

SMALL ZEROS OF QUADRATIC FORMS OVER NUMBER FIELDS. II

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ABSTRACT. Let F be a nontrivial quadratic form in N variables with coefficients in a number field k and let \mathcal{Z} be a subspace of k^N of dimension M , $1 \leq M \leq N$. If F restricted to \mathcal{Z} vanishes on a subspace of dimension L , $1 \leq L < M$, and if the rank of F restricted to \mathcal{Z} is greater than $M - L$, then we show that F must vanish on $M - L + 1$ distinct subspaces $\mathcal{Z}_0, \mathcal{Z}_1, \dots, \mathcal{Z}_{M-L}$ in \mathcal{Z} each of which has dimension L . Moreover, we show that for each pair $\mathcal{Z}_0, \mathcal{Z}_l$, $1 \leq l \leq M - L$, the product of their heights $H(\mathcal{Z}_0)H(\mathcal{Z}_l)$ is relatively small. Our results generalize recent work of Schlickewei and Schmidt.

1. INTRODUCTION

Let

$$(1.1) \quad F(\mathbf{x}, \mathbf{y}) = \sum_{m=1}^N \sum_{n=1}^N \varphi_{mn} y_m x_n$$

be a symmetric bilinear form with coefficients $\varphi_{mn} = \varphi_{nm}$ in an algebraic number field k . We write $\Phi = (\varphi_{mn})$ for the associated $N \times N$ matrix and $F(\mathbf{x}) = F(\mathbf{x}, \mathbf{x})$ for the associated quadratic form. As in our earlier paper [14] we will consider F restricted to a fixed subspace $\mathcal{Z} \subseteq k^N$ of dimension M , $1 \leq M \leq N$, and define

$$\mathcal{Z}^{(0)} = \{\mathbf{z} \in \mathcal{Z} : F(\mathbf{z}) = 0\}.$$

A basic problem in this situation is to show that if $\mathcal{Z}^{(0)}$ is not trivial then it necessarily contains vectors or subspaces of small height. Beginning with a result of Cassels [3, 4], the papers [1, 5–10, 12–15], are all directed at this type of problem. In case $k = \mathbb{Q}$ and $M = N$ Schlickewei and Schmidt [10] have recently proved a theorem which includes most of the previous results. Our purpose in the present paper is to generalize the work of Schlickewei and Schmidt to an arbitrary number field k and to the case where \mathcal{Z} may be a proper subspace of k^N . As we have already noted in [14], the introduction of the subspace \mathcal{Z} is equivalent to considering the simultaneous zeros of the

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quadratic form F and a system of $N - M$ independent linear forms. If A is an $(N - M) \times N$ matrix over k with $\text{rank}(A) = N - M$ and

$$\mathcal{Z} = \{\mathbf{x} \in k^N : A\mathbf{x} = \mathbf{0}\},$$

then by a basic duality principle (equation (2.6) of [14]) the subspace \mathcal{Z} and the matrix A have the same height. For this reason all of our results which we state for a quadratic form F restricted to \mathcal{Z} have obvious analogs for the set of simultaneous solutions in k^N of $F(\mathbf{x}) = 0$ and $A\mathbf{x} = \mathbf{0}$.

We suppose that the number field k has degree d over \mathbb{Q} . Our notation for places, completions of k , normalized absolute values, heights and measures will be identical to that which was already described in [14, §2]. Also, we will assume that F restricted to \mathcal{Z} has rank r . Our main result which generalizes Theorem 1 in [10] is as follows.

Theorem 1. *Suppose that $\mathcal{Z}^{(0)}$ contains an L -dimensional subspace, $1 \leq L < M$, and F restricted to \mathcal{Z} has rank $r > M - L$. Then there exist $M - L + 1$ distinct L -dimensional subspaces $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_{M-L}$ in $\mathcal{Z}^{(0)}$ with the following properties:*

- (i) *for each $l = 1, 2, \dots, M - L$ the subspace $\mathcal{Z}_0 \cap \mathcal{Z}_l$ has dimension $L - 1$;*
- (ii) *the union $\mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \dots \cup \mathcal{Z}_{M-L}$ spans the subspace \mathcal{Z} ;*
- (iii) *for each $l = 1, 2, \dots, M - L$,*

$$(1.2) \quad H(\mathcal{Z}_0)^2 \leq H(\mathcal{Z}_0)H(\mathcal{Z}_l) \leq \{2c_k(M - L)^2 \mathcal{H}(\Phi)\}^{M-L} \{2c_k(1)H(\mathcal{Z})\}^2.$$

(For each positive integer n , $c_k(n)$ is a field constant defined in [14, §2] and in (2.2) below.)

In our previous result [14, Theorem 1] we made no assumption concerning the rank of F restricted to \mathcal{Z} but instead we assumed that the integer L was maximal. That is, we assumed that L was the largest integer such that $\mathcal{Z}^{(0)}$ contains a subspace of dimension L . We note that in the present paper we are *not* assuming that L is maximal. Of course $\mathcal{Z}^{(0)}$ must contain the subspace

$$\mathcal{Z}^\perp = \{\mathbf{z} \in \mathcal{Z} : F(\boldsymbol{\zeta}, \mathbf{z}) = 0 \text{ for all } \boldsymbol{\zeta} \in \mathcal{Z}\},$$

and $\dim(\mathcal{Z}^\perp) = M - r$. Thus our hypothesis in Theorem 1 concerning the rank of F restricted to \mathcal{Z} could be stated this way: we assume that $\mathcal{Z}^{(0)}$ contains a subspace of dimension L with $L > \dim(\mathcal{Z}^\perp)$. It turns out that the method used to prove Theorem 1 also provides a bound on the height of the subspace \mathcal{Z}^\perp .

Theorem 2. *If $1 \leq r < M$ then*

$$(1.3) \quad H(\mathcal{Z}^\perp) \leq c_k(r)^r \mathcal{H}(\Phi)^{r/2} H(\mathcal{Z}).$$

Suppose that in Theorem 1 the quadratic form F restricted to \mathcal{Z} has rank $r = M$. Then we may take $L = 1$. We find that if F has a nontrivial zero

in \mathcal{Z} , then there is a basis $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{M-1}\}$ of \mathcal{Z} such that $F(\mathbf{x}_m) = 0$, $m = 0, 1, \dots, M - 1$, and

$$(1.4) \quad H(\mathbf{x}_0)H(\mathbf{x}_l) \leq \{2c_k(M - 1)^2 \mathcal{H}(\Phi)\}^{M-1} \{2c_k(1)H(\mathcal{Z})\}^2,$$

for $l = 1, 2, \dots, M - 1$. Obviously (1.4) implies that

$$(1.5) \quad \begin{aligned} &H(\mathbf{x}_0)^{M-1} H(\mathbf{x}_1)H(\mathbf{x}_2) \cdots H(\mathbf{x}_{M-1}) \\ &\leq \{2c_k(M - 1)^2 \mathcal{H}(\Phi)\}^{(M-1)^2} \{2c_k(1)H(\mathcal{Z})\}^{2(M-1)}. \end{aligned}$$

The bounds (1.4) and (1.5) generalize results of Schulze-Pillot [13, Theorem 2] and Chalk [5].

If F and \mathcal{Z} satisfy the hypotheses in Theorem 1 then $\mathcal{Z}^{(0)}$ contains the L -dimensional subspace \mathcal{Z}_0 , and the height $H(\mathcal{Z}_0)$ is bounded by (1.2). In fact the slightly sharper bound

$$(1.6) \quad H(\mathcal{Z}_0) \leq \{2c_k(M - L)^2 \mathcal{H}(\Phi)\}^{(M-L)/2} H(\mathcal{Z})$$

follows immediately from Theorem 4 and equation (5.11) below. Theorem 2 and (1.6) provide a sharper and more general formulation of our previous result [14, Theorem 1].

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2. PRELIMINARY LEMMAS

If v is a place of k we write k_v for the completion of k at v . Let $\mathcal{A} \subseteq (k)^N$ be a subspace of dimension L , $1 \leq L < N$, and let $\mathcal{A}_v \subseteq (k_v)^N$ be the completion of \mathcal{A} in $(k_v)^N$. We shall make frequent use of the $N \times N$ projection matrices $P_v = P_v(\mathcal{A}_v)$ which were defined in [14, §4]. Here we simply summarize the main results concerning $P_v(\mathcal{A}_v)$ which we will need. Complete proofs and further details can be found in [14].

At each place v the matrix P_v is a projection operator in the usual sense:

$$P_v \mathbf{x} \in \mathcal{A}_v \quad \text{for all } \mathbf{x} \in (k_v)^N,$$

and

$$P_v \mathbf{x} = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathcal{A}_v.$$

Let $\mathbf{b} \in (k)^N \setminus \mathcal{A}$ so that

$$\mathcal{B} = \text{span}_k\{\mathcal{A}, \mathbf{b}\}$$

is a subspace of dimension $L + 1$. The height of the subspaces \mathcal{A} and \mathcal{B} are related by the identity

$$(2.1) \quad H(\mathcal{B}) = H(\mathcal{A}) \prod_v H_v\{(1_N - P_v)\mathbf{b}\}.$$

This follows from [14, Lemma 4].

If $\mathcal{A} \subseteq k^N$ we shall sometimes simplify our notation and write $P_v(\mathcal{A})$ for projection onto the completion of \mathcal{A} in $(k_v)^N$.

Next we require a lemma whose proof uses methods from geometry of numbers over adèle spaces. The relevant definitions concerning this subject are contained in [2, pp. 16–18]. We write k_A for the adèle ring of the number field k and if v is a finite place of k then

$$O_v = \{x \in k_v : |x|_v \leq 1\}$$

denotes the maximal compact subring of k_v . Since the additive group of k_v is locally compact we may select a normalized Haar measure β_v on k_v as follows:

- (i) If $v \mid p$, where p is a (finite) rational prime, we require that $\beta_v(O_v) = |\mathcal{D}_v|_v^{d/2}$. Here \mathcal{D}_v is the local different of k at v .
- (ii) If $k_v = \mathbb{R}$ then β_v is ordinary Lebesgue measure on \mathbb{R} .
- (iii) If $k_v = \mathbb{C}$ then β_v is Lebesgue measure on the complex plane multiplied by 2.

The product measure $\beta = \prod_v \beta_v$ then induces a normalized Haar measure (also denoted by β) on k_A . If $(k_A)^N$ is an N -fold product of adèle spaces we write V for the product Haar measure β^N on $(k_A)^N$.

At each infinite place v we define a positive real number $r_v(N)$ so that

$$\beta_v^N(\{\mathbf{u} \in (k_v)^N : \|\mathbf{u}\|_v < r_v(N)\}) = 1.$$

The exact value of $r_v(N)$ is given in [14, §2]. We also define

$$(2.2) \quad c_k(N) = \left\{ 2|\Delta_k|^{1/2d} \prod_{v|\infty} (r_v(N))^{d_v/d} \right\},$$

where Δ_k is the discriminant of k and $d_v = [k_v : \mathbb{Q}_v]$ is the local degree. The quantity $c_k(N)$ will occur as a field constant.

Now suppose that A_v is an $N \times M$ matrix over k_v with

$$\text{rank}(A_v) = M \leq N.$$

Let $\xi_v \neq \mathbf{0}$ be a vector in $(k_v)^N$. If $v \mid \infty$ we set

$$S_v = \{\mathbf{u} \in (k_v)^M : \|A_v \mathbf{u}\|_v < \|\xi_v\|_v\}.$$

It follows easily that

$$\beta_v^M(S_v) = r_v(M)^{-Md_v} \|\xi_v\|_v^{Md_v} H_v(A_v)^{-d}.$$

If $v \nmid \infty$ we write

$$S_v = \{\mathbf{u} \in (k_v)^M : \|A_v \mathbf{u}\|_v \leq \|\xi_v\|_v\}.$$

By using the v -adic cube slicing identity, which is (4.8) and (4.9) of [2], the Haar measure of S_v is given by

$$\beta_v^M(S_v) = |\mathcal{D}_v|_v^{Md/2} \|\xi_v\|_v^{Md_v} H_v(A_v)^{-d}.$$

If we assume that $S_v = (O_v)^M$ at almost all finite places v , then the set

$$\mathcal{S} = \prod_v S_v$$

is a subset of the M -fold product $(k_A)^M$. Hence the Haar measure of \mathcal{S} is given by

$$(2.3) \quad V(\mathcal{S}) = \prod_v \beta_v^M(S_v) = |\Delta_k|^{-M/2} \left\{ \prod_v |\xi_v|_v^M H_v(A_v)^{-1} \right\}^d \left\{ \prod_{v|\infty} r_v(M)^{-Md_v} \right\}.$$

(In (2.3) we have used the identity

$$\prod_{v \nmid \infty} |\mathcal{D}_v|_v^{d/2} = |\Delta_k|^{-1/2}.)$$

If $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_M < \infty$ denote the successive minima of \mathcal{S} then by the adèlic form of Minkowski's second theorem (this is Theorem 3 of [2]) we have

$$(\lambda_1 \lambda_2 \dots \lambda_M)^d V(\mathcal{S}) \leq 2^{dM}.$$

Using (2.3) this may be written as

$$(2.4) \quad (\lambda_1 \lambda_2 \dots \lambda_M) \leq c_k(M)^M \left(\prod_v |\xi_v|_v^{-M} H_v(A_v) \right).$$

To simplify the statement of the following lemma we write

$$G(\mathbf{x}) = \prod_v H_v\{(1_N - P_v)\mathbf{x}\}$$

for $\mathbf{x} \in k$ and $P_v = P_v(\mathcal{A}_v)$.

Lemma 3. *Let ξ and ζ be linearly independent vectors in $(k)^N \setminus \mathcal{A}$. Then there exists a scalar $\alpha \neq 0$ in k such that*

$$G(\alpha\zeta + \xi) \leq 2c_k(1) \max\{G(\zeta), G(\xi)\}.$$

Proof. We use the inequality (2.4) with $M = 1$, $A_v = (1_N - P_v)\zeta$, and $\xi_v = (1_N - P_v)\xi$. For almost all finite places we have

$$\|A_v\|_v = \|\xi_v\|_v = 1$$

and therefore $S_v = (O_v)^M$ at almost all finite places. Hence the inequality (2.4) holds. It follows that there exists $\alpha \neq 0$ in k such that $\alpha \in \lambda\mathcal{S}$ for all $\lambda > \lambda_1$. That is,

$$\|(1_N - P_v)\alpha\zeta\|_v \leq \lambda_1 \|(1_N - P_v)\xi\|_v$$

if $v | \infty$ and

$$\|(1_N - P_v)\alpha\zeta\|_v \leq \|(1_N - P_v)\xi\|_v$$

if $v \nmid \infty$. Thus we have

$$\prod_v |(1_N - P_v)(\alpha\boldsymbol{\zeta} + \boldsymbol{\xi})|_v \leq \left\{ \prod_{v|\infty} (1 + \lambda_1)^{d_v/d} \right\} \prod_v |(1_N - P_v)\boldsymbol{\xi}|_v,$$

or

$$(2.5) \quad G(\alpha\boldsymbol{\zeta} + \boldsymbol{\xi}) \leq (1 + \lambda_1)G(\boldsymbol{\xi}).$$

Of course (2.4) may be written as

$$\lambda_1 \leq c_k(1)G(\boldsymbol{\zeta})^{-1}G(\boldsymbol{\xi})$$

and therefore (2.5) implies that

$$G(\alpha\boldsymbol{\zeta} + \boldsymbol{\xi}) \leq G(\boldsymbol{\xi}) + c_k(1)G(\boldsymbol{\zeta}) \leq 2c_k(1) \max\{G(\boldsymbol{\zeta}), G(\boldsymbol{\xi})\}.$$

This proves the lemma.

Let $F(\mathbf{x}, \mathbf{y})$ be the bilinear form defined in (1.1) by the $N \times N$ symmetric matrix $\Phi = (\varphi_{mn})$. At each place v of k the local height \mathcal{H}_v is defined by

$$\mathcal{H}_v(\Phi) = \max_{1 \leq m, n \leq N} |\varphi_{mn}|_v \quad \text{if } v \nmid \infty,$$

$$\mathcal{H}_v(\Phi) = \left\{ \sum_{m=1}^N \sum_{n=1}^N \|\varphi_{mn}\|_v^2 \right\}^{d_v/2d} \quad \text{if } v | \infty.$$

The global height of Φ is then given by

$$\mathcal{H}(\Phi) = \prod_v \mathcal{H}_v(\Phi).$$

If $\boldsymbol{\xi}_v$ and $\boldsymbol{\zeta}_v$ are vectors in $(k_v)^N$ we have

$$(2.6) \quad |F(\boldsymbol{\xi}_v, \boldsymbol{\zeta}_v)|_v \leq \mathcal{H}_v(\Phi) |\boldsymbol{\xi}_v|_v |\boldsymbol{\zeta}_v|_v.$$

This follows immediately from the ultrametric inequality if $v \nmid \infty$ and from the Cauchy-Schwarz inequality if $v | \infty$.

Now suppose that $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are vectors in k^N , F vanishes identically on $\mathcal{A} \subseteq \mathcal{Z} \subseteq k^N$ and $\boldsymbol{\zeta} \in \mathcal{A}^\perp$, where

$$\mathcal{A}^\perp = \{\mathbf{z} \in \mathcal{Z} : F(\mathbf{a}, \mathbf{z}) = 0 \text{ for all } \mathbf{a} \in \mathcal{A}\}.$$

By continuity F vanishes identically on the completion \mathcal{A}_v of \mathcal{A} in $(k_v)^N$. It follows that $F(P_v\boldsymbol{\xi}, \boldsymbol{\zeta}) = 0$ and therefore

$$(2.7) \quad F(\boldsymbol{\xi}, \boldsymbol{\zeta}) = F((1_N - P_v)\boldsymbol{\xi}, \boldsymbol{\zeta}).$$

If $F(\xi, \zeta) \neq 0$, then (2.6), (2.7) and the product formula imply that

$$\begin{aligned} 1 &= \prod_v |F(\xi, \zeta)|_v \\ &= \prod_v |F((1_N - P_v)\xi, \zeta)|_v \\ &\leq \prod_v \{ \mathcal{H}_v(\Phi) H_v((1_N - P_v)\xi) H_v(\zeta) \} \\ &= \mathcal{H}(\Phi) \left\{ \prod_v H_v((1_N - P_v)\xi) \right\} H(\zeta). \end{aligned}$$

This type of argument will be used frequently.

3. TWO BASIC THEOREMS

Our proof of Theorem 1 divides naturally into two parts and each part can be easily formulated as a separate result. We assume throughout §§3–6 of this paper that the hypotheses in Theorem 1 hold. Then among all L -dimensional subspaces contained in $\mathcal{Z}^{(0)}$ let \mathcal{X} be an L -dimensional subspace with minimal height. As the set of L -dimensional subspaces of \mathcal{Z} having height less than a positive constant is a finite set, such a subspace \mathcal{X} clearly exists. We define

$$\mathcal{X}^\perp = \{z \in \mathcal{Z} : F(x, z) = 0 \text{ for all } x \in \mathcal{X}\}.$$

Since F restricted to \mathcal{X} has rank $r > M - L$, it follows that \mathcal{X}^\perp is a proper subspace of \mathcal{Z} .

Theorem 4. *Suppose that ξ is a vector in $\mathcal{Z} \setminus \mathcal{X}^\perp$ and let \mathcal{Y} be the $(L + 1)$ -dimensional subspace*

$$\mathcal{Y} = \text{span}_k \{ \mathcal{X}, \xi \}.$$

Then there exists a subspace $\mathcal{X}' \subseteq \mathcal{Y}$ such that

- (i) F vanishes identically on \mathcal{X}' ,
- (ii) $\dim(\mathcal{X}') = L$ and $\dim(\mathcal{X} \cap \mathcal{X}') = L - 1$,
- (iii) $H(\mathcal{X}')^2 \leq H(\mathcal{X})H(\mathcal{X}') \leq 2\mathcal{H}(\Phi)H(\mathcal{Y})^2$,
- (iv) $1 \leq 2\mathcal{H}(\Phi) \{ \prod_v H_v((1_N - P_v)\xi) \}^2$,

where $P_v = P_v(\mathcal{X}_v)$ is projection onto the completion \mathcal{X}_v of \mathcal{X} in $(k_v)^N$.

The proof of this result is similar to our proof of [14, Theorem 1]. In fact we have made some technical simplifications which lead to the sharper inequality (1.6).

Theorem 5. *There exist $M - L$ linearly independent vectors z_1, z_2, \dots, z_{M-L} in $\mathcal{Z} \setminus \mathcal{X}^\perp$ such that*

$$(3.1) \quad \mathcal{Z} = \text{span}_k \{ \mathcal{X}, z_1, z_2, \dots, z_{M-L} \},$$

and each of the subspaces

$$(3.2) \quad \mathcal{Y}_l = \text{span}_k \{ \mathcal{X}, z_l \}, \quad l = 1, 2, \dots, M - L,$$

satisfies

$$(3.3) \quad H(\mathcal{Y}_1) \leq 2c_k(1)\{c_k(M-L)\}^{M-L}\{2\mathcal{H}(\Phi)\}^{(M-L-1)/2}H(\mathcal{Z}).$$

In order to prove Theorem 1 from these results we set $\mathcal{X} = \mathcal{X}_0$. Then with $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{M-L}$ as in Theorem 5 we apply Theorem 4 with $\xi = \mathbf{z}_1$. It follows that the subspace $\mathcal{Y}_1 = \text{span}\{\mathcal{X}_0, \mathbf{z}_1\}$ contains an L -dimensional subspace $\mathcal{X}_1 = \mathcal{X}'_1$ such that $\mathcal{X}_1 \subseteq \mathcal{Z}^{(0)}$, $\dim(\mathcal{X}_0 \cap \mathcal{X}_1) = L - 1$, and

$$\text{span}\{\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_{M-L}\} = \mathcal{Z}.$$

Using (iii) of Theorem 4 we have

$$(3.4) \quad H(\mathcal{X}_0)^2 \leq H(\mathcal{X}_0)H(\mathcal{X}_1) \leq 2\mathcal{H}(\Phi)H(\mathcal{Y}_1)^2.$$

Then (3.3) and (3.4) combine to give exactly the estimate (1.2) in the statement of Theorem 1.

4. PROOF OF THEOREM 4

Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_L\}$ be a basis for \mathcal{X} . Since $\xi \in \mathcal{Z} \setminus \mathcal{X}^\perp$ it follows that

$$(4.1) \quad F(\mathbf{x}_l, \xi) \neq 0 \quad \text{for some } l, 1 \leq l \leq L.$$

Obviously $\{\mathbf{x}_1, \dots, \mathbf{x}_L, \xi\}$ is a basis for \mathcal{Y} . Define $\mathcal{X}' \subseteq \mathcal{Y}$ by

$$\mathcal{X}' = \left\{ \sum_{l=1}^L \alpha_l \mathbf{x}_l + \beta \xi : \alpha_l \in k, \beta \in k, \text{ and } \sum_{l=1}^L \alpha_l F(\mathbf{x}_l, \xi) + \frac{1}{2} \beta F(\xi) = 0 \right\}.$$

In view of (4.1), \mathcal{X}' is a subspace of \mathcal{Y} with

$$\dim(\mathcal{X}') = L \quad \text{and} \quad \dim(\mathcal{X} \cap \mathcal{X}') = L - 1.$$

In particular, \mathcal{X} and \mathcal{X}' are distinct subspaces. If

$$(4.2) \quad \mathbf{y} = \mathbf{x} + \beta \xi, \quad \mathbf{x} \in \mathcal{X}, \beta \in k,$$

is a vector in \mathcal{Y} then

$$(4.3) \quad \begin{aligned} F(\mathbf{y}) &= 2F(\mathbf{x}, \beta \xi) + F(\beta \xi) \\ &= 2\beta F(\mathbf{x} + \frac{1}{2} \beta \xi, \xi). \end{aligned}$$

This shows that $\beta = 0$ is equivalent to $\mathbf{y} \in \mathcal{X}$ and $F(\mathbf{x} + \frac{1}{2} \beta \xi, \xi) = 0$ is equivalent to $\mathbf{y} \in \mathcal{X}'$. Therefore (i) and (ii) of Theorem 4 hold for the subspace \mathcal{X}' .

At each place v of k let $\mathcal{X}_v \subseteq (k_v)^N$ be the completion of \mathcal{X} and let $\mathcal{X}'_v \subseteq (k_v)^N$ be the completion of \mathcal{X}' . Then set

$$P_v = P_v(\mathcal{X}_v), \quad P'_v = P_v(\mathcal{X}'_v),$$

and

$$Q_v = \frac{1}{2}(1_N - P_v).$$

By continuity F must vanish identically on $\mathcal{X}'_v \cup \mathcal{X}''_v$. With \mathbf{y} given by (4.2) we have

$$\begin{aligned}
 (4.4) \quad F(\mathbf{x} + \frac{1}{2}\beta\boldsymbol{\xi}, \boldsymbol{\xi}) &= F(\mathbf{x}, \boldsymbol{\xi}) + \frac{1}{2}\beta F(\boldsymbol{\xi}) \\
 &= F(\mathbf{x}, (1_N - P_v)\boldsymbol{\xi}) + \frac{1}{2}\beta F((1_N + P_v)\boldsymbol{\xi}, (1_N - P_v)\boldsymbol{\xi}) \\
 &= F(Q_v\mathbf{x}, (1_N - P_v)\boldsymbol{\xi}) + F(\beta Q_v\boldsymbol{\xi}, (1_N - P_v)\boldsymbol{\xi}) \\
 &= F(Q_v\mathbf{y}, (1_N - P_v)\boldsymbol{\xi}).
 \end{aligned}$$

If we combine (4.3) and (4.4) we find that

$$(4.5) \quad F(\mathbf{y}) = 2\beta F(Q_v\mathbf{y}, (1_N - P_v)\boldsymbol{\xi})$$

at each place v of k . It follows that $\mathbf{y} \in \mathcal{X}'$ if and only if

$$F(Q_v\mathbf{y}, (1_N - P_v)\boldsymbol{\xi}) = 0.$$

Now select $\mathbf{y} \in \mathcal{Y} \setminus (\mathcal{X} \cup \mathcal{X}')$. Then $P'_v\mathbf{y} \in \mathcal{X}'_v$ and so at each place v

$$(4.6) \quad F(Q_v P'_v\mathbf{y}, (1_N - P_v)\boldsymbol{\xi}) = 0.$$

Combining (4.5) and (4.6) we obtain

$$(4.7) \quad 0 \neq F(\mathbf{y}) = 2\beta F(Q_v(1_N - P'_v)\mathbf{y}, (1_N - P_v)\boldsymbol{\xi}).$$

Next we apply the product formula to (4.7) and use the basic inequality (2.6) at each place. This leads to

$$\begin{aligned}
 (4.8) \quad 1 &= \prod_v |(2\beta)^{-1} F(\mathbf{y})|_v \\
 &\leq \mathcal{H}(\Phi) \left\{ \prod_v H_v(Q_v(1_N - P'_v)\mathbf{y}) \right\} \left\{ \prod_v H_v((1_N - P_v)\boldsymbol{\xi}) \right\}.
 \end{aligned}$$

By [14, Lemma 8] we may remove the operator Q_v from the right-hand side of (4.8) while compensating with an extra factor of 2. In this way we find that

$$(4.9) \quad 1 \leq 2\mathcal{H}(\Phi) \left\{ \prod_v H_v((1_N - P'_v)\mathbf{y}) \right\} \left\{ \prod_v H_v((1_N - P_v)\boldsymbol{\xi}) \right\}.$$

Finally we multiply both sides of (4.9) by $H(\mathcal{X})H(\mathcal{X}')$ and apply (2.1). This establishes the inequality

$$\begin{aligned}
 (4.10) \quad H(\mathcal{X})H(\mathcal{X}') &\leq 2\mathcal{H}(\Phi)H(\text{span}_k\{\mathcal{X}', \mathbf{y}\})H(\text{span}_k\{\mathcal{X}, \boldsymbol{\xi}\}) \\
 &= 2\mathcal{H}(\Phi)H(\mathcal{Y})^2,
 \end{aligned}$$

which is exactly the second inequality in (iii) of the theorem.

Now $\mathcal{X} \subseteq \mathcal{X}^{(0)}$ has minimal height among all L -dimensional subspaces contained in $\mathcal{X}^{(0)}$. Therefore (4.10) immediately implies that

$$\begin{aligned}
 (4.11) \quad H(\mathcal{X})^2 &\leq H(\mathcal{X})H(\mathcal{X}') \leq 2\mathcal{H}(\Phi)H(\mathcal{Y})^2 \\
 &= 2\mathcal{H}(\Phi)H(\mathcal{X})^2 \left\{ \prod_v H_v((1_N - P_v)\boldsymbol{\xi}) \right\}^2.
 \end{aligned}$$

Of course (4.11) is (iv) and the first inequality in (iii) of the theorem.

5. A NESTED SEQUENCE OF SUBSPACES

In this section we identify several objects connected with F , \mathcal{Z} and \mathcal{Z} which will be used in our proof of Theorem 5. Again we suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_L\}$ is a basis for \mathcal{Z} and we write $P_v = P_v(\mathcal{Z}_v)$ for the projection operator which maps $(k_v)^N$ onto the completion \mathcal{Z}_v of \mathcal{Z} at each place v . To simplify some expressions we write

$$G(\mathbf{z}) = \prod_v H_v((1_N - P_v)\mathbf{z})$$

if $\mathbf{z} \in \mathcal{Z}$. By [14, Theorem 10] there exists a basis $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_L, \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{M-L}\}$ of \mathcal{Z} such that

$$(5.1) \quad \prod_{l=1}^{M-L} G(\mathbf{y}_l) \leq \{c_k(M-L)\}^{M-L} H(\mathcal{Z})H(\mathcal{Z})^{-1}.$$

Also, by reordering if necessary, we may assume that the numbers $G(\mathbf{y}_l)$ are increasing for $l = 1, 2, \dots, M-L$. We use the vectors in $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{M-L}\}$ to define a nested sequence of subspaces

$$\mathcal{A}_l = \text{span}_k\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_L, \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_l\},$$

for $l = 0, 1, 2, \dots, M-L$. Thus we have

$$\mathcal{Z} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots \subseteq \mathcal{A}_{M-L} = \mathcal{Z}.$$

Obviously $\dim(\mathcal{A}_l) = L + l$. For each choice of $l = 0, 1, 2, \dots, M-L$ we set

$$\mathcal{A}_l^\perp = \{\mathbf{z} \in \mathcal{Z} : F(\mathbf{a}, \mathbf{z}) = 0 \text{ for all } \mathbf{a} \in \mathcal{A}_l\}.$$

Clearly each \mathcal{A}_l^\perp is a subspace of \mathcal{Z} and

$$\mathcal{Z}^\perp = \mathcal{A}_{M-L}^\perp \subseteq \mathcal{A}_{M-L-1}^\perp \subseteq \dots \subseteq \mathcal{A}_1^\perp \subseteq \mathcal{A}_0^\perp = \mathcal{Z}^\perp \subseteq \mathcal{Z}.$$

Concerning the dimension of \mathcal{A}_l^\perp we have

$$\dim(\mathcal{A}_l) + \dim(\mathcal{A}_l^\perp) = \dim(\mathcal{Z}) + \dim(\mathcal{A}_l \cap \mathcal{Z}^\perp).$$

Since $r > M-L$ it follows that

$$(5.2) \quad \begin{aligned} \dim(\mathcal{A}_l^\perp) &\leq \dim(\mathcal{Z}) + \dim(\mathcal{Z}^\perp) - \dim(\mathcal{A}_l) \\ &= M + (M-r) - (L+l) \\ &\leq M-l-1. \end{aligned}$$

The quadratic form F vanishes identically on $\mathcal{Z} = \mathcal{A}_0$ but does not vanish identically on $\mathcal{Z} = \mathcal{A}_{M-L}$. Hence there exists a unique integer s , $0 \leq s \leq M-L-1$, such that F vanishes identically on \mathcal{A}_s but F does not vanish identically on \mathcal{A}_{s+1} . For the subspace \mathcal{A}_s we must have $\mathcal{A}_s \subseteq \mathcal{A}_s^\perp$, and therefore by (5.2)

$$L+s = \dim(\mathcal{A}_s) \leq \dim(\mathcal{A}_s^\perp) \leq M-s-1.$$

This shows that $0 \leq s \leq \frac{1}{2}(M-L-1)$.

Lemma 6. *If $\mathbf{y}_{s+1} \in \mathcal{A}_s^\perp$ then*

$$1 \leq \mathcal{H}(\Phi)G(\mathbf{y}_{s+1})^2.$$

Proof. By the definition of s , F does not vanish identically on

$$\mathcal{A}_{s+1} = \text{span}_k\{\mathcal{A}_s, \mathbf{y}_{s+1}\}.$$

But

$$(5.3) \quad F(\mathbf{a}) = 0 \quad \text{and} \quad F(\mathbf{a}, \mathbf{y}_{s+1}) = 0$$

for all $\mathbf{a} \in \mathcal{A}_s$. It follows that $F(\mathbf{y}_{s+1}) \neq 0$. Also, (5.3) implies that at each place v

$$F(P_v \mathbf{y}_{s+1}) = 0 \quad \text{and} \quad F(P_v \mathbf{y}_{s+1}, \mathbf{y}_{s+1}) = 0.$$

Therefore, we have

$$\begin{aligned} F(\mathbf{y}_{s+1}) &= F((1_N - P_v)\mathbf{y}_{s+1}, \mathbf{y}_{s+1}) \\ &= F((1_N - P_v)\mathbf{y}_{s+1}). \end{aligned}$$

We apply (2.6) and the product formula to conclude that

$$\begin{aligned} 1 &= \prod_v |F(\mathbf{y}_{s+1})|_v \\ &= \prod_v |F((1_N - P_v)\mathbf{y}_{s+1})|_v \\ &\leq \mathcal{H}(\Phi)G(\mathbf{y}_{s+1})^2. \end{aligned}$$

Lemma 7. *Suppose that $1 \leq s$ and ξ is a vector in $\mathcal{Z} \setminus \mathcal{A}_s^\perp$. Let m be the smallest integer in the set $\{1, 2, \dots, s\}$ such that ξ is contained in $\mathcal{Z} \setminus \mathcal{A}_m^\perp$. Then we have*

$$(5.4) \quad 1 \leq \mathcal{H}(\Phi)G(\mathbf{y}_m)G(\xi).$$

Proof. First we assume that $m = 1$. Then $\xi \in \mathcal{Z} \setminus \mathcal{A}_1^\perp$ and therefore the linear form $\mathbf{a} \rightarrow F(\mathbf{a}, \xi)$ is not trivial on \mathcal{A}_1 . It follows that

$$\mathcal{B}_0 = \{\mathbf{a} \in \mathcal{A}_1 : F(\mathbf{a}, \xi) = 0\}$$

is an L -dimensional subspace of \mathcal{A}_1 . As $1 \leq s$, F vanishes identically on \mathcal{A}_1 and also on \mathcal{B}_0 . Since $\mathcal{Z} = \mathcal{A}_0$ has minimal height among the L -dimensional subspaces in $\mathcal{Z}^{(0)}$, we have

$$(5.5) \quad H(\mathcal{A}_0) \leq H(\mathcal{B}_0).$$

Now let $\zeta \in \mathcal{A}_1 \setminus \mathcal{B}_0$ so that $F(\zeta, \xi) \neq 0$. From the definition of \mathcal{B}_0 we have

$$(5.6) \quad F(P_v(\mathcal{B}_0)\zeta, \xi) = 0$$

at each place v . Since F vanishes identically on \mathcal{A}_1 we also have

$$(5.7) \quad F((1_N - P_v(\mathcal{B}_0))\zeta, P_v(\mathcal{A}_0)\xi) = 0.$$

Using (5.6) and (5.7) we obtain the identity

$$\begin{aligned} F(\zeta, \xi) &= F((1_N - P_v(\mathcal{B}_0))\zeta, \xi) \\ &= F((1_N - P_v(\mathcal{B}_0))\zeta, (1_N - P_v(\mathcal{A}_0))\xi) \end{aligned}$$

at each place v . Again we apply (2.6) and the product formula to conclude that

$$\begin{aligned} (5.8) \quad 1 &= \prod_v |F(\zeta, \xi)|_v \\ &\leq \mathcal{H}(\Phi) \left\{ \prod_v H_v((1_N - P_v(\mathcal{B}_0))\zeta) \right\} \left\{ \prod_v H_v((1_N - P_v(\mathcal{A}_0))\xi) \right\}. \end{aligned}$$

Next we multiply both sides of (5.8) by $H(\mathcal{B}_0)$ and use (2.1) and (5.5). In this way we obtain

$$\begin{aligned} H(\mathcal{A}_0) \leq H(\mathcal{B}_0) &\leq \mathcal{H}(\Phi)H(\mathcal{A}_1) \left\{ \prod_v H_v((1_N - P_v(\mathcal{A}_0))\xi) \right\} \\ &= \mathcal{H}(\Phi)H(\mathcal{A}_0) \left\{ \prod_v H_v((1_N - P_v(\mathcal{A}_0))\mathbf{y}_1) \right\} \left\{ \prod_v H_v((1_N - P_v(\mathcal{A}_0))\xi) \right\}, \end{aligned}$$

which is (5.4) when $m = 1$.

To complete the proof we consider the case $2 \leq m \leq s$. Then $\xi \in \mathcal{Z} \setminus \mathcal{A}_m^\perp$ and $\xi \in \mathcal{A}_{m-1}^\perp$. Again the linear form $\mathbf{a} \rightarrow F(\mathbf{a}, \xi)$ is not trivial on \mathcal{A}_m so that

$$\{\mathbf{a} \in \mathcal{A}_m : F(\mathbf{a}, \xi) = 0\}$$

is an $(L+m-1)$ -dimensional subspace of \mathcal{A}_m . On the other hand, $\mathcal{A}_{m-1} \subseteq \mathcal{A}_m$, $\xi \in \mathcal{A}_{m-1}^\perp$ and therefore $\mathbf{a} \rightarrow F(\mathbf{a}, \xi)$ is identically zero on \mathcal{A}_{m-1} . In other words,

$$\mathcal{A}_{m-1} = \{\mathbf{a} \in \mathcal{A}_m : F(\mathbf{a}, \xi) = 0\}.$$

The vector \mathbf{y}_m is in $\mathcal{A}_m \setminus \mathcal{A}_{m-1}$ and therefore $F(\mathbf{y}_m, \xi) \neq 0$. As in the first part of the proof we have

$$\begin{aligned} (5.9) \quad F(\mathbf{y}_m, \xi) &= F((1_N - P_v(\mathcal{A}_0))\mathbf{y}_m, \xi) \\ &= F((1_N - P_v(\mathcal{A}_0))\mathbf{y}_m, (1_N - P_v(\mathcal{A}_0))\xi). \end{aligned}$$

Using (2.1), (5.9) and the product formula we find that

$$1 = \prod_v |F(\mathbf{y}_m, \xi)|_v \leq \mathcal{H}(\Phi)G(\mathbf{y}_m)G(\xi).$$

This proves the lemma.

Lemma 8. *The vectors $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{M-L-1}$ satisfy*

$$(5.10) \quad 1 \leq \{2\mathcal{H}(\Phi)\}^{(M-L-1)/2} \prod_{l=1}^{M-L-1} G(\mathbf{y}_l).$$

The vector \mathbf{y}_{M-L} satisfies

$$(5.11) \quad G(\mathbf{y}_{M-L}) \leq \{c_k(M-L)\}^{M-L} \{2\mathcal{H}(\Phi)\}^{(M-L-1)/2} H(\mathcal{Z})H(\mathcal{Z})^{-1}.$$

Proof. The inequalities (5.1) and (5.10) clearly imply that (5.11) holds. Therefore it suffices to establish only (5.10). First we consider the case $s = 0$. If $\mathbf{y}_1 \in \mathcal{A}_0^\perp$ then

$$1 \leq \mathcal{H}(\Phi)G(\mathbf{y}_1)^2$$

holds by Lemma 6. If $\mathbf{y}_1 \in \mathcal{Z} \setminus \mathcal{A}_0^\perp$ we may appeal to Theorem 4 part (iv) with $\xi = \mathbf{y}_1$. We find that

$$(5.12) \quad 1 \leq 2\mathcal{H}(\Phi)G(\mathbf{y}_1)^2.$$

Thus (5.12) holds generally if $s = 0$. Since the numbers $G(\mathbf{y}_l)$ are increasing for $l = 1, 2, \dots, M-L$, the desired inequality (5.10) follows immediately.

For the remainder of the proof we assume that $1 \leq s$. Let $0 \leq l \leq s$ so that

$$\mathcal{A}_l \subseteq \mathcal{A}_s \subseteq \mathcal{A}_s^\perp \subseteq \mathcal{A}_l^\perp.$$

In particular,

$$\mathcal{A}_s = \text{span}_k \{\mathcal{A}_0, \mathbf{y}_1, \dots, \mathbf{y}_s\} \subseteq \mathcal{A}_l^\perp$$

and $\dim(\mathcal{A}_l^\perp) \leq M-l-1$ by (5.2). It follows that at most $M-L-s-l-1$ of the vectors in the set $\{\mathbf{y}_{s+1}, \mathbf{y}_{s+2}, \dots, \mathbf{y}_{M-L}\}$ are also in \mathcal{A}_l^\perp . Alternatively, at least $l+1$ vectors in the set $\{\mathbf{y}_{s+1}, \mathbf{y}_{s+2}, \dots, \mathbf{y}_{M-L}\}$ are also in $\mathcal{Z} \setminus \mathcal{A}_l^\perp$. Hence we may select distinct integers i_0, i_1, \dots, i_s in $\{s+1, s+2, \dots, M-L\}$ so that

$$\{\mathbf{y}_{i_0}, \mathbf{y}_{i_1}, \dots, \mathbf{y}_{i_s}\} \subseteq \mathcal{Z} \setminus \mathcal{A}_l^\perp$$

for each $l = 0, 1, 2, \dots, s$. By removing \mathbf{y}_{M-L} if necessary, we may select distinct integers j_1, j_2, \dots, j_s in the set $\{s+1, s+2, \dots, M-L-1\}$ so that

$$\{\mathbf{y}_{j_1}, \mathbf{y}_{j_2}, \dots, \mathbf{y}_{j_s}\} \subseteq \mathcal{Z} \setminus \mathcal{A}_l^\perp$$

for $l = 1, 2, \dots, s$. Since $\mathcal{Z} \setminus \mathcal{A}_l^\perp \subseteq \mathcal{Z} \setminus \mathcal{A}_s^\perp$ the hypotheses of Lemma 7 hold with $\xi = \mathbf{y}_{j_l}$. The corresponding value of m plainly satisfies $1 \leq m \leq l$. Thus by Lemma 7 we have

$$(5.13) \quad 1 \leq \mathcal{H}(\Phi)G(\mathbf{y}_m)G(\mathbf{y}_{j_l}) \leq \mathcal{H}(\Phi)G(\mathbf{y}_l)G(\mathbf{y}_{j_l}),$$

for each $l = 1, 2, \dots, s$.

Next we claim that

$$(5.14) \quad 1 \leq \mathcal{H}(\Phi)G(\mathbf{y}_{s+1})^2.$$

If $\mathbf{y}_{s+1} \in \mathcal{A}_s^\perp$ then this follows from Lemma 6. If $\mathbf{y}_{s+1} \in \mathcal{Z} \setminus \mathcal{A}_s^\perp$ we may apply Lemma 7 with $\xi = \mathbf{y}_{s+1}$. Then we use the trivial inequality $G(\mathbf{y}_m) \leq G(\mathbf{y}_{s+1})$ (where $1 \leq m \leq s$) to deduce that (5.14) holds in this case as well.

Now suppose that t is an integer, $s+1 \leq t \leq M-L-1$, but t is *not* in the set $\{j_1, j_2, \dots, j_s\}$. Obviously $G(\mathbf{y}_{s+1}) \leq G(\mathbf{y}_t)$ and therefore (5.14) implies that

$$(5.15) \quad 1 \leq \mathcal{H}(\Phi)G(\mathbf{y}_t)^2.$$

There are $M-L-2s-1$ such integers t so that (5.15) leads to the inequality

$$(5.16) \quad 1 \leq \mathcal{H}(\Phi)^{(M-L-1)/2-s} \prod_t G(\mathbf{y}_t).$$

From (5.13) we have

$$(5.17) \quad 1 \leq \mathcal{H}(\Phi)^s \left\{ \prod_{l=1}^s G(\mathbf{y}_l)G(\mathbf{y}_{j_l}) \right\}.$$

Finally, the inequalities (5.16) and (5.17) combine to establish (5.10).

6. PROOF OF THEOREM 5

We have already seen that there exists an integer i_0 , $s+1 \leq i_0 \leq M-L$, such that \mathbf{y}_{i_0} is contained in $\mathcal{Z} \setminus \mathcal{Z}_0^\perp = \mathcal{Z} \setminus \mathcal{Z}^\perp$. Using \mathbf{y}_{i_0} and the linearly independent vectors $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{M-L}\}$ we define a second set of linearly independent vectors $\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{M-L}\}$ as follows.

- (i) If $\mathbf{y}_l \in \mathcal{Z} \setminus \mathcal{Z}^\perp$ we set $\mathbf{z}_l = \mathbf{y}_l$.
- (ii) If $\mathbf{y}_l \in \mathcal{Z}^\perp$ we select a scalar $\alpha_l \neq 0$ in k such that

$$G(\alpha_l \mathbf{y}_l + \mathbf{y}_{i_0}) \leq 2c_k(1) \max\{G(\mathbf{y}_l), G(\mathbf{y}_{i_0})\}.$$

That such a scalar $\alpha_l \neq 0$ exists follows from Lemma 3. Then we set

$$\mathbf{z}_l = \alpha_l \mathbf{y}_l + \mathbf{y}_{i_0}.$$

It is clear that

$$\text{span}_k\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{M-L}\} = \text{span}_k\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{M-L}\}$$

and therefore (3.1) holds. Also, each vector \mathbf{z}_l is in $\mathcal{Z} \setminus \mathcal{Z}^\perp$.

To complete the proof we note that

$$(6.1) \quad G(\mathbf{z}_l) \leq 2c_k(1)G(\mathbf{y}_{M-L})$$

for each l , $1 \leq l \leq M-L$. If \mathcal{Y}_l is defined by (3.2) then

$$(6.2) \quad H(\mathcal{Y}_l) = H(\mathcal{Z})G(\mathbf{z}_l).$$

The inequality (3.3) plainly follows from (5.11), (6.1) and (6.2).

7. PROOF OF THEOREM 2

Since the rank of F restricted to \mathcal{Z} is r there exists a subspace $\mathcal{Y} \subseteq \mathcal{Z}$ such that \mathcal{Y} has dimension r and F restricted to \mathcal{Y} is nonsingular. By Theorem 10 of [14] there exists a basis $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_r\}$ of \mathcal{Y} for which

$$(7.1) \quad \prod_{l=1}^r \left\{ \prod_v H_v\{(1_N - P_v(\mathcal{Z}^\perp))\mathbf{y}_l\} \right\} \leq c_k(r)^r H(\mathcal{Z})H(\mathcal{Z}^\perp)^{-1}.$$

Let

$$Y = (y_1 y_2 \cdots y_r)$$

denote the $N \times r$ matrix having y_l as its l th column, $1 \leq l \leq r$. We must have

$$\det\{Y^T \Phi Y\} \neq 0,$$

and therefore when we expand the determinant

$$(7.2) \quad \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{l=1}^r F(y_l, y_{\sigma(l)}) \neq 0.$$

In (7.2) the sum is over all permutations σ of the set $\{1, 2, \dots, r\}$. It follows that there exists a permutation τ of the set $\{1, 2, \dots, r\}$ such that

$$\prod_{l=1}^r F(y_l, y_{\tau(l)}) \neq 0.$$

Using (2.1) and the product formula we find that

$$(7.3) \quad \begin{aligned} 1 &= \prod_{l=1}^r \left\{ \prod_v |F(y_l, y_{\tau(l)})|_v \right\} \\ &= \prod_{l=1}^r \left\{ \prod_v |F((1_N - P_v(\mathcal{Z}^\perp))y_l, (1_N - P_v(\mathcal{Z}^\perp))y_{\tau(l)})|_v \right\} \\ &\leq \prod_{l=1}^r \{ \mathcal{H}(\Phi) G(y_l) G(y_{\tau(l)}) \} \\ &= \mathcal{H}(\Phi)^r \prod_{l=1}^r G(y_l)^2, \end{aligned}$$

where we have written

$$G(\mathbf{x}) = \prod_v H_v \{ (1_N - P_v(\mathcal{Z}^\perp))\mathbf{x} \}.$$

Finally, from (7.1) and (7.3) we obtain the inequality

$$1 \leq \mathcal{H}(\Phi)^r \{ c_k(r)^r H(\mathcal{Z}) H(\mathcal{Z}^\perp)^{-1} \}^2,$$

which is the statement of the theorem.

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