|_* < \delta_1$ , we have  $w^2 \omega^2$  and  $w^2 \omega^{-2} \in A_2$ , so the right-hand side of (4.43) is bounded by a constant times  $\|f\|_{L(0)}^2$ , where the constant depends only on  $w$ . Thus  $\|\beta\|_* < \delta_1$  implies that  $\|T_{1,n}^{\gamma,\delta}\|_{\mathcal{L}(0)}$  is bounded by a constant independent of  $n, \beta$ .

By virtue of the decomposition (4.17), the proof is complete.  $\square$

The following corollaries are immediate:

**Corollary 4.2.1.**  $\langle P_n^{\gamma, \delta} \rangle$  and  $\langle Q_n^{\gamma, \delta} \rangle$  are uniformly holomorphic in a neighborhood of 0 in **BMO** whenever  $\gamma, \delta \geq -\frac{1}{2}$ .  $\square$

**Corollary 4.2.2.** For  $\gamma, \delta \geq -\frac{1}{2}$ , there exists a constant  $C(\gamma, \delta)$  such that for all  $\beta \in \mathbf{BMO}$ ,

$$(4.44) \quad \| [M_\beta, S_n^{\gamma, \delta}(0)] \|_{\mathcal{L}^{\gamma, \delta}(0)} \leq C(\gamma, \delta) \|\beta\|_* . \quad \square$$

5. AN APPLICATION TO THE TODA FLOW

Let  $d\mu$  be a nonnegative measure on  $[-1, 1]$  which is absolutely continuous with respect to Lebesgue measure; for example,  $d\mu$  may be Lebesgue measure weighted by a Jacobi weight. Following the notation of §2, for  $n \in \mathbf{N}$ , let  $H_n(0)$  denote the set of polynomials of degree at most  $n$ , considered as a subspace of  $L(0) = L^2([-1, 1], d\mu)$ . Let  $\omega$  be a fixed nonnegative real-valued function such that  $\omega^2 + \omega^{-2} \in L^1([-1, 1], d\mu)$ ; write  $\beta = \log \omega \in L^1([-1, 1], d\mu)$ . For each  $t$  in a neighborhood of 0 in  $\mathbf{R}$ , we consider the Gram-Schmidt procedure in the space  $L(t\beta) = L^2([-1, 1], \omega^{2t} d\mu)$ . Specifically, we let  $\langle p_{n,t}(x) : n \in \mathbf{N} \rangle$  denote the orthogonal polynomial system on  $L(t\beta)$  obtained by applying the Gram-Schmidt procedure to  $\langle 1, x, x^2, \dots \rangle$ . For  $t \neq 0$ , it is easily seen that  $\langle p_{n,t} \rangle$  arises also by applying Gram-Schmidt to  $\langle p_{n,0} \rangle$ ; and, in fact,

$$(5.1) \quad p_{n,t} = \frac{\{S_n(t\beta) - S_{n-1}(t\beta)\} p_{n,0}}{\| \{S_n(t\beta) - S_{n-1}(t\beta)\} p_{n,0} \|_{L(t\beta)}} .$$

The polynomials  $\langle p_{n,t} \rangle$  satisfy the following three-term recurrence (see, for example, [10, §3.2]):

$$(5.2) \quad x p_{n,t}(x) = A_{n-1}(t) p_{n-1,t}(x) + B_n(t) p_{n,t}(x) + A_n(t) p_{n+1,t}(x), \quad n \in \mathbf{N},$$

where we let  $p_{-1,t}(x) \equiv 0 \equiv A_{-1}(t)$ , and, for  $n \in \mathbf{N}$ ,

$$(5.3) \quad A_n(t) = \int_{-1}^1 x (M_{\omega^t} p_{n,t})(x) (M_{\omega^t} p_{n+1,t})(x) d\mu(x),$$

$$(5.4) \quad B_n(t) = \int_{-1}^1 x [(M_{\omega^t} p_{n,t})(x)]^2 d\mu(x).$$

The Gram-Schmidt process can be done so that  $A_n(t) > 0$  for all  $n \in \mathbf{N}$ . It is easy to see that, for  $n \in \mathbf{N}$  and  $t \in \mathbf{R}$ ,  $|A_n(t)|, |B_n(t)| \leq 1$ . We note for future reference that, by (5.1),

$$(5.5) \quad M_{\omega^t} p_{n,t} = \frac{\{Q_n(t\beta) - Q_{n-1}(t\beta)\} (M_{\omega^t} p_{n,0})}{\| \{Q_n(t\beta) - Q_{n-1}(t\beta)\} (M_{\omega^t} p_{n,0}) \|_{L(0)}} .$$

Let  $l_+^2$  denote the complex Hilbert space of square summable sequences; i.e., a sequence  $\langle a_n \rangle$  is in  $l_+^2$  if and only if

$$(5.6) \quad \|\langle a_n \rangle\|_{l_+^2} \equiv \sum_{n=0}^{\infty} |a_n|^2 < \infty;$$

the inner product on  $l_+^2$  is given by

$$(5.7) \quad (\langle a_n \rangle, \langle b_n \rangle) = \sum_{n=0}^{\infty} a_n \bar{b}_n.$$

The mapping  $L_t: L(t\beta) \rightarrow L(t\beta)$  given by  $L_t f(x) = x f(x)$  induces a bounded linear transformation on  $l_+^2$  given by the matrix

$$(5.8) \quad J(t) = \begin{pmatrix} B_0(t) & A_0(t) & 0 & 0 & \cdots & \cdots \\ A_0(t) & B_1(t) & A_1(t) & 0 & \cdots & \cdots \\ 0 & A_1(t) & B_2(t) & A_2(t) & \cdots & \cdots \\ 0 & 0 & A_2(t) & B_3(t) & A_3(t) & \cdots \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix};$$

$J(t)$  is an infinite, symmetric, tridiagonal matrix with strictly positive off-diagonal elements, i.e., a *Jacobi matrix*. The mapping  $t \mapsto J(t)$  defines a flow on the space of Jacobi matrices, which is a generalized infinite-dimensional Toda flow of the type studied by Deift, Li, and Tomei in [4], especially §5. Deift et. al. have asked for a characterization of those functions  $\omega$  for which the flow  $J$  is analytic in a neighborhood of 0 in  $\mathbf{R}$ . As an application of our earlier work, we can now give a partial answer to their question. We remark that the analyticity of the flow  $J$  is essentially equivalent to the analyticity of the Gram-Schmidt process relative to measures of the form  $\omega^{2t} d\mu$  on  $[-1, 1]$ .

Let us be more explicit about the operator  $J(t)$  on  $l_+^2$ . Suppose that  $\hat{f} = \langle \hat{f}_0, \hat{f}_1, \hat{f}_2, \dots \rangle, \hat{g} = \langle \hat{g}_0, \hat{g}_1, \hat{g}_2, \dots \rangle \in l_+^2; \hat{f}$  and  $\hat{g}$  give the Fourier coefficients for functions  $f_t, g_t \in L(t\beta)$  defined by

$$(5.9) \quad f_t = \sum_{k=0}^{\infty} \hat{f}_k p_{k,t}, \quad g_t = \sum_{k=0}^{\infty} \hat{g}_k p_{k,t}.$$

Then we have

$$(5.10) \quad \begin{aligned} (J(t)\hat{f}, \hat{g}) &= B_0(t)\hat{f}_0\hat{g}_0 + A_0(t)\hat{f}_1\hat{g}_0 \\ &+ \sum_{k=0}^{\infty} \{A_k(t)\hat{f}_k\hat{g}_{k+1} + B_{k+1}\hat{f}_{k+1}\hat{g}_{k+1} + A_{k+1}(t)\hat{f}_{k+1}\hat{g}_{k+1}\}. \end{aligned}$$

We state our question about the analyticity of  $J$  precisely, as follows. We would like to know: under what conditions on  $\omega$  is it possible to extend  $J$  to

a neighborhood  $U$  of 0 in  $\mathbb{C}$ , in such a way as to insure that the extension  $\tilde{J}$  is a holomorphic map from  $U$  to the space  $\mathcal{L}(l_+^2)$  of bounded linear operators on  $l_+^2$ ? By virtue of (5.10), we see that it suffices to obtain conditions on  $\omega$  which will insure that there is a neighborhood  $U$  of 0 in  $\mathbb{C}$  to which, for each  $n \in \mathbb{N}$ , the functions  $A_n$  and  $B_n$  can be holomorphically extended to functions of modulus  $\leq 1$ .

The extension of  $A_n$  and  $B_n$  to complex values is easily effected by defining, for  $z \in \mathbb{C}$ ,

$$(5.11) \quad M_{\omega^z p_n, z} = \frac{\{Q_n(z\beta) - Q_{n-1}(z\beta)\}(M_{\omega^z p_n, 0})}{\|\{Q_n(z\beta) - Q_{n-1}(z\beta)\}(M_{\omega^z p_n, 0})\|_{L(0)}};$$

the extension is meaningful whenever the extension of  $\langle Q_n \rangle$  to complex-valued functions is meaningful. Since the  $L(0)$ -norm of (5.11) is 1, it follows from (5.3) and (5.4) with  $z$  in place of  $t$  that  $|A_n(z)|, |B_n(z)| \leq 1$ .

In fact, it suffices to obtain conditions on  $\omega$  which will guarantee the existence of a neighborhood  $U$  of 0 in  $\mathbb{C}$  such that the mapping  $z \mapsto M_{\omega^z p_n, z}$  is a holomorphic map from  $U$  to  $L(0)$  for each  $n \in \mathbb{N}$ . In the remainder of this section, we will do this in the special case in which  $d\mu$  is Lebesgue measure weighted by a Jacobi weight  $\omega_{\gamma, \delta}^2$ , where  $\gamma, \delta \geq -\frac{1}{2}$ .

We shall make use of the notation established in §4. For  $n \in \mathbb{N}$  and  $x \in [-1, 1]$ , we let

$$(5.12) \quad p_n^{(\gamma, \delta)}(x) = \frac{P_n^{(\gamma, \delta)}(x)}{\|P_n^{(\gamma, \delta)}\|_{L_{\gamma, \delta}}(0)}$$

denote the  $n$ th normalized Jacobi polynomial. We begin with the following result:

**Lemma 5.1.** *Suppose  $\beta \in BMO$  and let  $\omega = e^\beta$ . For  $n \in \mathbb{N}$ ,  $\gamma, \delta \geq -\frac{1}{2}$ , and  $z \in \mathbb{C}$ , let*

$$(5.13) \quad F_n^{\gamma, \delta}(z) = M_{\omega^z p_n^{(\gamma, \delta)}}.$$

*Then there exists a neighborhood  $U$  of 0 in  $\mathbb{C}$ , and a constant  $K > 0$ , such that for each  $n \in \mathbb{N}$ ,  $F_n^{\gamma, \delta}$  is an analytic function from  $U$  to  $L_{\gamma, \delta}(0)$ , and for all  $z \in U$ ,*

$$(5.14) \quad \|F_n^{\gamma, \delta}(z)\|_{L_{\gamma, \delta}(0)} \leq K.$$

*Proof.* By virtue of (4.20), (4.21), and [10, equation (4.3.4), p. 68], we see that there is a constant  $K(\gamma, \delta)$  such that, for  $n \in \mathbb{N}$ ,

$$(5.15) \quad |p_n^{(\gamma, \delta)}(x)| \leq \begin{cases} K(\gamma, \delta)(1+x+n^{-2})^{-\delta/2-1/4}, & x \in [-1, 0], \\ K(\gamma, \delta)(1-x+n^{-2})^{-\gamma/2-1/4}, & x \in [0, 1]. \end{cases}$$

Now note that if  $r, s > 1$  and  $1/r + 1/s = 1$ , we have

$$(5.16) \quad \|F_n^{\gamma, \delta}(z)\|_{L_{\gamma, \delta}(0)} \leq \left( \int_{-1}^1 |\omega(x)^{2zr}| (1-x)^\gamma (1+x)^\delta dx \right)^{1/r} \\ \times \left( \int_{-1}^1 |p_n^{(\gamma, \delta)}(x)|^{2s} (1-x)^\gamma (1+x)^\delta dx \right)^{1/s}$$

by Hölder's inequality. Now we have, by (5.15), for  $n \geq 1$ ,

$$(5.17) \quad \int_0^1 |p_n^{(\gamma, \delta)}(x)|^{2s} (1-x)^\gamma (1+x)^\delta dx \\ \leq 2^\delta K(\gamma, \delta)^{2s} \int_0^1 (1-x+n^{-2})^{-\gamma s - s/2} (1-x)^\gamma dx \\ \leq 2^{\delta+1} K(\gamma, \delta)^{2s} \left\{ \int_0^{1-n^{-2}} (1-x)^{-\gamma s - s/2 + \gamma} dx + \int_{1-n^{-2}}^1 n^{2\gamma s + s} (1-x)^\gamma dx \right\} \\ = 2^{\delta+1} K(\gamma, \delta)^{2s} \left\{ \int_{n^{-2}}^1 y^{-\gamma s - s/2 + \gamma} dy + n^{2\gamma s + s} \int_0^{n^{-2}} y^\gamma dy \right\}.$$

If  $\gamma = -\frac{1}{2}$ , the rightmost expression in (5.17) becomes

$$(5.18) \quad 2^{\delta+1} K(\gamma, \delta)^{2s} \int_0^1 y^\gamma dy = (\gamma+1)^{-1} 2^{\delta+1} K(\gamma, \delta)^{2s}.$$

If  $\gamma > -\frac{1}{2}$  and  $1 < s < (2+2\gamma)(1+2\gamma)^{-1}$ , we have  $-\gamma s - s/2 + \gamma > -1$ , so that the rightmost expression in (5.17) is dominated by

$$(5.19) \quad 2^{\delta+1} K(\gamma, \delta)^{2s} \left\{ \frac{2}{2\gamma + 2 - 2\gamma s - s} + \frac{1}{\gamma + 1} \right\}.$$

Thus it is not difficult to see that for  $\gamma, \delta \geq \frac{1}{2}$  and for  $s$  satisfying  $1 < s < \min\{(2+2\gamma)(1+2\gamma)^{-1}, (2+2\delta)(1+2\delta)^{-1}\}$ , there is a constant  $K_1(\gamma, \delta, s)$  such that for all  $n \in \mathbf{N}$ ,

$$(5.20) \quad \left( \int_{-1}^1 |p_n^{(\gamma, \delta)}(x)|^{2s} (1-x)^\gamma (1+x)^\delta dx \right)^{1/s} \leq K_1(\gamma, \delta, s).$$

For any such choice of  $s$ , let  $r$  be the conjugate exponent to  $s$ . By Lemma 4.1, there exists a neighborhood  $U_0$  of 0 in  $\mathbf{C}$  such that, for all  $z \in U_0$ ,

$$(5.21) \quad |\omega(x)^{2zr}| (1-x)^\gamma (1+x)^\delta$$

is an  $A_2$  weight, hence integrable on  $[-1, 1]$ ; in fact, using the characterization of  $A_2$  given in the proof of Lemma 4.1, it follows that for  $z \in U_0$ ,

$$(5.22) \quad \int_{-1}^1 |\omega(x)^{2zr}| (1-x)^\gamma (1+x)^\delta dx$$

is bounded above by a constant independent of  $z$ . Thus there is a constant  $K$  such that for all  $z \in U_0$ , and for all  $n \in \mathbf{N}$ , we obtain (5.14).

Now note that

$$(5.23) \quad \frac{d}{dz} F_n^{\gamma, \delta}(z) = M_\beta M_{\omega \cdot p_n^{(\gamma, \delta)}};$$

as before, if  $1 < s < \min\{(2+2\gamma)(1+2\gamma)^{-1}, (2+2\delta)(1+2\delta)^{-1}\}$  and  $1/r+1/s = 1$ , we obtain

$$(5.24) \quad \left\| \frac{d}{dz} F_n^{\gamma, \delta}(z) \right\|_{L_{\gamma, \delta}(0)} \leq K_1(\gamma, \delta, s) \left( \int_{-1}^1 |\beta(x)^{2zr} \omega(x)^{2zr}| (1-x)^\gamma (1+x)^\delta dx \right)^{1/r} \leq K_1(\gamma, \delta, s) \left( \int_{-1}^1 \exp(4r|z\beta(x)|) \cdot (1-x)^\gamma (1+x)^\delta dx \right)^{1/r}.$$

Again, there is a neighborhood  $V_0$  of 0 in  $\mathbb{C}$  such that, for all  $z \in V_0$ ,

$$(5.25) \quad \int_{-1}^1 \exp(4r|z\beta(x)|) \cdot (1-x)^\gamma (1+x)^\delta dx$$

is bounded above by a constant independent of  $z$ . If we take  $U = U_0 \cap V_0$ , the lemma follows.  $\square$

Now suppose  $\beta$  is a fixed function in  $BMO$ ,  $\omega = e^\beta$ ,  $\gamma, \delta \geq -\frac{1}{2}$ . For  $z \in \mathbb{C}$ , define

$$(5.26) \quad M_{\omega \cdot p_{n,z}^{(\gamma, \delta)}} = \frac{\{Q_n^{\gamma, \delta}(z\beta) - Q_{n-1}^{\gamma, \delta}(z\beta)\}(M_{\omega \cdot p_n^{(\gamma, \delta)}})}{\|\{Q_n^{\gamma, \delta}(z\beta) - Q_{n-1}^{\gamma, \delta}(z\beta)\}(M_{\omega \cdot p_n^{(\gamma, \delta)}})\|_{L_{\gamma, \delta}(0)}}.$$

By Corollary 4.2.1 and Lemma 5.1, there is a neighborhood  $U_1$  of 0 in  $\mathbb{C}$ , and a constant  $K_1 > 0$ , such that for  $n \in \mathbb{N}$ , the map

$$(5.27) \quad z \mapsto \{Q_n^{\gamma, \delta}(z\beta) - Q_{n-1}^{\gamma, \delta}(z\beta)\}(M_{\omega \cdot p_n^{(\gamma, \delta)}})$$

is an analytic function from  $U_1$  to  $L_{\gamma, \delta}(0)$ , and for all  $z \in U_1$ ,

$$(5.28) \quad \|\{Q_n^{\gamma, \delta}(z\beta) - Q_{n-1}^{\gamma, \delta}(z\beta)\}(M_{\omega \cdot p_n^{(\gamma, \delta)}})\|_{L_{\gamma, \delta}(0)} \leq K_1.$$

In particular, this implies that the family of maps (5.27) is continuous on  $U_1$  uniformly in  $n$ . Now note that

$$(5.29) \quad \{Q_n^{\gamma, \delta}(0) - Q_{n-1}^{\gamma, \delta}(0)\}(p_n^{(\gamma, \delta)}) = p_n^{(\gamma, \delta)}$$

so that, for  $z = 0$ , the denominator in (5.26) is identically 1. Thus there is a neighborhood  $U_2$  of 0 in  $\mathbb{C}$ , and a constant  $K_2 > 0$ , such that for all  $z \in U_2$  and for all  $n \in \mathbb{N}$ ,

$$(5.30) \quad \|\{Q_n^{\gamma, \delta}(z\beta) - Q_{n-1}^{\gamma, \delta}(z\beta)\}(M_{\omega \cdot p_n^{(\gamma, \delta)}})\|_{L_{\gamma, \delta}(0)} \geq K_2.$$

From this, then, it follows that there is a neighborhood  $U$  of 0 in  $\mathbb{C}$  such that the mapping

$$(5.31) \quad z \mapsto M_{\omega \cdot p_{n,z}^{(\gamma, \delta)}}$$

is holomorphic from  $U$  to  $L_{\gamma, \delta}(0)$  for each  $n \in \mathbb{N}$ . Consequently, we obtain

**Proposition 5.2.** *Let  $\gamma, \delta \geq -\frac{1}{2}$ , let  $\beta \in BMO$ , and let  $\omega = e^\beta$ . Let  $t \mapsto J_{\gamma, \delta}(t)$  denote the Toda flow corresponding to the Gram-Schmidt process relative to the measure*

$$(5.32) \quad \omega(x)^{2t}(1-x)^\gamma(1+x)^\delta dx$$

*on  $[-1, 1]$ . Then there is a neighborhood  $U$  of 0 in  $\mathbf{C}$  to which  $J_{\gamma, \delta}$  may be extended to a holomorphic map  $\tilde{J}_{\gamma, \delta}: U \rightarrow \mathcal{L}(l_+^2)$ .  $\square$*

### 6. UNIFORM ANALYTICITY ON THE CIRCLE, REVISITED

In §4, we showed that, for  $\gamma, \delta \geq -\frac{1}{2}$ , the family  $\langle P_n^{\gamma, \delta} \rangle$  of conjugated partial sum operators for Jacobi series is uniformly holomorphic in a neighborhood of 0 in **BMO**. We conjecture that, in fact, **BMO** is the space of uniform holomorphy at 0 for  $\langle P_n^{\gamma, \delta} \rangle$ . Owing in part to the complicated form which the kernel of  $P_n^{\gamma, \delta}$  takes, this conjecture will be somewhat more difficult to establish than the analogous results for the partial sums of Fourier series which we obtained in §3.

There are a number of classical results on the equiconvergence of Jacobi series with cosine series (for example, [10, Theorem 9.1.2]), which lead us to consider, as a preliminary step, the problem of determining the space of uniform holomorphy at 0 for the conjugated partial sums of cosine series. In this section, we shall show that **BMO**( $[0, \pi]$ )—i.e., the space of even functions of bounded mean oscillation on **T**—is the space of uniform holomorphy at 0 for conjugated partial sums of cosine series.

We will use the notation established in §3, with some additions and modifications. Let  $\tilde{L}(0) = L^2([0, \pi], d\theta)$  and let  $\omega$  be a nonnegative weight function on  $[0, \pi]$  such that  $\omega, \omega^{-1} \in \tilde{L}(0)$ ; write  $\beta = \log \omega$ . For each  $n \in \mathbf{N}$ , let  $\tilde{H}_n(0)$  be the span of  $\langle 1, \cos \theta, \cos 2\theta, \dots, \cos n\theta \rangle$  in  $\tilde{L}(0)$ . We define  $\tilde{L}(\beta) = L^2([0, \pi], \omega^2(\theta)d\theta)$  and let  $\tilde{H}_n(\beta)$  denote the closure of  $\tilde{H}_n(0)$  in  $\tilde{L}(\beta)$ . We let  $\tilde{S}_n(\beta)$  be the self-adjoint projection of  $\tilde{L}(\beta)$  onto  $\tilde{H}_n(\beta)$ , and then define

$$(6.1) \quad \tilde{Q}_n(\beta) = M_\omega \tilde{S}_n(\beta) M_\omega^{-1},$$

$$(6.2) \quad \tilde{P}_n(\beta) = M_\omega \tilde{S}_n(0) M_\omega^{-1}.$$

A function  $b \in L^1([0, \pi], d\theta)$  is an element of **BMO**( $[0, \pi], d\theta$ ) if and only if

$$(6.3) \quad \|b\|_* \equiv \sup_I |I|^{-1} \int_I |b(x) - m_I(b)| dx = \sup_I m_I(|b - m_I(b)|)$$

is finite, where the supremum is taken over all subintervals  $I$  of  $[0, \pi]$ . We shall abbreviate **BMO**( $[0, \pi], d\theta$ ) to **BMO**<sub>e</sub> (the subscript ‘e’ stands for ‘even’).

Now suppose that  $g \in \tilde{L}(0)$ , and let  $\tilde{g}$  be its even extension to  $[-\pi, \pi]$ . Then

$$(6.4) \quad S_n(0)\tilde{g} \Big|_{[0, \pi]} = \tilde{S}_n(0)g$$

so that, for  $\theta \in [0, \pi]$ ,

$$(6.5) \quad \{\tilde{S}_n(0)g\}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(\theta, \psi)\tilde{g}(\psi)d\psi = \frac{1}{2\pi} \int_0^{\pi} \tilde{D}_n(\theta, \psi)g(\psi)d\psi$$

where, for  $\psi \in [0, \pi]$ ,

$$(6.6) \quad \begin{aligned} \tilde{D}_n(\theta, \psi) &= D_n(\theta, -\psi) + D_n(\theta, \psi) \\ &= \frac{\sin[(2n+1)\frac{\theta-\psi}{2}]}{\sin(\frac{\theta-\psi}{2})} + \frac{\sin[(2n+1)\frac{\theta+\psi}{2}]}{\sin(\frac{\theta+\psi}{2})}. \end{aligned}$$

Letting  $\tilde{\mathcal{F}}(0)$  denote the space of bounded linear operators on  $\tilde{L}(0)$ , we obtain the following as an immediate consequence of our work in §3:

**Proposition 6.1.**  $\langle \tilde{P}_n \rangle$  and  $\langle \tilde{Q}_n \rangle$  are uniformly holomorphic families of mappings from a neighborhood of 0 in  $\mathbf{BMO}_e$  to  $\tilde{\mathcal{F}}(0)$ .  $\square$

To prove that  $\mathbf{BMO}_e$  is actually the space of uniform holomorphy at 0 for  $\langle \tilde{P}_n \rangle$  and  $\langle \tilde{Q}_n \rangle$ , it suffices by Corollary 2.2.2 to show that  $\beta \in \mathbf{BMO}_e$  is a necessary and sufficient condition for boundedness of the set  $\{ \| [M_\beta, \tilde{S}_n(0)] \|_{\tilde{\mathcal{F}}(0)} : n \in \mathbf{N} \}$ . To gain some intuition for this problem, we first consider a somewhat simpler problem involving “partial sums” of Fourier transforms on  $\mathbf{R}$ .

For  $f \in L^1(\mathbf{R})$ , we define the Fourier transform and its inverse according to the normalization

$$(6.7) \quad \tilde{f}(\xi) = \int_{\mathbf{R}} e^{-ix\xi} f(x)dx, \quad \check{f}(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{ix\xi} f(\xi)d\xi.$$

For each positive integer  $n$ , we define the operator  $T_n(0): L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$  by setting

$$(6.8) \quad T_n(0)f = (\hat{f}\chi_{[-n, n]})^\vee$$

for  $f \in L^2(\mathbf{R})$ .  $T_n(0)$  is a convolution operator, with kernel

$$(6.9) \quad K_n(x) = \frac{1}{\pi x} \sin nx.$$

We define the operator  $\tilde{T}_n(0): L^2([0, \infty)) \rightarrow L^2([0, \infty))$  as follows: for  $g \in L^2([0, \infty))$ , let  $\tilde{g}$  denote its even extension to  $\mathbf{R}$ , and define

$$(6.10) \quad \tilde{T}_n(0)g = T_n(0)\tilde{g} \Big|_{[0, \infty)}.$$

For  $x \in [0, \infty)$ , we have

$$(6.11) \quad \{\tilde{T}_n(0)\}g(x) = \int_{\mathbf{R}} K_n(x-y)\tilde{g}(y)dy = \int_0^\infty \tilde{K}_n(x, y)g(y)dy$$

where

$$(6.12) \quad \begin{aligned} \tilde{K}_n(x, y) &= K_n(x - y) + K_n(x + y) \\ &= \frac{1}{\pi} \left\{ \frac{(x + y) \sin n(x - y) + (x - y) \sin n(x + y)}{x^2 - y^2} \right\}. \end{aligned}$$

The operators  $T_n(0)$  and  $\tilde{T}_n(0)$  are the continuous analogues of  $S_n(0)$  and  $\tilde{S}_n(0)$ , respectively. Now suppose that  $\omega = e^\beta$  is a nonnegative weight function on  $\mathbf{R}$  such that  $\omega^2 + \omega^{-2} \in L^1_{\text{loc}}(\mathbf{R})$ , and define  $T_n(\beta) = M_\omega T_n(0) M_\omega^{-1}$ . We obtain the following continuous analogue of Proposition 3.1:

**Proposition 6.2.** *The quantity  $\sup\{\|T_n(\beta)\|_{\mathcal{L}(L^2(\mathbf{R}))} : n = 1, 2, 3, \dots\}$  is finite if and only if  $\omega^2 \in A_2$ .*

*Proof.* This is the content of Corollary 3.1.2, Chapter 4 of [5].  $\square$

**Corollary 6.2.1.**  *$\langle T_n \rangle$  is a uniformly holomorphic family of mappings from a neighborhood of 0 in  $\mathbf{BMO}(\mathbf{R})$  to  $\mathcal{L}(L^2(\mathbf{R}))$ .*

*Proof.* Completely analogous to that of Corollary 3.1.1.  $\square$

**Corollary 6.2.2.**  *$\langle \tilde{T}_n \rangle$  is a uniformly holomorphic family of mappings from a neighborhood of 0 in  $\mathbf{BMO}([0, \infty))$  to  $\mathcal{L}(L^2([0, \infty)))$ .  $\square$*

It is left as a straightforward exercise for the reader to prove the continuous analogue of Proposition 3.2 (i.e., with  $T_n(0)$  in place of  $S_n(0)$ ). It is the proof of the corresponding result for  $\tilde{T}_n(0)$  that is of greatest interest to us here. We shall begin with an extremely useful lemma.

Let  $1 < p < \infty$ . A nonnegative weight function  $w$  on  $\mathbf{R}$  is said to belong to the class  $A_p$  if and only if both  $w$  and  $w^{-1/(p-1)}$  are locally integrable, and there is a constant  $C > 0$  such that for all subintervals  $I$  of  $\mathbf{R}$ ,

$$(6.13) \quad \left( |I|^{-1} \int_I w \right) \left( |I|^{-1} \int_I w^{-1/(p-1)} \right)^{p-1} \leq C.$$

The smallest constant for which (6.13) holds is called the  $A_p$  constant of  $w$ . It is worth noting here that

$$(6.14) \quad A_1 = \bigcap_{1 \leq p < \infty} A_p, \quad A_\infty = \bigcup_{1 \leq p < \infty} A_p$$

(see, for example, [5, Chapter 4, Theorem 1.14 and Corollary 2.13]). Our lemma is as follows:

**Lemma 6.3.** *Suppose  $w \in A_\infty$  and  $b \in L^1_{\text{loc}}(\mathbf{R})$ . Suppose, moreover, that there is a constant  $K(b)$  such that, for all subintervals  $I$  of  $\mathbf{R}$ ,*

$$(6.15) \quad w(I)^{-1} \int_I |b(x) - m_I(b)| w(x) dx \leq K(b)$$

where

$$(6.16) \quad w(I) = \int_I w(x) dx.$$

Then  $b \in \mathbf{BMO}(\mathbf{R})$ , and  $\|b\|_* \leq C(w)K(b)$ , where  $C(w)$  is a constant depending only upon  $w$ .

*Proof.* Since  $w \in A_\infty$ , it follows from (6.14) that there is a  $p \in (1, \infty)$  such that  $w \in A_p$ . Now let  $I$  be a subinterval of  $\mathbf{R}$ ; we have, by Hölder's inequality,

$$(6.17) \quad \int_I |b(x) - m_I(b)| dx \leq \left( \int_I |b(x) - m_I(b)|^p w(x) dx \right)^{1/p} \left( \int_I w(x)^{-1/(p-1)} dx \right)^{(p-1)/p}.$$

It is not difficult to see that there is a constant  $C_1$  depending only upon  $p$  such that

$$(6.18) \quad \left( \int_I |b(x) - m_I(b)|^p w(x) dx \right)^{1/p} \leq C_1 w(I)^{1/p} K(b)$$

(see, for example, [5, Chapter 2, Corollary 3.10]). Moreover, if  $C_2$  is the  $A_p$  constant of  $w$ , we have

$$(6.19) \quad \left( \int_I w(x)^{-1/(p-1)} dx \right)^{(p-1)/p} \leq C_2^{1/p} |I| w(I)^{-1/p}.$$

Combining (6.17)–(6.19), and letting  $C(w) = C_1 C_2^{1/p}$ , we have

$$(6.20) \quad \int_I |b(x) - m_I(b)| dx \leq |I| C(w) K(b)$$

from which the result follows.  $\square$

We make use of the lemma to prove

**Proposition 6.4.** *There exist constants  $C_1, C_2 > 0$  such that*

$$(6.21) \quad C_1 \|\beta\|_* \leq \sup_n \|[M_\beta, \tilde{T}_n(0)]\|_{\text{op}} \leq C_2 \|\beta\|_*$$

for all  $\beta \in L^1_{\text{loc}}([0, \infty))$ , where  $\|\cdot\|_*$  denotes the norm on  $\mathbf{BMO}([0, \infty))$ , and  $\|\cdot\|_{\text{op}}$  denotes the norm as an operator on  $L^2([0, \infty))$ .

*Proof.* The existence of  $C_2$  with the requisite property is immediate from Corollary 6.2.2. To establish the existence of  $C_1$ , we make use of Lemma 6.3. For  $x \in [0, \infty)$ , let  $w(x) = x^2$ ;  $w$  is an  $A_\infty$  weight. We shall show that there exists a constant  $\mu > 0$  such that for all subintervals  $I$  of  $[0, \infty)$ , and for all  $\beta \in L^1_{\text{loc}}([0, \infty))$ ,

$$(6.22) \quad w(I)^{-1} \int_I |\beta(x) - m_I(\beta)|^2 w(x) dx \leq \mu C^2,$$

where

$$(6.23) \quad C = \sup_n \|[M_\beta, \tilde{T}_n(0)]\|_{\text{op}}.$$

The result is then immediate from the  $[0, \infty)$ -version of Lemma 6.3.

Let  $I$  be a subinterval of  $[0, \infty)$  and let  $x \in [0, \infty)$ . Define the function  $f_x = f_{I, x, n}$  by setting,

$$(6.24) \quad f_x(y) = \chi_I(y) \pi(x^2 - y^2) \cos ny = \chi_I(y) \frac{\pi}{2}(x^2 - y^2)(e^{iny} + e^{-iny})$$

for  $y \in [0, \infty)$ . Note that, for  $y \in I$ ,

$$(6.25) \quad \begin{aligned} \frac{1}{\pi} \frac{(x+y) \sin n(x-y)}{x^2 - y^2} f_x(y) &= \frac{1}{4i}(x+y)[e^{in(x-y)} - e^{-in(x-y)}](e^{iny} + e^{-iny}) \\ &= \frac{1}{4i}(x+y)[e^{inx} - e^{-inx} - e^{in(2y-x)} + e^{-in(2y-x)}] \\ &= \frac{1}{2}(x+y) \sin nx + \frac{1}{2}(x+y) \sin n(x-2y), \end{aligned}$$

$$(6.26) \quad \begin{aligned} \frac{1}{\pi} \frac{(x-y) \sin n(x+y)}{x^2 - y^2} f_x(y) &= \frac{1}{4i}(x-y)[e^{in(x+y)} - e^{-in(x+y)}](e^{iny} + e^{-iny}) \\ &= \frac{1}{4i}(x-y)[e^{inx} - e^{-inx} - e^{-in(2y+x)} + e^{in(2y+x)}] \\ &= \frac{1}{2}(x-y) \sin nx + \frac{1}{2}(x-y) \sin n(x+2y), \end{aligned}$$

so that, by (6.12),

$$\begin{aligned} K_n(x, y) f_x(y) &= \{x \sin nx + \frac{1}{2}(x+y) \sin n(x-2y) \\ &\quad + \frac{1}{2}(x-y) \sin n(x+2y)\} \chi_I(y) \\ &= \{x \sin nx + x \sin nx \cos 2ny - y \sin 2ny \cos nx\} \chi_I(y). \end{aligned}$$

Combining (6.11) and (6.27), we obtain

$$(6.28) \quad \begin{aligned} \{[M_\beta, \tilde{T}_n(0)]f_x\}(x) &= x \sin nx |I|(\beta(x) - m_I(\beta)) \\ &\quad + x \sin nx \int_0^\infty \cos 2ny(\beta(x) - \beta(y)) \chi_I(y) dy \\ &\quad - \cos nx \int_0^\infty y \sin 2ny(\beta(x) - \beta(y)) \chi_I(y) dy. \end{aligned}$$

As in the proof of Proposition 3.2, we apply the Riemann-Lebesgue lemma and Fatou's lemma to see that

$$(6.29) \quad \begin{aligned} \overline{\lim}_{n \rightarrow \infty} \int_I |\{[M_\beta, \tilde{T}_n(0)]f_x\}(x)|^2 dx &= \overline{\lim}_{n \rightarrow \infty} \int_I x^2 \sin^2 nx |I|^2 |\beta(x) - m_I(\beta)|^2 dx \\ &= \frac{1}{2} |I|^2 \int_I |\beta(x) - m_I(\beta)|^2 w(x) dx \end{aligned}$$

where we have used the fact that  $\sin^2 nx = \frac{1}{2} - \frac{1}{2} \cos 2nx$ .

Now suppose  $x_0$  is the midpoint of  $I$ , and suppose further that  $x_0 > 2|I|$ . By (6.24), we may write

$$(6.30) \quad f_x(y) = (x^2 - x_0^2) \frac{\pi}{2} \chi_I(y) \cos ny + (x_0^2 - y^2) \frac{\pi}{2} \chi_I(y) \cos ny$$

so that

$$(6.31) \quad \{[M_\beta, \tilde{T}_n(0)]f_x\}(x) = (x^2 - x_0^2)\frac{\pi}{2}\{[M_\beta, \tilde{T}_n(0)]h_1\}(x) + \frac{\pi}{2}\{[M_\beta, \tilde{T}_n(0)]h_2\}(x),$$

where

$$(6.32) \quad h_1(y) = \chi_I(y) \cos ny, \quad h_2(y) = (x_0^2 - y^2)\chi_I(y) \cos ny.$$

If  $x_0 > 2|I|$ , then, for  $y \in I$ ,

$$(6.33) \quad \frac{3x_0}{4} \leq x_0 - \frac{|I|}{2} \leq y \leq x_0 + \frac{|I|}{2} \leq \frac{5x_0}{4},$$

so that, in particular,

$$(6.34) \quad (y^2 - x_0^2)^2 = (y - x_0)^2(y + x_0)^2 \leq (|I|^2/4)(81x_0^2/16) \leq 2x_0^2|I|^2.$$

Consequently,

$$(6.35) \quad \|h_1\|_2^2 \leq |I|,$$

$$(6.36) \quad \|h_2\|_2^2 \leq \int_I 2x_0^2|I|^2 dx = 2x_0^2|I|^3,$$

$$(6.37) \quad w(I) = \int_I y^2 dy \geq \frac{9}{16}x_0^2|I|.$$

Thus, by (6.23), (6.31), and (6.34)–(6.37), we have

$$(6.38) \quad \int_I |\{[M_\beta, \tilde{T}_n(0)]f_x\}(x)|^2 dx \leq \pi^2 x_0^2 |I|^3 C^2 \leq \frac{16}{9} \pi^2 C^2 w(I) |I|^2.$$

Thus, by (6.29)

$$(6.39) \quad w(I)^{-1} \int_I |\beta(x) - m_I(\beta)|^2 w(x) dx \leq \frac{32}{9} \pi^2 C^2.$$

If, on the other hand,  $\frac{1}{2}|I| \leq x_0 \leq 2|I|$ , we write

$$(6.40) \quad f_x(y) = x^2 \frac{\pi}{2} \chi_I(y) \cos ny - y^2 \frac{\pi}{2} \chi_I(y) \cos ny$$

so that

$$(6.41) \quad \{[M_\beta, \tilde{T}_n(0)]f_x\}(x) = x^2 \frac{\pi}{2} \{[M_\beta, \tilde{T}_n(0)]h_1\}(x) - \frac{\pi}{2} \{[M_\beta, \tilde{T}_n(0)]wh_1\}(x).$$

Now we have

$$(6.42) \quad \begin{aligned} w(I) = \int_I y^2 dy &= \frac{1}{3} \left\{ (x_0 + \frac{1}{2}|I|)^3 - (x_0 - \frac{1}{2}|I|)^3 \right\} \\ &= \frac{1}{3} \left\{ 3x_0^2|I| + \frac{1}{4}|I|^3 \right\} \\ &\geq \frac{1}{12}|I|^3, \end{aligned}$$

$$(6.43) \quad \|h_1\|_2^2 \leq |I|,$$

$$(6.44) \quad \|wh_1\|_2^2 \leq \int_I y^4 dy \leq \frac{1}{5}|I|^5.$$

Noting that, for  $x \in I$ ,  $x \leq \frac{5}{2}|I|$ , we have

$$(6.45) \quad \int_I | \{ [M_\beta, \tilde{T}_n(0)] f_x \} (x) |^2 dx \leq 10\pi^2 C^2 |I|^5 \leq 120\pi^2 C^2 w(I) |I|^2,$$

by (6.23) and (6.41)–(6.44). Hence, by (6.29),

$$(6.46) \quad w(I)^{-1} \int_I |\beta(x) - m_I(\beta)|^2 w(x) dx \leq 240\pi^2 C^2.$$

Thus, combining (6.39) and (6.46), we obtain (6.22) with  $\mu = 240\pi^2$ . This completes the proof.  $\square$

**Corollary 6.4.1.**  $\mathbf{BMO}([0, \infty))$  is the space of uniform holomorphy at 0 for  $(\tilde{T}_n)$ .  $\square$

The proof of the periodic analogue of Proposition 6.4 is only slightly more complicated, but the basic idea is the same:

**Proposition 6.5.** Suppose  $\beta \in L^2([0, \pi])$ . Then

$$(6.47) \quad C = \sup_n \| [M_\beta, \tilde{S}_n(0)] \|_{\tilde{\mathcal{L}}(0)} < \infty$$

if and only if  $\beta \in \mathbf{BMO}_e$ .

*Proof.* The sufficiency of  $\beta \in \mathbf{BMO}_e$  is an immediate consequence of Proposition 6.1. We shall prove that (6.47) implies  $\beta \in \mathbf{BMO}_e$  by using the  $[0, \pi]$ -version of Lemma 6.3.

For  $\theta \in [0, \pi]$ , let  $w(\theta) = \sin^2 \theta/2$ ;  $w$  is an  $A_\infty$  weight on  $[0, \pi]$ . We shall show that there exists a constant  $\mu > 0$  such that whenever  $I$  is a subinterval of  $[0, \pi]$  with  $|I| < \pi/400$ , and whenever  $\beta \in L^1([0, \pi])$ , there is a constant  $c_I(\beta)$  such that

$$(6.48) \quad w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \leq \mu C^2.$$

The result then follows from the  $[0, \pi]$ -version of Lemma 6.3 together with [5, Chapter 1, Lemmas 9.4 and 9.5].

Let  $I$  be a subinterval of  $[0, \pi]$  with  $|I| < \pi/400$ , and let  $\theta \in I$ . Define the function  $f_\theta = f_{I, \theta, n}$  by setting, for  $\psi \in [0, \pi]$ ,

$$(6.49) \quad \begin{aligned} f_\theta(\psi) &= 2\chi_I(\psi) \cos \left[ (2n+1) \frac{\psi}{2} \right] \sin \left( \frac{\theta - \psi}{2} \right) \sin \left( \frac{\theta + \psi}{2} \right) \\ &= \chi_I(\psi) \left\{ \exp \left[ i(2n+1) \frac{\psi}{2} \right] + \exp \left[ -i(2n+1) \frac{\psi}{2} \right] \right\} \\ &\quad \times \sin \left( \frac{\theta - \psi}{2} \right) \sin \left( \frac{\theta + \psi}{2} \right). \end{aligned}$$

A straightforward calculation using (6.6) shows that

$$(6.50) \quad \tilde{D}_n(\theta, \psi) f_\theta(\psi) = 2 \sin \left[ (2n + 1) \frac{\theta}{2} \right] \sin \frac{\theta}{2} \cos \frac{\psi}{2} + \Phi(n, \theta, \psi)$$

where

$$(6.51) \quad \begin{aligned} \Phi(n, \theta, \psi) = & \sin \left[ (2n + 1) \left( \frac{\theta - 2\psi}{2} \right) \right] \sin \left( \frac{\theta + \psi}{2} \right) \\ & + \sin \left[ (2n + 1) \left( \frac{\theta + 2\psi}{2} \right) \right] \sin \left( \frac{\theta - \psi}{2} \right). \end{aligned}$$

For  $\psi \in [0, \pi]$ , let  $\rho(\psi) = \cos \psi/2$ , and set

$$(6.52) \quad \rho(I) = \int_I \rho(\psi) d\psi.$$

Then, by (6.5) and (6.50),

$$(6.53) \quad \begin{aligned} \{[M_\beta, \tilde{S}_n(0)]f_\theta\}(\theta) &= \frac{1}{2\pi} \int_0^\pi \tilde{D}_n(\theta, \psi) f_\theta(\psi) (\beta(\theta) - \beta(\psi)) d\psi \\ &= \frac{1}{\pi} \sin \left[ (2n + 1) \frac{\theta}{2} \right] \sin \frac{\theta}{2} \rho(I) [\beta(\theta) - c_I(\beta)] \\ &\quad + \frac{1}{2\pi} \int_I \Phi(n, \theta, \psi) (\beta(\theta) - \beta(\psi)) d\psi, \end{aligned}$$

where

$$(6.54) \quad c_I(\beta) = \rho(I)^{-1} \int_I \beta(\psi) \rho(\psi) d\psi$$

and, by the Riemann-Lebesgue lemma,

$$(6.55) \quad \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_I \Phi(n, \theta, \psi) (\beta(\theta) - \beta(\psi)) d\psi = 0.$$

Applying the Riemann-Lebesgue lemma and Fatou's lemma in by now familiar fashion, we obtain

$$(6.56) \quad \begin{aligned} \overline{\lim}_{n \rightarrow \infty} \int_I |\{[M_\beta, \tilde{S}_n(0)]f_\theta\}(\theta)|^2 d\theta \\ = \overline{\lim}_{n \rightarrow \infty} \frac{1}{\pi^2} \rho(I)^2 \int_I \sin^2 \left[ (2n + 1) \frac{\theta}{2} \right] |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \\ = \frac{1}{2\pi^2} \rho(I)^2 \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \end{aligned}$$

where we have used the fact that  $\sin^2[(2n + 1)\frac{\theta}{2}] = \frac{1}{2} - \frac{1}{2} \cos[(2n + 1)\theta]$ .

Now suppose  $\psi_0$  is the midpoint of  $I$ . We shall consider five cases:

- (i)  $\pi/200 > 2|I| \geq \psi_0$ ;
- (ii)  $\pi/200 > \psi_0 > 2|I|$ ;
- (iii)  $\pi/200 \leq \psi_0 \leq 199\pi/200$ ;
- (iv)  $\pi/200 > 2|I| \geq \pi - \psi_0$ , i.e.,  $199\pi/200 < \pi - 2|I| \leq \psi_0$ ;
- (v)  $\pi/200 > \pi - \psi_0 > 2|I|$ , i.e.,  $199\pi/200 < \psi_0 < \pi - 2|I|$ .

We begin with:

Case (i):  $\pi/200 > 2|I| \geq \psi_0$ . The addition formula for sines show that

$$(6.57) \quad \sin\left(\frac{\theta + \psi}{2}\right) \sin\left(\frac{\theta - \psi}{2}\right) = \sin^2 \frac{\theta}{2} \cos^2 \frac{\psi}{2} - \sin^2 \frac{\psi}{2} \cos^2 \frac{\theta}{2} \\ = w(\theta)\rho^2(\psi) - w(\psi)\rho^2(\theta).$$

If we let

$$(6.58) \quad \lambda_n(\psi) = 2 \cos\left[(2n+1)\frac{\psi}{2}\right] \chi_I(\psi)$$

then, by (6.49),

$$(6.59) \quad f_\theta(\psi) = w(\theta)\rho^2(\psi)\lambda_n(\psi) - \rho^2(\theta)w(\psi)\lambda_n(\psi)$$

so that

$$(6.60) \quad \{[M_\beta, \tilde{S}_n(0)]f_\theta\}(\theta) = w(\theta)\{[M_\beta, \tilde{S}_n(0)]\rho^2\lambda_n\}(\theta) - \rho^2(\theta)\{[M_\beta, \tilde{S}_n(0)]w\lambda_n\}(\theta).$$

It is easily seen that there exist constants  $C_1, C_2, C_3 > 0$  such that, for  $\psi \in I$ ,

$$(6.61) \quad C_1|\psi|^2 \leq w(\psi) \leq C_2|\psi|^2 \quad \text{and} \quad C_3 \leq \rho(\psi) \leq 1.$$

Thus we have

$$(6.62) \quad |I| \geq \rho(I) \geq C_3|I|,$$

$$(6.63) \quad \|\rho^2\lambda_n\|_2^2 \leq \int_I \rho(\psi)^4 d\psi \leq |I|,$$

$$(6.64) \quad \|w\lambda_n\|_2^2 \leq C_2^2 \int_I \psi^4 d\psi = C_2^2 \frac{1}{5} \left(\frac{5}{2}|I|\right)^5 \leq 20C_2^2 |I|^5,$$

$$(6.65) \quad w(I) \geq C_1 \int_I \psi^2 d\psi \geq \frac{1}{12} C_1 |I|^3,$$

where the estimation (6.65) is exactly like (6.42). Note also that, for  $\psi \in I$ ,

$$(6.66) \quad w(\psi) \leq C_2|\psi|^2 \leq \frac{25}{4} C_2 |I|^2.$$

Then we have

$$(6.67) \quad \int_I \{[M_\beta, \tilde{S}_n(0)]f_\theta\}(\theta)^2 d\theta \\ \leq \int_I |w(\theta)|^2 \{[M_\beta, \tilde{S}_n(0)]\rho^2\lambda_n\}(\theta)^2 d\theta + \int_I \{[M_\beta, \tilde{S}_n(0)]w\lambda_n\}(\theta)^2 d\theta \\ \leq \left[\frac{25}{4} C_2 |I|^2\right]^2 \cdot C^2 \cdot \|\rho^2\lambda_n\|_2^2 + C^2 \cdot \|w\lambda_n\|_2^2 \\ \leq K_1 w(I) \rho(I)^2 C^2$$

where  $K_1$  is independent of  $\beta, I$ , and  $n$ . Combining (6.56) and (6.67), we obtain

$$(6.68) \quad w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \leq 2\pi^2 K_1 C^2.$$

Case (ii):  $\pi/200 > \psi_0 > 2|I|$ . In this case we write

$$(6.69) \quad \begin{aligned} \sin\left(\frac{\theta - \psi}{2}\right) &= \sin\left(\frac{\theta - \psi_0 + \psi_0 - \psi}{2}\right) \\ &= \sin\left(\frac{\theta - \psi_0}{2}\right) \cos\left(\frac{\psi_0 - \psi}{2}\right) + \sin\left(\frac{\psi_0 - \psi}{2}\right) \cos\left(\frac{\theta - \psi_0}{2}\right), \end{aligned}$$

$$(6.70) \quad \begin{aligned} \sin\left(\frac{\theta + \psi}{2}\right) &= \sin\frac{\theta}{2} \cos\frac{\psi}{2} + \cos\frac{\theta}{2} \sin\frac{\psi}{2} \\ &= w^{1/2}(\theta)\rho(\psi) + \rho(\theta)w^{1/2}(\psi). \end{aligned}$$

If we let  $w_0^{1/2}(\theta) = \sin(\frac{\theta - \psi_0}{2})$ ,  $\rho_0(\theta) = \cos(\frac{\theta - \psi_0}{2})$ , then we have

$$(6.71) \quad \begin{aligned} \sin\left(\frac{\theta + \psi}{2}\right) \sin\left(\frac{\theta - \psi}{2}\right) &= [w^{1/2}(\theta)\rho(\psi) + \rho(\theta)w^{1/2}(\psi)][w_0^{1/2}(\theta)\rho_0(\psi) - w_0^{1/2}(\psi)\rho_0(\theta)] \\ &= (ww_0)^{1/2}(\theta)(\rho\rho_0)(\psi) - (w^{1/2}\rho_0)(\theta)(\rho w_0^{1/2})(\psi) \\ &\quad + (w_0^{1/2}\rho)(\theta)(\rho_0 w^{1/2})(\psi) - (\rho\rho_0)(\theta)(ww_0)^{1/2}(\psi) \end{aligned}$$

so that, by (6.49), (6.58), and (6.71), we have

$$(6.72) \quad \begin{aligned} \{[M_\beta, \tilde{S}_n(0)]f_\theta\}(\theta) &= (ww_0)^{1/2}(\theta)\{[M_\beta, \tilde{S}_n(0)]\lambda_n\rho\rho_0\}(\theta) \\ &\quad - (w^{1/2}\rho_0)(\theta)\{[M_\beta, \tilde{S}_n(0)]\lambda_n\rho w_0^{1/2}\}(\theta) \\ &\quad + (w_0^{1/2}\rho)(\theta)\{[M_\beta, \tilde{S}_n(0)]\lambda_n\rho_0 w^{1/2}\}(\theta) \\ &\quad - (\rho\rho_0)(\theta)\{[M_\beta, \tilde{S}_n(0)]\lambda_n w^{1/2} w_0^{1/2}\}(\theta). \end{aligned}$$

Since  $\psi_0$  and  $|I|$  are small, (6.61) continues to hold for  $\psi \in I$ , and hence (6.62) holds as well. Moreover, (6.61) holds for  $w_0, \rho_0$  in place of  $w, \rho$ . As in (6.33), we obtain

$$(6.73) \quad \frac{3\psi_0}{4} \leq \psi \leq \frac{5\psi_0}{4}, \quad \psi \in I.$$

Thus we have:

$$(6.74) \quad w(I) \geq \frac{9}{16} C_1 \psi_0^2 |I|.$$

For  $\psi \in I$ , we have

$$(6.75) \quad |(ww_0)^{1/2}(\psi)| \leq \frac{5}{8} C_2 \psi_0 |I|,$$

$$(6.76) \quad |(w^{1/2}\rho_0)(\psi)| \leq \frac{5}{4} C_2^{1/2} \psi_0,$$

$$(6.77) \quad |(w_0^{1/2}\rho)(\psi)| \leq \frac{1}{2} C_2^{1/2} |I|,$$

$$(6.78) \quad |(\rho\rho_0)(\psi)| \leq 1,$$

and so we have

$$(6.79) \quad \|\lambda_n(w w_0)^{1/2}\|_2^2 \leq \frac{25}{64} C_2^2 \psi_0^2 |I|^3,$$

$$(6.80) \quad \|\lambda_n(w^{1/2} \rho_0)\|_2^2 \leq \frac{25}{16} C_2 \psi_0^2 |I|,$$

$$(6.81) \quad \|\lambda_n(w_0^{1/2} \rho)\|_2^2 \leq \frac{1}{4} C_2 |I|^3,$$

$$(6.82) \quad \|\lambda_n(\rho \rho_0)\|_2^2 \leq |I|.$$

Thus, by (6.62), (6.72), and (6.74)–(6.82) we have

$$(6.83) \quad \int_I |\{[M_\beta, \tilde{S}_n(0)]f_\theta\}(\theta)|^2 d\theta \leq K C^2 |I|^3 \psi_0^2 \leq K_2 C^2 w(I) \rho(I)^2$$

where  $K, K_2$  are independent of  $\beta, I, n$ . Combining (6.56) and (6.83), we obtain

$$(6.84) \quad w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \leq 2\pi^2 K_2 C^2.$$

Case (iii):  $\pi/200 \leq \psi_0 \leq 199\pi/200$ . Exactly as in Case (ii), we obtain the expression (6.72). For this case, we observe that the functions  $w, \rho_0$ , and  $\rho$  behave essentially as constants. Moreover, there is a constant  $C_4$  such that for  $\psi \in I$ ,

$$(6.85) \quad |w_0(\psi)| \leq C_4 |I|^2.$$

Thus we have

$$(6.86) \quad w(I) \sim |I|,$$

$$(6.87) \quad \rho(I)^2 \sim |I|^2,$$

$$(6.88) \quad \|\lambda_n(w w_0)^{1/2}\|_2^2 \lesssim |I|^2,$$

$$(6.89) \quad \|\lambda_n(w_0^{1/2} \rho_0)\|_2^2 \lesssim |I|,$$

$$(6.90) \quad \|\lambda_n(w_0^{1/2} \rho)\|_2^2 \lesssim |I|^2,$$

$$(6.91) \quad \|\lambda_n(\rho \rho_0)\|_2^2 \lesssim |I|$$

so that, combining (6.72) and (6.85)–(6.91) we have

$$(6.92) \quad \int_I |\{[M_\beta, \tilde{S}_n(0)]f_\theta\}(\theta)|^2 d\theta \leq K_3 C^2 w(I) \rho(I)^2$$

where  $K_3$  is independent of  $\beta, I, n$ . Combining (6.56) and (6.92), we have

$$(6.93) \quad w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \leq 2\pi^2 K_3 C^2.$$

Case (iv):  $\pi/200 > 2|I| \geq \pi - \psi_0$ . This case is in certain respects analogous to Case (i). Using the fact that  $\sin x = \sin(\pi - x)$ , we write, as in (6.57),

$$(6.94) \quad \begin{aligned} \sin\left(\frac{\theta + \psi}{2}\right) \sin\left(\frac{\theta - \psi}{2}\right) &= \sin\left[\frac{(\pi - \theta) + (\pi - \psi)}{2}\right] \sin\left[\frac{(\pi - \theta) - (\pi - \psi)}{2}\right] \\ &= w_\pi(\theta) \rho_\pi^2(\psi) - w_\pi(\psi) \rho_\pi^2(\theta) \end{aligned}$$

where  $w_\pi(\psi) = w(\pi - \psi)$ ,  $\rho_\pi(\psi) = \rho(\pi - \psi)$ . Thus we obtain (6.60) with  $w_\pi$  in place of  $w$  and  $\rho_\pi$  in place of  $\rho$ . The estimate involving  $w_\pi$  and  $\rho_\pi$  are essentially the same as those involving  $w$  and  $\rho$  in Case (i). Moreover, it is easy to see that  $w(I) \sim |I|$  while  $\rho(I)^2 \gtrsim |I|^4$ , so that  $w(I)\rho(I)^2 \gtrsim |I|^5$ . Thus, by essentially the same argument as in Case (i), we have

$$(6.95) \quad w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \leq 2\pi^2 K_4 C^2$$

where  $K_4$  is independent of  $\beta, I, n$ .

Case (v):  $\pi/200 > \pi - \psi_0 > 2|I|$ . This case is analogous to Case (ii). We write

$$(6.96) \quad \sin\left(\frac{\theta - \psi}{2}\right) = w_0^{1/2}(\theta)\rho_0(\psi) - w_0^{1/2}(\psi)\rho_0(\theta)$$

$$(6.97) \quad \sin\left(\frac{\theta + \psi}{2}\right) = \sin\left[\frac{(\pi - \theta)}{2} + \frac{(\pi - \psi)}{2}\right] = w_\pi^{1/2}(\theta)\rho_\pi(\psi) + \rho_\pi(\theta)w_\pi^{1/2}(\psi)$$

and, as in Case (ii), we obtain (6.72) with  $w_\pi$  in place of  $w$  and  $\rho_\pi$  in place of  $\rho$ .

Geometrically,  $I$  is a "mirror image" across  $\psi = \pi/2$  of an interval of the type considered in Case (ii). Consequently, we obtain estimates of the form

$$(6.98) \quad \rho(I) \gtrsim (\pi - \psi_0)|I|, \quad w(I) \sim |I|,$$

$$(6.99) \quad |(w_\pi w_0)^{1/2}(\psi)| \lesssim |I| \quad \text{for } \psi \in I,$$

$$(6.100) \quad |(w_\pi^{1/2} \rho_0)(\psi)| \leq 1 \quad \text{for } \psi \in I,$$

$$(6.101) \quad |(w_0^{1/2} \rho_\pi)(\psi)| \lesssim (\pi - \psi_0)|I| \quad \text{for } \psi \in I,$$

$$(6.102) \quad |(\rho_\pi \rho_0)(\psi)| \lesssim (\pi - \psi_0) \quad \text{for } \psi \in I$$

so that

$$(6.103) \quad \|\lambda_n(w_\pi w_0)^{1/2}\|_2^2 \lesssim |I|^3,$$

$$(6.104) \quad \|\lambda_n(w_\pi^{1/2} \rho_0)\|_2^2 \leq |I|,$$

$$(6.105) \quad \|\lambda_n(w_0^{1/2} \rho_\pi)\|_2^2 \lesssim (\pi - \psi_0)^2 |I|^3,$$

$$(6.106) \quad \|\lambda_n(\rho_\pi \rho_0)\|_2^2 \lesssim (\pi - \psi_0)^2 |I|.$$

Thus we have

$$(6.107) \quad \int_I | \{ [M_\beta, \tilde{S}_n(0)] f_\theta \}(\theta) |^2 d\theta \leq K' C^2 |I|^3 (\pi - \psi_0)^2 \leq K_5 C^2 w(I) \rho(I)^2$$

where  $K', K_5$  are independent of  $\beta, I, n$ . Combining (6.56) and (6.107), we obtain

$$(6.108) \quad w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \leq 2\pi^2 K_5 C^2.$$

We have now considered all possible cases. If we take  $\delta = 2\pi^2 \max_{1 \leq j \leq 5} K_j$ , we obtain (6.48), and the proof is complete.  $\square$

**Corollary 6.5.1.**  $\mathbf{BMO}_e$  is the space of uniform holomorphy at 0 for  $\langle \tilde{P}_n \rangle$  and  $\langle \tilde{Q}_n \rangle$ .  $\square$

We also obtain the following result for partial sum operators on  $[-1, 1]$ :

**Corollary 6.5.2.** In the notation of § 4,  $\mathbf{BMO}$  is the space of uniform holomorphy at 0 for  $\langle P_n^{-1/2, -1/2} \rangle$  and  $\langle Q_n^{-1/2, -1/2} \rangle$ .

*Proof.* By Corollary 4.2.1, it suffices to show that

$$(6.109) \quad C = \sup_n \| [M_\beta, S_n^{-1/2, -1/2}(0)] \|_{\mathcal{L}_{-1/2, -1/2}(0)} < \infty,$$

for  $\beta \in L^1([-1, 1], (1-x^2)^{-1/2} dx)$ , only if  $\beta \in \mathbf{BMO}$ .

For  $f \in L_{-1/2, -1/2}(0)$  and  $\theta \in [0, \pi]$ , define  $Uf(\theta) = f \circ \cos(\theta)$ . It is easy to see that  $U$  is an isometry from  $L_{-1/2, -1/2}(0)$  to  $\tilde{L}(0)$ . By virtue of this isometry, we see that the family  $\langle t_n; n \in \mathbf{N} \rangle$  of Chebyshev polynomials, defined by

$$(6.110) \quad t_n(x) = \cos n\theta, \quad x = \cos \theta$$

is an orthogonal polynomial system on  $[-1, 1]$  relative to  $(1-x^2)^{-1/2}$  (cf. [10], §§1.12 and 2.4). In particular, we see that

$$(6.111) \quad S_n^{-1/2, -1/2}(0) = U^{-1} \tilde{S}_n(0) U$$

so that

$$(6.112) \quad [M_\beta, S_n^{-1/2, -1/2}(0)] = U^{-1} [M_{\beta \circ \cos}, \tilde{S}_n(0)] U.$$

Since  $U, U^{-1}$  are isometries, (6.109) and (6.112) imply that  $\beta \circ \cos \in \mathbf{BMO}_e$ .

Now suppose  $I = [a, b]$  is any subinterval of  $[-1, 1]$ , and let  $\omega(x) = (1-x^2)^{-1/2}$ ; clearly  $\omega$  is an  $A_\infty$  weight on  $[-1, 1]$ . Let  $J = [\arccos b, \arccos a]$ ; then we have

$$(6.113) \quad \begin{aligned} \omega(I)^{-1} \int_I |\beta(x) - m_J(\beta \circ \cos)| \omega(x) dx \\ = |J|^{-1} \int_J |\beta \circ \cos(\theta) - m_J(\beta \circ \cos)| d\theta \leq \|\beta \circ \cos\|_* . \end{aligned}$$

Thus  $\beta \in \mathbf{BMO}$  by the  $[-1, 1]$ -version of Lemma 6.3.  $\square$

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