TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 313, Number 2, June 1989

EXTENDING HOMEOMORPHISMS AND APPLICATIONS TO METRIC LINEAR SPACES WITHOUT COMPLETENESS

TADEUSZ DOBROWOLSKI

ABSTRACT. A method of extending homeomorphisms between compacta metric spaces is presented. The main application is that homeomorphisms between compacta of an infinite-dimensional locally convex metric linear space extend to the whole space. A lemma used in the proof of this fact together with the method of absorbing sets is employed to show that every σ-compact normed linear space is homeomorphic to a dense linear subspace of a Hilbert space. A discussion of the relative topological equivalence of absorbing sets in noncomplete spaces is included. The paper is concluded with some controlled versions of an isotopy extension theorem.

INTRODUCTION

The first part of the paper is devoted to a question of whether homeomorphisms between two sets K and L of a space X extend to the whole X. In the second part, we apply our results concerning this question to show the relative topological equivalence of certain σ-compact linear spaces in their completions. From this point of view, the most important case arises if K and L are compact and X is a metric linear space. The first significant contribution in answering the question was a result of Klee [14]. It states that infinite-dimensional Hilbert spaces have the homeomorphism extension property for compacta, i.e., every homeomorphism between compact of an infinite-dimensional Hilbert space extends to the whole space. Since then the homeomorphism extension property has become central in investigations and applications of infinite-dimensional topology. Our objective is to establish the homeomorphism extension property for certain spaces X without completeness.

The method we use goes back to the following "open action principle": if an action of a complete group on a complete space is almost open (i.e., closures of images of all open sets have nonempty interiors) then the action is open and transitive provided each of its orbits is dense. (Conversely, transitive actions

Received by the editors March 29, 1985 and, in revised form, February 19, 1988. Partially presented in the Special Session on Infinite Dimensional Topology at the annual meeting of the American Mathematical Society in Anaheim, California, January 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57N17; Secondary 54C20, 46A05.

Key words and phrases. Homeomorphism extension property, almost extension property, locally convex linear space, skeleton sets.

of separable metric groups are almost open on complete spaces.) The fact was discovered and communicated to the author by H. Toruńczyk (unpublished). It is an abstraction on the convergence procedure developed successively by Homma [13], Gluck [12], Bryant [5] and Štanko [16] which has its ancestor in the classical proof of the Banach open mapping principle due to Schauder. In our setting, we consider the natural action $c: H(X) \times E(K, X) \to E(K, X)$ of a group $H(X)$ of all homeomorphisms of X on the space $E(K, X)$ of all embeddings of K into X. To express the fact that c is almost open and has each orbit dense we introduce the notion of the almost extension property. It requires, for every map $f: K \to X$ with $d(f, \mathrm{id}) < \mathrm{dist}(K, X\backslash U)$, where U is any neighborhood of K, and every $\varepsilon > 0$, the existence of $h \in H(X)$ with $d(h|K, f) < \varepsilon$, $d(h, \mathrm{id}) < d(f, \mathrm{id}) + \varepsilon$ and $h|X\backslash U = \mathrm{id}$. Clearly, if X is an infinite-dimensional linear space, $H(X)$ is not metrizable and the result on transitive actions is not directly applicable. However, under certain assumptions on X, a procedure of a proof of the above mentioned result can be adopted to show that c is transitive. This yields the homeomorphism extension property of X. In the case of noncomplete X, one can consider its completion $\check{X}$ and require additionally that h of the almost extension property be a homeomorphism of $\check{X}$ preserving X. With this extra hypothesis we are able to extend a homeomorphism between K and L of X to a homeomorphism of $\check{X}$ preserving X. Summarizing, a verification of the almost extension property is crucial in order to obtain the homeomorphism extension property. We have done this in the case of locally convex metric linear spaces X. The main result on extending homeomorphisms is that every homeomorphism between compacta K and L of an infinite-dimensional locally convex metric linear space X extends to a homeomorphism h of the completion $\check{X}$ such that h preserves X. A smooth variation of this fact says that if additionally X is a pre-Hilbert space, then the homeomorphism h may be taken as a diffeomorphism of $\check{X}\backslash K$ onto $\check{X}\backslash L$.

In the application part of the paper we mostly deal with infinite-dimensional σ-compact linear spaces. We show that in such spaces, the homeomorphism extension property for the class of all Z-sets is true. Another application is an attempt to describe topological types of σ-compact linear spaces that are universal for a class of compacta. We prove that if locally convex metric linear spaces E_1 and E_2 are universal for a family of compacta $\mathscr{C}$ and E_1 and E_2 are countable unions of elements of $\mathscr{C}$, then they are homeomorphic. This generalizes the fact that maximal (resp. minimal) types of σ-compact linear spaces, i.e. universal for all compacta (resp., all finite-dimensional compacta), are homeomorphic, see [1, 3, 17 and 10]. As a corollary, we obtain the result that every infinite-dimensional locally convex metric linear space (resp., every infinite-dimensional normed linear space) which is σ-compact is homeomorphic to a dense linear subspace of the countable product of lines (resp., of a Hilbert space).

The next part of the paper deals with the problem of the relative topological equivalence of skeletoids in the sense of Bessaga-Pełczyński [3] (equivalently:

absorbing sets in the sense of Anderson [1]) in noncomplete spaces X. In contrast to the case of complete X, we provide simple examples of homeomorphic skeletoids in standard σ-compact normed linear spaces X which are not homeomorphic via a homeomorphism of the whole X. We have shown that, in certain instances, the theorem on the relative topological equivalence of skeletoids in noncomplete spaces holds true. For example, if a σ-compact locally convex metric linear space E contains a Hilbert cube, then any σ-compact linear spaces E_1 and E_2 dense in E, which do not contain a Hilbert cube and are homeomorphic, are also homeomorphic via a homeomorphism of E, provided each $E \backslash E_i$ is σ-compact. (The spaces E_1 and E_2 are skeletoids for the class of all compacta which are embeddable in E_1 and E_2, respectively.) The basic method of constructing homeomorphisms in the application part is the standard method of skeletoids [3] (or absorbing sets [1]). Consequently, all homeomorphisms which we construct can be extended to the completions of spaces which we deal with.

In the last section of the paper we prove controlled versions of an isotopy extension theorem for infinite-dimensional locally convex metric linear spaces. It enables us to show that such spaces have the homeomorphism extension property for local compacta.

I am grateful to the referee for the suggestion to add a section on the controlled isotopy extension theorems. I also thank H. Toruńczyk for substantial discussions during the preparation of the final version of the paper.

1. Extending homeomorphisms. An abstract scheme

Let (X, d) be a metric space. Let us recall that X is a neighborhood extensor for a class $\mathscr{A}$ of closed subsets of X if

NE($\mathscr{A}$): for every $A \in \mathscr{A}$ and every map $f: A \to X$, there exists a map $\bar{f}$ which is defined on a neighborhood of A and which extends f.

Throughout this paper we will deal with a question of whether a homeomorphism $h: A \to B$ between subsets of X extends to a homeomorphism of the whole X. Let G be a subgroup of the group $H(X)$ of all homeomorphisms of X. The space X is said to have the $G - \mathscr{A}$ extension property if every homeomorphism between subsets of $\mathscr{A}$ extends to a homeomorphism $h \in G$, see [4, p. 124]. It means that, for every $A \in \mathscr{A}$, the restriction map $h \to h|A$, $h \in G$, is onto. If the restriction map is an open map in the limitation topologies [20] on suitable function spaces, then X is said to have the $G - \mathscr{A}$ estimated extension property, see [4]. We will be interested in the following version of the extension property for a class $\mathscr{K}$ of compacta of X. We say that X has the $G - \mathscr{K}$ almost extension property if

AEP($G - \mathscr{K}$): for every $K \in \mathscr{K}$, its neighborhood U in X, a map $f: K \to X$ with $d(f, \mathrm{id}) < \operatorname{dist}(K, X \backslash U)$ and every $\varepsilon > 0$, there exists a homeomorphism $h \in G$ such that $d(h|K, f) < \varepsilon$, $d(h, \mathrm{id}) < d(f, \mathrm{id}) + \varepsilon$ and $h|X \backslash U = \mathrm{id}$.

The above property is an estimated variation of the almost openness of the restriction map $h \to h|K$.

Given a subfamily $\mathscr{F}$ of the class $\mathscr{K}$, we say that X has the $G-\mathscr{K}-\mathscr{F}$ limit flattening property if

LFP($G-\mathscr{K}-\mathscr{F}$): for every $K \in \mathscr{K}$, its neighborhood U in X and every $\delta > 0$, there exists a sequence $\{\phi_n\}$ of homeomorphisms of X such that the uniform limit $\lim \phi_n|K = \phi_\infty$ is a map with $\phi_\infty(K) \in \mathscr{F}$ and, for every n, $d(\phi_n, \mathrm{id}) < \delta$ and $\phi_n|K\backslash U = \mathrm{id}$.

In infinite-dimensional topology the idea to "flatten" compact sets goes back to Klee [14], who observed that a compact subset of an infinite-dimensional Hilbert space can be considered as (a subset) of the graph of some map. Then the graph can be easily transformed via a homeomorphism of the whole space into a linear subspace of a nontrivial defect. This process is referred to as flattening a compact set. In our setting, the condition LFP($G-\mathscr{K}-\mathscr{F}$) guarantees flattening of each element of $\mathscr{K}$ to an element of $\mathscr{F}$.

Our main objective of this section is to establish the $G-\mathscr{K}$ extension property of a space X with the $G-\mathscr{K}$ almost extension property. Consequently, the importance of detecting the $G-\mathscr{K}$ almost extension property arises. In many instances we find it easier to show AEP($G-\mathscr{F}$) for some subclass $\mathscr{F}$ of $\mathscr{K}$. It turns out that the $G-\mathscr{K}-\mathscr{F}$ limit flattening property reduces verifying AEP($G-\mathscr{K}$) to verifying AEP($G-\mathscr{F}$) together with NE($\mathscr{K}$). Our second goal of this section is to show this reduction.

We will start by noting the following remarks.

1.1. *Remark.* If X satisfies AEP($G-\mathscr{K}$), then for every $K \in \mathscr{K}$, a map $f: K \to X$ can be arbitrarily closely approximating by restrictions $h|K, h \in G$. (We simply agree $\mathrm{dist}(K, \varnothing) = \infty$.)

1.2. *Remark.* The condition AEP($G-\mathscr{K}$) implies the following:

(*) for every $K \in \mathscr{K}$, its neighborhood U and a map $\varepsilon: X \to (0, \infty)$, there exists α such that for every map $f: K \to X$ with $d(f, \mathrm{id}) < \alpha$ can be arbitrarily closely approximated by restrictions $h|K$ with $h \in G, h|X\backslash U = \mathrm{id}$ and $d(h(x), x) < \varepsilon(x)$.

The main result of this section reads as follows:

1.3. **Theorem.** *Let (X, d) be a metric space. Suppose X has the $G-\mathscr{K}$ almost extension property for some class $\mathscr{K}$ of compacta of X and some subgroup G of $H(X)$. Then, every homeomorphism $h: A_1 \to A_2$, where $A_1, A_2 \in \mathscr{K}$, extends to a homeomorphism $\bar{h} \in G$. Moreover, $\bar{h}$ can be chosen so that, locally, $\bar{h}|X\backslash A_1$ and $(\bar{h})^{-1}|X\backslash A_2$ are finite compositions of elements of G.*

Proof. We shall inductively find two sequences $\{h_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ of G such that the sequences $\{(g_{n+1} \circ \cdots \circ g_1 \circ g_0)^{-1} \circ h_n \circ \cdots \circ h_1 \circ h_0\}$ and $\{(h_n \circ \cdots \circ h_1 \circ h_0)^{-1} \circ g_{n+1} \circ \cdots \circ g_1 \circ g_0\}$ are uniformly Cauchy with respect to the metric d and stabilize outside A_1 and A_2, respectively. Moreover, we shall ensure that $\{(g_{n+1} \circ \cdots \circ g_1 \circ g_0)^{-1} \circ (h_n \circ \cdots \circ h_1 \circ h_0)|A_1\}$ converges to h and $\{(h_n \circ \cdots \circ h_1 \circ h_0)^{-1} \circ (g_{n+1} \circ \cdots \circ g_1 \circ g_0)|A_2\}$ converges to h^{-1}. We shall finally set

$$\bar{h}(x) = \lim(g_{n+1} \circ \cdots \circ g_1 \circ g_0)^{-1} \circ h_n \circ \cdots \circ h_1 \circ h_0(x), \qquad x \in X.$$

Write

$$F_k^1 = \{x \in X: \operatorname{dist}(x, A_1) \geq 2^{-k}\}, \qquad F_k^2 = \{x \in X: \operatorname{dist}(x, A_2) \geq 2^{-k}\},$$

$$G_k^1 = \{x \in X: \operatorname{dist}(x, h_k \circ \cdots \circ h_1 \circ h_0(A_1)) \geq 2^{-k}\}$$

and

$$G_k^2 = \{x \in X: \operatorname{dist}(x, g_k \circ \cdots \circ g_1 \circ g_0(A_2)) \geq 2^{-k}\},$$

for $k = 1, 2, \ldots$, and let $F_0^1 = F_0^2 = G_0^1 = G_0^2 = \varnothing$. To unify the inductive construction we fix the following notation. Let $\Phi_n = h_k \circ \cdots \circ h_1 \circ h_0$, $F_n = F_k^1$ and $F_n' = G_k^1$, for $n = 2k$, and let $\Phi_n = g_k \circ \cdots \circ g_1 \circ g_0$, $F_n = F_k^2$ and $F_n' = G_k^2$, for $n = 2k-1$, $k = 0, 1, \ldots$. We also write $u_1 = h: A_1 \to A_2$ and $u_{-1} = \mathrm{id}: A_1 \to A_1$. With this notation we shall construct a sequence $\{\Phi_n\}_{n=-1}^{\infty}$ of G to satisfy

$$d(\Phi_{n-1}^{-1} \circ \Phi_n \circ u_{(-1)^{n-1}}, u_{(-1)^n}) < 2^{-n}$$

and

$$d(\Phi_n \circ u_{(-1)^{n-1}}, \Phi_{n-1} \circ u_{(-1)^n}) < \alpha_n,$$

where α_n is that of $(*)$ (of 1.2) applied with

$$K = \Phi_{n-1} \circ u_{(-1)^n}(A_1), \qquad U = X \backslash (\Phi_{n-1}(F_{n-1}) \cup F_{n-1}')$$

and

(1) $$\varepsilon = \varepsilon_{n-1}: X \to (0, \infty)$$

such that $g \in H(X)$, $d(g(x), x) < \varepsilon(x)$ imply

$$d^*(g \circ \Phi_{n-1}, \Phi_{n-1}) = d(g \circ \Phi_{n-1}, \Phi_{n-1}) + d((g \circ \Phi_{n-1})^{-1}, \Phi_{n-1}^{-1}) < 2^{-n}$$

and

$$d(\Phi_{n-2}^{-1} \circ \Phi_{n-1}, \Phi_{n-2}^{-1} \circ g \circ \Phi_{n-1}) < 2^{-n},$$

for $n = 0, 1, 2, \ldots (\Phi_{-2} = \mathrm{id})$.

(2) $$\Phi_n | F_{n-2} \cap \Phi_{n-2}^{-1}(F_{n-2}') = \Phi_{n-2} | F_{n-2} \cap \Phi_{n-2}^{-1}(F_{n-2}),$$

for $n = 1, 2, \ldots$,

(3) $$d^*(\Phi_n, \Phi_{n-2}) = d(\Phi_n, \Phi_{n-2}) + d(\Phi_n^{-1}, \Phi_{n-2}^{-1}) < 2^{-n+1},$$

for $n = 1, 2, \ldots$,

(4) $$d(\Phi_{n-3}^{-1}\circ\Phi_{n-2},\Phi_{n-3}^{-1}\circ\Phi_n)<2^{-n+1},\quad \text{for } n=2,3,\dots.$$

Inductive construction. Set $\Phi_{-1}=\mathrm{id}$ and let $n=0$. Pick α_0 of the condition $(*)$ applied to $K=\Phi_{-1}\circ u_1(A_1)=h(A_1)$, $U=X$ and $\varepsilon_{-1}(x)\equiv\frac{1}{2}$. By Remark 1.1, there exists $\Phi\in G$ such that $d(\Phi|A_1,h)<\alpha_0$. Letting $\Phi_0=\Phi$, we see that the condition (1) is satisfied for $n=0$. Suppose that $\Phi_{-1},\Phi_0,\dots,\Phi_{n-1}$ has been constructed for $n\geq 1$. Let α_n be that of $(*)$ applied with $\Phi_{n-1}\circ u_{(-1)^n}(A_1)$, $U=X\backslash(\Phi_{n-1}(F_{n-1})\cup F'_{n-1})$ and $\varepsilon=\varepsilon_{n-1}\colon X\to(0,\infty)$ such that $g\in H(X)$, $d(g(x),x)<\varepsilon(x)$ imply $d^*(g\circ\Phi_{n-1},\Phi_{n-1})<2^{-n}$ and $d(\Phi_{n-2}^{-1}\circ\Phi_{n-1},\Phi_{n-2}^{-1}\circ g\circ\Phi_{n-1})<2^{-n}$. We assume that α_n is so small to fulfill the condition: $a\in\Phi_{n-1}\circ u_{(-1)^n}(A_1)$, $d(x,a)<\alpha_n$ imply $d(\Phi_{n-1}^{-1}(x),\Phi_{n-1}^{-1}(a))<2^{-n}$. Writing

$$K_{n-1}=\Phi_{n-2}\circ u_{(-1)^{n-1}}(A_1)\quad\text{and}$$
$$u_{n-1}=\Phi_{n-1}\circ u_{(-1)^{n-2}}\circ(\Phi_{n-2}\circ u_{(-1)^{n-1}})^{-1}|K_{n-1},$$

according to (1), we have $d(u_{n-1},\mathrm{id})<\alpha_{n-1}$. We also have $K_{n-1}\cap\Phi_{n-2}(F_{n-2})\cap F'_{n-2}=\varnothing$. Thus by the property of α_{n-1}, there exists a homeomorphism $g\in G$ such that

(a) $d(g|K_{n-1},u_{n-1})<\alpha_n$,
(b) $g|\Phi_{n-2}(F_{n-2})\cup F'_{n-2}=\mathrm{id}$,
(c) $d(g(x),x)<\varepsilon_{n-2}(x)$.

We set $\Phi_n=g\circ\Phi_{n-2}$. The condition (a) yields

$$d(\Phi_n\circ u_{(-1)^{n-1}},\Phi_{n-1}\circ u_{(-1)^n})<\alpha_n;$$

hence, by the choice of α_n,

$$\begin{aligned}d(\Phi_{n-1}^{-1}&\circ\Phi_n\circ u_{(-1)^{n-1}},\Phi_{n-1}^{-1}\circ\Phi_{n-1}\circ u_{(-1)^n})\\&=d(\Phi_{n-1}^{-1}\circ\Phi_n\circ u_{(-1)^{n-1}},u_{(-1)^n})<2^{-n}.\end{aligned}$$

This, together with (b), shows that Φ_n satisfies (1) and (2). By the choice of ε_{n-2}, Φ_n also satisfies (3) and (4) (for $n\geq 2$). The inductive construction is complete.

We shall show that the sequences $\{\Phi_{2k+1}^{-1}\circ\Phi_{2k}(x)\}$ and $\{\Phi_{2k}^{-1}\circ\Phi_{2k+1}(x)\}$ converge d-uniformly in X with respect to $x\in X$. Clearly then,

$$\lim(\Phi_{2k+1}^{-1}\circ\Phi_{2k})\quad\text{and}\quad\lim(\Phi_{2k}^{-1}\circ\Phi_{2k+1})$$

are inverse maps of one another; and consequently $\bar{h}=\lim(\Phi_{2k+1}^{-1}\circ\Phi_{2k})$ is a homeomorphism of X, see [4, p. 121]. Moreover, from the condition (1), we get $d(\Phi_{2k}^{-1}\circ\Phi_{2k+1}\circ h,\mathrm{id})<2^{-2k-1}$. This yields $\bar{h}|A_1=h$. A strategy to show that uniform limits of $\{\Phi_{2k+1}^{-1}\circ\Phi_{2k}\}$ and $\{\Phi_{2k}^{-1}\circ\Phi_{2k+1}\}$ are maps of X into X is to prove that the sequences: (i) are uniformly Cauchy, (ii) locally stabilize outside A_1 and A_2, and (iii) converge to h and h^{-1} on A_1 and A_2, respectively. Using (3) and (4), we estimate

$$\begin{aligned}
d(\Phi_{2k+1}^{-1}\circ\Phi_{2k}, \Phi_{2k+3}^{-1}\circ\Phi_{2k+2}) &\le d(\Phi_{2k+1}^{-1}\circ\Phi_{2k}, \Phi_{2k-1}^{-1}\circ\Phi_{2k}) \\
&\quad + d(\Phi_{2k-1}^{-1}\circ\Phi_{2k}, \Phi_{2k-1}^{-1}\circ\Phi_{2k+2}) + d(\Phi_{2k-1}^{-1}\circ\Phi_{2k+2}, \Phi_{2k+3}^{-1}\circ\Phi_{2k+2}) \\
&= d(\Phi_{2k+1}^{-1}, \Phi_{2k-1}^{-1}) + d(\Phi_{2k-1}^{-1}\circ\Phi_{2k}, \Phi_{2k-1}^{-1}\circ\Phi_{2k+2}) + d(\Phi_{2k-1}^{-1}, \Phi_{2k+3}^{-1}) \\
&\le 2^{-2k} + 2^{-2k-1} + d(\Phi_{2k-1}^{-1}, \Phi_{2k+1}^{-1}) + d(\Phi_{2k+1}^{-1}, \Phi_{2k+3}^{-1}) \\
&< 2^{-2k} + 2^{-2k-1} + 2^{-2k} + 2^{-2k-2} < 5\cdot 2^{-2k}.
\end{aligned}$$

This shows that $\{\Phi_{2k+1}^{-1}\circ\Phi_{2k}(x)\}$ is a Cauchy sequence uniformly with respect to $x\in X$. To conclude that $\{\Phi_{2k}^{-1}\circ\Phi_{2k+1}(x)\}$ is a Cauchy sequence we similarly observe, via (3) and (4), that

$$\begin{aligned}
&d(\Phi_{2k}^{-1}\circ\Phi_{2k+1}, \Phi_{2k+2}^{-1}\circ\Phi_{2k+3}) \\
&\quad\le d(\Phi_{2k}^{-1}\circ\Phi_{2k+1}, \Phi_{2k}^{-1}\circ\Phi_{2k+3}) + d(\Phi_{2k}^{-1}\circ\Phi_{2k+3}, \Phi_{2k+2}^{-1}\circ\Phi_{2k+3}) \\
&\quad\le 2^{-2k-2} + d(\Phi_{2k}^{-1}, \Phi_{2k+2}^{-1}) \le 2^{-2k-2} + 2^{-2k-1} < 2^{-2k}.
\end{aligned}$$

It was already noticed that $\{\Phi_{2k+1}^{-1}\circ\Phi_{2k}|A_2\}$ converges uniformly to h^{-1}. We have

$$\begin{aligned}
d(\Phi_{2k+1}^{-1}\circ\Phi_{2k}|A_1, h) \le\ & d(\Phi_{2k+1}^{-1}\circ\Phi_{2k}, \Phi_{2k+3}^{-1}\circ\Phi_{2k+2}) \\
&+ d(\Phi_{2k+3}^{-1}\circ\Phi_{2k+2}, \Phi_{2k+1}^{-1}\circ\Phi_{2k+2}) \\
&+ d(\Phi_{2k+1}^{-1}\circ\Phi_{2k+2}|A_1, h).
\end{aligned}$$

The above estimate together with (1) yields the convergence of $\{\Phi_{2k+1}^{-1}\circ\Phi_{2k}|A_1\}$ to h.

Thus, it remains to show that $\{\Phi_{2k+1}^{-1}\circ\Phi_{2k}\}$ and $\{\Phi_{2k}^{-1}\circ\Phi_{2k+1}\}$ locally stabilize off A_1 and A_2, respectively. Pick $x_0\notin A_1$. By (2), there exists a neighborhood U_0 of x_0 with $U_0\cap A_1=\varnothing$, and n_0 such that $h_n\circ\cdots\circ h_{n_0}\circ\cdots\circ h_1\circ h_0|U_0 = h_{n_0}\circ\cdots\circ h_1\circ h_0|U_0$, we claim that there exists a neighborhood $V_0\subset h_{n_0}\circ\cdots\circ h_1\circ h_0(U_0)$ of $y_0=h_{n_0}\circ\cdots\circ h_1\circ h_0(x_0)$ such that $V_0\subset G_k^2$ for almost all k. Otherwise, there would exist a sequence $\{y_{k_l}\}\subset h_{n_0}\circ\cdots\circ h_1\circ h_0(U_0)$ convergent to y_0 and such that $d(y_{k_l}, g_{k_l}\circ\cdots\circ g_1\circ g_0(a_{k_l})) < 2^{-k_l}$ for some $a_{k_l}\in A_2$. By the compactness of A_2 we would assume that $\{a_{k_l}\}$ converges to some $a\in A_2$. Consequently, the sequence $\{g_{k_l}\circ\cdots\circ g_1\circ g_0(a_{k_l})\}$ converges to y_0. Therefore for k_l large, we have

$$\begin{aligned}
&(h_{k_l-1}\circ\cdots\circ h_1\circ h_0)^{-1}\circ(g_{k_l}\circ\cdots\circ g_1\circ g_0)(a_{k_l}) \\
&\qquad = (h_{n_0}\circ\cdots\circ h_1\circ h_0)^{-1}\circ(g_{k_l}\circ\cdots\circ g_1\circ g_0)(a_{k_l});
\end{aligned}$$

the latter sequence convergent to x_0. On the other hand

$$\{(h_{k_l-1}\circ\cdots\circ h_1\circ h_0)^{-1}\circ(g_{k_l}\circ\cdots\circ g_1\circ g_0)(a_{k_l})\}$$

converges to $h^{-1}(a)$, contradicting the fact that $x_0 \notin A_2$. Thus, for $k \geq k_0$, we have $V_0 \subset h_{n_0} \circ \cdots \circ h_1 \circ h_0(U_0) \cap G_k^2$ and, by virtue of (2), $g_k | V_0 = \mathrm{id}$ for $k \geq k_0$. Hence, for every $y \in V_0$ and $k \geq k_0 + 1$, we have $g_k \circ \cdots \circ g_{k_0+1} \circ g_{k_0}(y) = y$; so letting $z = (g_{k_0-1} \circ \cdots \circ g_0)^{-1}(y)$, we get $g_k \circ \cdots \circ g_1 \circ g_0(z) = y$. Thus, for $y = h_{k_0} \circ \cdots \circ h_1 \circ h_0(x) \in V_0$, $x \in U_0$, we obtain

$$\begin{aligned} z &= (g_{k_0+1} \circ \cdots \circ g_1 \circ g_0)^{-1} h_{k_0} \circ \cdots \circ h_1 \circ h_0(x) \\ &= (g_{k+1} \circ \cdots \circ g_1 \circ g_0)^{-1} \circ h_k \circ \cdots \circ h_1 \circ h_0(x) \end{aligned}$$

provided $k_0 \geq n_0$. The same argument shows that $\{(h_k \circ \cdots \circ h_1 \circ h_0)^{-1} \circ g_{k+1} \circ \cdots \circ g_1 \circ g_0\}$ locally stabilizes outside A_2. From this it follows that, locally $\bar{h}$ and $(\bar{h})^{-1}$ are finite compositions of elements of G on $X \setminus A_1$ and $X \setminus A_2$, respectively. The proof is complete.

In the case where (X, d) is a complete metric space the above proof can be simplified. In this case the uniform limits of $\{h_n \circ \cdots \circ h_1 \circ h_0\}$ and $\{g_n \circ \cdots \circ g_1 \circ g_0\}$ will be homeomorphisms of X and $\lim g_n \circ \cdots \circ g_1 \circ g_0(h(a)) = \lim h_n \circ \cdots \circ h_1 \circ h_0(a)$ for $a \in A_1$. Hence $(\lim g_n \circ \cdots \circ g_1 \circ g_0)^{-1} \circ \lim h_n \circ \cdots \circ h_1 \circ h_0$ is a required homeomorphism $\bar{h}$. To show that $\lim(h_n \circ \cdots \circ h_1 \circ h_0)$ and $\lim(g_n \circ \cdots \circ g_1 \circ g_0)$ are homeomorphisms we do not need the stabilizing condition (2). Hence the extension $\bar{h} \in H(X)$ exists even if the stabilizing condition of 1.2 is dropped. However to construct homeomorphisms $h \in H(X)$ with $d(h(x), x) < \varepsilon(x)$ for arbitrary $\varepsilon : X \to (0, \infty)$ and noncompact X we usually require h to be the identity off some open subset of X.

1.4. *Note.* Suppose X_0 is a subspace of X such that $g(X_0) = X_0$ for every $g \in G$ (i.e., X_0 is G-invariant). Then, in 1.3 we additionally obtain that the pairs $(X_0, X_0 \setminus A_1)$ and $((X_0 \setminus A_2) \cup h(A_1 \cap X_0), X_0 \setminus A_2)$ are homeomorphic. Moreover, by the last part of 1.3, $\bar{h} | X_0 \setminus A_1$ and $(\bar{h})^{-1} | X_0 \setminus A_2$ are locally finite compositions of $g | X_0$ with $g \in G$. Consequently, if $A_1, A_2 \subset X_0$ then $\bar{h} | X_0$ is a homeomorphism of X_0 extending h.

1.5. **Theorem.** *Let (X, d) be a metric space, G be a subgroup of $H(X)$, $\mathscr{K}$ a class of compacta of X and $\mathscr{F}$ its subclass. Assume X has the G–$\mathscr{K}$–$\mathscr{F}$ limit flattening property and is a neighborhood extensor for $\mathscr{K}$. If X has the $G - \mathscr{F}$ almost extension property, then it has the $G - \mathscr{K}$ almost extension property.*

Proof. Let $K \in \mathscr{K}$, its neighborhood U and $\varepsilon > 0$ be given. Consider a map $f : K \to X$ such that $d(f, \mathrm{id}) < d_0 = \mathrm{dist}(K, X \setminus U)$. We may assume that f extends to $\bar{f}$ over U. There exists $\delta > 0$ with $\delta < \varepsilon/4$ and such that for $k \in K$, $d(x, k) \leq \delta$ implies $d(\bar{f}(x), f(k)) < \varepsilon/4$. With this δ, pick a sequence $\{\varphi_n\}$ given by the condition $\mathrm{LFP}(G - \mathscr{K} - \mathscr{F})$. Assume

$\varepsilon < d_0 - d(f, \mathrm{id})$ and write $f' = \bar{f}|F$, where $F = \varphi_\infty(K) \in \mathscr{F}$. We have

$$\begin{aligned} d(f', \mathrm{id}) &\le d(\bar{f}\varphi_\infty, f) + d(f, \mathrm{id}) + d(\mathrm{id}, \varphi_\infty) \\ &\le \varepsilon/4 + d(f, \mathrm{id}) + \varepsilon/4 < d_0 - \varepsilon/4 \\ &< \mathrm{dist}(F, X\backslash U). \end{aligned}$$

Now, the condition $\mathrm{AEP}(G{-}\mathscr{F})$ is applicable with (F, U, f'). So, there exists $h' \in G, h'|X\backslash U = \mathrm{id}$ and $d(h', \mathrm{id}) < d(f', \mathrm{id}) + \varepsilon/4$. We shall find a required homeomorphism h of $\mathrm{AEP}(G-\mathscr{K})$ in the form $h = h_n = h' \circ \varphi_n$. First of all observe that

$$\begin{aligned} d(h_n, \mathrm{id}) = d(h'\varphi_n, \mathrm{id}) &\le d(h'\varphi_n, \varphi_n) + d(\varphi_n, \mathrm{id}) \\ &< d(h', \mathrm{id}) + \varepsilon/4 < d(f', \mathrm{id}) + \varepsilon/2 \\ &\le d(f, \mathrm{id}) + \varepsilon/2 + \varepsilon/2 = d(f, \mathrm{id}) + \varepsilon. \end{aligned}$$

Next, by the uniform convergence of $\{\varphi_n|K\}$ there exists n_0 such that for every $n \ge n_0$, $d(h'\varphi_n|K, h'\varphi_\infty) < \varepsilon/4$. This enables us to estimate for $n \ge n_0$

$$\begin{aligned} d(h_n|K, f) &= d(h'\varphi_n|K, f) \\ &\le d(h'\varphi_n|K, h'\varphi_\infty) + d(h'|K, f') + d(\bar{f}\varphi_\infty, f) \\ &\le \varepsilon/4 + \varepsilon/4 + \varepsilon/4 < \varepsilon. \end{aligned}$$

Since both h' and φ_{n_0} are the identity outside U, h_{n_0} also is. We complete the proof, letting $h = h_{n_0}$.

1.6. *Note.* An inspection of the proof of 1.6 yields the following observation: Assume $K \in \mathscr{K}$ and its neighborhood U satisfy $\mathrm{LFP}(G{-}\mathscr{K}{-}\mathscr{F})$, and suppose every map K into X admits a neighborhood extension. Then, if for every $F \in \mathscr{F}$ with $F \subset U$, mapping $f'\colon F \to U$, and $\varepsilon > 0$ one can find $h' \in G$ with $d(h'|F, f') < \varepsilon$, $h'|X\backslash U = \mathrm{id}$, then for every $f\colon K \to U$ and $\varepsilon > 0$ one can find $h \in G$ with $d(h|K, f) < \varepsilon$, $h|X\backslash U = \mathrm{id}$ and $d(h, \mathrm{id}) < d(f, \mathrm{id}) + \varepsilon$.

2. Applications—The topological case

In our applications of the results 1.3 and 1.5 X will be a metric linear space (briefly: m.l.s.). In the case where X is a locally convex metric linear space (briefly: l.c.m.l.s.) we are able to show the homeomorphism extension property for compacta. We have done very little in the case where the local convexity is dropped. In the applications, $\mathscr{K}$ will be the class of all compacta of X and $\mathscr{F}$ will be the class of all compacta contained in finite-dimensional *linear* subspaces of X. We also use the class $\mathscr{F}d$ of all finite-dimensional compacta of X. Note that $\mathscr{F} \subset \mathscr{F}d \subset \mathscr{K}$. We always consider X with a metric d induced by an F-norm; hence, d is translation invariant, and $d(tx, 0) \le d(x, 0)$ for every $x \in X$ and $|t| \le 1$. If X_0 is a dense linear subspace of X, then by $H(X|X_0)$ we denote the subgroup of $H(X)$ of homeomorphisms preserving *every* linear subspace of X containing X_0.

We start with

2.1. **Lemma.** *Let X be an infinite-dimensional m.l.s. and X_0 its dense linear subspace. Then, X has the $G - \mathscr{F}$ almost extension property, where $G = H(X|X_0)$ and $\mathscr{F}$ is the class of all compacta contained in linear finite-dimensional subspaces of X_0.*

Proof. Let $F \in \mathscr{F}$, its neighborhood U and $f: F \to X$ with $d(f, \mathrm{id}) < d_0 = \mathrm{dist}(F, X\backslash U)$ be given. First, we easily show that f can be arbitrarily closely approximated by embeddings $u: F \to X_0$ with $u(F) \in \mathscr{F}$. Approximate f by f' such that $\mathrm{span}(\mathrm{im}\, f') = X' \subset X_0$. Then, since X_0 is infinite-dimensional, there is a finite-dimensional linear subspace X'' of X_0 admitting embeddings $v: F \to X''$ with $d(v, 0)$ as small as we wish and such that $X'' \cap X' = \{0\}$. It is clear that u can be taken as $f' + v$.

Consider a finite-dimensional linear subspace X_1 of X_0 with $X' + X'' \subset X_1$ and $\dim(X_1) \geq 2\dim(X' + X'') + 3$. If u is close to f, then $d(u, \mathrm{id}) < \mathrm{dist}(F, X\backslash U)$ and $u_t(x) = (1-t)x + tu(x)$, $0 \leq t \leq 1$, joins id_F with u in $X_1 \cap U$. The space X_1 has the estimated homeomorphism extension property for compacta $A \subset X_1$ with $\dim(\mathrm{span}\, A) \leq \dim(X' + X'') + 1$. Therefore, given $\varepsilon > 0$ there is an isotopy (Φ_t) of X_1 such that $\Phi_0 = \mathrm{id}$, $d(\Phi_1|F, u) < \varepsilon$, $d(\Phi_t, \mathrm{id}) < d(u, \mathrm{id}) + \varepsilon$ and $\Phi_t|X\backslash U = \mathrm{id}$ for all $0 \leq t \leq 1$.

We shall extend Φ_1 to a homeomorphism of X preserving every linear subspace containing X_0. By a result of Michael [15], there exists a continuous right inverse $\alpha: X/X_1 \to X$ for the quotient map $\kappa: X \to X/X_1$ such that $\alpha(0) = 0$. Moreover, the formula $h_0(x) = (\kappa(x), x - \alpha\kappa(x))$ defines a homeomorphism of X onto $(X/X_1) \times X_1$ transforming X_1 identically onto $\{0\} \times X_1$. Also, we have $h_0(Y) = (Y/X_1) \times X_1$ for every linear subspace Y of X with $Y \supset X_1$. Consider neighborhoods V of the origin in X/X_1 and U' of $\mathrm{im}(u_t)$ in X_1 such that $V \times U' \subset h_0(U)$. Pick a map $\lambda: X/X_1 \to [0,1]$ such that $\lambda(0) = 1$ and $\lambda(y) = 0$ for $y \notin V$. For $(y, x) \in (X/X_1) \times X_1$, write $\Phi(y, x) = (y, \Phi_{\lambda(y)}(x))$. Clearly, Φ is a homeomorphism extending $h_0 u h_0^{-1}$ and Φ is the identity outside $h_0(U)$. Consequently, $h = h_0^{-1}\Phi h_0$ extends u and is the identity outside U. We complete the proof by noting that $d(h, \mathrm{id}) < d(u, \mathrm{id}) + \varepsilon$ and that $h \in H(X|X_0)$.

2.2. *Note.* Under the assumptions of 2.1 one can fulfill the condition $\mathrm{AEP}(G - \mathscr{K})$ with the requirement $d(f, \mathrm{id}) < \mathrm{dist}(K, X\backslash U)$ dropped, provided U is the complement of a compact subset L of X. This follows from the fact that L is a Z-set in X. Consequently, there exists a homotopy $(f_t): F \to U = X\backslash K$ with $f_0 = \mathrm{id}$ and f_1 as close to f as we wish. Next, we approximate (f_t) by embeddings $u: F \times [0,1] \to X\backslash K$ with $u_0 = \mathrm{id}$ and $\mathrm{im}(u) \in \mathscr{F}$. Now we may follow the argument of the proof of 2.1.

2.3. **Lemma.** *Let E be an infinite-dimensional l.c.m.l.s. and E_0 its dense linear subspace. Then, E has the $H(E|E_0) - \mathscr{K} - \mathscr{F}$ limit flattening property,* *where $\mathscr{K}$ is the class of all compacta of E and $\mathscr{F}$ is the class of all compacta contained in finite-dimensional linear subspaces of E_0.*

Proof. Fix $K \in \mathscr{K}$, its neighborhood U and $\delta > 0$. We may assume that $\delta < \operatorname{dist}(K, E\backslash U)$. Pick a finite-dimensional linear subspace E_1 of E_0 with $K \subset E_1 + \{x \in E : d(x, 0) < \delta/4\}$. Consider a continuous right inverse α for the quotient map $\kappa: E \to E/E_1$ [15]. By the local convexity we may additionally require $d(\alpha(y), 0) \leq 2\overline{d}(y, 0)$ for all $y \in E/E_1$, where $\overline{d}$ is the quotient metric of d. Write $h_0(x) = (\kappa(x), x - \alpha\kappa(x))$ for $x \in E$. For every $k \in K$ and every $x \in E$ with $d(\kappa(x), 0) \leq d(\kappa(k), 0)$, we have $(\kappa(x), k - \alpha\kappa(k)) \in h_0(U)$. Otherwise $h_0^{-1}(\kappa(x), k - \alpha\kappa(k)) \in E\backslash U$ and we would get

$$\begin{aligned} d(\alpha\kappa(x) + k - \alpha\kappa(k), k) &= d(\alpha\kappa(x) - \alpha\kappa(k), 0) \\ &\leq 4\overline{d}(\kappa(k), 0) < \delta < \operatorname{dist}(K, E\backslash U), \end{aligned}$$

a contradiction. The above shows that, using a partition of unity argument, we can construct a map $\omega: E_1 \to [0, \delta/4]$ with

$$h_0(K) \subset \{(y, x) \in (E/E_1) \times E_1 : \overline{d}(y, 0) < \omega(x)\} \subset h_0(U).$$

Let $\alpha_t: [0, \infty] \to (0, 1]$, $0 \leq t < 1$, be a homotopy with $\alpha_0 \equiv 1$, α_t monotone, $\alpha_t|[1, \infty] \equiv 1$ for each t, and for every $s < 1$, $\lim_{t \to 1} \alpha_t(s) = 0$. Define an isotopy (g_t) of $(E/E_1) \times E_1$, $0 \leq t < 1$, by letting $g_t(y, x) = (\alpha_t(\overline{d}(y, 0)/\omega(x))y, x)$ for $y \neq 0$ and $g_t(0, x) = (0, x)$. Observe that for each $0 \leq t < 1$, $\Lambda_t = h_0^{-1} g_t h_0 \in H(E|E_0)$, $d(\Lambda_t, \mathrm{id}) < \delta$, $\Lambda_t|E\backslash U = \mathrm{id}$, and

$$\lim_{t \to 1} \Lambda_t(k) = k - \alpha\kappa(k) \in E_1$$

uniformly with respect to $k \in K$. We set $\varphi_n = \Lambda_{1-(1/n)}$ to fulfill $\mathrm{LFP}(G - \mathscr{K} - \mathscr{F})$.

Now, we shall prove our main result on extending homeomorphisms.

2.4. **Theorem.** *Let E be an infinite-dimensional l.c.m.l.s. and $\check{E}$ its completion. Every homeomorphism $h: K \to L$ between compacta of E extends to a homeomorphism $h \in H(\check{E}|E)$.*

Proof. Let $\mathscr{K}$ be the class of all compacta and $\mathscr{F}$ the class of all compacta contained in finite-dimensional subspaces of E. The space $\check{E}$ satisfies

$$\mathrm{AEP}(H(E|E) - \mathscr{F}) \quad \text{and} \quad \mathrm{LFP}(H(E|E) - \mathscr{K} - \mathscr{F})$$

by 2.1 and 2.3, respectively. The local convexity of $\check{E}$ implies that $\check{E}$ is an AR (see [4, p. 67]) and consequently it satisfies $\mathrm{NE}(\mathscr{K})$. Hence, Theorem 1.5 is applicable and $\check{E}$ satisfies $\mathrm{AEP}(H(\check{E}|E) - \mathscr{K})$. Now, the assertion of theorem follows from 1.3 and 1.4.

For the purpose of the future citation let us note the following facts.

2.5. **Proposition.** *Let E be an infinite-dimensional l.c.m.l.s. and E_0 its dense linear subspace. For every compact set $K \subset E$, its neighborhood U and a map $\varepsilon: E \to (0, \infty)$, there exists a $\delta > 0$ such that every map $f: K \to E$ with $d(f, \mathrm{id}) < \delta$ can be arbitrarily closely approximated by restrictions $h|K$, where $h \in H(E|E_0)$ satisfies $d(h(x), x) < \varepsilon(x)$ and $h|E \setminus U = \mathrm{id}$.*

Proof. An argument of the proof of Theorem 2.4 shows that E satisfies $\mathrm{AEP}(H(E|E_0) - \mathscr{K})$. This yields the assertion via Remark 1.2.

2.6. **Proposition** (cf. [10]). *Let E be an infinite-dimensional l.c.m.l.s. and E_0 its dense linear subspace. For every disjoint pair of compacta K and L of E, mapping $f: K \to E$, and $\varepsilon > 0$, there exists a homeomorphism $h \in H(E|E_0)$ with $d(f, h|K) < \varepsilon$, $d(h, \mathrm{id}) < d(f, \mathrm{id}) + \varepsilon$ and $h|L = \mathrm{id}$.*

Proof. Follow the proof of 2.5. The needed changes are those of 2.2 and 1.6.

We do not know whether Lemma 2.3 remains true without the local convexity. Observe that the condition $\mathrm{LFP}(\mathscr{K} - \mathscr{F})$ (or more generally: $\mathrm{LFP}(\mathscr{K} - \mathscr{F}d)$) implies $\mathrm{NE}(\mathscr{K})$ for any m.l.s. X. For, if X satisfies $\mathrm{LFP}(\mathscr{K} - \mathscr{F}d)$, then for every $K \in \mathscr{K}$, id_K is the uniform limit of a sequence $\phi_n: K \to X$ with $\phi_n(K) \in \mathscr{F}$; consequently, by a result of [8], X is an absolute extensor for compacta. Since, in spite of few attempts, the problem of whether all m.l.s. X are extensors for compacta is still unsettled, we suspect that also verifying $\mathrm{LFP}(\mathscr{K} - \mathscr{F}d)$ is not simple. It is reasonable to ask

2.7. **Question.** Assume an infinite-dimensional m.l.s. X is an absolute extensor for the class of all compacta $\mathscr{K}$. Does X satisfy $\mathrm{LFP}(\mathscr{K} - \mathscr{F}d)$?

Summarizing, not only do we not know if all m.l.s. X have the homeomorphism extension property for compacta, but we do not know this for the class $\mathscr{F}d$ of all finite-dimensional compacta either. It is not clear how to verify the condition $\mathrm{LFP}(\mathscr{F}d - \mathscr{F})$ for arbitrary X. In the case where X is complete, the homeomorphism extension property for compacta implies that X is an absolute extensor for compacta. Below we show that the complete X has the homeomorphism extension property for the class $\mathscr{F}d$.

2.8. **Theorem.** *Every infinite-dimensional complete m.l.s. X satisfies the following version of the $\mathscr{F}d - H(X)$ estimated homeomorphism extension property: for every $K \in \mathscr{F}d$, its neighborhood U and an embedding $v: K \to X$ with $d(v, \mathrm{id}) < \mathrm{dist}(K, X \setminus U)$ and $\varepsilon > 0$, there exists $h \in H(X)$ extending v, and satisfying $d(h, \mathrm{id}) < d(v, \mathrm{id}) + \varepsilon$ and $h|X \setminus U = \mathrm{id}$.*

We shall need:

2.9. **Lemma.** *Every infinite-dimensional complete m.l.s. X has the homeomorphism extension property for finite-dimensional compacta.*

Sketch of proof. Let $K, L \in \mathscr{F}d$ and let $h: K \to L$ be a homeomorphism. Since $\mathrm{span}(K \cup L)$ is σ-compact and X is complete, there exists a finite-dimensional linear subspace E of X with $\dim(E) \geq 2\dim(K) + 1$ and such that $E \cap \operatorname{span}(K \cup L) = \{0\}$. Consider the homeomorphism

$$h_0(x) = (\kappa(x), x - \alpha\kappa(x))$$

of X onto $(X/E)\times E$, where $\kappa: X \to X/E$ is the quotient map and $\alpha: X/E \to X$ its cross section (we use a Michael result of [15]). Clearly, $h_0(K \cup L)$ is the graph of a map of a finite-dimensional compact subset of X/E into E. Now, the result follows in a standard way by using Klee's method [14] (see also [7]). (Note that, under our assumptions on the dimension of E, L can be embedded in E.)

Proof of 2.8. We shall employ a method of Anderson and McCharen of [2]. First, observe that the homotopy $u_t = (1-t)\,\mathrm{id} + tv$, $0 \le t \le 1$, joins id_K with v in U with $d(u_t, \mathrm{id}) \le d(v, \mathrm{id})$. Since $\operatorname{span}(\operatorname{im}(u_t))$ is σ-compact we can approximate (u_t) by (v_t) such that $v_0 = \mathrm{id}$, $v_1 = v$ and $(k, t) \to (v_t(k), t)$ is an embedding. Pick a finite-dimensional linear subspace of X with $\dim(E) \ge 2\dim(K) + 1$ and a vector $x \in X\backslash E$. Let $Y = X/E \oplus \mathbf{R}x$ and h_0 be a homeomorphism of X onto $Y \times E \oplus \mathbf{R}x = Y \times E \times \mathbf{R}$ given in the same fashion as in the proof of 2.6. If $i: K \to E$ is any embedding, write $e(k, t) = (i(k), t)$ for an embedding of $K \times [0, 1]$ into $\{0\} \times E \times R \subset Y \times E \times R$. By virtue of 2.9, there eixsts a homeomorphism $\overline{h}: X \to E \times \mathbf{R} \times Y$ such that $\overline{h}v(k, t) = e(k, t)$, $(k, t) \in K \times [0, 1]$. Given $\varepsilon > 0$, let

$$U(k) = \bigcup_{0 \le t \le 1} \{x \in U : d(x, v(k, t)) < \varepsilon/2\}.$$

Clearly, we have $e(\{k\} \times [0, 1] \subset \overline{h}^{-1}(U(k))$ for every $k \in K$. Using the $\mathbf{R}$-coordinate only, we construct a homeomorphism Φ of $Y \times E \times \mathbf{R}$ with $\Phi(y, e(k, 0)) = (y, e(k, 1))$, $(y, k) \in Y \times K$ and such that $\Phi = \mathrm{id}$ outside $\bigcup_{k\in K} \overline{h}(U(k))$. Moreover, we can require that for each $x \in \bigcup_k U(k) = U'$ there corresponds $k_0 \in K$ with $\overline{h}(x) \in \overline{h}(U(k_0))$ and $\Phi(\overline{h}(x)) \in \overline{h}(U(k_0))$. We set $h = \overline{h}^{-1}\Phi\overline{h}$. Since $h(x) = x$ for every $x \notin U'$, $h|X\backslash U = \mathrm{id}$. Furthermore, if $x \in U'$ then for some $k_0 \in K$ we have $\overline{h}^{-1}\Phi\overline{h}(x)$, $x \in U(k_0)$; consequently $d(x, h(x)) < \operatorname{diam}(U(k_0)) \le \varepsilon + d(v(k_0, t), v(k_0, s))$. Thus, if (v_t) is close enough to (u_t), we get $d(h, \mathrm{id}) < \varepsilon + d(v, \mathrm{id})$. To finish the proof observe that $h|K = v$.

2.10. *Remark.* Every infinite-dimensional complete m.l.s. X has the $\mathscr{F}d - H(X)$ almost extension property. This can be easily derived from 2.8, because by the σ-compactness of $f(K)$, every map $f: K \to X$, $K \in \mathscr{F}d$, can be arbitrarily closely approximated by embeddings.

Note that an affirmative answer to 2.7, together with 2.10, 1.5, and 1.3 will imply the extension homeomorphism property for compact of X.

We close this section by proving that every infinite-dimensional l.c.m.l.s. E has the $\mathscr{K} - H(\check{E}|E)$ estimated homeomorphism extension property (in the version of [4]). Note that to prove this, we cannot apply the Anderson-McCharen technique [2] because we do not know whether compact sets are deficient in possibly noncomplete E. Instead, we shall make use of Toruńczyk's result [17]. A set $K \subset E \subset \check{E}$ is said to be $H(\check{E}|E)$-thin [4] if, for every neighborhood U, the identity id_E can be arbitrarily closely approximated by $h \in H(\check{E}|E)$ with $h|\check{E}\backslash U = \mathrm{id}$ and $h(K) \cap K = \varnothing$.

2.11. **Lemma.** *Every compact subset K of an infinite-dimensional l.c.m.l.s. E is $H(\check{E}|E)$-thin.*

Proof. Since the compact subsets of every infinite-dimensional m.l.s. are Z-sets, id_K can be approximated by maps $K \to (E \cap U)\backslash K$. Now, the result follows from the $\mathscr{K} - H(\check{E}|E)$ almost extension property of $\check{E}$ (see 2.4).

A result of [17] together with 2.11 yields

2.12. **Theorem.** *Every infinite-dimensional l.c.m.l.s. E has the $\mathscr{K} - H(\check{E}|E)$ estimated homeomorphism extension property.*

2.13. *Note.* The version of the estimated homeomorphism property formulated in 2.8 also holds true in the above case. We could prove it without referring to Toruńczyk's theorem. Observe that once we choose h_0 in the proof of 1.3 we determine the inductive process in the sense that $h_1, h_2, \ldots$ and $g_0, g_1, \ldots$ can be chosen as close to the identity as we please.

3. Applications—The smooth case

In this section we shall be interested in the question of whether a homeomorphism $h: K \to L$ between compacta of a normed linear space E (briefly: n.l.s.) extends to a homeomorphism $\overline{h}$ of E such that $\overline{h}: E\backslash K \to E\backslash L$ is a diffeomorphism. The technique of Theorem 1.3 enables us to discuss this question without referring to the Inverse Function Theorem. Consequently, it allows us to drop the completeness assumption on E in [7] and to prove the first result in this area without completeness.

We start with a lemma whose proof is a smooth analogue of that of 2.8.

3.1. **Lemma.** *Let E be an infinite-dimensional n.l.s. and E_0 be its dense linear subspace. If E admits a C^p norm, then E has the* $\mathrm{Diff}^p(E|E_0) - \mathscr{F}$ *almost extension property, where $\mathscr{F}$ is a class of all compacta contained in finite-dimensional linear subspaces of E_0 and* $\mathrm{Diff}^p(E|E_0)$ *is the group of all C^p diffeomorphisms of E preserving every linear space containing E_0.*

Proof. We shall make the necessary changes in the proof of 2.8 that are required by the smooth case. Let $K \in \mathscr{F}$, its neighborhood U and a map $f: K \to E$ with $d(f, \mathrm{id}) < \mathrm{dist}(K, E\backslash U) = d_0$ be given. Since $\mathrm{span}(K)$ is finite-dimensional there exists a C^∞ manifold M with $K \subset M$ and $K \cap \partial M = \varnothing$ such that f admits an extension $\overline{f}: M \to E_0$ with $d(\overline{f}, \mathrm{id}) < d_0$ and $\overline{f}(M) \in \mathscr{F}$. We may additionally require that $M \subset U$. Consider a homotopy $f(m, t) = (1-t)m + t\overline{f}(m)$, for $m \in M$ and $-\delta < t < 1+\delta$, with $\delta > 0$ chosen so that $\mathrm{im} f \subset U$. Include $M \cup \overline{f}(M)$ in a finite-dimensional linear subspace

E_1 of large dimension. If $3(\dim E_1 + 1) > \dim(M)$, then $f: M \times (-\delta, 1+\delta) \to U \cap E_1$ can be closely approximated by C^∞ embeddings $g: M \times (-\delta, 1+\delta) \to U \cap E_1$ with $g|M \times \{0\} = \mathrm{id}$. Hence, we may assume that $\overline{f} = f(\cdot, 1)$ and $g(\cdot, 1)$, and $d(g_t, \mathrm{id})$ and $d(f_t, \mathrm{id})$ are as close as we wish, respectively. Split $E_1 = Y \oplus E' \oplus \mathbf{R}$ in such a way that M admits a C^∞ embedding $i: M \to E'$. Write $e(m, t) = (0, i(m), t)$ for a C^∞ embedding of $M \times (-\delta, 1+\delta)$ into $Y \oplus E \oplus \mathbf{R}$. We may assume that $\dim(E_1)$ is so large that embeddings g and e are C^∞ diffeotopic in E_1. Then, by the classic theorem of Thom there exists a C^∞ diffeomorphism $h': E_2 \to Y \times E' \oplus \mathbf{R}$ such that $h'g = e$. Consider any splitting $E = E' \oplus E_1$ and write $h_0(x' + x_1) = x' + h(x_1)$ for $x' + x_1 \in E' \oplus E_1$. Next, as in the proof of 2.8, construct a homeomorphism Φ of $E' \oplus Y \oplus E \oplus \mathbf{R}$ with $\Phi(y, m, 0) = (y, m, 1)$, $d(h_0^{-1}\Phi h_0(x), x) < d(g_t, \mathrm{id}) + \varepsilon$ and $h_0^{-1}\Phi h_0|E \backslash U = \mathrm{id}$. Since E admits a C^p norm, we can easily perform this construction to obtain Φ as a C^p diffeomorphism. The proof is completed because $h_0^{-1}\Phi h_0|M = g(\cdot, 1)$ is close to $\overline{f}$ and $\overline{f}|K = f$.

3.2. **Lemma.** *Let H be an infinite-dimensional pre-Hilbert space and H_0 its linear dense subspace. Then H has the* $\mathrm{Diff}^\infty(H|H_0) - \mathscr{K} - \mathscr{F}$ *limit flattening property, where $\mathscr{K}$ is the class of all compacta of H and $\mathscr{F}$ is the class of all compacta contained in finite-dimensional linear subspaces of H_0.*

Proof. We shall follow the proof of 2.3. Let $K \in \mathscr{K}$, its neighborhood U and $\delta > 0$ be given. Assume $\delta < \mathrm{dist}(K, H \backslash U)$. There is an orthogonal splitting $H = H_1 \oplus H'$ such that a finite-dimensional space H_1 is contained in H_0 and $x_1 + x_2 \in K$, $x_1 \in H_1$, $x_2 \in H'$ imply $\|x_2\| < \delta/2$. Using the fact that Hilbert spaces admit smooth partitions of unity [18], we construct a C^∞ map $\omega: H_1 \to [0, \delta/2]$ with

$$K \subset \{x_1 + x_2 \in H_1 \oplus H' : \|x_2\| < \omega(x_1)\} \subset U.$$

Consider the homotopy (α_t) of the proof of 2.3. We additionally may require that each α_t is C^∞ smooth and $\alpha_t \equiv 0$ on a neighborhood of 0. Write $g_t(x_1 + x_2) = x_1 + \alpha_t(\|x_2\|/\omega(x_1))x_2$ for $x_2 \neq 0$ and $g_t(x_1) = x_1$. Since each $g_t, 0 \le t < 1$, is a C^∞ diffeomorphism, we set $\varphi_n = g_{1-(1/n)}$ to fulfill $\mathrm{LFP}(\mathrm{Diff}^\infty(H|E_0) - \mathscr{K} - \mathscr{F})$.

3.3. **Lemma.** *Let E be a Banach space with a Schauder basis. Assume E admits a C^p norm. Then E has the* $\mathrm{Diff}^p(E) - \mathscr{K} - \mathscr{F}$ *limit flattening property.*

Proof. Let $K \in \mathscr{K}$, its neighborhood U and $\delta > 0$ be given. Let $\{(e_i, x_i^*)\}_{i=1}^\infty$ be a Schauder basis of E. With $\delta < \mathrm{dist}(K, E \backslash U)$, there exists n such that $\|\sum_{k \ge n} x_k^*(x)e_k\| < \delta/2$ for every $x \in K$. Let $P_1(x) = \sum_{k=1}^n x_k^*(x)e_k$ and $P_2(x) = x - P_1(x)$. Replacing the splitting $H = H_1 \oplus H'$ by $E = P_1(E) \oplus P_2(E)$ we may now proceed as in the proof of 3.2.

Our main result on smooth extension of homeomorphism is

3.4. **Theorem.** *Let H be an infinite-dimensional pre-Hilbert space and $\check{H}$ its completion. Every homeomorphism $h: K \to L$ between compacta of H can be extended to $\overline{h} \in H(\check{H}|H)$ such that $\overline{h}$ is a C^∞ diffeomorphism on $\check{H}\backslash K$.*

Proof. This follows from 3.1, 3.2 and 1.5 applied to 1.3 (see also 1.4).

As a consequence we obtain the following results on C^∞ extracting of compact sets from pre-Hilbert spaces.

3.5. **Corollary.** *Every compact subset K of an infinite-dimensional pre-Hilbert space H is C^∞ negligible in H, i.e., there exists a C^∞ diffeomorphism $h: H\backslash K \xrightarrow[\text{onto}]{} H$.*

Proof. The result is that of [7] if H is complete. So, with $\check{H}$ to be the completion of H, we may assume $x \in \check{H}\backslash H$. Consider the map $k \to x + k$ of K into $\check{H}\backslash H$. Using 3.4, this map can be extended to a homeomorphism $\overline{h}$ of $\check{H}$ such that $\overline{h}|\check{H}\backslash K$ is a C^∞ diffeomorphism. Note that $\overline{h}$ sends $H\backslash K$ onto $H\backslash(x + K) = H$ (see 1.4).

Employing 3.3 instead of 3.2 in the proof of 3.4 we obtain

3.6. **Theorem.** *Let E be a Banach space with a Schauder basis. Assume E admits a C^p norm. Then, every homeomorphism $h: K \to L$ between compacta of E extends to a homeomorphism $\overline{h}$ of E such that $\overline{h}$ is a C^p diffeomorphism of $E\backslash K$ onto $E\backslash L$.*

4. Minimal types of m.l.s. that are universal for a class of compacta are homeomorphic

Every infinite-dimensional m.l.s. which is a countable union of finite-dimensional compacta is universal for the class of finite-dimensional compacta. Every such space is homeomorphic to l_2^f, a linear subspace of l_2 consisting of all finite sequences; see [3]. Also every σ-compact l.c.m.l.s. which is universal for the class of all compacta is homeomorphic to Σ, the linear span of the Hilbert cube $Q = \{(x_i) \in l_2: |x_i| \leq 1/i\}$ in l_2; see [10]. The following fact generalizes these two results.

4.1. **Theorem.** *Let E_1 and E_2 be l.c.m.l.s. universal for a class of compacta $\mathscr{T}$. The spaces E_1 and E_2 are homeomorphic iff $E_1, E_2 \in \mathscr{T}_\sigma$, where $\mathscr{T}_\sigma$ is the class of sets which are countable unions of elements of $\mathscr{T}$.*

We shall prove a slightly general fact.

4.2. **Proposition.** *Expressing $E_i = \bigcup_1^\infty A_n^i$ as unions of compacta, assume each A_n^1 can be embedded in E_2 and each A_n^2 can be embedded in E_1. Then, there exists a homeomorphism $h: \check{E}_1 \to \check{E}_2$ of the completions of E_i's with $h(E_1) = E_2$.*

The proof will employ the skeleton technique of [1, 3 and 17]. We shall need the following lemma which is a refinement of a result of [10].

4.3. **Lemma.** *Let (A, B) be a pair of compacta such that A admits an embedding in a l.c.m.l.s. E. Given a compact set $K \subset E$, $\varepsilon > 0$ and a map $f: (A, B) \to (\check{E}, E)$ such that $f|B$ is an embedding, there exists an embedding $v: A \to E$ with $v|B = f|B$, $d(v, f) < \varepsilon$ and $v(A \backslash B) \cap K = \varnothing$.*

Proof (cf. [10]). Since $\check{E} \backslash E$ is locally homotopy negligible in $\check{E}$, $f: A \to \check{E}$ can be arbitrarily closely approximated by maps equal to f on B whose ranges are in E. Therefore, we shall assume $f(A) \subset E$. We also shall assume that $A \subset E$, possibly with $A \cap (K \cup f(A)) = \varnothing$. Represent $A \backslash B = \bigcup_0^\infty A_i$, where $\varnothing = A_0 \subset A_1 \subset A_2 \subset \cdots$ is a tower of compacta. Given $\varepsilon > 0$, we shall construct a sequence $\{h_n\}_{n=0}^\infty \subset H(E)$ such that for $n \geq 1$ we have

(1) $d(h_n, h_{n-1}) < 2^{-n+1}\varepsilon$;
(2) $h_n|A_{n-1} = h_{n-1}|A_{n-1}$ and $h_n|f(A) \cup K = \mathrm{id}$;
(3) $d(h_n|B, f|B) < 2^{-n-1}\varepsilon$.

To construct h_0 apply 2.6 with $K \equiv A$ and $L = f(A) \cup K$. If $h_0, h_1, \ldots, h_{n-1}$ $(n \geq 1)$ are already constructed, by applying 2.6 with $fh_{n-1}^{-1}|h_{n-1}(B)$, $L = h_{n-1}(A_{n-1}) \cup f(A) \cup K$, we obtain a homeomorphism h. Clearly, $h_n = h \circ h_{n-1}$ satisfies (1)–(3).

Set $v(a) = \lim h_n(a)$, $a \in A$. Conditions (1)–(3) imply that v is a continuous one-to-one map of A into E with $v|B = f|B$. By (2) and the fact that $A \cap (K \cap f(A)) = \varnothing$, we infer that $v(A \backslash B) \cap K = \varnothing$.

Proof of 4.2. By the Kadec-Anderson-Toruńczyk Theorem [20], the completions $\check{E}_1$ and $\check{E}_2$ are homeomorphic. Let d_1 and d_2 be metrics of $\check{E}_1$ and $\check{E}_2$, respectively. The method of skeletoids [3] requires us to construct sequences of compacta $\{C_n^1\}_{n=1}^\infty$ of E_1 and $\{C_n^2\}_{n=1}^\infty$ of E_2 and a sequence of homeomorphisms $h_n: \check{E}_1 \to \check{E}_2$, $n = 1, 2, \ldots$, such that

(1) $A_n^1 \subset C_n^1$ and $A_n^2 \subset h_n^1(C_n^1) \subset C_n^2$,
(2) $h_n|C_{n-1}^1 = h_{n-1}|C_{n-1}^1$;
(3) $d^*(h_n, h_{n-1}) = d_2(h_n, h_{n-1}) + d_1(h_n^{-1}, h_{n-1}^{-1}) \leq 2^{-n}$ $(n \geq 2)$.

Clearly, the condition (3) implies that $h = \lim h_n$ and $g = \lim h_n^{-1}$ exist and g and h are inverse maps of one another. Hence, h is a homeomorphism of $\check{E}_1$ onto $\check{E}_2$ sending E_1 onto E_2 (use (2) and (3)).

The inductive construction. With $A_1^1 = C_1^1 = A_1^2 = C_1^2 = \varnothing$, let h_1 be any homeomorphism of $\check{E}_1$ onto $\check{E}_2$. Assume $h_1, h_2, \ldots, h_{n-1}$ $(n \geq 2)$ have been constructed. Apply Lemma 4.3 with $A = A_n^1, B = C_{n-1}^1 \cap A_n^1$, $K = h_{n-1}(C_{n-1}^1)$ and $f = h_{n-1}|A_n^1$, to approximate f by an embedding $v: A_n^1 \to E_2$ satisfying $v|B = f|B$ and $v(A_n^1 \backslash C_{n-1}^1) \cap h_{n-1}(C_{n-1}^1) = \varnothing$. Thus, by letting $v|C_{n-1}^1 = h_{n-1}|C_{n-1}^1$, v extends to an embedding over the whole C_{n-1}^1. Applying the estimated homeomorphism extension property [4, p. 153] (see also 2.5), we find a homeomorphism $h: \check{E}_1 \to \check{E}_2$ such that $h|C_{n-1}^1 = h_{n-1}^1|C_{n-1}^1$, $h(C_{n-1}^1 \cup A_n^1) = v(C_{n-1}^1 \cup A_n^1) \subset E_2$ and $d^*(h, h_n) < 2^{-n-1}$. Next apply

Lemma 4.3 with $A = A_n^2$, $B = A_n^2 \cap h(C_{n-1}^1 \cup A_n^2)$, $K = h(C_{n-1}^1 \cup A_n^2)$ and $f = h^{-1}|A_n^2$, to approximate f by an embedding $v: A_n^2 \to E_2$ with $v|B = h^{-1}|B$ and $v(A_n^2 \setminus h(C_{n-1}^1 \cup A_n^1)) \cap (C_{n-1}^1 \cup A_n^1) = \varnothing$. Again by letting $v|h(C_{n-1}^1 \cup A_n^1) = h^{-1}|h(C_{n-1}^1 \cup A_n^1)$, v extends to an embedding over $h(C_{n-1}^1 \cup A_n^1)$. By using the estimated homeomorphism extension property we extend v to a homeomorphism $g: \check{E}_2 \to \check{E}_1$ such that $d^*(g, h^{-1}) < 2^{-n-1}$. Finally, we set $C_n^1 = C_{n-1}^1 \cup A_n^1 \cup g(A_n^2)$, $C_n^2 = h(C_{n-1}^1 \cup A_n^1) \cup A_n^2$ and $h_n = g^{-1}$. Conditions (1)–(3) are fulfilled.

The following fact is a direct consequence of 4.2.

4.4. **Corollary.** *Let E_1 and E_2 be infinite-dimensional σ-compact l.c.m.l.s. and $\check{E}_1, \check{E}_2$ their completions. If there exists a one-to-one map of E_1 onto E_2, then the pairs $(\check{E}_1, E_1)$ and $(\check{E}_2, E_2)$ are homeomorphic.*

4.5. **Corollary.** *Every infinite-dimensional σ-compact l.c.m.l.s. [resp. n.l.s.] E is homeomorphic to a dense linear subspace of the countable product of lines $\mathbf{R}^N$ [resp. of the Hilbert space l_2]. In either case there exists a homeomorphism which can be extended to the completions.*

Proof. Let $\{x_n^*\}_{n=1}^{\infty}$ be a sequence of continuous linear functionals separating the points of E. Consider $T(x) = (x_i^*(x))$, $x \in E$. Clearly, T is a continuous one-to-one map of E into $\mathbf{R}^N$. By 4.5, E is homeomorphic to $T(E)$. In the case where E is a normed space, we may require additionally that $\|x_n^*\| \leq 1/n$, $n = 1, 2, \dots$, so that T will map E into l_2. Now, the first assertion of 4.5 follows from the fact that every closed infinite-dimensional linear subspace of $\mathbf{R}^N$ [resp. of l_2] is isomorphic to $\mathbf{R}^N$ [resp. to l_2] itself. The second assertion is a consequence of 4.4.

Below, we apply the skeleton technique to show the homeomorphism extension property for Z-sets in every l.c.m.l.s. which is σ-compact.

4.6. **Theorem.** *Every homeomorphism $h: F_1 \to F_2$ between Z-sets of an infinite-dimensional σ-compact l.c.m.l.s. E extends to a homeomorphism $\overline{h}$ of E.*

Proof. Let $\check{E}$ be the completion of E. It follows from the Lavrientieff theorem that h extends to $\tilde{h}: \widetilde{F}_1 \to \widetilde{F}_2$, where $\widetilde{F}_i$ is a G_δ subset of $\mathrm{cl}(F_i) \subset \check{E}$, $i = 1, 2$. There is a G_δ subset $\widetilde{E}$ of $\check{E}$ with $E \subset \widetilde{E}$, and $\widetilde{F}_1$ and $\widetilde{F}_2$ are closed in $\widetilde{E}$. (Let ρ be a complete metric on $\widetilde{F}_1 \cup \widetilde{F}_2$. Since $\widetilde{F}_1 \cup \widetilde{F}_2$ is closed in $\widetilde{F}_1 \cup \widetilde{F}_2 \cup E$, by a Hausdorff theorem [4, p. 62] ρ extends to $\widetilde{F}_1 \cup \widetilde{F}_2 \cup E$. The completion $(\widetilde{F}_1 \cup \widetilde{F}_2 \cup E, \rho)$ may serve as a suitable $\widetilde{E}$.) The space $\widetilde{E}$ as a complement of a σZ-set in $\check{E}$ is homeomorphic to $\mathbf{R}^N$ [4, p. 166]. Clearly, $\widetilde{F}_1$ and $\widetilde{F}_2$ are Z-sets in $\widetilde{E}$. Therefore, by a theorem of Anderson [4, p. 165] $\tilde{h}$ can be extended to a homeomorphism h_0 of $\widetilde{E}$.

We shall modify h_0 in order to get $h_0(E) = E$. Represent $E \setminus F_1 = \bigcup_1^\infty A_n^1$ and $E \setminus F_2 = \bigcup_1^\infty A_n^2$, where A_n^i are compacta. We shall inductively construct sequences of compacta $\{C_n^1\}_{n=1}^\infty$ and $\{C_n^2\}_{n=1}^\infty$ and a sequence of homeomorphisms $\{h_n\}_{n=1}^\infty$ of $\widetilde{E}$ such that for $n \geq 1$ we have

(1) $A_n^1 \subset C_n^1 \subset E\backslash F_1$, $A_n^2 \subset h_n(C_n^1) \subset C_n^2 \subset E\backslash F_2$;

(2) $h_n|C_{n-1}^1 = h_{n-1}|C_{n-1}^1$ and $h_n|\widetilde{F}_1 = h_0|\widetilde{F}_1$;

(3) $d^*(h_n, h_{n-1}) = d(h_n, h_{n-1}) + d(h_n^{-1}, h_{n-1}^{-1}) \leq 2^{-n}$, where d is a complete metric on $\widetilde{E}$.

Finally, we set $\overline{h} = \lim h_n|E$.

With $A_1^i = C_1^i = \varnothing$ $(i = 1, 2)$, we let $h_1 = h_0$. Suppose $h_1, \ldots, h_{n-1}$ $(n \geq 2)$ have been constructed. We make use of 4.3 to approximate $h_{n-1}|C_{n-1}^1 \cup A_n^1$ by embeddings $v: C_{n-1}^1 \cup A_n^1 \to E$. If v is sufficiently close to $h_{n-1}|C_{n-1}^1 \cup A_n^1$, then $v(C_{n-1}^1 \cup A_n^1) \cap h_0(F_1) = \varnothing$. Using the estimated homeomorphism extension property for compacta of $\widetilde{E}$, we extend v to a homeomorphism h of $\widetilde{E}$ with $d^*(h, h_{n-1}) < 2^{-n-2}$. We repeat the same argument to $h^{-1}|h(C_n^1 \cup A_n^1) \cup A_n^2$ (see the proof of 4.2) to obtain g with $g|h(C_n^1 \cup A_n^1) \cup A_n^2 = h^{-1}|h(C_n^1 \cup A_n^1) \cup A_n^2$ and $d^*(g, h^{-1}) < 2^{-n-2}$. Finally, we let $h_n = g^{-1}$.

5. Relative topological equivalence of skeletoids in noncomplete spaces

The importance of skeleton sets follows from their relative topological equivalence in complete metric spaces X, i.e., if X_1 and X_2 are skeletoids then there exists $h \in H(X)$ with $h(X_1) = X_2$. This is no longer true if one drops the completeness assumption of X. To show this, let us consider the simplest example of a skeletoid, a countable dense subset of a separable m.l.s. X which is a skeleton set for the class of all finite sets of X.

5.1. **Example.** For each n, let $E_n = \{(x_i) \in l_2^f : x_i = 0 \text{ for } i \geq n+1\}$. Pick a minimal $1/n$-net A_n in $E_n\backslash E_{n-1}$ $(n \geq 1)$. Then the set $A = \bigcup_1^\infty A_n$ is a countable dense subset of l_2^f. Let B be a countable dense subset of l_2^f such that $B \cap E_1$ is dense in E_1. There exists a homeomorphism h of l_2 carrying A onto B. However, there is no homeomorphism h of l_2^f with $h(A) = B$. This is a consequence of the fact that $l_2^f\backslash A$ being the union of $(E_n\backslash E_{n-1})\backslash A_n$ is σ-compact but $l_2^f\backslash B$ is not. If $l_2^f\backslash B$ were σ-compact, then $E_1\backslash B$ would be also σ-compact, contradicting the Baire category theorem.

Let us recall that every m.l.s. spanned by countable many linearly independent vectors is a skeletoid (for the class of finite-dimensional compacta) in its completion [3]. Below we show that two such skeletoids are no longer relatively topological equivalent if they are contained in Σ, the linear span of the Hilbert space l_2.

5.2. **Example.** Consider a linearly independent subset $\{x_i\}_{i=1}^\infty$ of l_2 such that $E_1 = \operatorname{span}\{x_i\}_{i=1}^\infty$ is dense in l_2 and $E_1 \cap \Sigma = \{0\}$. Let $\Sigma' = \Sigma + E$. Both

Σ and Σ' are skeletoids for the class of compacta in l_2; consequently there eixsts $h \in H(l_2)$ with $h(\Sigma) = \Sigma'$; see [17]. However, no homeomorphism of Σ onto Σ' sends $E_0 = l_2^f \subset \Sigma$ onto $E_1 \subset \Sigma'$. The reason is that every $\Sigma_n = (\Sigma + \operatorname{span}\{x_1, x_2, \dots, x_n\}) \setminus \operatorname{span}\{x_1, x_2, \dots, x_n\}$ is σ-compact and so is their union $\Sigma' \setminus E_1 = \bigcup_1^n \Sigma_n$, but $\Sigma \setminus E_0$ is not σ-compact. If $\Sigma \setminus E_0$ were σ-compact, $\{(x_i) \in l_2 : |x_i| \le 1/i\} \setminus E_1$ would also be σ-compact, contradicting the Baire category theorem.

The following result shows that the σ-compactness of complements of skeletoids considered in 5.1 suffices to obtain their relative topological equivalence in l_2^f.

5.3. **Theorem.** *Let A and B be countable dense subsets of l_2^f. If $l_2^f \setminus A$ and $l_2^f \setminus B$ are σ-compact, then there exists $h \in H(l_2)$ with $h(l_2^f) = l_2^f$ and $h(A) = B$.*

We shall need the following.

5.4. **Lemma.** *Let A be a countable dense subset of l_2^f. Given a pair of finite-dimensional compacta (K, L), every map $F: (K, L) \to (l_2, l_2^f)$ such that $f|L$ is an embedding into $l_2^f \setminus A$ can be arbitrarily closely approximated by embeddings $v: K \to l_2^f \setminus A$ with $v|L = f|L$.*

Proof. First, since $l_2 \setminus l_2^f$ is locally homotopy negligible in l^2 [19], we can assume that $f(A) \subset l_2^f$. Consider any embedding $i: K \to [0,1]^q \times \{0\} \subset [0,1]^q \times [0,1]$ for a suitable integer q. Pick a map $\alpha: [0,1]^q \to [0,1]$ with $\alpha^{-1}(\{0\}) = i(L)$. Write $M = \{(p, t) \in [0,1]^q \times [0,1] : t \le \alpha(p)\}$ and let $\overline{f}: M \to l_2^f$ be given by $\overline{f}(p, t) = f \circ i^{-1}(p)$. By a property of l_2^f there exists an embedding $\overline{v}: M \to l_2^f$ with $\overline{v}|i(L) = \overline{f}|i(L)$ and as close to $\overline{f}$ as we wish [6]. Consider a parametric family of embeddings $v_t(k) = \overline{v}(i(k), t\alpha(i(k)))$, $0 \le t \le 1$, $k \in K$. Since for $t \ne s$, $v_t(K \setminus L) \cap v_s(K \setminus L) = \varnothing$, for some t we have $v_t(K) \cap A = \varnothing$. We set $v = v_t$.

In the proof below we slightly modify the technique of skeleton sets. Namely, we deal with two pairs of skeletoids $(A, l_2^f \setminus A)$ and $(B, l_2^f \setminus A)$, and while A and B are skeletoids for the class of finite sets, $l_2^f \setminus A$ and $l_2^f \setminus B$ are skeletoids for the class of finite-dimensional compacta. Our goal is to construct $h \in H(l_2)$ sending $(A, l_2^f \setminus A)$ onto $(B, l_2^f \setminus B)$.

Proof of 5.3. Let $l_2^f \setminus A = \bigcup_1^\infty A_n^1$ and $l_2^f \setminus B = \bigcup_1^\infty A_n^2$, where A_n^i are compacta. Let, moreover, $A = \{a_n^1\}_{n=2}^\infty$ and $B = \{a_n^2\}_{n=1}^\infty$. Inductively, we shall construct sequences of finite-dimensional compacta $\{C_n^1\}$ of $l_2^f \setminus A$ and $\{C_n^2\}$ of $l_2^f \setminus B$ finite sets $F_n^1 \subset A$ and $F_n^2 \subset B$, and a sequence $\{h_n\}_1^\infty$ of homeomorphisms of l_2 such that for $n \ge 2$

(1) $A_n^1 \cup \{a_i^1\}_{i=2}^n \subset C_n^1 \cup F_n^1$ and $A_2^n \cup \{a_i^2\}_{i=2}^n \subset h_n(C_n^1 \cup F_n^1) \subset C_n^2 \cup F_n^2$;

(2) $h_n|C^1_{n-1} \cup F^1_{n-1} = h_{n-1}|C^1_{n-1} \cup F^1_{n-1}$;
(3) $d^*(h_n, h_{n-1}) = d(h_n, h_{n-1}) + d(h_n^{-1}, h_{n-1}^{-1}) \le 2^{-n}$.
Then, clearly $h = \lim h_n$ will be a required homeomorphism.

Inductive construction. With $A^i_1 = C^i_1 = F^i_1 = \varnothing$ $(i = 1, 2)$ we may set $h_1 = \mathrm{id}$. Assume that $h_1, h_2, \ldots, h_{n-1}$ $(n \ge 2)$ exist. By 5.4, $h_{n-1}|C^1_{n-1} \cup A^1_n$ can be approximated by an embedding $v: C^1_{n-1} \cup A^1_n \to l^f_2\backslash B$ with $v|C^1_{n-1} = h_{n-1}|C^1_{n-1}$. We pick a point $v(a^1_n) \in B$ as close to $h_{n-1}(a^1_n)$ as we wish and such that $(h_{n-1}|F^1_n) \cup v(a^1_n)$ is one-to-one. By the estimated homeomorphism extension property applied to l_2, there is $h \in H(l_2)$ which extends v, i.e., $h(C^1_{n-1} \cup A^1_n) \subset l^f_2\backslash B$, $h(F^1_{n-1} \cup \{a^1_n\}) \subset B$, $h|C^1_{n-1} \cup F^1_{n-1} = h_{n-1}|C^1_{n-1} \cup F^1_{n-1}$ and $d^*(h, h_{n-1}) < 2^{-n-2}$. Next, applying 5.4, $h^{-1}|h(C^1_{n-1} \cup A^1_n) \cup A^2_n$ can be approximated by an embedding v into $l^f_2\backslash A$ with $v|h(C^1_{n-1} \cup A^1_n) = h^{-1}|h(C^1_{n-1} \cup A^1_n)$. We can find a point $v(a^2_n) \in A$ as close to $h^{-1}(a^2_n)$ as we wish and such that $h^{-1}|h(F^1_n \cup \{a^1_n\}) \cup v(a^2_n)$ is one-to-one. We extend v to a homeomorphism g of l_2 with $d^*(g, h^{-1}) < 2^{-n-2}$. Finally, we set $C^1_n = C^1_{n-1} \cup A^1_n \cup g^{-1}(A^2_n)$, $C^2_n = h(C^1_{n-1} \cup A^1_n) \cup A^2_n$, $F^1_n = F^1_n \cup \{a^1_n\} \cup \{g^{-1}(a^2_n)\}$, $F^2_n = h(F^1_n \cup \{a^1_n\}) \cup \{a^2_n\}$ and $h_n = g^{-1}$.

5.5. *Remark.* The assertion of 5.3 holds true if one replaces l^f_2 by Σ.

An argument of 5.3 can be adopted to prove the following fact.

5.6. **Proposition.** *Let E be an infinite-dimensional σ-compact l.c.m.l.s. and let E_1 and E_2 be two homeomorphic dense linear subspaces of E with σ-compact complements. Suppose that $E\backslash E_i$ can be represented as a union of compacta, $E\backslash E_i = A^i_n$, in such a way that each A^1_n embeds in $E\backslash E_2$ and each A^2_n embeds in $E\backslash E_1$. Then, there exists a homeomorphism h of the completion $\check{E}$ of E with $h(E) = E$ and $h(E_1) = E_2$.*

We need the following modification of 4.3.

5.7. **Lemma.** *Let E be an infinite-dimensional l.c.m.l.s. and E_0 its dense linear subspace. Let (A, B) be a pair of compacta such that A embeds in $E\backslash E_0$. Given a compact set K, $\varepsilon > 0$ and a map $f: (A, B) \to (\check{E}, E\backslash E_0)$ such that $f|B$ is an embedding, there exists an embedding $v: A \to E\backslash E_0$ with $v|B = f|B$, $d(v, f) < \varepsilon$ and $v(A\backslash B) \cap K = \varnothing$.*

Proof. We may assume that $f(A) \subset E$ (see the proof of 4.3) and $A \subset E\backslash E_0$ with $A \cap (f(A) \cup K) = \varnothing$. Let $A\backslash B = \bigcup_1^\infty A_i$, where $\{A_i\}$ is a tower of compacta. Applying 2.6, we construct a sequence $\{h_n\}_0^\infty \subset H(E|E_0)$ such that (1)–(3) of 4.3 are satisfied. As in the proof of 4.3 $v = \lim h_n|A$ is an embedding with $v|B = f|B$ and $v(A\backslash B) \cap K = \varnothing$. Since $A_n \subset E\backslash E_0$ and h_n preserves E_0, $v(A_n) = h_n(A_n) \subset E\backslash E_0$ $(n = 1, 2, \ldots)$.

Proof of 5.3. Let $E_1 = \bigcup_1^\infty B_n^1$, $E_2 = \bigcup_1^\infty B_n^2$, where B_n^i are compacta. We construct sequences of compacta $\{C_n^1\}$ of $E \backslash E_1$, $\{C_n^2\}$ of $E \backslash E_2$, $\{D_n^1\}$ of E_1 and $\{D_n^2\}$ of E_2 and a sequence of homeomorphisms $\{h_n\}_{n=1}^\infty$ of $\check{E}$ such that for $n \geq 2$

(1) $A_n^1 \cup B_n^1 \subset C_n^1 \cup D_n^1$ and $A_n^2 \cup B_n^2 \subset h_n(C_n^1 \cup D_n^1) \subset C_n^2 \cup D_n^2$;

(2) $h_n | C_{n-1}^1 \cup D_{n-1}^1 = h_{n-1} | C_{n-1}^1 \cup D_{n-1}^1$;

(3) $d^*(h_n, h_{n-1}) < 2^{-n}$.

We omit the inductive construction. To perform it, follow the proof of 4.2 and make use of the result of 5.5. We complete the proof by letting $h = \lim h_n$.

5.8. **Theorem.** *Let E be a σ-compact l.c.m.l.s. containing a Hilbert cube (by a result of* [10] *E is homeomorphic to Σ). Let E_1 and E_2 be homeomorphic σ-compact dense linear subspaces of E. If E_i contains no Hilbert cube and $E \backslash E_i$ $(i = 1, 2)$ are σ-compact, then there exists a homeomorphism h of the completion $\check{E}$ of E such that $h(E) = E$ and $h(E_1) = E_2$.*

Proof. Let Q denote a Hilbert cube contained in E. Since each $Q \backslash E_i$ is nonempty and σ-compact, by the Baire category theorem, it contains an open subset of Q. Consequently, each $E \backslash E_i$ is universal for compacta. Therefore, the assertion follows from 5.6.

A direct conclusion of 5.8 is the following.

5.9. **Corollary.** *Let E_1 and E_2 be dense linear subspaces which are countable unions of finite-dimensional compacta of Σ. If $E \backslash E_i$ are σ-compact $(i = 1, 2)$, then there exists a homeomorphism of l_2 with $h(\Sigma) = \Sigma$ and $h(E_1) = E_2$.*

5.10. *Note.* If, additionally in 5.9, $A \subset E_1$ and $B \subset E_2$ are countable and dense sets such that both $E_1 \backslash A$ and $E_2 \backslash B$ are σ-compact, then there exists a homeomorphism h of l_2 with $h(\Sigma) = \Sigma$, $h(E_1) = E_2$ and $h(A) = B$.

In the case where the complements $E \backslash E_i$ are not σ-compact, we have the following partial result only.

5.11. **Theorem.** *Let E be a l.c.m.l.s. such that $E = \bigcup_1^\infty C_n$, where each C_n is a compact convex set and C_n is a Z-set in C_{n+1}. Let E_1 and E_2 be dense linear subspaces which are countable unions of finite-dimensional compacta of E. If each $C_n \cap E_i$ $(i = 1, 2)$ is dense in C_n, then there exists a homeomorphism h of the completion $\check{E}$ of E such that $h(E) = E$ and $h(E_1) = E_2$.*

We start with a lemma.

5.12. **Lemma.** *Let $C_1 \subset C_2 \subset \cdots \subset C_k$ be a tower of compact convex sets in an infinite-dimensional l.c.m.l.s. E. Let E_0 be a linear dense subspace of E with each $C_i \cap E_0$ dense in C_i and assume C_i is a Z-set in C_{i+1}, $i = 1, \ldots, k-1$. Then, every embedding of a pair of finite-dimensional subcompacta (A, B) of C_k $u: (A, B) \to (C_k, C_k \cap E_0)$ with $u(A \cap (C_i \backslash C_{i-1})) \subset C_i \setminus C_{i-1}$ $(C_0 = \varnothing)$ can be arbitrarily closely approximated by embeddings $v: A \to C_k$ with $v(A \cap C_1) \subset E_0$, $v|B = u|B$ and $v(A \cap (C_i \backslash C_{i-1})) \subset C_i \backslash C_{i-1}$.*

Proof. By virtue of [9], a convex set $C_1 \cap E_0$ is homeomorphic to l_2^f. Hence, by a property of l_2^f, $u|A \cap C_1$ can be approximated by embeddings $v_1 : A \cap C_1 \to C_1 \cap E_0$ such that $v_1|B \cap C_1 = u|B \cap C_1$. There is a map $f_1 : A \cap C_2 \to C_2$ with $f_1|A \cap C_1 = v_1|A \cap C_1$ and $f_1|B \cap C_2 = u|B \cap C_2$. We can find f_1 as close to $u|A \cap C_2$ as we wish, provided v_1 is close enough to $u|A \cap C_1$. Consider $C_2 \cap E_0$, another copy of l_2^f. Since $C_1 \cap E_0$ is a Z-set in $C_2 \cap E_0$, the map f_1 can be approximated by embeddings $v_2 : A \cap C_2 \to C_2 \cap E_0$ satisfying $v_2(A \cap (C_2 \backslash C_1)) \cap C_2 \backslash C_1$ and $v_2|(A \cap C_1) \cup (B \cap C_2) = f_1|(A \cap C_1) \cup (B \cap C_2)$. Continuing this process, we find embeddings $v_i : A \cap C_i \to C_i \cap E_0$ such that $v_i|A \cap C_{i-1} = v_{i-1}|A \cap C_{i-1}$, $v_i(A \cap (C_i \backslash C_{i-1})) \subset C_i \backslash C_{i-1}$ and v_i to be as close to $u|A \cap C_i$ as we wish. Clearly, $v = v_k$ fulfills the assertion of 5.12.

Proof of 5.11. It is easily seen that there are sequences of finite-dimensional compacta $\{A_n^1\}_1^\infty$ and $\{A_n^2\}_1^\infty$ with $A_n^i \subset C_n$ and such that $\bigcup_1^\infty A_n^i = E_i$, $i = 1, 2$. We shall construct inductively sequences of finite-dimensional compacta $\{C_n^i\}_1^\infty$, a sequence of integers $1 = k(1) < k(2) < \cdots$, and a sequence of homeomorphisms $\{h_n\}_1^\infty$ of E such that for $n \geq 2$,

(1) $A_n^1 \subset C_n^1 \subset C_{k(n)}$ and $A_n^2 \subset h_n(C_n^1) \subset C_n^2 \subset C_{k(n)}$;
(2) $h_n|C_{n-1}^1 = h_{n-1}|C_{n-1}^1$;
(3) $h_n(D_{k(i)}) = D_{k(i)}$ with $D_j = C_j \backslash C_{j-1}$, $i = 1, \ldots, n$;
(4) $d^*(h_n, h_{n-1}) = d(h_n, h_{n-1}) + d(h_n^{-1}, h_{n-1}^{-1}) < 2^{-n}$.

Suppose $\{h_n\}_1^\infty$ has been constructed. Then, by (4), $h = \lim h_n$ is a homeomorphism of $\check{E}$ satisfying, by (1) and (2), $h(E_1) = E_2$. Condition (3) implies $h(C_{k(i)}) = C_{k(i)}$ for every i. Therefore, we additionally obtain $h(E) = E$.

Inductive construction. To construct h_1, consider $C_1 \cap E_2$, a convex copy of l_2^f; see [8]. Since $C_1 \cap E_2$ is dense in C_1, the identity on A_1^1 can be approximated by embeddings $v : A_1^1 \to C_1 \cap E_2$. The estimated homeomorphism extension property [4, p. 155] of C_1, homeomorphic to the Hilbert cube [4, p. 100], implies that the identity on C_1 can be approximated by homeomorphisms $\bar{h}$ of C_1 with $\bar{h}|A_1^1 = v|A_1^1$. Next, we extend $\bar{h}$ to a homeomorphism h of $\check{E}$; moreover, the estimated homeomorphism extension property of $\check{E}$ yields h with $d^*(h, \mathrm{id}) < 1/4$ if $\bar{h}$ is sufficiently close to id_{C_1}. Let $A = h(A_1^1) \cup A_1^2$ and $B = h(A_1^1)$, and let $u = h^{-1}|A$. Now $C_1 \cap E_1$ is a dense convex copy of l_2^f in C_1. Hence, u can be approximated by embeddings $v : A \to C_1 \cap E_1$ with $v|B = u|B$. As above, we extend v to a homeomorphism g of $\check{E}$ satisfying $g(C_1) = C_1$ and $d^*(g, h^{-1}) < 1/4$. We set $h_1 = g^{-1}$ together with $C_1^1 = A_1^1 \cup g(A_1^2)$ and $C_1^2 = h(A_1^1) \cup A_1^2$.

Assume that (1)–(4) are valid for $n \geq 2$. Consider $A = C_n^1 \cup A_{n+1}^1$, $B = C_n^1$, and $u = h_n|A$. There exists an integer $k(n+1) > k(n)$ and an embedding $v_0 : A \to C_{k(n+1)}$ as close to u as we wish with $v_0|A \cap C_{k(n)} = u|A \cap C_{k(n)}$.

Applying 5.12 to a tower $C_{k(1)} \subset C_{k(2)} \subset \cdots \subset C_{k(n+1)}$, we find an embedding $v: A \to C_{k(n+1)}$ such that $v(A) \subset E_2$, $v|B = v_0|B = h_n|B$, and $v(A \cap D_{k(i)}) \subset D_{k(i)}$, $i = 1, 2, \ldots, n+1$. Since each $C_{k(i)}$ is a Z-set in $C_{k(i+1)}$ and $C_{k(i+1)}$ is a Z-set in $\check{E}$, the estimated homeomorphism extension properties of each $C_{k(i+1)}$ and $\check{E}$ yield that there exists a homeomorphism h of $\check{E}$ preserving each $C_{k(i)}$, $i = 1, 2, \ldots, n+1$, and satisfying $h|A = v|A$ and $d^*(h, h_n) < 2^{-n-2}$. Next, let $A = h(C_n^1 \cup A_{n+1}^1) \cup A_{n+1}^2$, $B = h(C_n^1 \cup A_{n+1}^1)$ and $u = h^{-1}|A$. We have $A \subset C_{k(n+1)}$ and $u(A \cap D_{k(i)}) \subset D_{k(i)}$, $i = 1, 2, \ldots, n+1$. Applying 5.12, we approximate u by an embedding $v: A \to C_{k(n+1)}$ with $v(A) \subset E_1$ and $v(A \cap D_{k(i)}) \subset D_{k(i)}$ for $i = 1, 2, \ldots, n+1$. Finally, we extend v in the above fashion to a homeomorphism g of $\check{E}$ which preserves each $C_{k(i)}$, $i = 1, 2, \ldots, n+1$, and $d^*(g, h^{-1}) < 2^{-n-2}$. To finish the inductive construction we set $h_{n+1} = g^{-1}$, $C_{n+1}^1 = C_n^1 \cup A_{n+1}^1 \cup g^{-1}(A_{n+1}^2)$, $C_{n+1}^2 = h(C_n^1 \cup A_{n+1}^1) \cup A_{n+1}^2$.

5.13. *Note.* The proof of 5.11 works in the case where $\bigcup_1^\infty C_i = C$ is a nontrivial convex subset of E. If C^1 $(= E_1)$ and C^2 $(= E_2)$ are convex subsets of C and each $C_k \cap C^i$ is σ-finite-dimensional dense in C, then there exists a homeomorphism h of the closure $\overline{C}$ of C with $h(C) = C$ and $h(C^1) = C^2$ (see [6]).

5.14. **Example.** Let $Q = [-1, 1]^N$ be the Hilbert cube. Consider $C = \bigcup_1^\infty \{(x_i) \in Q: |x_i| < 1 - \frac{1}{n},\ i \geq n+1\}$, a convex dense subset of Q, and let $I_n = \{(x_i) \in Q: x_i = 0,\ i \geq n+1\}$ and $\mathring{I}_n = \{(x_i) \in I_n: |x_i| < 1,\ i = 1, 2, \ldots, n\}$. We infer from 5.13 that there exists $h \in H(Q)$ with $h(C) = C$ and $h(\bigcup_n I_n) = \bigcup_n \mathring{I}_n$.

6. Extending homeomorphisms in an isotopy setting

It is possible to give an isotopy version of our abstract theorem concerning the homeomorphism extension property. To obtain that both notions, the almost extension and the limit flattening properties, have to be changed. We are not going to present this in full generality. We shall only discuss the case of extending homeomorphisms in locally convex metric linear spaces. Our controlled isotopy extension theorems are formulated in a similar way to that of Anderson-McCharen [2]. The results, as the referee suggested, may have applications in investigating manifolds modelled on metric linear spaces without completeness. We include here, as a direct application of that, the homeomorphism extension theorem for local compacta in such spaces.

Before we formulate a controlled isotopy extension theorem we explain the following notation concerning homotopies and isotopies. We say that a homotopy $h = (h_t): Z \to E$ is limited by a collection $\mathscr{U}$ of subsets of E if each set $h(z \times I)$, $z \in Z$, is either contained in some member of $\mathscr{U}$ or it consists of one point. We write I for the interval $[0, 1]$ and $\mathscr{U}^+ = \bigcup \mathscr{U}$. Together with a homotopy (h_t), $h_0 = \mathrm{id}$, we shall consider the set $\mathrm{Fix}(h) = \{z \in Z : h(z \times I) = \{z\}\}$. Clearly we have $\mathscr{U}^+ \supset h((Z \setminus \mathrm{Fix}(h)) \times I)$. By an isotopy $\overline{h} = (\overline{h}_t)$ of E we mean a homotopy $\overline{h}: E \times I \to E$ with the property that $(x, t) \to (\overline{h}(x, t), t)$ is a homeomorphism of $E \times I$.

6.1. **Theorem.** *Let E be an infinite-dimensional l.c.m.l.s. and let K be its compact subset. Let $h = (h_t)$ be a homotopy of K into E so that $h_0 = \mathrm{id}_K$ and $h_1: K \to E$ is an embedding. Assume that h is limited by a collection $\mathscr{U}$ of open subsets of E. If $\mathscr{U}^+$ contains the closure of $K \setminus \mathrm{Fix}(h)$ then there exists an isotopy $\overline{h} = (\overline{h}_t)$ of E such that $\overline{h}_0 = \mathrm{id}$, $\overline{h}_1 | K = h_1$, $\overline{h}$ is limited by $\mathscr{U}$ and $\overline{h}(k, t) = k$ for all $k \in \mathrm{Fix}(h)$ and $t \in I$.*

First we modify the homotopy h in order to have $h((K \setminus \mathrm{Fix}(h)) \times I) \cap \mathrm{Fix}(h) = \varnothing$.

6.2. **Lemma.** *Write $K_0 = \mathrm{Fix}(h)$. There exists a homotopy $g = (g_1)$ of K which is limited by $\mathscr{U}$ and such that $g = h$ on $K_0 \times I \cup (K \setminus K_0) \times \{0, 1\}$ and $g((K \setminus K_0) \times I) \cap K_0 = \varnothing$.*

Proof. Write $\overline{\varepsilon}(k) = \frac{1}{2} \sup\{\mathrm{dist}(h(k \times I), E \setminus U) : U \in \mathscr{U}\}$ and notice that $\overline{\varepsilon}$ is lower-semicontinuous on $K \cap \mathscr{U}^+$. There exists a continuous mapping ε with $0 < \varepsilon(k) \le \overline{\varepsilon}(k)$, $k \in K \cap \mathscr{U}^+$. Correct ε to have $\varepsilon(k_n) \to 0$ whenever $d(k_n, K_0) \to 0$. Since K_0 is a Z-set in E, one can find a homotopy $\phi = (\phi_s)$ on $(K \setminus K_0) \times I$ into I so that $\phi_s((K \setminus K_0) \times I) \cap K_0 = \varnothing$ for all $0 < s \le 1$, $\phi_0 = h$ on $(K \setminus K_0) \times I$ and $d(\phi_s(k, t), h(k, t)) < \varepsilon(k)$ for all $k \in K \setminus K_0$ and $(t, s) \in I \times I$, see [19]. The latter inequality yields: $d(\phi_s(k, t), h(k, t)) < \mathrm{dist}(h(k \times I), E \setminus U)$ for some $U \in \mathscr{U}$ containing $h(k \times I)$; i.e., $\phi_s(k, t) \in U$ for $(s, t) \in I \times I$. Pick a steering map $\lambda: (K \setminus K_0) \times I \to I$ such that $\lambda^{-1}(0) \in (K \setminus K_0) \times \{0, 1\}$ and put $g(k, t) = \phi((k, t), \lambda(k, t))$ for $(k, t) \in (K \setminus K_0) \times I$. It is clear that this map can be continuously extended to the whole $K \times I$ by putting $g = h$ on $K_0 \times I$. Now we show that $\mathscr{U}$ can be replaced by a collection $\{\mathrm{st}(h(k \times I), \mathscr{V}) : k \in K \cap \mathscr{U}^+\}$ for some collection $\mathscr{V}$; here $\mathrm{st}(A, \mathscr{V})$ stands for $\bigcup\{V \in \mathscr{V} : V \cap A = \varnothing\}$.

6.3. **Lemma.** *There exists a collection $\mathscr{V}$ consisting of open sets of E such that $\mathscr{V}^+$ contains $h((K \cap \mathscr{U}^+) \times I)$ and $\mathrm{st}((k \times I), \mathscr{V})$ is contained in some element of $\mathscr{U}$ for each $k \in K \cap \mathscr{U}^+$.*

Proof. Let ε be a function of the proof of 6.2. Write $B(x, r) = \{y : d(x, y) < r\}$. We shall pick for each $x \in h((K \cap \mathscr{U}^+) \times I)$ a number $\delta(x) > 0$ such that whenever $y \in \bigcup\{B(x, \delta(x)) : B(x, \delta(x)) \cap h(k \times I) = 0\}$ then

$$d(y, h(k \times I)) < \varepsilon(k).$$

From this it easily follows that $\mathscr{V} = \{B(x, \delta(x)) : x \in h((K \cap \mathscr{U}^+) \times I)\}$ has the required property. Write $\overline{\delta}(x) = \inf\{\frac{1}{4}\varepsilon(k) : x \in h(k \times I)\}$. By the compactness of K we infer that $\overline{\delta}$ is lower-semicontinuous and $\overline{\delta}(x) > 0$ for $x \in h((K \cap \mathscr{U}^+) \times I)$. Thus there exists a continuous map $\overline{\overline{\delta}}$ such that $0 < \overline{\delta}(x) \le \overline{\overline{\delta}}(x)$. Now, by the continuity of $\overline{\overline{\delta}}$ there is a continuous map δ such that

$0 < \delta(x) \le \bar{\bar{\delta}}(x)$ and that $d(y,x) < 2\delta(x)$ implies $|\bar{\bar{\delta}}(y) - \bar{\bar{\delta}}(x)| < \bar{\bar{\delta}}(x)$. Assume now that $y \in B(x,\delta(x))$, $x \in h((K \cap \mathscr{U}^+) \times I)$ with $B(x,\delta(x)) \cap h(k \times I) = 0$. Then for some $t \in I$ we have $d(x, h_t(k)) < \delta(x)$, and consequently $|\bar{\bar{\delta}}(x) - \bar{\bar{\delta}}(h_t(k))| < \bar{\bar{\delta}}(h_t(k))$. This yields $\bar{\bar{\delta}}(x) \le 2\bar{\bar{\delta}}(h_t(k))$. We can estimate $d(y, h_t(k)) \le d(y,x) + d(x, h_t(k)) \le 2\delta(x) \le 4\bar{\bar{\delta}}(h_t(k)) \le \varepsilon(k)$. A proof of 6.3 is completed.

The following fact will be our main step in proving 6.1. It replaced the condition of the almost extension property.

6.4. **Proposition.** *Let (K, K_0) be a pair of compacta of E, U an open neighborhood of $K\backslash K_0$ in E and $\varepsilon > 0$. Suppose $h = (h_t)$ is a homotopy of K in $K_0 \cup U$ such that $h_0 = \mathrm{id}$, $h_1 : K \to E$ is an embedding, $h((K\backslash K_0) \times I) \cap K_0 = \varnothing$ and $h_t | K_0 = \mathrm{id}$ for all $t \in I$. Then for every collection $\mathscr{V}$ of open subsets of E with $\mathscr{V}^+ \supset h((K\backslash K_0) \times I)$ there exists an isotopy (ϕ_t) of E satisfying*

(1) $\phi_0 = \mathrm{id}$, $d(\phi_1 | K, h_1) < \varepsilon$,

(2) $\phi_t | K_0 \cup (E\backslash U) = \mathrm{id}$ *for all* $t \in I$,

(3) (ϕ_t) *is limited by the collection* $\{\mathrm{st}(h(k \times I), \mathscr{V}) : k \in K\backslash K_0\}$.

6.5. **Lemma.** *The assertion of* 6.4 *holds true if we additionally assume that* $K_0 = \varnothing$.

Proof. Let $\delta(x)$ be the map of 6.3 chosen for the collection $\mathscr{U} = \{\mathrm{st}(h(k \times I), \mathscr{V}) : k \in K\}$. There is a $\delta > 0$ so that $\delta \le \delta(x)$ for all $x \in h(K \times I)$. Thus $\mathscr{U}$ may be replaced by $\{B(h(k \times I), \delta) : k \in K\}$, where $B(A,\delta) = \{x \in E : d(x,A) < \delta\}$. A required isotopy (ϕ_t) will be given as the composition of an isotopy (Λ_t), $0 \le t < t_0$, such that $\Lambda_0 = \mathrm{id}$ with another isotopy (h'_t), $t_0 \le t \le 1$ with $h'_{t_0} = \mathrm{id}$. Observe that whenever (Λ_t) is a $\delta/2$-isotopy and (h'_t) is an isotopy limited by $\{B(h(k \times I), \delta/2) : k \in K\}$ then the isotopy (ϕ_t) equal to (Λ_t) for $0 \le t \le t_0$ and $\phi_t = h'_t \circ \Lambda_{t_0}$ for $t_0 \le t \le 1$ will be limited by $\{B(h(k \times I), \delta) : k \in K\}$. The isotopy (Λ_t) that we choose is that of the proof of 2.3. We recall that $\Lambda_0 = \mathrm{id}$ and $\Lambda_t | E\backslash U = \mathrm{id}$, $t \in I$. The isotopy (h'_t) is given by $h'_t = h_0^{-1} \circ \Phi_t \circ h_0$, where h_0 and $\Phi = (\Phi_t)$ are those from the proof of 2.1. We recall that $h'_t(x) = \alpha\kappa(x) + \Phi(x - \alpha\kappa(x), t\lambda\kappa(x))$, where α, κ and λ are those of 2.1 and Φ is an isotopy of a finite-dimensional linear subspace X_1 of E. We additionally require that (h'_t) be limited by $\{B(h(k \times I), \delta/4) : k \in K\}$ and $\lambda(y) = 0$ for $d(\alpha(y), 0) \ge \delta/4$. This yields that (h'_t) is limited by $\{B(h(k \times I), \delta/2) : k \in K\}$. Both isotopies (Λ_t) and (h'_t) are taken in such a way that suitable estimates of the proof of 1.5 can be carried over in order to get $d(\phi_1 | K, h_1) < \varepsilon$.

Proof of 6.4. Using the condition that $h((K\backslash K_0) \times I) \cap K_0 = \varnothing$ we infer that for every neighborhood U_0 of K_0 in K there exists an open neighborhood V_0 of K_0 such that $k' \in U_0$ provided $k \in V_0$ and $d(k, h_t(k')) < \delta$. Consider $U_0 = \{k \in K : d(h_t(k), k) < \varepsilon/4,\ t \in I\}$ and pick $V_0 \subseteq U_0$ and $\delta > 0$, $\delta < \varepsilon/4$, to fulfill the above condition. With $L = K\backslash V_0$, $(h_t)|L$ and $\{B(x, \delta(x)):$

$x \in h(L \times I)\}$ apply 6.5 to get an isotopy (ϕ_t) of E such that $\phi_0 = \mathrm{id}$, $d(\phi_1|L, h_1|L) < \varepsilon/2$, $\phi_t|(E\backslash U) \cup K_0 = \mathrm{id}$ for all $t \in I$ and (ϕ_t) is limited by $\{B(h(k \times I), \delta/2): k \in L\}$. We may assume that δ is so small that $\{B(h(k \times I), \delta): k \in L\}$ refines the collection $\{\mathrm{st}(h(k \times I), \mathscr{V}): k \in K\backslash K_0\}$. Therefore it is enough to check that $d(\phi_1|K, h_1) < \varepsilon$. Pick $k \in V_0\backslash K_0$. Then $\phi(K \times I)$ is contained in $B(h(k' \times I), \delta/2)$ for some $k' \in L$ and thus $d(k, h_t(k')) < \delta/2$. By the choice of δ we infer that $k' \in U_0$ which implies that $\mathrm{diam}(h(k' \times I)) < \varepsilon/2$. Finally we have

$$\begin{aligned} d(\phi_1(k), h_1(k)) &\leq d(\phi_1(k), k) + d(k, h_1(k)) \\ &\leq \mathrm{diam}(h(k' \times I)) + \delta + \varepsilon/4 < \varepsilon. \end{aligned}$$

The following consequence of Proposition 6.4 corresponds to the condition $(*)$ of 1.2.

6.6. **Corollary.** *For every pair of compacta (K, K_0) of E, a neighborhood U of the closure of $K\backslash K_0$ and a map $\varepsilon: U \to (0, \infty)$ there exists $\alpha > 0$ such that every embedding $h: K \to E$ with $h|K_0 = \mathrm{id}$ and $d(h, \mathrm{id}) < \alpha$ can be arbitrary closely approximated by restrictions $\overline{h}_1|K$, where $(\overline{h}_t)$ is an isotopy of E such that $\overline{h}_0 = \mathrm{id}$, $\overline{h}_t|K_0 \cup (E\backslash U) = \mathrm{id}$ and $d(\overline{h}_t(x), x) < \varepsilon(x)$ for all $x \in U$ and $t \in I$.*

We shall employ the following simple fact.

6.7. **Lemma.** *Suppose that an isotopy (ϕ_t) of E, $\phi_0 = \mathrm{id}$, is limited by $\{\mathrm{st}(h(K \times I), \mathscr{W}): k \in K\backslash K_0\}$, where $\mathscr{W}$ is a collection of open subsets of E such that $h((K\backslash K_0) \times I) \subset \mathscr{W}^+$. Then there exists a map $\varepsilon: \mathscr{W}^+ \to (0, \infty)$ such that for every isotopy (ψ_t) of E with $\psi_0 = \mathrm{id}$, $\psi_t|E\backslash\mathscr{W}^+ = \mathrm{id}$ and $d(\psi_t(x), x) < \varepsilon(x)$ for all $x \in \mathscr{W}^+$ and $t \in I$, the isotopy $(\psi_t \circ \phi_t)$ is limited by $\{\mathrm{st}(h(k \times I), \mathrm{st}(\mathscr{W})): k \in K\backslash K_0\}$.*

Proof. Choose $\varepsilon: \mathscr{W}^+ \to (0, \infty)$ with the property that whenever $h \in H(E)$, $h|E\backslash\mathscr{W}^+ = \mathrm{id}$ and $d(h(x), x) < \varepsilon(x)$, $x \in \mathscr{W}^+$, then h is limited by $\mathscr{W}$; i.e., for each x either $\{x, h(x)\}$ is contained in a member of $\mathscr{W}$ or it is one point. Now from $d(\psi_t(x), x) < \varepsilon(x)$ for some $t \in I$ it follows that $\{\psi_t(\phi_t(x)), \phi_t(x)\} \subset W$, $W \in \mathscr{W}$. Since $\phi(x \times I) \subset \mathrm{st}(h(k \times I), \mathscr{W})$ for some $k \in K\backslash K_0$ we infer that $\{\psi_t(\phi_t(x)): t \in I\} \subset \mathrm{st}(h(k \times I), \mathrm{st}(\mathscr{W}))$.

Proof of Theorem 6.1. Using 6.2 we may assume that the homotopy (h_t) satisfies $h((K\backslash K_0) \times I) \cap K_0 = \varnothing$. By 6.3 there is a collection $\mathscr{V}$ of open subsets of E such that $h((K \cap \mathscr{U}^+) \times I) \subset \mathscr{V}^+$ and $\{\mathrm{st}(h(k \times I), \mathscr{V}): k \in K\backslash K_0\}$ refines $\mathscr{U}$. Pick $\mathscr{W}$ open cover of $\mathscr{V}^+$ such that $\mathrm{st}(\mathscr{W})$ refines $\mathscr{V}$. A required isotopy can be found as $\overline{h}_t = \psi_t \circ \phi_t$ with (ϕ_t) and (ψ_t) satisfying the assumptions of 6.7. Consequently $\overline{h} = (\overline{h}_t)$ will be limited by $\mathscr{U}$. The isotopies (ϕ_t) and (ψ_t) will be chosen so that $\phi_1 = h_0$ and

$$\psi_1 = \lim(g_0^{-1} \circ \cdots \circ g_n^{-1} \circ g_{n+1}^{-1} \circ h_n \circ \cdots \circ h_1),$$

where $h_0, h_1, \ldots$ and $g_0, g_1, \ldots$ are those of the proof of 1.3. The isotopy (ϕ_t) is that of Proposition 5.4 applied to the quadruple $((K, K_0), U = \mathscr{W}^+, \varepsilon = \alpha, \mathscr{W})$, where α is that of 6.6 applied to the triple $((h_1(K), h_1(K_0)), U = \mathscr{W}^+, \varepsilon(x)/4)$. Notice that U contains the closure of $(h_1(K)\backslash h_1(K_0))$. Using 6.6 we proceed by the inductive construction of 1.3 to get isotopies $g^0, g^1, \ldots$ and $h^1, h^2, \ldots$ of E. Finally we let $\phi_t = \lim((g_t^{n+1} \circ \cdots \circ g_t^0)^{-1} \circ h_t^n \circ \cdots \circ h_t^1)$. The control functions $\varepsilon_n(x)$ are taken so small to get $d(\psi_t(x)) < \varepsilon(x)$ for some $x \in \mathscr{W}^+$. Since the convergence in the proof of 1.3 is uniform, the maps $(t, x) \to \psi_t(x)$ and $(t, x) \to \psi_t^{-1}(x)$ are continuous. Note that $\psi_t^{-1}(x) = \lim(h_t^n \circ \cdots \circ h_t^1)^{-1} \circ g_t^{n+1} \circ \cdots \circ g_t^0(x)$. Hence the proof of 6.1 is completed.

Now we present another result concerning the controlled isotopy extension. We drop the assumption that $\mathscr{U}^+$ contains the closure of $K\backslash \mathrm{Fix}(h)$.

6.8. **Theorem.** *Suppose that a homotopy (h_t) satisfies the hypothesis of* 6.1. *Then there exists an isotopy $(\overline{h}_t)$ limited by* $\mathrm{st}(\mathscr{U})$ *and fulfilling the assertion of* 6.1.

We need the following lemma in which we slightly strengthen the fact that open subsets U of E have the discrete approximation property (see [11]), i.e., for every map $f: \bigoplus_{n=1}^\infty K_n \to U$ of disjoint union of compacta (K_n) and for each open cover $\mathscr{W}$ of U there exists a map $g: \bigoplus_{n=1}^\infty K_n \to U$ such that the collection $\{g(K_n)\}$ is discrete in U and the family $\{f(x), g(x)\}$, $x \in \bigoplus_{n=1}^\infty K_n$, refines $\mathscr{W}$.

6.9. **Lemma.** *Let V be an open subset of E with $U \subseteq V$. Assume that the closure of* $\mathrm{im}(f)$ *in V is contained in U and that $\{f(L_n)\}$ is discrete in V, where L_n is a closed subset of K_n, $n = 1, 2, \ldots$. Then there exists $\overline{f}: \bigoplus_{n=1}^\infty K_n \to U$ such that $\{\overline{f}(K_n)\}$ is discrete in V, $\overline{f}|L_n = f|L_n$, $n = 1, 2, \ldots$, and $\{f(x), \overline{f}(x)\}$, $x = \bigoplus_{n=1}^\infty K_n$, refines $\mathscr{W}$.*

Proof. Let $\mathscr{U}' = \mathscr{U} \cup (V\backslash\overline{\mathrm{im}(f)})$. There exists $g: \bigoplus_{n=1}^\infty K_n \to V$ such that $\{(g(K_n)\}$ is discrete in V and $d(f(x), g(x)) < \varepsilon(f(x))$, where $\varepsilon: V \to (0, \infty)$ is a map such that $d(f(x), \phi(x)) < \varepsilon(f(x))$ implies $\{f(x), \phi(x)\}$ refines $\mathscr{U}$, see [11]. Since $\{f(L_n)\}$ is discrete in V we may assume that $\varepsilon(y) < 1/n$ for all $y \in f(L_n)$. Consider an open set $G_n = \{x \in K_n : d(f(x), g(x)) < 1/n$ and $\mathrm{dist}(f(x), f(L_n)) < 1/n\}$. We see that $L_n \subset G_n$, so there exists a map $\lambda: \bigoplus_{n=1}^\infty K_n \to [0, 1]$ with $\lambda|L_n = 1$ and $\lambda|K_n\backslash G_n = 0$, $n = 1, 2, \ldots$. Write $\overline{f}(x) = \lambda(x)f(x) + (1 - \lambda(x))g(x)$ for $x \in \bigoplus_{n=1}^\infty K_n$. Then

$$d(f(x), \overline{f}(x)) = d((1 - \lambda(x))(f(x) - g(x)), 0) \leq d(f(x), g(x)) < \varepsilon(f(x)).$$

From this it follows that $\{f(x), \overline{f}(x)\}$ refines $\mathscr{U}$ and $\mathrm{im}(\overline{f}) \subseteq U$. Let $\{\overline{f}(x_k)\}$, $x_k \in K_{n_k}$ and $n_1 < n_2 < \cdots$, be a sequence convergent in V. Since $\overline{f}|K_n\backslash G_n = g|K_n\backslash G_n$ we may assume that $x_k \in G_{n_k}$. The inequality $d(f(x), \overline{f}(x)) \leq d(f(x), g(x))$ yields the convergence of $\{f(x_k)\}$. Since $\mathrm{dist}(f(x_k), f(L_{n_k})) < 1/n_k$ the above inequality contradicts the fact that $\{f(L_n)\}$ was discrete.

Proof of Theorem 6.8. First, according to 6.2, we replace the original homotopy by a homotopy h such that $h((K\backslash K_0)\times I)\cap K_0=\varnothing$, where $K_0=\mathrm{Fix}(h)$. Pick a sequence $\{U_n\}_{n=1}^{\infty}$ of open sets of K such that $K=U_0\supset\overline{U}_1\supset U_1\supset\overline{U}_2\supset U_2\supset\cdots$ and $\bigcap_{n=1}^{\infty}U_n=K_0$. Write $A_i=\overline{U}_{2i-2}\backslash U_{2i-1}$ and $B_i=\overline{U}_{2i-1}\backslash U_{2i}$, $i=1,2,\ldots$. We are going to construct a homotopy $h':K\times I\to E$ satisfying the following conditions: $h'|K\times\{0,1\}=h|K\times\{0,1\}$, both collections $\{h'(A_i\times I)\}$ and $\{h'(B_i\times I)\}$ are discrete in $E\backslash K_0$ and h' is limited by $\mathscr{U}$. Let $\mathscr{V}$ be a collection of open sets in E such that $\mathscr{V}^+$ contains $h((K\backslash K_0)\times I)$ and $\{\mathrm{st}(h(k\times I),\mathscr{V}):k\in K\backslash K_0\}$ refines $\mathscr{U}$. There exists an open cover $\mathscr{W}$ such that $\mathrm{st}^2(\mathscr{W})$ refines $\mathscr{V}$. We apply 6.9 and approximate $\{h|A_i\times I\}$ by $\{g_1|A_i\times I\}$ so that $h=g_1$ on $A_i\times\{0,1\}$, the collection $\{g_1(A_i\times I)\}$ is discrete in $E\backslash K_0$ and $\{h(x),g_1(x)\}$, $x\in\bigcup A_i\times I$, refines $\mathscr{W}$. Note that the closure of $h((K\backslash K_0)\times I)$ in $E\backslash K_0$ is contained in $\mathscr{W}^+$. Next $\{h|B_i\times I\}$ can be approximated by $\{g_2|B_i\times I\}$ so that $g_2=h$ on $B_i\times\{0,1\}$ and $g_2=g_1$ on $(\partial B_i)\times I$, where ∂B_i is the boundary of B_i in K, $\{g_2(B_i\times I)\}$ is discrete in $E\backslash K_0$ and $\{h(x),g_2(x)\}$, $x\in\bigcup B_i\times I$, refines $\mathrm{st}^2(\mathscr{W})$. To make this possible first we find a map $f:(K\backslash K_0)\times I\to E$ so that $f=g_1$ on $\bigcup(A_i\times I)\cup(K\backslash K_0)\times\{0,1\}$ and $\{h(x),f(x)\}$, $x\in(K\backslash K_0)\times I$, refines $\mathrm{st}(\mathscr{W})$, and then we apply 6.9 to the map $f|\bigcup B_i\times I$, we see that $h'|(K\backslash K_0)\times I$ given by $h'|A_i\times I=g_1|A_i\times I$ and $h'|B_i\times I=g_2|B_i\times I$ has the required properties. We may choose g_1 and g_2 in order that h' can be continuously extended to $K_0\times I$ by letting $h'(x,t)=x$ for $(x,t)\in K_0\times I$. Further on use h to denote the homotopy h'. We enlarge the elements of $\{h(A_i\times I)\}$ and $\{h(B_i\times I)\}$ in order that the collections $\{B(h(A_i\times I),\delta_i)\}$ and $\{B(h(B_i\times I),\delta_i)\}$ remain discrete in $E\backslash K_0$; here $\delta_i>0$. For each i, let $\lambda_i:K\to I$ be a map satisfying $\lambda_i|A_i=1$ and $h(x,\lambda_i(x))\in B(h(A_i\times I),\delta_i/2)$ provided that $\lambda_i(x)\neq 0$. Each map λ_i determines a homotopy $f^i:K\times I\to E$ by $f^i(x,t)=h(x,\lambda_i(x)\cdot t)$. Clearly, we have $f_0^i=\mathrm{id}$, $f_1^i(x,t)=h(x,\lambda_i(x))$, $f^i=h$ on $A_i\times I$, the family $\{f^i((K\backslash\mathrm{Fix}(f^i))\times I)\}$ is discrete in $E\backslash K_0$, and f^i is limited by $\mathscr{U}$. It is easily seen that the family (f^i) determines a homotopy $f:K\times I\to E$ equal to f^i on $(K\backslash\mathrm{Fix}(f^i))\times I$ and the identity elsewhere. Next we approximate $f_1:K\to E$ by an embedding $u:K\to E$ such that $u|A_i=f_1^i|A_i=h_i|A_i$ and $u|\mathrm{Fix}(f)=\mathrm{id}$. We can do it applying 4.3 to each f^i. Consider the segment homotopy $u_t(x)=tu(x)+(1-t)f_1(x)$ connecting f_1 with the embedding $u_1=u$. A homotopy $g:K\times I\to E$ which is the composition of f and (u_t) joins the identity with u. Consider the homotopy $g^i:K\times I\to E$ equal to (u_t) on $K\backslash\mathrm{Fix}(F^i)$ and $u_i(x)=x$ elsewhere. If u is close enough to f_1 then $g^i((K\backslash\mathrm{Fix}(g^i))\times I)\subset B(h(A_i\times I),\delta_i)$ and therefore the collection $\{g^i((K\backslash\mathrm{Fix}(g^i))\times I)\}$ is discrete in $E\backslash K_0$. Moreover if u is sufficiently close to f_1 then g, and consequently g^i, will be limited by $\mathscr{U}$. The next step is to connect homotopically g_1 with h_1. Observe that the homotopy (u_{1-t}) joins u with $f_1(x) = h(x, \lambda(x))$ while the homotopy $(x, t) \to h(x, (1-t)\lambda(x) + t)$ joins f_1 with h_1. Thus their composition (p_t') joins u with h_1. Consider $C_i = u(B_i)$ and let $p_t^i(y) = p_t'(u^{-1}(y))$ for $y \in C_i$ and $i = 1, 2, \ldots$. Then (p_t^i) joins id_{C_i} with $h_1 \circ u^{-1}|C_i$. Since $p_t^i(y) = y$ for $y \in \partial C_i$, the boundary ∂C_i is taken in $u(K)$, the homotopy (p_t^i) can be extended to the whole $u(K)$ by letting $p_t^i(y) = y$ off C_i. If the map u is sufficiently close to f_1 then $p^i((u(K)\setminus \mathrm{Fix}(p^i)) \times I) \subset B(h(B_i \times I), \delta_i)$ and thus the collection $\{p^i((u(K)\setminus \mathrm{Fix}(p^i)) \times I)\}$ will be discrete in $E\setminus K_0$, moreover p^i will be limited by $\mathscr{U}$. Now to each homotopy g^i and p^i we apply Theorem 6.1. There are isotopies G^i and P^i of E such that $G_0^i = \mathrm{id}, P_0^i = \mathrm{id}$, $G_1^i = u$ on $K\setminus \mathrm{Fix}(f^i)$, $P_1^i = p_1^i$ and $\mathrm{Fix}(P^i) = \mathrm{Fix}((p_t^i))$. The isotopies G^i and P^i are limited by a collection $\mathscr{V}$ finer than $\mathscr{U}$ in order that the families $\{E\setminus \mathrm{Fix}(G^i)\}$ and $\{E\setminus \mathrm{Fix}(P^i)\}$ are discrete in $E\setminus K_0$ and the sequences $\sup\{\mathrm{dist}(y, E\setminus \mathrm{Fix}(g^i)) : y \notin \mathrm{Fix}(G^i)\}$ and $\sup\{\mathrm{dist}(y, E\setminus \mathrm{Fix}(p^i)) : y \notin \mathrm{Fix}(P^i)\}$ tend to zero. We see that the isotopies (G^i) and (P^i) determine isotopies G and P on $E\setminus K_0$ given by $G(x) = G^i(x)$ for $x \in E\setminus \mathrm{Fix}(G^i)$ and $P(x) = P^i(x)$ for $x \in E\setminus \mathrm{Fix}(P^i)$. It can be easily checked that, letting $G(x, t) = P(x, t) = x$ for $(x, t) \in K_0 \times I$, G and P extend to isotopies of the whole E. Finally the composition of G and P defines a required isotopy $\overline{h}$ which is limited by $\mathrm{st}(\mathscr{U})$.

6.10. *Remark.* Let $K' = K \times I \sim$, where $(x_1, t) \sim (x_2, s)$ iff $x_1 = x_2 \in \mathrm{Fix}(h)$. Furthermore, let $h' : K' \to E$ be defined by $h'([(x, t)]) = h_t(x)$. (Note that K' is a compact metrizable space.) If we assume that K' admits an embedding into E then in the assertion of 6.8 we may additionally require that the isotopy $\overline{h}$ is limited by $\mathscr{U}$. In this case we may replace the original homotopy by a homotopy h such that $h|(K\setminus K_0) \times I$ is an embedding. Then use the fact that for every $\varepsilon > 0$, there exists $\delta_n > 0$ such that $\sup\{d(h_s(x), h_s(y)) : s \in I\} < \varepsilon$ provided $d(h_t(x), h_u(y)) < \delta_n$ for some $t, u \in I$ and every $x, y \in A_n \cup B_n \cup A_{n-1}$.

Here is our main result concerning controlled isotopy extension for local compacta.

6.11. **Theorem.** *Let E be an infinite-dimensional l.c.m.l.s. and let A be its closed locally compact subset. Let $h = (h_t)$ be a homotopy of A into E such that $h_0 = \mathrm{id}$ and $h_1 : A \to h_1(A)$ is a closed embedding. Assume that h is limited by a collection $\mathscr{U}$ of open subsets of E. If the closure of $h(A\setminus \mathrm{Fix}(h))$ is contained in $\mathscr{U}^+$ then there exists an isotopy $(\overline{h}_t)$ of E limited by $\mathrm{st}(\mathscr{U})$ with $\overline{h}_0 = \mathrm{id}$, $\overline{h}_1|A = h_1$ and $\overline{h}_t|\mathrm{Fix}(h) = \mathrm{id}$, $t \in I$.*

Proof. We repeat the argument of 6.8. We replace the original homotopy by a homotopy h with $h((A\setminus A_0) \times I) \cap A_0 = \varnothing$, where $A_0 = \mathrm{Fix}(h)$. Then we pick a sequence $\{U_n\}_{n=0}^{\infty}$ of open subsets of A such that each $A\setminus U_n$ is compact, $A = U_0 \supset \overline{U}_1 \supset U_1 \supset \overline{U}_2 \supset U_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} U_n = \varnothing$. Define the sets A_i

and B_i as in the proof of 6.8. The condition that $\mathrm{cl}(h((A\backslash A_0) \times I)) \subset \mathscr{U}^+$ allows us to modify Lemma 6.9 so that we are able to construct a homotopy $h': A \times I \to E$ which is limited by $\mathscr{U}$ and such that $h'|A \times \{0,1\} = h|A \times \{0,1\}$ and the collections $\{h'(A_i \times I)\}$ and $\{h'(B_i \times I)\}$ are discrete in the whole space E. Further on we proceed as in the proof of 6.8. While constructing G^i and P^i we additionally require that $G_t^i|A_0 = P_t^i|A_0 = \mathrm{id}$, $t \in I$ and $i = 1, 2, \ldots$. This time the families $\{E \backslash \mathrm{Fix}(G^i)\}$ and $\{E \backslash \mathrm{Fix}(P^i)\}$ are discrete in the whole E. Hence the step of extending G and P to A_0 can be omitted. An isotopy $\overline{h}$ is the composition of G and P.

An immediate consequence of Theorem 6.11 is the following generalization of 2.12.

6.12. **Corollary.** *Every infinite-dimensional locally convex metric linear space E has the homeomorphism extension property for local compacta, i.e., every homeomorphism between closed local compacta of E extends to a homeomorphism of E.*

Let $\overline{E}$ be an infinite-dimensional l.c.m.l.s. containing E as a dense linear subspace. All the isotopies $(\overline{h}_t)$ appearing in 6.4, 6.6 and in the proofs of 6.1, 6.8, 6.11 and 6.12 can be taken to satisfy $\overline{h}_t \in H(\overline{E}|E)$, $t \in I$. This makes it possible to claim

6.13. *Remark.* In Theorems 6.1, 6.8, 6.11 and 6.12 we may additionally require that $\overline{h}_t \in H(\overline{E}|E)$, $t \in I$, and $\overline{h} \in H(\overline{E}|E)$, respectively.

References

1. R. D. Anderson, *On sigma-compact subsets of infinite-dimensional spaces*, unpublished manuscript.
2. R. D. Anderson and J. D. McCharen, *On extending homeomorphisms to Fréchet manifolds*, Proc. Amer. Math. Soc. **25** (1970), 283–289.
3. C. Bessaga and A. Pełczyński, *The estimated extension theorem homogeneous collections and skeletons and their applications to the topological classification of linear metric spaces and convex sets*, Fund. Math. **69** (1970), 153–190.
4. ——, *Selected topics in infinite-dimensional topology*, Polish Scientific Publishers, Warsaw, 1975.
5. J. Bryant, *On embeddings of compacta in euclidean space*, Proc. Amer. Math. Soc. **23** (1969), 46–51.
6. D. Curtis, T. Dobrowolski and J. Mogilski, *Some applications of the topological characterization of the sigma-compact spaces l_2^f and Σ*, Trans. Amer. Math. Soc. **284** (1984), 837–846.
7. T. Dobrowolski, *Smooth and R-analytic negligibility of subsets and extension of homeomorphisms in Banach spaces*, Studia Math. **65** (1979), 115–139.
8. ——, *On extending mappings into nonlocally convex linear metric spaces*, Proc. Amer. Math. Soc. **93** (1985), 555–560.
9. ——, *The compact Z-set property in convex sets*, Topology Appl. **23** (1986), 163–172.
10. T. Dobrowolski and J. Mogilski, *Sigma-compact locally convex metric linear spaces universal for compacta are homeomorphic*, Proc. Amer. Math. Soc. **78** (1982), 653–658.

11. T. Dobrowolski and H. Toruńczyk, *Separable complete ANRs admitting group structures are Hilbert manifolds*, Topology Appl. **12** (1981), 229–235.
12. H. Gluck, *Embeddings in trivial range*, Ann. of Math. (2) **81** (1965), 195–210.
13. T. Homma, *On the embedding of compacta in manifolds*, Yokohama Math. J. **10** (1962), 5–10.
14. V. L. Klee, *Some topological properties of convex sets*, Trans. Amer. Math. Soc. **78** (1955), 30–45.
15. E. Michael, *Convex structures and continuous selections*, Canad. J. Math. **11** (1954), 556–575.
16. M. A. Štanko, *The embedding of compacta in euclidean space*, Mat. Sb. **12** (1970), 234–254.
17. H. Toruńczyk, *Skeletonized sets in complete metric spaces and homeomorphisms of the Hilbert cube*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **18** (1970), 119–126.
18. ——, *Smooth partitions of unity in some non-separable Banach spaces*, Studia Math. **46** (1973), 43–51.
19. ——, *Concerning locally homotopy negligible sets and characterizations of l_2-manifolds*, Fund. Math. **101** (1978), 93–110.
20. ——, *Characterizing Hilbert space topology*, Fund. Math. **111** (1981), 247–262.

INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, PKiN IX P., 00-901 WARSAW, POLAND