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THE SPACE OF HARMONIC MAPS OF S^2 INTO S^4

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ABSTRACT. Every branched superminimal surface of area $4\pi d$ in S^4 is shown to arise from a pair of meromorphic functions (f_1, f_2) of bidegree (d, d) such that f_1 and f_2 have the same ramification divisor. Conditions under which branched superminimal surfaces can be generated from such pairs of functions are derived. For each $d \geq 1$ the space of harmonic maps (i.e branched superminimal immersions) of S^2 into S^4 of harmonic degree d is shown to be a connected space of complex dimension $2d + 4$.

INTRODUCTION

In a study of minimal surfaces in euclidean spheres, Calabi showed that every minimal immersion of S^2 in S^n arises from an isotropic map to projective space [4], [5]. This work was used by Bryant who showed that every compact Riemann surface can be superminimally immersed in S^4. There exist Calabi-type theorems representing harmonic maps of S^2 into other locally symmetric spaces in essentially algebro-geometric terms. These are of interest to people studying σ-models in physics. In this paper, we study the space of branched superminimal immersions of compact Riemann surfaces into S^4.

In §I, we characterize branched superminimal surfaces in S^4 by pairs of meromorphic functions with the same ramification divisor. This is done by constructing a contact map between $\tilde{\mathbf{P}}^3$ and $\mathbf{P}T(\mathbf{CP}^1 \times \mathbf{CP}^1)$ where $\tilde{\mathbf{P}}^3$ is the blow-up of $\mathbf{CP}^3$ along 2 skew lines. The bidegree of such a pair is related to the degree of the canonical lift of the surface in $\mathbf{CP}^3$. We then show that if in addition the surface is linearly full (i.e. not contained in any strict subspace of $\mathbf{R}^5$) then the pair of meromorphic functions has bidegree (d, d) where $d \geq 3$ and where the 2 functions do not differ by a Möbius transformation.

In §II, we analyze the space of harmonic maps of S^2 into S^4. By examining the projective geometry of certain Grassmann varieties, we show that the space $\mathfrak{H}_d$ of harmonic maps of S^2 into S^4 of degree d is a *connected* space of complex dimension $2d + 4$. We also construct examples of *unbranched* superminimal surfaces of genus 0 in S^4 of area $4\pi d$ for $d \geq 3$.

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In §III, we consider branched superminimal surfaces of genus g. We discuss conditions under which a pair of meromorphic functions on a Riemann surface Σ can give rise to a branched superminimal immersion of Σ into S^4.

This paper is based on the author's Ph.D thesis [13]. The author would like to thank Blaine Lawson for all the help and advice he has given me.

Preliminaries

Let Σ be a compact Riemann surface and $\psi : \Sigma \looparrowright S^4$ an immersion into the unit 4-sphere. Let B denote the second fundamental form of ψ. Then ψ is a *minimal immersion* if the mean curvature $H :=$ trace B vanishes identically. More generally, ψ is a *branched minimal immersion* if it is minimal away from the set of isolated singular points. These are precisely the nonconstant conformal harmonic maps. Observe that any harmonic map $\psi : S^2 \to S^4$ is automatically conformal. Thus, branched minimal immersions of S^2 in S^4 are just the nonconstant harmonic maps from S^2 to S^4 (Eells-Lemaire [7]).

Let $\psi : \Sigma \looparrowright S^4$ be a (branched) minimal immersion of a compact Riemann surface in S^4. Let x and y denote the local isothermal coordinates on Σ. Consider the holomorphic quartic form $\Phi \in H^0(\Sigma; (\Omega^1)^4)$ defined by $\Phi := \varphi \cdot \varphi \, dz^4$ where

$$\varphi = \frac{1}{2}\left\{ B\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - iB\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right\},$$

and where "$\cdot$" is the complex bilinear extension of the dot product to $\mathbb{C}^5$. We say that ψ is a (*branched*) *superminimal immersion* if Φ vanishes identically. This means that ψ has a holomorphic horizontal lift, $\tilde{\psi}$, to $\mathbb{CP}^3$ (Bryant [3], Chern-Wolfson [6], Lawson [10]). Observe that since S^2 has no nontrivial holomorphic quartic differentials, every branched minimal immersion (i.e. harmonic map) of S^2 into S^4 is automatically branched superminimal.

Consider the Calabi-Penrose fibration $\pi : \mathbb{CP}^3 \to S^4 = \mathbf{HP}^1$. This fibration can be obtained via a quotient of 2 Hopf maps. Choose homogeneous coordinates (z_0, z_1, z_2, z_3) for $\mathbb{CP}^3$. Consider $\mathbb{C}^4 \cong \mathbf{H}^2$ as a quaternion vector space with left scalar multiplication, where the identification is given by $(z_0, z_1, z_2, z_3) \mapsto (z_0 + z_1 j, z_2 + z_3 j)$. The Kähler form of the Fubini-Study metric is given by $\omega = \partial\bar{\partial} \log \|z\|^2$. The Calabi-Penrose fibration is then given by the quotient

$$\begin{array}{ccc}
\mathbb{C}^4 - \{0\} & = & \mathbf{H}^2 - \{0\} \\
\mathrm{Hopf}_{\mathbb{C}} \big\downarrow & & \big\downarrow \mathrm{Hopf}_{\mathbf{H}} \\
\mathbb{CP}^3 & \xrightarrow{\ \pi\ } & \mathbf{HP}^1
\end{array}$$

with fiber $\mathbf{CP}^1$. The horizontal 2-plane field $\mathscr{H}$ for π is given by a 1-form whose lifting to $\mathbf{C}^4 - \{0\}$ is

$$\Omega := \frac{1}{\|z\|^2}(z_0\,dz_1 - z_1\,dz_0 + z_2\,dz_3 - z_3\,dz_2).$$

Superminimal surfaces in S^4 are just the projections to S^4 of nonsingular holomorphic curves in $\mathbf{CP}^3$ which are integral curves of $\mathscr{H}$. Unfortunately, it is difficult to find integral curves of $\mathscr{H}$ directly. Our search for superminimal surfaces would be vastly simplified if we can find a contact manifold $(M, \mathscr{F})$ birationally equivalent to $\mathbf{CP}^3$, where it is easy to find integral curves of the contact plane field $\mathscr{F}$. Robert Bryant has found a birational correspondence between $\mathbf{CP}^3$ and the projectivized tangent bundle of $\mathbf{CP}^2$ carrying $\mathscr{H}$ to the contact plane field of $\mathbf{P}T(\mathbf{CP}^2)$. Using that, he was able to prove the following result:

Theorem (Bryant [3]). *Every compact Riemann surface admits a superminimal immersion into* S^4.

In this paper, I will be using another contact manifold—$\mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1)$. From now on, I will let $\mathbf{P}^n$ denote $\mathbf{CP}^n$.

I. Some projective geometry

1. Holomorphic contact structures. Let V be a complex $(2n+1)$-manifold. A *holomorphic contact structure* on V is a nondegenerate holomorphic distribution $\mathscr{F}$ of hyperplanes on V (i.e. the orthogonal spaces of some twisted holomorphic 1-form). (cf. Arnold [1], LeBrun [12]).

Let M be a complex n-manifold. Then the projectivized cotangent bundle of M has a canonical holomorphic contact structure. Now let $\pi : \mathbf{P}T^*M \to M$ denote the projection map onto the base space. A point $\varphi \in \mathbf{P}T^*M$ defines a hyperplane P_φ in $T_{\pi(\varphi)}M$. The contact hyperplane at φ is given by $(\pi_*^{-1})_\varphi(P_\varphi)$. Thus the canonical contact 2-plane field $\mathscr{K}$ at a point $y \in \mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1) \cong \mathbf{P}T^*(\mathbf{P}^1 \times \mathbf{P}^1)$ is given by $(\pi_*^{-1})_y(L_y)$ where L_y denotes the tangent line at $\pi(y)$ corresponding to y.

The Calabi-Penrose fibration $p : \mathbf{P}^3 \to S^4$ has a contact 2-plane field $\mathscr{H}$ orthogonal to the fibers of p with respect to the Fubini-Study metric. The 2-plane field $\mathscr{H}$ for p is given by a 1-form whose lifting to $\mathbf{C}^4 - \{0\}$ is $\Omega = \|z\|^{-2}(z_0\,dz_1 - z_1\,dz_0 + z_2\,dz_3 - z_3\,dz_2)$. Let $\omega := dz_0 \wedge dz_1 + dz_2 \wedge dz_3$ denote the standard holomorphic symplectic form on $\mathbf{C}^4$. Let

$$\xi := z_0\frac{\partial}{\partial z_0} + z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}.$$

Then $\Omega = \|z\|^{-2}\,\xi \lrcorner\, \omega$.

2. Projection to $\mathbf{P}^1 \times \mathbf{P}^1$. Consider the two distinguished skew lines in $\mathbf{P}^3$ defined by $L_1 := p^{-1}(N) = \{[0,0,z_2,z_3] \mid [z_2,z_3] \in \mathbf{P}^1\}$ and $L_2 := p^{-1}(S) = \{[z_0,z_1,0,0] \mid [z_0,z_1] \in \mathbf{P}^1\}$, where N and S denote the north and south poles of S^4 respectively.

Lemma 1.1. *There is a well-defined projection map* $\mathrm{pr}\colon \mathbf{P}^3 - (L_1 \cup L_2) \to \mathbf{P}^1 \times \mathbf{P}^1$ *with* $\mathbf{P}^1$ *as fiber.*

Proof. It suffices to show that there is a unique line L through each point $x \in \mathbf{P}^3 - (L_1 \cup L_2)$ which intersects L_1 and L_2. The intersection of L with L_1 and L_2 (identifying $L_1 \times L_2$ with $\mathbf{P}^1 \times \mathbf{P}^1$) gives us the desired projection map. For each $x \in \mathbf{P}^3 - (L_1 \cup L_2)$ consider the planes P_1 and P_2 in $\mathbf{P}^3$ defined by $P_1 = \mathrm{span}(x, L_1)$ and $P_2 = \mathrm{span}(x, L_2)$. Since L_1 and L_2 are skew, P_1 and P_2 intersect in a line L which contains the point x and which intersects both L_1 and L_2. $\square$

Proposition 1.2. *The fibers of* $\mathrm{pr}\colon \mathbf{P}^3 - (L_1 \cup L_2) \to \mathbf{P}^1 \times \mathbf{P}^1$ *are horizontal with respect to* p *(i.e. the fibers of* pr *are integral curves of* $\mathscr{H}$*).*

Proof. Let $(x, y) \in L_1 \times L_2$. Let L denote the line through x and y, i.e. $L = \mathrm{pr}^{-1}(x, y)$. Denote the inverse images of L, L_1, L_2, x and y to $\mathbf{C}^4 - \{0\}$ by P, P_1, P_2, l_x and l_y respectively.

Note. P_1 and P_2 are orthogonal with respect to ω. Let $A \in P_1$ and $B \in P_2$. Then $A = (0,0,a,b)$ and $B = (c,d,0,0)$ for some $a,b,c,d \in \mathbf{C}$. It is clear from the definition of ω that $\omega(A,B) = 0$. Since ω is skew, we also have $\omega(A,A) = \omega(B,B) = 0$.

Now pick nonzero vectors $X \in l_x \subset P_1$ and $Y \in l_y \subset P_2$. Observe that P is spanned by X and Y. Now let $V_1 = \alpha X + \beta Y$ and $V_2 = \gamma X + \delta Y$ be 2 vectors in P. Then by the note, $\omega(V_1, V_2) = 0$. Thus ω vanishes on P. Let $\rho\colon \mathbf{C}^4 - \{0\} \to \mathbf{P}^3$. Since ξ is tangent to the fibers of ρ and $\Omega|_L = \|z\|^{-2}(\xi \lrcorner\, \omega)|_P$, we see that Ω vanishes on L. Thus L is horizontal with respect to p. $\square$

3. The contact map. Let X denote the blow up of $\mathbf{P}^3$ along L_1 and L_2, i.e. $X := \{([z_0,z_1,z_2,z_3],[y_0,y_1],[y_2,y_3]) \mid z_0y_1 = z_1y_0,\ z_2y_3 = z_3y_2\}$. Note that X is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1 \times \mathbf{P}^1$: $\tilde{\pi}\colon X \to \mathbf{P}^1 \times \mathbf{P}^1$ where

$$\tilde{\pi}([z_0,z_1,z_2,z_3],[y_0,y_1],[y_2,y_3]) = ([y_0,y_1],[y_2,y_3]) .$$

For ease of notation, let Y denote $\mathbf{P}T^*(\mathbf{P}^1 \times \mathbf{P}^1) \cong \mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1)$. Let $\psi\colon X \to Y$ be defined by

$$\begin{aligned}&\psi([z_0,z_1,z_2,z_3],[y_0,y_1],[y_2,y_3]) \\ &\quad = ([y_0,y_1],[y_2,y_3],[z_0\,dy_1 - z_1\,dy_0, z_2\,dy_3 - z_3\,dy_2]) .\end{aligned}$$

We have the following diagram:

$$\begin{array}{ccccc}
\mathbf{P}^3 & \xleftarrow{\beta} & X & \xrightarrow{\psi} & Y \\
{\scriptstyle p}\downarrow & & {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\
S^4 & & \mathbf{P}^1 \times \mathbf{P}^1 & = & \mathbf{P}^1 \times \mathbf{P}^1
\end{array}$$

Observe that $\mathscr{H}$ extends to all of X, and for $x \in X$, $\tilde{\pi}_*(\mathscr{H}_x)$ is a tangent line in $T_{\tilde{\pi}(x)}(\mathbf{P}^1 \times \mathbf{P}^1)$, i.e. $\tilde{\pi}_*(\mathscr{H}_x) \in \mathbf{P}T_{\tilde{\pi}(x)}(\mathbf{P}^1 \times \mathbf{P}^1)$. Furthermore, $\tilde{\pi} = \pi \circ \psi$ where π is the projection to $\mathbf{P}^1 \times \mathbf{P}^1$. Now let $l := \tilde{\pi}_*(\mathscr{H}_x)$. Then $\pi_*^{-1}(l)$ is the contact plane at $l \in Y$. Now $l = \tilde{\pi}_*(\mathscr{H}_x) = (\pi \circ \psi)_*(\mathscr{H}_x) = \pi_* \circ \psi_*(\mathscr{H}_x)$. Thus, $\pi_*^{-1}(l) = \psi_*(\mathscr{H}_x)$. We thus have

Lemma 1.3. *ψ is a contact map, i.e. ψ_* sends the horizontal plane field $\mathscr{H}$ in X to the contact plane field $\mathscr{K}$ in Y.*

The blow ups, σ_1 and σ_2, of the two distinguished skew lines $L_1, L_2 \in \mathbf{P}^3$ are given by

$$\sigma_1 := \left\{([0, 0, z_2, z_3], [y_0, y_1], [z_2, z_3]) \mid [y_0, y_1] \in \mathbf{P}^1 \text{ and } [z_2, z_3] \in \mathbf{P}^1\right\}$$

and

$$\sigma_2 := \left\{([z_0, z_1, 0, 0], [z_0, z_1], [y_2, y_3],) \mid [z_0, z_1] \in \mathbf{P}^1 \text{ and } [y_2, y_3] \in \mathbf{P}^1\right\}.$$

We observe that

$$\psi(\sigma_1) = \left\{([y_0, y_1], [z_2, z_3], [1, 0]) \mid [y_0, y_1] \in \mathbf{P}^1 \text{ and } [z_2, z_3] \in \mathbf{P}^1\right\}$$

and

$$\psi(\sigma_2) = \left\{([z_0, z_1], [y_2, y_3], [0, 1]) \mid [z_0, z_1] \in \mathbf{P}^1 \text{ and } [y_2, y_3] \in \mathbf{P}^1\right\}.$$

Proposition 1.4. *ψ is a branched 2-fold covering map. It is branched precisely along σ_1 and σ_2*

This proposition will be proved in the next subsection.

4. The involutions on X and S^4. We first define an involution $\alpha \colon X \to X$ by

$$\alpha([z_0, z_1, z_2, z_3], [y_0, y_1], [y_2, y_3]) = ([z_0, z_1, -z_2, -z_3], [y_0, y_1], [y_2, y_3]).$$

(Actually, α is an involution on $\mathbf{P}^3$ which is extended to X in a trivial manner.)

Note.

(1) $\alpha|_{\sigma_1} = \mathrm{Id}$, $\alpha|_{\sigma_2} = \mathrm{Id}$ and $\alpha^*\Omega = \Omega$.

(2) By Note 1, α_* maps the horizontal plane $\mathscr{H}_x$ at $x \in X$ to the horizontal plane $\mathscr{H}_{\alpha(x)}$ at $\alpha(x)$.

(3) Let $u \in L_1$ and $v \in L_2$. Denote by l_{uv} the line in $\mathbf{P}^3$ uniquely defined by u and v. Since $\alpha(u) = u$ and $\alpha(v) = v$, we have $\alpha(l_{uv}) = l_{uv}$.

Consequently, $\tilde{\pi} \circ \alpha = \tilde{\pi}$. (This actually follows immediately from the definition of α and $\tilde{\pi}$.)

(1) Since $\tilde{\pi}_*(\mathscr{H}_x) = \pi_* \circ \psi_*(\mathscr{H}_x) = \psi(x)$, we have

$$\begin{aligned} \psi(\alpha(x)) = \tilde{\pi}_*(\mathscr{H}_{\alpha(x)}) &= \tilde{\pi}_*(\alpha_*\mathscr{H}_x) \qquad \text{by Note 2} \\ &= (\tilde{\pi} \circ \alpha)_*(\mathscr{H}_x) \\ &= \tilde{\pi}_*(\mathscr{H}_x) \qquad \text{by Note 3} \\ &= \psi(x). \end{aligned}$$

Thus $\psi \circ \alpha = \psi$, i.e. ψ is α-invariant.

Notes 1–4 imply that ψ is at least 2 to 1 except along σ_1 and σ_2. From the definition of ψ, it is clear that ψ is 1-to-1 on σ_1 and σ_2. Let us now examine the map ψ explicitly in local coordinates. Assume that $x \notin \sigma_1 \cup \sigma_2$. We can then set $z_i = y_i$ for $i = 0, 1, 2, 3$. Without loss of generality, we can suppose that $z_0 = y_0 = 1$ and $z_2 \neq 0$. Set $s = y_1$ and $t = y_3/y_2$. Then $ds = dy_1$ and $dt = z^{-2}(z_2\,dy_3 - z_3\,dy_2)$. Thus, $z_2^2\,dt = z_2\,dy_3 - z_3\,dy_2$. Hence, $\psi([1, z_1, z_2, z_3], s, t) = (s, t, [ds, z_2^2\,dt])$. We also have

$$\psi([1, z_1, -z_2, -z_3], s, t) = (s, t, [ds, z_2^2\,dt]).$$

From the above local coordinate expression for ψ, it is clear that ψ is 2-to-1 away from σ_1 and σ_2. Now, ψ is a holomorphic map with finite fibers between compact complex 3-folds. Thus, it is a branched covering map of degree 2. This proves Proposition 1.4.

Let us now examine the inverse image of ψ locally. Choose a point $y \in Y - (S_1 \cup S_2)$ where S_1 and S_2 are the images under ψ of σ_1 and σ_2 respectively. Locally, y has coordinates (s, t, a). Recall that $\psi([1, z_1, z_2, z_3],\ s, t) = (s, t, [ds, z_2^2\,dt])$ where $s = z_1$ and $t = z_3/z_2$. Then

$$\psi^{-1}(y) = \psi^{-1}(s, t, a) = ([1, s, \sqrt{a}, \sqrt{a}t], s, t).$$

The involution α on X corresponds to a permutation of the roots. Thus,

Proposition 1.5. *$\psi: X \to Y$ is equivalent to the projection map $p: X \to X/\mathbb{Z}_2$ where the $\mathbb{Z}_2$-action on X is given by the involution α.*

The involution on $\mathbf{P}^3$ descends to an involution on S^4. Identifying S^4 with $\mathbf{HP}^1$, the stereographic projections to $\mathbf{R}^4 = \mathbf{H}^1$ from the south and north poles are respectively given by $\varphi_1([q_1, q_2]) = q_1^{-1}q_2$ and $\varphi_2([q_1, q_2]) = q_2^{-1}q_1$, with transition functions $q \mapsto q^{-1}\|q\|^{-2}\bar{q}$. Now $p([z_0, z_1, z_2, z_3]) = [z_0 + z_1 j, z_2 + z_3 j] \in \mathbf{HP}^1$, where $[z_0, z_1, z_2, z_3] \in \mathbb{C}\mathbf{P}^3$. Thus,

$$p(\alpha[z_0, z_1, z_2, z_3]) = p([z_0, z_1, -z_2, -z_3]) = [z_0 + z_1 j, -(z_2 + z_3 j)].$$

The involution α thus descends to an involution on $S^4 = \mathbf{HP}^1$ as follows: $\alpha([q_1, q_2]) = [q_1, -q_2]$ for all $[q_1, q_2] \in \mathbf{HP}^1$. (We will let α denote the involution on both X as well as S^4.)

Now, $\varphi_1 \circ \alpha([q_1, q_2]) = \varphi_1([q_1, -q_2]) = -q_1^{-1}q_2$ and $\varphi_2 \circ \alpha([q_1, q_2]) = \varphi_2([q_1, -q_2]) = -q_2^{-1}q_1$. Hence the action of α on a point $x \in S^4$ is just the antipodal map on the $S^3 \subset S^4$ obtained by the intersection of the horizontal 4-plane through x with S^4. (This S^3 is the "latitudinal S^3".) Thus, the geodesic 3-sphere in S^4 passing through the north and south poles is invariant under α.

5. Some degree computations. We now compute the degree of the total preimage in $\mathbf{P}^3$ of a holomorphic curve in Y. Recall the diagram:

$$\begin{array}{ccccc} \mathbf{P}^3 & \xleftarrow{\beta} & X & \xrightarrow{\psi} & Y \\ {\scriptstyle p}\downarrow & & {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ S^4 & & \mathbf{P}^1 \times \mathbf{P}^1 & = & \mathbf{P}^1 \times \mathbf{P}^1 \end{array}$$

Let l_1 and l_2 (resp. l_1' and l_2') denote the preimages in X (resp. Y) of the first and second factors of $\mathbf{P}^1 \times \mathbf{P}^1$ respectively under the map $\tilde{\pi}: X \to \mathbf{P}^1 \times \mathbf{P}^1$ (resp. $\pi: Y \to \mathbf{P}^1 \times \mathbf{P}^1$). Let S_1 and S_2 denote the 2 distinguished sections of Y corresponding to lines tangent to the second and first factors of $\mathbf{P}^1 \times \mathbf{P}^1$ respectively. Recall that $\psi_*(\sigma_1) = S_1$ and $\psi_*(\sigma_2) = S_2$. Note that $\psi_*(l_i) = 2l_i'$, $i = 1, 2$. Let H be a hyperplane in $\mathbf{P}^3$. Then $\beta^* H = \sigma_1 + l_1 = \sigma_2 + l_2$. Thus $\sigma_1 - \sigma_2 = l_2 - l_1$. Also, $S_1 - S_2 = \psi_*(\sigma_1 - \sigma_2) = \psi_*(l_2 - l_1) = 2(l_2' - l_1')$. Hence, the Picard group of X and Y are given by

$$Pic(X) = \mathbf{Z}\{l_1, l_2, \sigma_1, \sigma_2\} / \langle \sigma_1 - \sigma_2 = l_2 - l_1 \rangle$$

and

$$Pic(Y) = \mathbf{Z}\{l_1', l_2', S_1, S_2\} / \langle S_1 - S_2 = 2(l_2' - l_1') \rangle.$$

Let Σ be a compact Riemann surface of genus g. Let $\phi: \Sigma \to \mathbf{P}^1$ be a holomorphic map of degree d. A point $x \in \Sigma$ is a *ramification point* of ϕ if $d\phi(x) = 0$, and its image $\phi(x) \in \mathbf{P}^1$ is called a *branch point* of ϕ. By the Riemann-Hurwitz Theorem the number of branch points of ϕ (counting multiplicities) is $2g + 2d - 2$. The *ramification divisor* of ϕ is the formal sum $\sum a_i p_i$ where p_i is a ramification point of ϕ with multiplicity a_i, and where the sum is taken over all ramification points of ϕ. We will let $\mathrm{Ram}(\phi)$ denote the ramification divisor of ϕ.

Let $F = (f_1, f_2): \Sigma \to \mathbf{P}^1 \times \mathbf{P}^1$ be a holomorphic map of bidegree (n, m). Then the curve $C = F(\Sigma)$ is of class (m, n). Let $\tilde{F}$ denote the canonical lift (i.e. Gauss lift) of F to Y and let $C' := \tilde{F}(\Sigma)$. (The lift of a point $x \in C$ is the tangent line to C at x.) If we assume that C is *nonsingular*, then

$$\deg \tilde{F}^*(l_1') = m, \qquad \deg \tilde{F}^*(l_2') = n,$$
$$\deg \tilde{F}^*(S_1) = \#\text{ branch points of } f_1 = 2g - 2 + 2n \text{ and}$$
$$\deg \tilde{F}^*(S_2) = \#\text{ branch points of } f_2 = 2g - 2 + 2m$$

where 'deg' refers to the intersection number of $\tilde{F}(\Sigma)$ with the relevant generators. Let $\tilde{C} := \psi^{-1}(C') \subset X$ and $\gamma := \beta_*(\tilde{C}) \subset \mathbf{P}^3$. Then for a generic hyperplane H in $\mathbf{P}^3$, we have

$$\begin{aligned}\deg \gamma &= H \cdot \beta_*(\tilde{C}) = \beta^* H \cdot \tilde{C} = (\sigma_1 + l_1) \cdot (\psi^{-1} C') \\ &= \psi_*(\sigma_1 + l_1) \cdot C' = (S_1 + 2l'_1) \cdot \tilde{F}_*(\Sigma) \\ &= \deg \tilde{F}^*(S_1 + 2l'_1) = 2g - 2 + 2n + 2m.\end{aligned}$$

Supose $\deg f_1 = \deg f_2 = d$ and $\operatorname{Ram}(f_1) = \operatorname{Ram}(f_2)$. Then the curve $C = F(\Sigma)$ has singular points with the property that $\deg \tilde{F}^*(S_1) = \deg \tilde{F}^*(S_2) = 0$. Consequently, $\deg \gamma = 2d$.

6. Conjugate branched superminimal surfaces. Let us suppose that $f: \Sigma \looparrowright S^4$ is a branched superminimal immersion of a compact Riemann surface in S^4. Generically, $f(\Sigma)$ misses a pair of antipodal points in S^4 (say the north and south poles). Also, generically, $\alpha(f(\Sigma)) \neq f(\Sigma)$, i.e. $f(\Sigma)$ is not α-invariant. Let $\tilde{f}: \Sigma \to \mathbf{P}^3$ be the holomorphic horizontal lift of f to $\mathbf{P}^3$.

Proposition 1.6. *A generic branched superminimal surface $f(\Sigma)$ in S^4 has the property that its lift $\tilde{f}(\Sigma)$ in $\mathbf{P}^3$ is not α-invariant.*

Proof. The proposition follows immediately from the definition of the involution α and the fact that α-invariance in $\mathbf{P}^3$ descends to α-invariance in S^4. □

Note. The converse is not necessarily true. For example, the totally geodesic S^2 of area 4π contained in the equator of S^4 is obviously α-invariant. However, its lift in $\mathbf{P}^3$ is a curve γ of degree 1 (and hence $\gamma \cong \mathbf{P}^1$) which avoids L_1 and L_2, and thus is not α-invariant. Observe that $\alpha(\gamma)$ projects down to the same geodesic S^2 (but with the opposite orientation).

Corollary 1.7. *Given a generic branched superminimal surface $f(\Sigma)$ in S^4, we obtain a conjugate branched superminimal surface, $\alpha \circ f(\Sigma)$, in S^4.*

Proof. Since $f(\Sigma)$ is generic, it avoids the poles and hence its lift $\tilde{f}(\Sigma)$ avoids L_1 and L_2. Thus, $\tilde{f}(\Sigma)$ is diffeomorphic to its image $\tilde{f}'(\Sigma)$ in X under the blow up of $\mathbf{P}^3$ along L_1 and L_2. Now by notes 1–4 in §I.4, we have $\tilde{\pi} \circ \tilde{f}'(\Sigma) = \tilde{\pi} \circ (\alpha \circ \tilde{f}'(\Sigma))$ and that $\alpha \circ \tilde{f}(\Sigma)$ is holomorphic and horizontal in $\mathbf{P}^3$ and thus projects to a branched superminimal surface in S^4, i.e. we obtain conjugate branched superminimal surfaces for free! □

7. Bidegrees and ramification divisors. Let $f(\Sigma)$ be a generic branched superminimal surface in S^4. Its lift $\tilde{f}(\Sigma)$ is a holomorphic horizontal curve γ in $\mathbf{P}^3$. The homology degree of $\gamma \subset \mathbf{P}^3$ is the fundamental class $[\gamma] \in H_2(\mathbf{P}^3; \mathbf{Z}) \cong \mathbf{Z}$. This degree is also the intersection number of γ with a generic $\mathbf{P}^2$ in $\mathbf{P}^3$ (i.e. homology degree = algebraic degree). Let $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2)$ denote the projection map of $\mathbf{P}^3 - (L_1 \cup L_2)$ to $\mathbf{P}^1 \times \mathbf{P}^1$. Define $f_1, f_2: \Sigma \to \mathbf{P}^1$ by $f_1 := \tilde{\pi}_1 \circ \tilde{f}$ and $f_2 := \tilde{\pi}_2 \circ \tilde{f}$.

Proposition 1.8. *Suppose that* $\deg(\gamma) = d$. *Then the holomorphic curve* $C = \tilde{\pi}\circ\tilde{f}(\Sigma)$ *in* $\mathbf{P}^1\times\mathbf{P}^1$ *has bidegree* (d, d), *i.e.* $\deg f_1 = \deg f_2 = d$. *Furthermore,* $\operatorname{Ram}(f_1) = \operatorname{Ram}(f_2)$.

Proof. Let $x_1 \in L_1$. The fiber $\tilde{\pi}_1^{-1}(x_1) \subset \mathbf{P}^3$ is the plane $P_1 = \operatorname{span}(x_1, L_2)$. Since $\deg\gamma = d$, P_1 has d intersection points with γ. Similarly, for $x_2 \in L_2$, the plane $P_2 = \tilde{\pi}_2^{-1}(x_2)$ has d intersection points with γ. Thus $C = \tilde{\pi}(\gamma)$ has bidegree (d, d).

Let z_0 be a ramification point of f_1. Let $p \in \gamma$ denote the point $\tilde{f}(z_0)$. Then the point $x := \tilde{\pi}_1(p)$ is a branch point of f_1. Let $y := \tilde{\pi}_2(p)$ and let L_{xy} denote the line in $\mathbf{P}^3$ through x and y. Finally, let H_x denote the plane $\{v \in T_p\mathbf{P}^3 \mid \tilde{\pi}_{1*}(v) = 0\}$. Now x is a branch point of f_1 and γ is an integral curve of $\mathscr{H}_p$, so the tangent line to the curve γ at p must be L_{xy}—the intersection of $\mathscr{H}_p$ and H_x. We thus have $\tilde{\pi}_{1*}(L_{xy}) = \tilde{\pi}_{2*}(L_{xy}) = 0$. Hence, y is a branch point of f_2 and so z_0 is in the ramification locus of both f_1 and f_2. By genericity, $\operatorname{Ram}(f_1) = \operatorname{Ram}(f_2)$. □

Lemma 1.9. *A holomorphic map* $F = (f_1, f_2)\colon \Sigma \to \mathbf{P}^1 \times \mathbf{P}^1$ *has a canonical Gauss lift* $\tilde{F}$ *to* $Y = \mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1)$.

Proof. First suppose $(df_1(z), df_2(z)) \neq (0, 0)$. Then the lift is given by $\tilde{F}(z) = (f_1(z), f_2(z), [f_1'(z), f_2'(z)])$. We are thus left with a finite set of singular points. Without loss of generality, suppose 0 is a singular point. Then $f_1'(z) = z^p g_1(z)$ and $f_2'(z) = z^q g_2(z)$ for some p, q and where $g_1(0) \neq 0$ and $g_2(0) \neq 0$. We may assume that $1 \leq p \leq q$. So

$$\tilde{F}(z) = \left(f_1(z), f_2(z), [g_1(z), z^{q-p} g_2(z)]\right)$$

for z in a neighborhood of 0. □

Proposition 1.10. *Suppose* $f\colon \Sigma \looparrowright S^4$ *is a generic superminimal immersion. Let* $\tilde{f}\colon \Sigma \to \mathbf{P}^3$ *be the holomorphic horizontal lift of* f, *and let* $f_1 := \tilde{\pi}_1 \circ \tilde{f}$ *and* $f_2 := \tilde{\pi}_2 \circ \tilde{f}$. *Suppose that* $\deg f_1 = \deg f_2 = d \geq 2$. *Then* $f_2 \neq A \circ f_1$ *for any* $A \in PSL(2, \mathbf{C})$.

Proof. Suppose $f_2 = A \circ f_1$ for some $A \in PSL(2, \mathbf{C})$. Then $F = (f_1, f_2) = (f_1, A \circ f_1)\colon \Sigma \to \mathbf{P}^1 \times \mathbf{P}^1$ factors through $\mathbf{P}^1$ as follows:

$$\Sigma \xrightarrow{f_1} \mathbf{P}^1 \xrightarrow{G=(\mathrm{Id}, A)} \mathbf{P}^1 \times \mathbf{P}^1.$$

Since G has bidegree $(1, 1)$, it is nonsingular and its canonical lift $\tilde{G}$ to Y avoids the two sections S_1 and S_2. The map f_1 is necessarily branched since $\deg f_1 \geq 2$. Hence, the canonical lift $\tilde{F}$ of F is a branched covering map of Σ into $\tilde{G}(\mathbf{P}^1) \cong \mathbf{P}^1$, i.e. $\tilde{F}(\Sigma)$ is branched. Consequently, its lift to $\mathbf{P}^3$, $\tilde{\tilde{F}}(\Sigma)$, is branched and hence projects to a branched superminimal surface in S^4. This contradicts the assumption that $f(\Sigma) \subset S^4$ is unbranched. □

Note that for $d = 1$, Σ must have genus zero and so $f(\Sigma)$ is totally geodesic in S^4.

We thus have

Theorem A. *Every superminimal immersion $f\colon \Sigma \looparrowright S^4$ arises from a pair of meromorphic functions f_1, f_2 on Σ such that*

(1) $\deg f_1 = \deg f_2 = d$ *for some integer* $d \geq 1$.
(2) $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$
(3) *For* $d \geq 2$, $f_1 \neq A \circ f_2$ *for any* $A \in PSL(2, \mathbb{C})$.

We would like to generate superminimal surfaces in S^4 by considering pairs of meromorphic functions on Σ which satisfy the three conditions in Theorem A. Suppose $F = (f_1, f_2)$ is such a pair. Let $\tilde{C} = \tilde{F}(\Sigma) \subset Y$. Our degree computations in §I.5 show that the total preimage curve $\gamma = \beta \circ \psi^{-1}(\tilde{C})$ in $\mathbf{P}^3$ has degree $2d$. Suppose γ consists of 2 connected (or irreducible) components γ_1 and γ_2. Then $\alpha(\gamma_1) = \gamma_2$ and consequently $\deg \gamma_1 = \deg \gamma_2 = d$. Under suitable conditions (to be discussed later), γ_1 and γ_2 will project to a conjugate pair of superminimal surfaces in S^4.

II. Genus zero

1. Meromorphic functions, Grassmannians and resultants. Let $f\colon \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic map of degree d (i.e. f is a meromorphic function of degree d). Then f can be expressed as a rational function of the form $P(z)/Q(z)$ where $P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$ and $Q(z) = b_d z^d + b_{d-1} z^{d-1} + \cdots + b_1 z + b_0$, $a_i, b_i \in \mathbb{C}$. Note that the map f is of degree d if $\min\{\deg P(z), \deg Q(z)\} = d$ and if the resultant of the 2 polynomials does not vanish. Let $P = (a_d, a_{d-1}, \ldots, a_1, a_0)$ and $Q = (b_d, b_{d-1}, \ldots, b_1, b_0)$ denote the coefficient vectors of $P(z)$ and $Q(z)$ respectively. Then the resultant $\mathscr{R}(P, Q)$ of $P(z)$ and $Q(z)$ is the determinant of the matrix

$$M = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} a_d & a_{d-1} & \ldots & a_1 \\ 0 & a_d & \ldots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_d \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_0 & 0 & \ldots & 0 \\ a_1 & a_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-1} & a_{d-2} & \ldots & a_0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} b_d & b_{d-1} & \ldots & b_1 \\ 0 & b_d & \ldots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_d \end{pmatrix}, \qquad B_2 = \begin{pmatrix} b_0 & 0 & \ldots & 0 \\ b_1 & b_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{d-1} & b_{d-2} & \ldots & b_0 \end{pmatrix}.$$

The resultant is a homogeneous polynomial of bidegree (d, d) in the a_i and the b_j. Furthermore, $\mathscr{R}(P, Q)$ is irreducible over any arbitrary field (cf. [18]). We thus require that $(P, Q) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathscr{R}$, where $\mathscr{R}$ is the irreducible resultant divisor. Observe that $(\lambda P, \lambda Q)$ describes the same function as (P, Q) for any $\lambda \in \mathbb{C}^*$. Thus the space of meromorphic functions of degree d is

$$M_d := \mathbf{P}(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathscr{R}) \subset \mathbf{P}^{2d+1}.$$

We next define an action of $GL(2, \mathbb{C})$ on $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ as follows:

$$g \cdot (P, Q) := (\alpha P + \beta Q, \gamma P + \delta Q) \quad \text{for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{C}).$$

Let $N_d := \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \Delta$ where $\Delta = \{(P, Q) \mid P \wedge Q = 0\}$. Observe that for $(P, Q) \in N_d$, $g \cdot (P, Q) = (\alpha P + \beta Q, \gamma P + \delta Q) = (P_1, Q_1)$, and $P_1 \wedge Q_1 = (\alpha P + \beta Q) \wedge (\gamma P + \delta Q) = (\alpha\delta - \beta\gamma) P \wedge Q \neq 0$. Thus, $GL(2, \mathbb{C})$ acts on N_d. In fact, we have a free action on N_d: $g \cdot (P, Q) = (\alpha P + \beta Q, \gamma P + \delta Q) = (P, Q)$ implies that $g = I$ since $P \wedge Q \neq 0$. Note that we can identify N_d with the Stiefel manifold of 2-frames in $\mathbb{C}^{d+1}$. For $(P, Q) \in N_d$, let $[P \wedge Q]$ denote the 2-plane in $\mathbb{C}^{d+1}$ spanned by P and Q. Let $P_1, Q_1 \in [P \wedge Q]$. Then $P_1 = \alpha P + \beta Q$ and $Q_1 = \gamma P + \delta Q$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. If $P_1 \wedge Q_1 \neq 0$, then $0 \neq P_1 \wedge Q_1 = (\alpha\delta - \beta\gamma) P \wedge Q$, i.e. $(\alpha\delta - \beta\gamma) \neq 0$. Thus, $GL(2, \mathbb{C})$ acts transitively on pairs of noncollinear vectors in $[P \wedge Q]$. It follows that $N_d / GL(2, \mathbb{C}) = G(2, d+1)$ and $\pi \colon N_d \to G(2, d+1)$ is a principal $GL(2, \mathbb{C})$-bundle (where $\pi(P, Q) = [P \wedge Q]$).

Lemma 2.1. $\mathscr{R}(g \cdot (P, Q)) = (\det g)^d \mathscr{R}(P, Q)$.

Proof. Let $(\tilde{P}, \tilde{Q})$ denote $g \cdot (P, Q)$. The resultant of $(\tilde{P}, \tilde{Q})$ is given by the determinant of the matrix

$$\tilde{M} = \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{B}_1 & \tilde{B}_2 \end{pmatrix}.$$

Since $(\tilde{P}, \tilde{Q}) = (\alpha P + \beta Q, \gamma P + \delta Q)$, we observe that

$$\tilde{A}_1 = \alpha A_1 + \beta B_1, \qquad \tilde{A}_2 = \alpha A_2 + \beta B_2,$$
$$\tilde{B}_1 = \gamma A_1 + \delta B_1, \qquad \tilde{B}_2 = \gamma A_2 + \delta B_2,$$

i.e.

$$\begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{B}_1 & \tilde{B}_2 \end{pmatrix} = \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix} \cdot \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}$$

where $I \in GL(d, \mathbb{C})$ is the identity matrix. It is straightforward to verify that

$$\det \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix} = (\alpha\delta - \beta\gamma)^d = (\det g)^d.$$

Thus, $\det \tilde{M} = (\det g)^d \cdot \det M$, i.e. $\mathscr{R}(g \cdot (P, Q)) = (\det g)^d \cdot \mathscr{R}(P, Q)$. □

It follows that $\mathscr{R} \subset \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ is fixed under the action of $GL(2, \mathbb{C})$. Let $\operatorname{Reg}(\mathscr{R})$ denote the regular part of $\mathscr{R}$. Since $\mathscr{R}$ is irreducible, $\operatorname{Reg}(\mathscr{R})$

is connected. Note that $\Delta = \{(P,Q) \mid P \wedge Q = 0\} \subset \mathscr{R}$ and that Δ has codimension d in $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$. So Δ cannot disconnect $\operatorname{Reg}(\mathscr{R})$ (which has dimension $2d+1$). Consequently, $(\operatorname{Reg}(\mathscr{R})) \cap N_d$ is connected, i.e. $\mathscr{R} \cap N_d$ is irreducible. For ease of notation, we shall let $\mathscr{R}$ to denote $\mathscr{R} \cap N_d$ also. By Lemma 2.1, $\dim(\mathscr{R}/GL(2,\mathbb{C})) = \dim(\pi(\mathscr{R})) = 2d-3$. Furthermore, since $\operatorname{Reg}(\mathscr{R})$ is connected and $\pi\colon N_d \to G(2,d+1)$ is a principal $GL(2,\mathbb{C})$-bundle, $\pi(\operatorname{Reg}(\mathscr{R})) = \operatorname{Reg}(\pi(\mathscr{R}))$ is connected. Thus, $\pi(\mathscr{R})$ is an irreducible divisor in $G(2,d+1)$.

Observe that the space of meromorphic functions of degree d is $M_d = \mathbf{P}(N_d - \mathscr{R})$. We thus have a free action of $PSL(2,\mathbb{C})$ on M_d. Furthermore, $M_d/PSL(2,\mathbb{C}) \subset G(2,d+1)$, the Grassmannian of 2-planes in $\mathbb{C}^{d+1}$.

2. The ramification divisor. Let $f\colon \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic map of degree d. Recall that $z_0 \in \mathbf{P}^1$ is a ramification point of f if $f_*(v) = 0$ for all $v \in T_{z_0}\mathbf{P}^1$. Expressing f as a rational function $P(z)/Q(z)$, we have $f'(z) = \big(Q(z)P'(z) - P(z)Q'(z)\big)/\big(Q(z)\big)^2$. Then the ramification points of f are given by the zero locus of $Q(z)P'(z) - P(z)Q'(z)$, a polynomial of degree $2d-2$. Observe that if $\deg\big(Q(z)P'(z) - P(z)Q'(z)\big) = k < 2d-2$, then ∞ is a ramification point of order $2d-2-k$.

Define a map $\Psi^d\colon M_d = \mathbf{P}(N_d - \mathscr{R}) \to \mathbf{P}^{2d-2}$ by

$$[(P,Q)] \mapsto [\operatorname{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}],$$

where $\operatorname{coeff}\{R(z)\}$ denotes the coefficient vector of the polynomial $R(z)$. The ramification map Ψ^d is well defined since

$$\begin{aligned}(\lambda P, \lambda Q) \mapsto & [\lambda^2 \cdot \operatorname{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}] \\ &= [\operatorname{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}],\end{aligned}$$

and if $Q(z)P'(z) - P(z)Q'(z) \equiv 0$, we have

$$\frac{P'(z)}{P(z)} = \frac{Q'(z)}{Q(z)}, \quad \textit{i.e.} \quad \log P(z) = \log Q(z) + C = \log(\tilde{C}Q(z)).$$

Thus $P(z) = \tilde{C}Q(z)$ and so $[(P,Q)] \notin M_d$.

Lemma 2.2. *$PSL(2,\mathbb{C})$ preserves the fibers of Ψ^d*

Proof. Let $g \in PSL(2,\mathbb{C})$. Let $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ be a representative of g. Then

$$\begin{aligned}\Psi^d(g \cdot [(P,Q)]) &= \Psi^d([\alpha P(z) + \beta Q(z), \gamma P(z) + \delta Q(z)]) \\ &= \big[\operatorname{coeff}\{\big(\gamma P(z) + \delta Q(z)\big)\big(\alpha P'(z) + \beta Q'(z)\big) \\ &\qquad - \big(\alpha P(z) + \beta Q(z)\big)\big(\gamma P'(z) + \delta Q'(z)\big)\}\big] \\ &= \big[\operatorname{coeff}\{(\alpha\delta - \beta\gamma)(Q(z)P'(z) - P(z)Q'(z)\}\big] \\ &= \big[\operatorname{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}\big] \\ &= \Psi^d([(P,Q)]). \quad \square\end{aligned}$$

Corollary 2.3. *$PSL(2,\mathbb{C})$ acts freely on the fibers of Ψ^d.*

Proof. $PSL(2,\mathbb{C})$ acts freely on $M_d = \mathbf{P}(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathscr{R})$, and by Lemma 2.2, it preserves fibers. □

We thus have an induced map $\Psi_d \colon G(2, d+1) \to \mathbf{P}^{2d-2}$ where

$$[P \wedge Q] \mapsto \left[\mathrm{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}\right].$$

This map is well defined.

Note that for $d = 2$, $G(2,3) \cong G(1,3) = \mathbf{P}^2$.

Proposition 2.4. *$\Psi_2 \colon G(2,3) \cong \mathbf{P}^2 \to \mathbf{P}^2$ is a biholomorphism.*

Proof. Let $[P \wedge Q] \in G(2,3)$. Then $[P \wedge Q]$ can be represented by one of the following matrices:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}, \quad \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where P and Q correspond to the rows of the matrices. For the first matrix, $P(z) = z^2 + a$, and $Q(z) = z + b$. Then

$$\begin{aligned} \Psi_2([P \wedge Q]) &= \left[\mathrm{coeff}\{Q(z)P'(z) - P(z)Q'(z)\}\right] \\ &= \left[\mathrm{coeff}\{(z+b)(2z) - (z^2 + a)\}\right] = [1, 2b, -a] \end{aligned}$$

i.e.

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix} \mapsto [1, 2b, -a].$$

Similarly, we have

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto [0, 2, a] \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto [0, 0, 1].$$

Note that in the second case, ∞ is a ramification point and that the third case is a degenerate case since $(P, Q) \in \mathscr{R}$. From the explicit computations, it is clear that Ψ_2 is one-to-one, nonsingular and is hence a biholomorphism. □

A consequence of the proposition is that $\Psi^2 \colon M_2 \to \mathbf{P}^2$ has connected fibers. Thus,

Corollary 2.5. *Let f be a meromorphic function of degree 2. Let g be any other meromorphic function of degree 2 with the property that* $\mathrm{Ram}(f) = \mathrm{Ram}(g)$. *Then $g = A \circ f$ for some $A \in PSL(2,\mathbb{C})$.*

Corollary 2.6. *There is no superminimal surface in S^4 whose lifting to $\mathbf{P}^3$ is a curve of degree 2.*

Proof. The genus 0 case follows immediately from Proposition 1.10 and Corollary 2.5. The following argument proves the general case. Let γ be a holomorphic horizontal curve in $\mathbf{P}^3$ of degree 2. Suppose γ is not a projective line. Pick any 3 noncollinear points A, B, C on γ. Let L_{AB} and L_{AC} denote the lines through $A\&B$ and $A\&C$ respectively. Let P denote the plane spanned by these two lines. Since $\deg(\gamma) = 2$ and P contains the points A, B and C, necessarily, $\gamma \subset P$, i.e. γ is planar. Since there are no horizontal planes in $\mathbf{P}^3$ (otherwise, that horizontal $\mathbf{P}^2$ would be diffeomorphic to S^4 !), P (and hence γ) is in fact a projective line. Since $\deg(\gamma) = 2$, γ is necessarily branched. (Nevertheless, γ projects to a totally geodesic surface in S^4.) $\square$

3. The orbits in the fibers of Ψ^d. Let $N = \frac{1}{2}(d+2)(d-1) = \binom{d+1}{2} - 1 = \dim\left(\mathbf{P}\left(\bigwedge^2 \mathbf{C}^{d+1}\right)\right)$. Let $P = (a_d, \ldots, a_0)$ and $Q = (b_d, \ldots, b_0)$ be two vectors in $\mathbf{C}^{d+1}$ which span a plane, $\binom{P}{Q}$, in $\mathbf{C}^{d+1}$. Then the Plücker embedding $G(2, d+1) \hookrightarrow \mathbf{P}^N = \mathbf{P}\left(\bigwedge^2 \mathbf{C}^{d+1}\right)$ is given by $\binom{P}{Q} \mapsto [P \wedge Q]$. Choose Plücker coordinates x_{ij} on $\mathbf{P}^N$ where $i > j$, $i = 1, \ldots, d$, $j = 0, \ldots, d-1$. Let $P(z) = a_d z^d + \cdots + a_1 z + a_0$ and $Q(z) = b_d z^d + \cdots + b_o$. Then

$$Q(z)P'(z) - P(z)Q'(z) = \alpha_{2d-2} z^{2d-2} + \cdots + \alpha_n z^n + \cdots + \alpha_1 z + \alpha_0$$

where

$$\alpha_n = \sum_{\substack{i+j=n+1 \\ i>j}} (i-j)x_{ij}, \qquad n = 0, \ldots, 2d-2.$$

Consider the linear map $L\colon \mathbf{C}^{N+1} \to \mathbf{C}^{2d-1}$ given by

$$(x_{ij}) \mapsto (\alpha_{2d-2}, \ldots, \alpha_n, \ldots, \alpha_0).$$

Observe that since α_n contains only the x_{ij}'s which satisfy the condition $i + j = n + 1$, L has maximal rank. Let K denote the kernel of L. Then $\dim K = \frac{1}{2}(d^2 + d) - 2d + 1 = \frac{1}{2}(d-2)(d-1)$. Let $\kappa := \mathbf{P}K$, a projective $\frac{1}{2}d(d-3)$-plane in $\mathbf{P}^N$. Note that the image of $G(2, d+1)$ in $\mathbf{P}^N$, G^{2d-2}, does not intersect κ by construction. Thus the map Ψ_d can be given in Plücker coordinates by

$$\Psi_d([P \wedge Q]) = [\alpha_{2d-2}, \ldots, \alpha_n, \ldots, \alpha_0].$$

So Ψ_d can be thought of as the restriction to G^{2d-2} of a "map" from $\mathbf{P}^N$ to $\mathbf{P}^{2d-2}$. We can extend Ψ_d to a map from $\mathbf{P}^N - \kappa$ to $\mathbf{P}^{2d-2}$. Let $\tilde{\mathbf{P}}^N$ denote the blow-up of $\mathbf{P}^N$ along κ. Let $q \in \mathbf{P}^{2d-2}$. Let $\tilde{\Psi}_d$ denote the map induced on $\tilde{\mathbf{P}}^N$. Then $\Lambda_q = (\tilde{\Psi}_d^{-1})(q)$ is a projective $\frac{1}{2}(d-2)(d-1)$-plane in $\mathbf{P}^N$, i.e. a plane of dimension complementary to that of G^{2d-2}. Therefore the number of points of intersection of Λ_q with G^{2d-2} is the degree of G^{2d-2} in $\mathbf{P}^N$, which is $(2d-2)!/(d-1)!d!$. As a consequence, there are generically $(2d-2)!/(d-1)!d!$ *distinct* $PSL(2, \mathbf{C})$-orbits of holomorphic maps of degree d from $\mathbf{P}^1$ to $\mathbf{P}^1$ which have the same ramification divisor. We thus have

Theorem B. *Let f be a generic meromorphic function of degree $d \geq 2$. Then, under the action of $PSL(2, \mathbf{C})$, there are $(2d-2)!/(d-1)!d!$ distinct orbits of meromorphic functions of degree d with ramification divisor* $\operatorname{Ram}(f)$.

Note that when $d = 2$ we have only 1 orbit. This is consistent with our previous result (Corollary 2.5).

4. The space $\mathfrak{H}_d$. Let $F = (f_1, f_2)\colon \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ be a holomorphic map of bidegree (d, d) such that $\operatorname{Ram}(f_1) = \operatorname{Ram}(f_2)$. By our previous results, the curve $\tilde{F}(\mathbf{P}^1) \subset Y = \mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1)$ avoids the 2 distinguished sections, S_1 and S_2 of Y. Since $\psi\colon \tilde{\mathbf{P}}^3 - (\sigma_1 \cup \sigma_2) \to Y - (S_1 \cup S_2)$ is a covering map of degree 2 and since $\pi_1(\mathbf{P}^1) = 0$, the map $\tilde{F}$ lifts to a map $\tilde{\tilde{F}}\colon \mathbf{P}^1 \to \tilde{\mathbf{P}}^3 - (\sigma_1 \cup \sigma_2)$. Let $\gamma_1 := \beta \circ \tilde{\tilde{F}}(\mathbf{P}^1)$ and $\gamma_2 := \beta \circ \alpha \circ \tilde{\tilde{F}}(\mathbf{P}^1) = \alpha(\gamma_1)$. Then γ_1 and γ_2 project to a conjugate pair of branched superminimal surfaces, Σ_1 and Σ_2, in S^4. If $\tilde{F}$ is an immersion, then the pair of surfaces are unbranched. We also showed that for $d \geq 2$, a necessary condition for Σ_1 and Σ_2 to be unbranched is that f_1 and f_2 belong to different orbits of $PSL(2, \mathbf{C})$. Our search for unbranched superminimal surfaces is thus aided by the following immediate consequence of Theorem B:

Theorem C. *For each $d \geq 3$, there is a branched superminimal surface of genus 0 in S^4 which arises from a pair of meromorphic functions (f_1, f_2), each of degree d such that* $\operatorname{Ram}(f_1) = \operatorname{Ram}(f_2)$ *and that f_1 and f_2 belong to distinct $PSL(2, \mathbf{C})$-orbits.*

Proof. By Theorem B, there are $(2d-2)!/(d-1)!d!$ distinct orbits for each generic ramification divisor. □

Recall that a branched superminimal immersion of S^2 into S^4 is just a harmonic map. Also, a (branched) superminimal surface of degree d in S^4 is a surface of area $4\pi d$ whose lifting to $\mathbf{P}^3$ is a holomorphic, horizontal curve of degree d. We say that a harmonic map $f\colon S^2 \to S^4$ has *harmonic degree* d if $f(S^2)$ has area $4\pi d$. Let $\mathfrak{H}_d$ denote the space of harmonic maps of S^2 into S^4 of harmonic degree d.

Theorem D. *For each $d \geq 1$, $\mathfrak{H}_d$ is parametrized by a space of complex dimension $2d + 4$.*

Proof. A meromorphic function of degree d is determined by $2d + 1$ complex parameters. The theorem follows immediately from the fact that the fibers of Ψ^d are 3-dimensional. □

Note. Theorem D is in agreement with the results of Verdier [17]. Verdier in fact shows that $\mathfrak{H}_d$ is naturally equipped with the structure of a complex algebraic variety of pure dimension $2d + 4$, and for $d \geq 3$, $\mathfrak{H}_d$ possesses three irreducible components. We will show that $\mathfrak{H}_d$ is connected.

5. Connectivity of $\mathfrak{H}_d$. Recall that a meromorphic function of degree d can be considered as an element of $M_d = \mathbf{P}(N_d) - \mathscr{R}$ where $N_d = \mathbf{C}^{d+1} \times \mathbf{C}^{d+1} - \{(P, Q) \mid P \wedge Q = 0\}$ and where $\mathscr{R}$ is the resultant divisor. We have a ramification map $\Psi^d\colon M_d \to \mathbf{P}^{2d-2}$. The action of $PSL(2, \mathbf{C})$ on M_d induces a map $\Psi_d\colon G(2,d+1)-\pi(\mathscr{R})\to \mathbf{P}^{2d-2}$, where $\pi(\mathscr{R})=\mathscr{R}/PSL(2,\mathbb{C})$ is an irreducible variety of codimension 1. For ease of notation, we will let $\mathscr{R}$ and $\mathscr{R}'$ denote $\pi(\mathscr{R})$ and $\Psi_d(\pi(\mathscr{R}))$ respectively for the rest of this section. Now, $\Psi_d\colon G(2,d+1)\to \mathbf{P}^{2d-2}$ is a branched covering map. Let $\mathfrak{R}$ and $\mathfrak{B}$ denote the ramification locus of Ψ_d and the branch locus of Ψ_d respectively. Then

$$\Psi_d\colon G(2,d+1)-\mathfrak{R}-\mathscr{R}\to \mathbf{P}^{2d-2}-\mathfrak{B}-\mathscr{R}'$$

is a covering map. Now consider the diagonal map

$$\delta\colon \mathbf{P}^{2d-2}\to \mathbf{P}^{2d-2}\times\mathbf{P}^{2d-2}.$$

Let $\mathscr{M}_d:=G(2,d+1)-\mathscr{R}$. From the diagram

$$\begin{array}{ccc} \delta^*(\mathscr{M}_d\times\mathscr{M}_d) & & \mathscr{M}_d\times\mathscr{M}_d \\ \downarrow & & \downarrow{\scriptstyle \Psi_d\times\Psi_d} \\ \mathbf{P}^{2d-2} & \xrightarrow[\delta]{} & \mathbf{P}^{2d-2}\times\mathbf{P}^{2d-2} \end{array}$$

we see that modulo the action of $PSL(2,\mathbb{C})$, an element of $\delta^*(\mathscr{M}_d\times\mathscr{M}_d)$ is a pair of meromorphic functions of degree d with the same ramification divisor. We will show that the space $\delta^*(\mathscr{M}_d\times\mathscr{M}_d)$ is connected and as a consequence $\mathfrak{H}_d$, the space of pairs of meromorphic functions of degree d with the same ramification divisor, is connected.

Lemma 2.7. *$\mathscr{R}$ is not a component of $\mathfrak{R}$. Thus,* $\dim(\mathfrak{R}\cap\mathscr{R})\le 2d-4$.

Proof. In §II.1, we showed that $\mathscr{R}$ is irreducible. Thus, it suffices to show that there exists an $x\in\mathscr{R}$ such that $x\notin\mathfrak{R}$. Now in ambient coordinates,

$$\Psi^d(P,Q)=\Psi^d(a_d,\dots,a_0,b_d,\dots,b_0)=(c_{2d-2},\dots,c_0)$$

where

$$\begin{aligned} c_m &= \sum_{j=0}^{m+1}(2j-m-1)a_jb_{m-j+1} \\ &= \sum_{k=0}^{m+1}(m-2k+1)a_{m-k+1}b_k, \qquad m=0,\dots,2d-2. \end{aligned}$$

Thus,

$$\frac{\partial c_m}{\partial a_j}=\begin{cases} (2j-m-1)b_{m-j+1}, & \text{for } j=0,\dots,m+1;m-j+1\le d, \\ 0, & \text{for } j>m+1, \end{cases}$$

and

$$\frac{\partial c_m}{\partial b_k}=\begin{cases} (m-2k+1)a_{m-k+1}, & \text{for } k=0,\dots,m+1;m-k+1\le d, \\ 0, & \text{for } k>m+1. \end{cases}$$

Let $P(z) = z^d + z^2$, $Q(z) = z$. Certainly $[P \wedge Q] \in \mathscr{R} \subset G(2, d+1)$. Then

$$\left.\frac{\partial c_m}{\partial a_j}\right|_{(P,Q)} \neq 0, \qquad \text{if } j = m = 0, 2, 3, \dots, d.$$

Also,

$$\left.\frac{\partial c_m}{\partial b_k}\right|_{(P,Q)} \neq 0, \quad \text{if } m = d + k - 1, \text{or } m = k + 1$$

i.e. this derivative does not vanish for $k = 0, m = 1$; $k = 0, m = d - 1$; $k = 1, m = d$; $\dots$; $k = d - 1, m = 2d - 2$. Consequently, $d\Psi^d|_{(P,Q)}$ has maximal rank. Thus, $[P \wedge Q] \notin \mathfrak{R}$. □

Recall that an element of $\delta^*(\mathscr{M}_d \times \mathscr{M}_d)$ is (up to a Möbius transformation) a pair of meromorphic functions of degree d with the same ramification divisor. Thus, if $q \in \mathscr{M}_d$, the diagonal pair (q, q) is obviously in $\delta^*(\mathscr{M}_d \times \mathscr{M}_d)$. Since $\mathscr{M}_d$ is connected, it is clear that a diagonal point $(q, q) \in \delta^*(\mathscr{M}_d \times \mathscr{M}_d)$ is path connected to any other diagonal point $(q', q') \in \delta^*(\mathscr{M}_d \times \mathscr{M}_d)$. Thus, to show that $\delta^*(\mathscr{M}_d \times \mathscr{M}_d)$ is path connected, it suffices to show that any point $(x, y) \in \delta^*(\mathscr{M}_d \times \mathscr{M}_d)$ is path connected to the point (y, y).

Now let $(x, y) \in \delta^*(\mathscr{M}_d \times \mathscr{M}_d)$. Let $\Psi_d(x) = \Psi_d(y) = \star \in \mathbf{P}^{2d-2} - \mathscr{R}'$. Without loss of generality, $\star \in \mathbf{P}^{2d-2} - \mathfrak{B} - \mathscr{R}'$, and so, $x, y \notin \mathfrak{R}$. (If $\star \in \mathfrak{B}$, we can find a path C in $\mathbf{P}^{2d-2} - \mathscr{R}'$ so that $C(0) = \star$ and $C(1) = \star' \notin \mathfrak{B}$). Since $G(2, d+2) - \mathscr{R} - \mathfrak{R}$ is connected, there is a path $\tilde{\gamma} \subset G(2, d+1) - \mathscr{R} - \mathfrak{R}$ so that $\tilde{\gamma}(0) = x$, $\tilde{\gamma}(1) = y$. Then $\gamma := \Psi_d(\tilde{\gamma})$ is a based loop in $\mathbf{P}^{2d-2} - \mathfrak{B} - \mathscr{R}'$, i.e. $[\gamma] \in \pi_1(\mathbf{P}^{2d-2} - \mathfrak{B} - \mathscr{R}', \star)$. Thus $\gamma \colon S^1 \to \mathbf{P}^{2d-2} - \mathfrak{B} - \mathscr{R}' \subset \mathbf{P}^{2d-2}$. Since $\mathbf{P}^{2d-2}$ is simply connected, we can extend γ to a map $\gamma' \colon D^2 \to \mathbf{P}^{2d-2}$. By Thom transversality and Lemma 2.7, we can make γ' transversal to $\operatorname{Reg}(\mathfrak{B})$, $\operatorname{Reg}(\mathscr{R}')$ and $\Psi_d(\mathfrak{R} \cap \mathscr{R}) = \mathfrak{B} \cap \mathscr{R}'$, i.e.

$$\gamma'(D^2) \cap \{\operatorname{Sing}(\mathfrak{B}) \cup \operatorname{Sing}(\mathscr{R}') \cup \{\mathfrak{B} \cap \mathscr{R}'\}\} = \varnothing.$$

Then $\gamma'(D^2)$ intersects $\operatorname{Reg}(\mathfrak{B})$ and $\operatorname{Reg}(\mathscr{R}')$ in a finite number of points, say $\gamma'(D^2) \cap \operatorname{Reg}(\mathfrak{B}) = \{z_1, \dots, z_n\}$ and $\gamma'(D^2) \cap \operatorname{Reg}(\mathscr{R}') = \{\zeta_1, \dots, \zeta_m\}$ where $z_i \neq \zeta_j$ for any i, j. Let σ_i and τ_j be tiny based loops around z_i and ζ_j respectively. Then γ is homotopic to a composition of the σ_i's and the τ_j's. Observe that the τ_j's act trivially on $F = \Psi_d^{-1}(\star)$. Let $x_1 := x$ and $x_{n+1} := y$. Since $[\gamma](x) = y$, we have $[\sigma_1](x_1) = x_2$, $[\sigma_2](x_2) = x_3$, $\dots$, $[\sigma_n](x_n) = x_{n+1} = y$ for some $x_2, \dots, x_n \in F$. Let $\tilde{\sigma}_i$ be the lifting of σ_i so that $\tilde{\sigma}_i(0) = x_i$ and $\tilde{\sigma}_i(1) = x_{i+1}$. As σ_i traces along the boundary of a tiny disc D_i around the branch point z_i, $\tilde{\sigma}_i$ traces a path around some ramification point $y_i \in \Psi_d^{-1}(z_i)$. Let $\tilde{D}_i$ denote the contractible disc in $G(2, d+1) - \mathscr{R}$ around y_i which projects to D_i. Suppose $\sigma_i(t)$ traces ∂D_i for $t \in [t_{\alpha_i}, t_{\beta_i}]$. Let $u_i = \tilde{\sigma}_i(t_{\alpha_i})$ and $v_i = \tilde{\sigma}_i(t_{\beta_i})$. Let $\tilde{\alpha}_i$ be a path from u_i to y_i and let $\tilde{\beta}_i$ be a path from y_i to v_i. Say $\tilde{\alpha}_i(t_{\alpha_i}) = u_i$, $\tilde{\beta}_i(t_{\beta_i}) = v_i$ and $\tilde{\alpha}_i(t_{\varepsilon_i}) = \tilde{\beta}_i(t_{\varepsilon_i}) = y_i$ for some $t_{\varepsilon_i} \in (t_{\alpha_i}, t_{\beta_i})$. Consider the modified path $\tilde{\sigma}'_i$ defined as follows:

$$\tilde{\sigma}'_i(t) = \begin{cases} \tilde{\sigma}_i(t), & \text{for } t \in [0, t_{\alpha_i}], \\ \tilde{\alpha}_i(t), & \text{for } t \in [t_{\alpha_i}, t_{\varepsilon_i}], \\ \tilde{\beta}_i(t), & \text{for } t \in [t_{\varepsilon_i}, t_{\beta_i}], \\ \tilde{\sigma}_i(t), & \text{for } t \in [t_{\beta_i}, 1]. \end{cases}$$

Let $\sigma'_i := \Psi_d(\tilde{\sigma}'_i)$. Observe that σ'_i is a homotopically trivial loop in $\mathbf{P}^{2d-2} - \mathscr{R}'$. Let $\tilde{\sigma}''_i$ denote the lifting of σ'_i so that $\tilde{\sigma}''_i(0) = \tilde{\sigma}''_i(1) = y$. Let γ_i denote the path $\left(\tilde{\sigma}'_i, \tilde{\sigma}''_i\right)$ in $\delta^*(\mathscr{M}_d \times \mathscr{M}_d)$ from (x_i, y) to (x_{i+1}, y). We have thus constructed a path $\gamma_n \circ \gamma_{n-1} \circ \cdots \circ \gamma_1$ in $\delta^*(\mathscr{M}_d \times \mathscr{M}_d)$ from (x, y) to (y, y). Thus,

Theorem E. *For each* $d \geq 1$, $\mathfrak{H}_d$ *is connected.*

6. Examples. Consider the map $F_d = (f_1, f_2)\colon \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ $(d > 2)$ where

$$f_1(z) = \frac{P_1(z)}{Q_1(z)} = \frac{z^d + dz + 1}{z^{d-1} + z + (d-2)} \qquad \text{and}$$

$$f_2(z) = \frac{P_2(z)}{Q_2(z)} = \frac{z^d - dz + 1}{z^{d-1} + z - (d-2)}.$$

We will show that for $d > 2$, F_d gives rise to a conjugate pair of unbranched superminimal surfaces in S^4.

Observe that f_1 and f_2 belong to different $PSL(2, \mathbb{C})$-orbits.

Lemma 2.8. *For* $d > 2$, F_d *has bidegree* (d, d). *Furthermore,* $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$.

Proof. We must first show that $P_i(z)$ and $Q_i(z)$ have no common zeroes ($i = 1, 2$).

Suppose ζ is a common zero of $P_1(z)$ and $Q_1(z)$. Certainly ζ must be a zero of $P(z) = zQ_1(z) - P_1(z) = z^2 - 2z - 1$. But $P(z)$ has roots $1 \pm \sqrt{2}$ which are certainly not roots of $P_1(z)$ or $Q_1(z)$. Thus, $\deg(f_1) = d$. A similar argument shows that $\deg(f_2) = d$. Now

$$f'_1(z) = \frac{R(z)}{Q_1^2(z)} = \frac{z^{2d-2} + (d-1)z^d - (d-1)z^{d-2} + d(d-2) - 1}{[z^{d-1} + z + (d-2)]^2}$$

and

$$f'_2(z) = \frac{R(z)}{Q_2^2(z)} = \frac{z^{2d-2} + (d-1)z^d - (d-1)z^{d-2} + d(d-2) - 1}{[z^{d-1} + z - (d-2)]^2}.$$

Thus, $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$. $\square$

Proposition 2.9. *The map F_d is generically one-to-one onto its image. Hence, it is not a branched covering map.*

Proof.

$$F_d(0) = \left(\frac{1}{d-2}, \frac{-1}{d-2}\right).$$

Note that 0 is not a ramification point of either f_1 or f_2. We shall compute

$$F_d^{-1}\left(\frac{1}{d-2}, \frac{-1}{d-2}\right).$$

This amounts to solving the simultaneous equations

$$\frac{z^d + dz + 1}{z^{d-1} + z + (d-2)} = \frac{1}{d-2} \qquad \text{and} \qquad \frac{z^d - dz + 1}{z^{d-1} + z - (d-2)} = \frac{-1}{d-2}.$$

We obtain

$$(d-2)(z^d + dz + 1) - (z^{d-1} + z + (d-2)) = 0 \qquad \text{and}$$
$$(d-2)(z^d - dz + 1) - (z^{d-1} + z - (d-2)) = 0.$$

Thus, we have to solve the simultaneous equations

$$g_1(z) = (d-2)z^d - z^{d-1} + (d(d-2) - 1)z = 0 \qquad \text{and}$$
$$g_2(z) = (d-2)z^d + z^{d-1} - (d(d-2) - 1)z = 0.$$

Observe that if ζ is a common zero of g_1 and g_2, then certainly it is a zero of $(g_1 + g_2)(z) = 2(d-2)z^d$ $(d > 2)$. But $g_1 + g_2$ has 0 as its only zero. Thus

$$F_d^{-1}\left(\frac{1}{d-2}, \frac{-1}{d-2}\right) = \{0\},$$

i.e. F_d is generically one to one onto its image. □

Proposition 2.10. *The map $\tilde{F}_d\colon \mathbf{P}^1 \to \mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1)$ is nonsingular.*

Proof. It suffices to show that $\tilde{F}_{d*}$ does not vanish at the ramification points. We will consider three cases.

Case 1. Assume that the zeroes of $Q_1(z)$ and $Q_2(z)$ are not ramification points. Then $\tilde{F}_d$ can be described locally by

$$\tilde{F}_d(z) = (f_1(z), f_2(z), G(z))$$

where

$$G(z) = \frac{f_1'(z)}{f_2'(z)} = \left(\frac{z^{d-1} + z - (d-2)}{z^{d-1} + z + (d-2)}\right)^2.$$

It suffices to show that G' does not vanish at the ramification points. Now

$$G'(z) = 2\left(\frac{z^{d-1} + z - (d-2)}{(z^{d-1} + z + (d-2))^3}\right) \cdot 2(d-2)h(z)$$

where $h(z) = (d-1)z^{d-2}+1$. Observe that $h(z)$ vanishes when $z^{d-2} = -1/(d-1)$. Let ζ be a $(d-2)$th root of $-1/(d-1)$. Then

$$\begin{aligned} R(\zeta) &= \zeta^{2d-2} + (d-1)\zeta^d - (d-1)\zeta^{d-2} + d(d-2) - 1 \\ &= \zeta^2(\zeta^{2(d-2)} + (d-1)\zeta^{d-2}) - (d-1)\zeta^{d-2} + d(d-2) - 1 \\ &= \zeta^2\left(\left(\frac{1}{d-1}\right)^2 - 1\right) + d(d-2) \neq 0. \end{aligned}$$

Thus, the zeroes of G' do not coincide with the ramification points, i.e. $\tilde{F}_d$ is nonsingular.

Case 2. Suppose ζ is a common zero of $R(z)$ and $Q_1(z)$. Let $\tilde{f}_1(z) = Q_1(z)/P_1(z)$. Then locally,

$$\tilde{F}_d(z) = (\tilde{f}_1(z), f_2(z), G(z)) \quad \text{where } G(z) = \frac{\tilde{f}_1'(z)}{f_2'(z)} = -\left(\frac{Q_2(z)}{P_1(z)}\right)^2.$$

Then $G'(z) = -2[Q_2(z)/P_1^3(z)] \cdot \Delta(z)$ where

$$\begin{aligned} \Delta(z) &= P_1(z)Q_2'(z) - Q_2(z)P_1'(z) \\ &= -z^{2d-2} + (1-d)z^d + d(2d-4)z^{d-1} + (d-1)z^{d-2} + d + d(d-2) + 1. \end{aligned}$$

Let $S(z) = R(z) + \Delta(z) = d(2d-4)z^{d-1} + 2d(d-2)$. First observe that $Q_1(z)$ and $Q_2(z)$ have no common zeroes since $Q_1(z) + Q_2(z) = 2(d-2) \neq 0$ for $d > 2$. Thus $G'(\zeta) = 0$ if and only if $\Delta(\zeta) = 0$. Suppose that ζ is a common zero of Δ and R. Then ζ must be a zero of S. But $S(z)$ vanishes when $z^{d-1} = -2d(d-2)/d(2d-4) = -1$. Then ζ must be a $(d-1)$th root of -1. But $Q_1(\zeta) = -1 + \zeta + (d-2) = \zeta + d - 3 \neq 0$ for $d > 2$, contradicting our assumption that ζ was a zero of $Q_1(z)$. Thus, $G'(\zeta) \neq 0$.

Case 3. Suppose ζ is a common zero of $R(z)$ and $Q_2(z)$. Let $\tilde{f}_2(z) = Q_2(z)/P_2(z)$. Then locally,

$$\tilde{F}_d(z) = (f_1(z), \tilde{f}_2(z), G(z)) \quad \text{where } G(z) = \frac{f_1'(z)}{\tilde{f}_2'(z)} = -\left(\frac{P_2(z)}{Q_1(z)}\right)^2.$$

Then $G'(z) = -2[P_2(z)/Q_1^3(z)] \cdot \Delta(z)$ where

$$\begin{aligned} \Delta(z) &= Q_1(z)P_2'(z) - P_2(z)Q_1'(z) \\ &= z^{2d-2} + (d-1)z^d + d(2d-4)z^{d-1} - (d-1)z^{d-2} - d(d-2) - 1. \end{aligned}$$

Let $S(z) = R(z) - \Delta(z) = -d(2d-4)z^{d-1} + 2d(d-2)$. If ζ is a common zero of Δ and R, certainly it is a zero of S. But $S(z)$ vanishes when $z^{d-1} = 2d(d-2)/d(2d-4) = 1$, i.e. ζ is a $(d-1)$th root of 1. But $Q_2(\zeta) = \zeta - (d+3) \neq 0$ for $d > 2$, a contradiction. Thus, $G'(\zeta) \neq 0$. □

Thus the total preimage $\beta \circ \psi^{-1}(\tilde{F}_d(\mathbf{P}^1))$ is a conjugate pair of nonsingular holomorphic, horizontal curves in $\mathbf{P}^3$ which project to a conjugate pair of superminimal surfaces, each of area $4\pi d$, in S^4 ($d \geq 3$).

III. Higher genus

We now consider branched superminimal immersions of a compact Riemann surface Σ of genus $g > 0$ into S^4.

Let $f: \Sigma \looparrowright S^4$ be a branched superminimal immersion such that $f(\Sigma)$ has area $4\pi d$. Recall that generically, $f(\Sigma)$ misses a pair of antipodal points on S^4, say the north and south poles. We have shown that f arises from a pair of meromorphic functions (f_1, f_2) of bidegree (d, d) such that $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$. Moreover, f is linearly full (i.e. $f(\Sigma)$ is not contained in any strict linear subspace of $\mathbf{R}^5$) provided $d \geq 3$ and $f_2 \neq A \circ f_1$ for any $A \in PSL(2, \mathbf{C})$. For each $d \geq 3$, we wish to construct linearly full branched superminimal immersions from such pairs of functions. Let $F = (f_1, f_2)$ be such a pair of functions. Let $\tilde{C}$ denote the curve $\tilde{F}(\Sigma)$. We require that $\psi^{-1}(\tilde{C})$ consist of two connected components, γ_1 and γ_2, such that $\alpha(\gamma_1) = \gamma_2$ and $\psi(\gamma_1) = \psi(\gamma_2) = \tilde{C}$. If this is the case, then the curves γ_1 and γ_2 project to a conjugate pair of (branched) superminimal surfaces in S^4.

Let $X := \tilde{\mathbf{P}}^3 - (\sigma_1 \cup \sigma_2) \cong \mathbf{P}^3 - (L_1 \cup L_2)$ and $Y := \mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1) - (S_1 \cup S_2)$. Note that $\pi_1 X = 0$ and $\psi: X \to Y$ is a covering map of degree 2. The maps that we are considering, $F = (f_1, f_2): \Sigma \to \mathbf{P}^1 \times \mathbf{P}^1$, are such that $\tilde{F}(\Sigma) \subset Y$. Observe that $\tilde{F}$ lifts to a map $\tilde{\tilde{F}}: \Sigma \to X$ if and only if $\tilde{F}_*(\pi_1\Sigma) = 0$. If $\tilde{F}_*(\pi_1\Sigma) = 0$, then we have 2 maps, $\tilde{\tilde{F}}$ and $\alpha \circ \tilde{\tilde{F}}$, from Σ to X. Thus

Theorem F. *Suppose $F = (f_1, f_2): \Sigma \to \mathbf{P}^1 \times \mathbf{P}^1$ is a holomorphic map of bidegree (d, d) of a compact Riemann surface of genus g to $\mathbf{P}^1 \times \mathbf{P}^1$ such that* $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$ *and $f_2 \neq A \circ f_1$ for any $A \in PSL(2, \mathbf{C})$. Let $\tilde{F}: \Sigma \to \mathbf{P}T(\mathbf{P}^1 \times \mathbf{P}^1) - (S_1 \cup S_2)$ be the canonical Gauss lift of F. Then F gives rise to a conjugate pair of linearly full branched superminimal surfaces of genus g in S^4 provided $\tilde{F}_*(\pi_1\Sigma) = 0$.*

Note. The condition $\tilde{F}_*(\pi_1\Sigma) = 0$ is automatically satisfied if Σ has genus 0. However, if $\tilde{F}_*(\pi_1\Sigma) \neq 0$, then we do not have a lift of Σ to X. Nevertheless, there is a two-fold cover $\tilde{\Sigma}$ of Σ which lifts to X (where genus$(\tilde{\Sigma}) = 2g - 1$). We then obtain a branched superminimal surface in S^4 of genus $2g - 1$.

An easy way to satisfy the lifting criterion is by factoring through $\mathbf{P}^1$:

$$F = (F_1, F_2): \xrightarrow{\varphi} \mathbf{P}^1 \xrightarrow{(f_1, f_2)} \mathbf{P}^1 \times \mathbf{P}^1$$

where φ is a holomorphic map of degree d_1 and (f_1, f_2) is a holomorphic map of bidegree (d_2, d_2) which gives rise to a linearly full branched superminimal immersion of $\mathbf{P}^1$ into S^4. Note that F has bidegree $(d_1 d_2, d_1 d_2)$. Certainly,

$\mathrm{Ram}(F_1) = \mathrm{Ram}(F_2)$ and $F_2 \neq A \circ F_1$ for any $A \in PSL(2, \mathbb{C})$ (since (f_1, f_2) is linearly full). Let $\tilde{F}: \Sigma \to Y$ be the canonical Gauss lift of F. Then $\tilde{F}_*(\pi_1 \Sigma) = 0$ and by Theorem F, $\tilde{F}$ lifts to a holomorphic horizontal map, $\tilde{\tilde{F}}$, to $\mathbf{P}^3$. Note however that $\tilde{\tilde{F}}(\Sigma)$ is necessarily branched. Nevertheless, it projects to a branched superminimal surface in S^4 of area $4\pi d_1 d_2$. We thus have lots of branched superminimal immersions of Σ into S^4.

The construction in the previous paragraph gives us superminimal surfaces of genus $g > 0$ in S^4 which were necessarily branched. It would be interesting if the map F can be deformed (in the space of branched superminimal immersions of Σ into S^4 of degree $d_1 d_2$) to a map F' so that F' gives rise to an *unbranched* superminimal surface in S^4.

It has come to the author's attention that Verdier has obtained a result similar to Theorem E (which was his conjecture in [17]).

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