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PROJECTIONS ONTO TRANSLATION-INVARIANT SUBSPACES OF $H^1(\mathbf{R})$

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ABSTRACT. Recently I. Klemes has characterized the complemented translation-invariant subspaces of $H^1(\mathbf{T})$. In this paper we investigate the case of $H^1(\mathbf{R})$. The main results are that the hull of a complemented translation-invariant subspace is ε-separated for some $\varepsilon > 0$, and that an ε-separated subset of $\mathbf{R}^+$ which is in the ring generated by cosets of closed subgroups of $\mathbf{R}$ (intersected with $\mathbf{R}^+$) and lacunary sequences is the hull of a complemented ideal.

0. INTRODUCTION

The initial investigation of complemented translation-invariant subspaces of $L^1(\mathbf{R})$ was conducted by Rosenthal [R1]. He showed that a complemented translation-invariant subspace must have its hull in the coset ring of $\mathbf{R}_d$, $\mathbf{R}$ with the discrete topology, and thus is of the form

$$\bigcup_{i=1}^{n} \alpha_i \mathbf{Z} + \beta_i \setminus F ,$$

where F is a finite set and, for each i, α_i and β_i are real numbers. Subsequently, Alspach and Matheson [AM] showed that such a set is the hull of a complemented translation-invariant subspace of $L^1(\mathbf{R})$ if and only if the α_i's are pairwise rationally dependent. Hence a natural first guess is that a complemented translation-invariant subspace of $H^1(\mathbf{R})$ must have a hull of the same form.

In his thesis Klemes [K] showed that a complemented translation-invariant subspace of $H^1(\mathbf{T})$ has hull in the ring generated by arithmetic progressions and lacunary sequences. (This is also sufficient for complementation.) The role of lacunary sequences in $H^1(\mathbf{R})$ is less clear at first glance but it will be shown that they provide complemented l^2's. Thus we are led to the following conjecture:

Conjecture. *Let $X \neq \{0\}$ be a translation-invariant subspace of $H^1(\mathbf{R})$. Then X is complemented if and only if the hull of X is in the ring generated by arithmetic progressions and lacunary sequences and is ε-separated for some $\varepsilon > 0$.*

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In this paper we prove the sufficiency of the condition and the necessity of the separation portion of the condition. We also give some partial results in support of the necessity of the other portion of the condition as well. In particular we will show that the conjecture is true provided the natural extension of Klemes' results to $H^1(\mathbb{T}^n)$ holds. In [A] the proof of the conjecture is completed.

Let us now fix some notational conventions and recall some basic definitions which will be used in the paper. $H^1(\mathbb{R})$ denotes the subspace of $L^1(\mathbb{R})$ consisting of functions with Fourier transform equal to zero on $\{t \in \mathbb{R}: t \leq 0\}$ and $H^1(\mathbb{T})$ denotes the subspace of $L^1(\mathbb{T})$ of functions with Fourier transform equal to zero on $\{n \in \mathbb{Z} : n < 0\}$. $H^1(\mathbb{T})$ and $H^1(\mathbb{R})$ are known to be isometric as Banach spaces [He, p. 103], but of course the translation-invariant structure is different. $H_0^1(\mathbb{T})$ denotes the subspace of $H^1(\mathbb{T})$ of functions with Fourier transform vanishing at zero. Let $e_t(s) = e^{its}$ for $s, t \in \mathbb{R}$. The pairing $\langle f, g\rangle$ with $f \in L^\infty(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ will be

$$\langle f, g\rangle = \int f(-s)g(s)\,ds\,.$$

Thus $\hat{g}(t) = \langle e_t, g\rangle$ and $f * g(0) = \langle f, g\rangle$. Often we will write t in place of e_t. For example, if $A \subset \mathbb{R}$, then $[A] = \overline{\operatorname{span}}\{e_a : a \in A\}$, or if T is an operator on $L_\infty(\mathbb{R})$ and $a \in \mathbb{R}$, then $Ta = Te_a$. We will say that a subset A of $\mathbb{R}$ is *ε-separated* if $\inf\{|s-u| : s, u \in A,\ s \neq u\} \geq \varepsilon > 0$. $\Omega_l(\mathbb{R})$ will denote the ring of subsets of $\mathbb{R}$ generated by lacunary sequences (possibly of finite length) and arithmetic progressions. By the term *arithmetic progression* we mean a set of the form $\alpha\mathbb{Z}+\beta$. We will use the term *lacunary* for sequences of positive numbers $(a_n)_{n\in M}$ where $M = \{1, 2, 3, \ldots, N\}$ or $M = \mathbb{N}$ and $\inf\{a_{n+1}/a_n : n \in M\} > 1$ and *generalized lacunary* for sequences satisfying the same ratio condition but indexed by a subset of $\mathbb{Z}$. Note that we do not consider $\mathbb{R}$ itself to be in $\Omega_l(\mathbb{R})$. We will use the notation $I(A)$ for both the subspace of $L^1(\mathbb{R})$ or $L^1(\mathbb{T})$ of functions with transform vanishing on A and its intersection with $H^1(\mathbb{R})$ or $H^1(\mathbb{T})$. If X is a translation-invariant subspace of $L^1(\mathbb{R})$, $Z(X)$ will be the hull of X, *i.e.*, $\{t: \hat{x}(t) = 0 \text{ for all } x \in X\}$. To avoid writing some parentheses we will assume that the algebraic operations take precedence over set theoretic operations, *e.g.*, $B \cup \alpha\mathbb{Z}+\beta = B \cup ((\alpha\mathbb{Z})+\beta)$. Other standard notation and facts may be found in [K, He, R1].

The remainder of the paper is divided into three sections. In the first we prove that lacunary sequences in $\hat{\mathbb{R}}$ correspond to Hilbert spaces contained in $H^1(\mathbb{R})$, just as lacunary sequences in $\mathbb{Z}$ correspond to Hilbert spaces contained in $H^1(\mathbb{T})$. We also show that arithmetic progressions in $\hat{\mathbb{R}}$ correspond to complemented copies of $H_0^1(\mathbb{T})$ in $H^1(\mathbb{R})$. These tools allow us to carry over the techniques of [AM] from $L^1(\mathbb{R})$ to $H^1(\mathbb{R})$ and prove that ε-separated sets which are in $\Omega_\ell(\mathbb{R})$ are hulls of complemented translation-invariant subspaces of $H^1(\mathbb{R})$. In the second section we prove the necessity of the ε-separation condition. The idea is similar to that used in [AM] to show that in the hull

of a complemented ideal in $L^1(\mathbf{R})$ the arithmetic progressions have rationally dependent periods but here we use lacunary sequences in place of arithmetic progressions. In the third we prove some partial results on the nature of the hull of a complemented translation-invariant subspace. In particular we show that operators on $H^1(\mathbf{R})$ with translation-invariant kernel can be averaged to yield operators on $\mathscr{HM}(b\mathbf{R})$, the space of "analytic" measures on the Bohr compactification of $\mathbf{R}$. This allows us to relate our problem to complemented translation-invariant subspaces of $H^1(\mathbf{T}^n)$.

1. Construction of projections

In this section we will show that if A is an ε-separated subset of $\mathbf{R}$ which is in the ring generated by arithmetic progressions and lacunary sequences then $I(A) = \{f \in H^1(\mathbf{R}) \colon \hat{f}(a) = 0 \text{ for all } a \in A\}$ is complemented in $H^1(\mathbf{R})$. The argument here is similar to that of [AM] but we must first show that the required liftings exist; *e.g.*, if $A = \mathbf{Z}$ then we must show that there is a subspace X of $H^1(\mathbf{R})$ such that the natural quotient map of $L^1(\mathbf{R})$ onto $L^1(\mathbf{T})$ carries X onto $H_0^1(\mathbf{T})$. First we consider the case of lacunary sequences.

Lemma 1.1. *Suppose that $(a_n)_{n\in M}$ is a generalized lacunary sequence of positive numbers. Then $\overline{[a_n : n \in M]}^{w*}$ (in $H^1(\mathbf{R})^*$) is isomorphic to $l^2(M)$ and the operator $Lf = (\hat{f}(a_n))$ is a bounded operator from $H^1(\mathbf{R})$ onto $l^2(M)$.*

Proof. The first assertion follows from the second. Without loss of generality we may assume $M = \mathbf{Z}$. To see the boundedness of L we need a version of Paley's inequality. This can be derived from a generalized Paley theorem for compact groups with ordered dual (see [Ru1, p. 213]) but we will show how to deduce the inequality directly from the well known Paley's inequality for $H^1(\mathbf{T})$.

For each $k \in \mathbf{N}$ we can find integers (j_n) such that $|a_n - j_n/k| < 1/k$. (Take $j_n = 0$ if $a_n < 1/k$.) Consider the quotient map Q_k from $L^1(\mathbf{R})$ into $L^1(\mathbf{T})$ (actually onto) defined by

$$Q_k f(t) = k \sum_{n=-\infty}^{\infty} [f](k(t + 2\pi n)) \ , \quad \text{for } t \in [0, 2\pi) \ .$$

It is easy to check that $Q_k^* e_{j_n} = e_{j_{n/k}}$. Note that $Q(H^1(\mathbf{R})) \subset H_0^1(\mathbf{T})$. Hence by Paley's inequality we have

$$\sum_{n=-\infty}^{\infty} |\hat{f}(j_n/k)|^2 = \sum_{n=-\infty}^{\infty} |Q_k f\hat{}(j_n)|^2 < C\|Q_k f\|_1^2 \le C\|f\|_1^2 \ , \quad \text{for all } f \in H^1(\mathbf{R}) .$$

(Strictly speaking we should take k sufficiently large else the nonzero portion of the sequence $\{j_n\}$ may not be lacunary.) Letting k go to infinity we see that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(a_n)|^2 \le C\|f\|_1^2 \ , \quad \text{for all } f \in H^1(\mathbf{R}) .$$

Thus L is bounded. In order to see that L is onto it is sufficient to show that there is a dense subset of the unit ball of $l^2(\mathbb{Z})$ with uniformly bounded inverse images. (This is a standard corollary of the proof of the open mapping theorem. See Lemma 4.13, [Ru2].) Let K_δ denote the Fejér kernel, *i.e.*, $\hat{K}_\delta(t) = 1 - |t|/\delta$ for $|t| < \delta$ and 0 otherwise. Then

$$\Big\| \sum_{n=-N}^{\infty} b_n e_{a_n} K_\delta \Big\|_1 = \int_{-\infty}^{\infty} \Big| \sum_{n=-N}^{\infty} b_n e_{a_n} \Big| K_\delta \, dt \le \Big\| \sum_{n=-N}^{\infty} b_n e_{a_n} \Big\|_{L^2(K_\delta \, dt)}$$

$$= \Big[\sum_{n=-N}^{\infty} |b_n|^2 \Big]^{1/2}$$

if δ is sufficiently small. Clearly $L \sum_{n=-N}^{\infty} b_n e_{a_n} K_\delta = (b_n)$ if δ is sufficiently small and the result follows. □

Lemma 1.2. *If $\{a_n\}$ is an ε-separated finite union of lacunary sequences in $\mathbb{R}^+$, then $I(\{a_n\})$ is complemented in $H^1(\mathbb{R})$. Moreover there is a constant K (depending only on $\sup_y \operatorname{card}[y, 2y) \cap \{a_n\}$) such that given any $\delta > 0$ there is a projection P with $\ker P = I(\{a_n\})$ such that the support of the Fourier transform of Pf is contained in $\bigcup_n (a_n - \delta, a_n + \delta)$ and $\|P\| \le K$.*

Proof. Because $\{a_n\}$ is a finite union of lacunary sequences it follows from Paley's inequality (Lemma 1.1) that there is a constant C' such that

$$\sum_{n=1}^{\infty} |\hat{f}(a_n)|^2 \le C' \|f\|_1^2 \,, \quad \text{for all } f \in H^1(\mathbb{R}) \,.$$

Now let K_δ be the Fejér kernel, as above. Then

$$\| \sum c_n e_{a_n} K_\delta \|_1 \le \| \sum c_n e_{a_n} \|_{L^2(K_\delta \, ds)} = \Big[\sum |c_n|^2 \Big]^{1/2} \,,$$

if $\delta < \inf\{|a_n - a_m| : n \ne m\}/2$. Hence the operator $S\colon H^1(\mathbb{R}) \to [e_{a_n} K_\delta]$ defined by $Sf = \sum \hat{f}(a_n) e_{a_n} K_\delta$ is a bounded projection with kernel $I(\{a_n\})$. □

We can now show that the sets in the ring, $\Omega_l(\mathbb{R})$, generated by the arithmetic progressions and lacunary sequences (intersected with $\mathbb{R}^+$), have many of the same properties that strong Ditkin (Wik) sets have. Observe that any set in $\Omega_l(\mathbb{R})$ is discrete and hence is a set of spectral synthesis.

Lemma 1.3. *Suppose that A and B are elements of $\Omega_l(\mathbb{R})$ then*

(a) $I(A) \cap I(B) = I(A \cup B)$,
(b) $I(A) + I(B) = I(A \cap B)$.

Proof. The first assertion is obvious. For the second we note that $\overline{I(A) + I(B)}$ is a translation-invariant subspace of $I(A \cap B)$ with the same discrete hull and thus the two are equal. To see that $I(A) + I(B)$ is closed we use a standard device formulated by Rudin to prove that a sum of two spaces is closed, [Ko, p.

202]. By that result it is sufficient to show that there is a sequence of operators $\{T_n\}$ from $H^1(\mathbf{R})$ to $I(A)$ such that

(i) $\sup \|T_n\| < \infty$,
(ii) $T_n(I(B)) \subset I(B)$ for all n,
(iii) If $f \in I(A)$ and $\varepsilon > 0$, there is an n with $\|T_n f - f\| < \varepsilon$.

Because $A \in \Omega_l(\mathbf{R})$, $A = (A_1 \setminus A_2) \cup A_3$ where A_2 and A_3 are finite unions of lacunary sequences, $A_1 \supset A_2$, $A_1 \cap A_3 = \varnothing$, and A_1 is a finite union of arithmetic progressions. Hence A_1 is a strong Ditkin set for $L^1(\mathbf{R})$, (see [AM or AMR]) and therefore there is a sequence of uniformly bounded measures $\{\mu_n\}$ such that

(1) $\hat{\mu}_n = 1$ on a neighborhood of A_1,
(2) If $f \in I(A_1)$, $\|\mu_n * f\| \to 0$,
(3) If C is a compact set which is disjoint from A_1, $\hat{\mu}_n = 0$ on C for all large n.

For each $k \in N$ there is a $\rho_k > 0$ such that $(A \cup B) \cap [0,k]$ is $2\rho_k$-separated. Let P_k be the projection given by Lemma 1.2 for $A_2 \cap [0,k]$ and ρ_k, and Q_k for $A_3 \cap [0,k]$ and ρ_k. (We may assume that $\rho_k \to 0$.) Consider the family of operators

$$S_{m,k} f = [I - Q_k][(\delta - \mu_m) * K_k * f + \mu_m * P_k K_k * f] \ , \quad \text{for } f \in H^1(\mathbf{R}) \ ,$$

where δ denotes the Dirac measure at 0. If m is sufficiently large the intersection of $[0,k]$ with the support of $\hat{\mu}_m$ is disjoint from $[(B \setminus A_1) \cup A_3] + [-\rho_k, \rho_k]$. Thus for such m, a simple computation shows that (i) and (ii) are satisfied by $S_{m,k}$. If $f \in I(A)$ then $(I - P_k)K_k * f \in I(A_1)$. Hence $\|\mu_m * (I - P_k)K_k * f\| \to 0$ and $(I - Q_k)K_k * f = K_k * f \to f$. Thus (iii) is also satisfied. Hence some ordering of the $S_{m,k}$ is the required sequence $\{T_n\}$, and (b) follows. □

We are now ready for our first complementation criterion. Note that for each $\alpha, \beta \in \mathbf{R}^+$ there is a natural quotient map $Q_{\alpha,\beta}$ from $L^1(\mathbf{R})$ onto $L^1(\mathbf{T})$ such that $Q_{\alpha,\beta} f\hat{\ }(n) = \hat{f}(\alpha n + \beta)$. This map is defined by

$$Q_{\alpha,\beta} f(t) = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} [e_{-\beta} f]\Big(\frac{t}{\alpha} + \frac{n2\pi}{\alpha}\Big) \ , \quad \text{for } t \in [0, 2\pi) \ .$$

Lemma 1.4. *Suppose that $B \in \Omega_l(\mathbf{R})$, α and $\beta \in \mathbf{R}$, and $\alpha\mathbf{Z} + \beta \cap B = \varnothing$. Then $I(B \cup \alpha\mathbf{Z} + \beta)$ is complemented in $I(B)$ if and only if there is a subspace X of $I(B)$ such that the quotient map $Q_{\alpha,\beta}\colon L^1(\mathbf{R}) \to L^1(\mathbf{T})$, is an isomorphism from X onto $H_0^1(\mathbf{T})$.*

Proof. First suppose that $I(B \cup \alpha\mathbf{Z} + \beta)$ is complemented in $I(B)$. Let X be the complement and let P be the projection onto X with kernel $I(B \cup \alpha\mathbf{Z} + \beta)$. Note that $Pf\hat{\ }(\alpha n + \beta) = \hat{f}(\alpha n + \beta)$ for all $n \in \mathbf{Z}$, and thus that $Qf = QPf$.

Because $B \in \Omega_l(\mathbb{R})$, it follows from Lemma 1.3 that

$$\begin{aligned} Q(I(B)) &= Q(I(B) + I(\alpha\mathbb{Z} + \beta)) \\ &= Q(I(B \cap \alpha\mathbb{Z} + \beta)) = Q(H^1(\mathbb{R})) = H_0^1(\mathbb{T}), \end{aligned}$$

and

$$\ker Q \cap I(B) = I(\alpha\mathbb{Z} + \beta) \cap I(B) = I(B \cup \alpha\mathbb{Z} + \beta) .$$

Hence $Q(X) = Q(I(B)) = H_0^1(\mathbb{T})$, and Q is one-to-one on X, as claimed.

Conversely, if such an X exists, define $Pf = [Q|_X]^{-1} Qf$. Clearly P is a projection from $I(B)$ onto X with kernel $I(B \cup \alpha\mathbb{Z} + \beta)$. □

Corollary 1.5. *Suppose that $B \in \Omega_l(\mathbb{R})$, α and $\beta \in \mathbb{R}$, A is a finite union of lacunary subsets of $\mathbb{Z}$, and $\alpha(\mathbb{Z} \setminus A) + \beta \cap B = \varnothing$. Then $I(B \cup \alpha(\mathbb{Z} \setminus A) + \beta)$ is complemented in $I(B)$ if and only if there is a subspace X of $I(B)$ such that the map $m_A Q_{\alpha,\beta}$ is an isomorphism from X onto $I(A)$, where m_A is the multiplier on $H^1(\mathbb{T})$ satisfying $(m_A f)\hat{} = 1_{\mathbb{Z}\setminus A}\hat{f}$.*

Proof. Use the fact that $I(A)$ is complemented in $H_0^1(\mathbb{T})$ and the proof of Lemma 1.4. □

This last result gives us a criterion for checking relative complementation which will be useful both for showing the existence of projections and for establishing the ε-separation result.

Proposition 1.6. *Suppose that $B \in \Omega_\ell(\mathbb{R})$ and A is a finite union of lacunary subsets of $\mathbb{Z}$. If there is an $\varepsilon > 0$ such that $d(B, \alpha(\mathbb{Z} \setminus A) + \beta) > \varepsilon$, then $I(B \cup \alpha(\mathbb{Z} \setminus A) + \beta)$ is complemented in $I(B)$.*

Proof. Because $\alpha\mathbb{Z} + \beta$ is a coset of a subgroup of $\mathbb{R}$ there is a measure μ on $\mathbb{R}$ such that $\hat{\mu} = 1$ on $\alpha\mathbb{Z} + \beta$ and $\hat{\mu} = 0$ on $[\alpha\mathbb{Z} + \beta + (-\rho, \rho)]^c$, where

$$\rho = \min(\varepsilon, |\alpha n + \beta| : n \in \mathbb{Z}),$$

[R2, AMR]. Let m be a function such that

$$|m(t)| \le \frac{C}{1 + t^2} ,$$

$$\operatorname{supp} \hat{m} \subset \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right), \quad \text{and} \quad \hat{m}(0) = 1 .$$

(For example, the Fejér kernel could be used.) Let

$$X = \left\{ f \in H^1(\mathbb{R}) : f = \sum_{n \in \mathbb{Z}\setminus A} a_n (e_{\alpha n + \beta}) m \right\} .$$

We claim that $\mu * X = X'$ is the required complement.

Clearly $X' \subset I(B)$. By the previous corollary it is sufficient to show that the map $m_A Q = m_A Q_{\alpha,\beta}$ is an isomorphism from X' onto $I(A)$. To see this let us compute $Qf\hat{}(k)$ for $f \in X'$.

$$\begin{aligned}
Qf\hat{}(k) &= \frac{1}{2\pi}\int_0^{2\pi} e_{-k}Qf = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\alpha}\sum_{n=-\infty}^{\infty} e_{-k}(t)[e_{-\beta}f]\Big(\frac{t}{\alpha}+\frac{n2\pi}{\alpha}\Big)\,dt \\
&= \int_{-\infty}^{\infty} e_{-\alpha k-\beta}(t)f(t)\,dt = \hat{f}(\alpha k+\beta) \\
&= \hat{\mu}(\alpha k+\beta)\sum_{j\in\mathbf{Z}\setminus A} a_j\big[(e_{\alpha j+\beta})m\big]\hat{}(\alpha k+\beta) \\
&= \sum_{j\in\mathbf{Z}\setminus A} a_j\delta_k^j = \begin{cases} a_k & \text{if } k\notin A, \\ 0 & \text{if } k\in A. \end{cases}
\end{aligned}$$

Hence $Qf \in I(A)$. Now suppose that $g \in I(A)$ and let $f = \sum_n \hat{g}(n)(e_{\alpha n+\beta})m$ then

$$\begin{aligned}
\|f\|_1 &= \Big\|\sum_n \hat{g}(n)(e_{\alpha n+\beta})m\Big\|_1 = \int_{-\infty}^{\infty}\Big|\sum_n \hat{g}(n)(e_{\alpha n+\beta})m\Big| \\
&= \int_{-\infty}^{\infty} |e_\beta(t)g(\alpha t)m(t)|\,dt \\
&= \sum_n \int_0^{2\pi/\alpha} |g(\alpha t)|\,\Big|m\Big(t+\frac{2\pi n}{\alpha}\Big)\Big|\,dt \\
&\le \frac{2\pi}{\alpha}\|g\|_1\,\Big\|\sum_n\Big|m\Big(t+\frac{2\pi n}{\alpha}\Big)\Big|\Big\|_\infty .
\end{aligned}$$

Because m satisfies $|m(t)| \le C/(1+t^2)$ for some $C < \infty$, the L^∞ norm above is finite; clearly $m_A Q(\mu * f) = g$. This shows that $m_A Q(X') = I(A)$. Obviously $m_A Q$ is one-to-one on X' and the result is proved. □

Remark 1.7. If $B = \varnothing$, and we choose $m(t) = m_\alpha(t) = \alpha m_1(\alpha t)$ then

$$\begin{aligned}
\frac{1}{\alpha}\Big\|\sum_n\Big|m\Big(t+\frac{2\pi n}{\alpha}\Big)\Big|\Big\|_\infty &= \Big\|\sum_n |m_1(\alpha t+2\pi n)|\Big\|_\infty \\
&= \Big\|\sum_n |m_1(x+2\pi n)|\Big\|_\infty .
\end{aligned}$$

Thus we see that the norm of the isomorphism does not depend on α or β.

We have now obtained the main positive result.

Theorem 1.8. *Suppose that B is an ε-separated subset of $\mathbf{R}$ which is in $\Omega_\ell(\mathbf{R})$. Then $I(B)$ is complemented in $H^1(\mathbf{R})$.*

Proof. Such a set B is a finite union of disjoint sets C_i, $i = 0,1,2,\dots,m$, where C_0 is a finite union of lacunary sets and each C_i, $i > 0$, is an arithmetic progression minus a finite union of lacunary sequences. Moreover, the sets C_i are ε-separated. Now $I(C_0)$ is complemented by Lemma 1.2. Letting B be $\bigcup_{i=0}^k C_i$ and $\alpha(\mathbf{Z}\setminus A)+\beta$ be C_{k+1} in Proposition 1.6, for $k = 0,1,2,\dots,$

$m-1$, we get a sequence of projections $P_0, P_1, P_2, \ldots, P_m$ such that P_i maps $I(\bigcup_{j=0}^{i-1} C_j)$ onto $I(\bigcup_{j=0}^{i} C_j)$. It follows then that $P = P_m P_{m-1} \cdots P_0$ is the required projection. □

2. The ε-separation condition

The first step in establishing that the hull of a complemented translation-invariant subspace is ε-separated will be to show that it is locally like the hull of a complemented translation-invariant subspace of $L^1(\mathbb{R})$. To do this we will need some basic facts from Banach space theory. We will simply list these as lemmas without proof. The first is an exercise, [DS, p. 513], and a proof of the second is contained in the proof of Lemma 1.4.

Lemma 2.1. *If X and Y are Banach spaces and $S: X^* \to Y^*$ is w^*-continuous then there is a bounded linear operator $T: Y \to X$ such that $T^* = S$. Moreover, if S is a w^*-isomorphism onto Y^*, then T is an isomorphism.*

Lemma 2.2. *If Z is a subspace of X then Z is complemented in X iff the quotient map $Q: X \to X/Z$ has a bounded right inverse T. Moreover the complement of Z can be taken to be $T(X/Z)$.*

The next lemma shows that there is a simple relation between annihilators of translation-invariant subspaces of $H^1(\mathbb{R})$ and $L^1(\mathbb{R})$ provided the hull is separated from 0 and ∞.

Lemma 2.3. *Suppose that A is a closed subset of $[a,b]$ with $0 < a < b < \infty$ and X is a translation-invariant subspace of $H^1(\mathbb{R})$ such that $Z(X) = A$. Let Y be the norm closure of $[X + \{f: f \in L_1(\mathbb{R})$ and $\hat{f}(t) = 0$ for all $t > a/2\}]$. Then $X^\perp \subseteq H^1(\mathbb{R})^*$ is w^*- isomorphic to $Y^\perp \subseteq L^1(\mathbb{R})^*$ via the natural quotient map $Q: L^\infty(\mathbb{R}) \to H^1(\mathbb{R})^*$.*

Proof. First observe that the quotient map $Q: L^\infty(\mathbb{R}) \to H^1(\mathbb{R})^*$ carries $Y^\perp$ into $X^\perp$. Indeed if $y \in Y^\perp$, $\langle Qy, f\rangle = \langle y, f\rangle = 0$ for all $f \in X$ because $X \subset Y$. Also Q is one-to-one on $Y^\perp$, because $\ker Q = H^1(\mathbb{R})^\perp$ and $H^1(\mathbb{R})^\perp \cap Y^\perp = \{0\}$. We claim that Q actually carries $Y^\perp$ onto $X^\perp$. Suppose that $x \in X^\perp$. Let F be an element of $H^1(\mathbb{R})$ such that supp $\hat{F} \subset [a/2, \infty)$ and $\hat{F}|_{[a-\varepsilon, b+\varepsilon]} = 1$, for some ε with $0 < \varepsilon < a/2$. Then if we define an element y of $L^\infty(\mathbb{R})$ by $\langle y, f\rangle = \langle x, F * f\rangle$ for all $f \in L^1(\mathbb{R})$, we have $\langle y, f\rangle = \langle x, f\rangle$ for $f \in H^1(\mathbb{R})$, because $f - F * f \in X$. In other words $Qy = x$. Because Q is w^*-continuous, the lemma is proved. □

Theorem 2.4. *If X is a complemented translation-invariant subspace of $H^1(\mathbb{R})$ and $Z(X) = A \subset [a,b]$, $0 < a < b < \infty$, then A is finite.*

Proof. We will show that $I(A)$ is complemented in $L^1(\mathbb{R})$. It will then follow from Theorem 1.6 of [R1] that A is a finite set.

In the notation of Lemma 2.3 the subspace Y of $L^1(\mathbf{R})$ has the property that $L^1(\mathbf{R})/Y$ is isomorphic to $H^1(\mathbf{R})/X$, by Lemma 2.1. Clearly $Z(Y) = A$. From Lemma 2.2 we know that the complement Z of X in $H^1(\mathbf{R})$ is isomorphic to $H^1(\mathbf{R})/X$ and hence to $L^1(\mathbf{R})/Y$. To establish the complementation of Y we will again use Lemma 2.2 and show that the quotient map R of $L^1(\mathbf{R})$ onto $L^1(\mathbf{R})/Y$ is an isomorphism from Z onto $L^1(\mathbf{R})/Y$. In particular we will show that $R = SQ_1 i$ where S is the isomorphism from $H^1(\mathbf{R})/X$ onto $L^1(\mathbf{R})/Y$, Q_1 is the quotient map from $H^1(\mathbf{R})$ onto $H^1(\mathbf{R})/X$, and i is the inclusion from $Z \subset L^1(\mathbf{R})$ into $H^1(\mathbf{R})$.

$$\begin{tikzcd}
L^1(\mathbf{R}) \arrow[r, "R"] & L^1(\mathbf{R})/Y & L^\infty(\mathbf{R}) \arrow[d] & Y^\perp \arrow[l, "R^*"'] \arrow[dd, "Q=S^*"] \\
Z \arrow[u, phantom, "\cup"] \arrow[d, "i"] & & L^\infty(\mathbf{R})/Z^\perp & \\
H^1(\mathbf{R}) \arrow[r, "Q_1"] & H^1(\mathbf{R})/X \arrow[uu, "S"'] & L^\infty(\mathbf{R})/H^1(\mathbf{R})^\perp \arrow[u, "i^*"'] & X^\perp \arrow[l, "Q_1^*"']
\end{tikzcd}$$

It is sufficient to check that the diagram on the left commutes. This is equivalent to checking the commutativity of the dual diagram on the right. However this is obvious. □

Our next result shows that there is a direct relation between complemented translation-invariant subspaces with hull in an interval and those with more general hulls. Below $\mathscr{M}(\mathbf{R})$ denotes the space of finite Radon measures on $\mathbf{R}$.

Lemma 2.5. *Suppose that* $P\colon H^1(\mathbf{R}) \to X$ *is a projection onto* X *and* X *is translation-invariant. Let* $A = Z(X)$. *If* $0 < a < b < c < d < \infty$, $[a,b] \cap A = [c,d] \cap A = \varnothing$, *and* $\mu \in \mathscr{M}(\mathbf{R})$ *such that* supp $\hat{\mu} \subset [a,d]$ *and* $\hat{\mu}(t) = 1$ *for all* $t \in [b,c]$, *then*

$$Uf = (I - P)\mu * f \ , \quad \textit{for} \ \ f \in H^1(\mathbf{R})$$

defines a projection with $\ker U = \overline{X + I([b,c])}$.

Proof. First we will show that $\ker U \supset X + I([b,c])$. If $f \in X$ then, because X is translation-invariant, $\mu * f \in X$. Hence $P\mu * f = \mu * f$ and thus $(I-P)\mu * f = 0$. If $f \in I([b,c])$ then $(\mu * f)\hat{} = 0$ except on $(a,b) \cup (c,d)$. Because the support of $(\mu * f)\hat{}$ is disjoint from A and compact there is an $F \in X$ such that $\hat{F} = 1$ on supp $(\mu * f)\hat{}$. (See [He, p. 150].) But then $F * \mu * f \in X$ and $F * \mu * f = \mu * f$. Hence $(I - P)\mu * f = 0$.

To complete the proof it is sufficient to check that $U^* z = z$ for all $z \in Z = (X + I([b,c]))^\perp$. Indeed range $U^* \subset \ker U^\perp$ and thus we get that U^* is the identity on its range, *i.e.*, U^* is a projection. Thus U is also a projection and $\ker U = Z_\perp$.

Now suppose that $z \in Z$ and $f \in L^1(\mathbf{R})$. Then

$$\begin{aligned}
\langle U^* z, f\rangle = \langle z, Uf\rangle = \langle z, (I - P)\mu * f\rangle &= \langle z, \mu * f\rangle - \langle z, P\mu * f\rangle \\
&= \langle z, \mu * f\rangle \quad (\text{because range } P \subset X \text{ and } z \in X^\perp) \\
&= \langle z, f\rangle \qquad (\text{because } f - \mu * f \in I([b,c])).
\end{aligned}$$

Thus $U^* z = z$ as claimed. □

Corollary 2.6. *If X is a complemented translation-invariant subspace of $H^1(\mathbb{R})$ and $b, c \notin Z(X)$, $0 < b < c < \infty$, then there is a complemented translation-invariant subspace Y of $H^1(\mathbb{R})$ such that $Z(Y) = Z(X) \cap [b, c]$. Consequently $Z(X) \cap [b, c]$ is finite.*

Proof. The previous lemma shows that $Y = \overline{X + I([b, c])}$ is complemented in $H^1(\mathbb{R})$. Clearly $Z(X) \cap [b, c] = Z(Y)$. □

To complete our proof that the hull of a complemented translation-invariant subspace of $H^1(\mathbb{R})$ is discrete we need to eliminate the possibility that the hull contains an interval with endpoint 0 or ∞. To do this we introduce a sequence of averages of an operator which will approximate the average over $\mathbb{R}$ (which generally will exist only in the w^* sense.)

Definition 2.7. If P is an operator on $H^1(\mathbb{R})$ define for each $N \in \mathbb{N}$,

$$ {}_N P f = \frac{1}{2N} \int_{-N}^{N} (P(f_\tau))_{-\tau} \, d\tau \quad \text{for } f \in H^1(\mathbb{R}), \quad \text{where } g_\tau(t) = g(t - \tau). $$

Because the map $\tau \to g_\tau$ is (norm) continuous the integral exists in the Bochner sense. Note that if P is a projection with translation-invariant range then ${}_N P$ is a projection with the same range and no larger norm.

The following lemma can be proved by a direct calculation:

Lemma 2.8. *If $F \in L^1(\mathbb{R})$ and T is a bounded operator on $H^1(\mathbb{R})$ then*

(a) $\|F * {}_N T f - {}_N T F * f\| \le a(N) \|f\|$ *and* $a(N) \to 0$ *as* $N \to \infty$.

(b) *If S is a w^*-operator limit of $(({}_N T)^*) = ({}_N (T^*))$, S is translation-invariant.*

Sketch of Proof. A computation shows that

$$ \|({}_N T f)_\tau - {}_N T(f_\tau)\| \le \min\left(2, \frac{|\tau|}{N}\right) \|T\| \, \|f\| \ , \quad \text{for all } \tau . $$

From this it follows that

$$ \|F * ({}_N T f) - {}_N T(F * f)\| \le \|T\| \, \|f\| \int_{-\infty}^{\infty} |F(\tau)| \min\left(2, \frac{|\tau|}{N}\right) d\tau . $$

which implies (a). Assertion (b) is a direct consequence of (a). □

Proposition 2.9. *If X is a nontrivial complemented translation-invariant subspace of $H^1(\mathbb{R})$ then $Z(X)$ does not contain an interval with endpoint 0 or ∞.*

Proof. Let $b = \sup\{x : [0, x] \subset Z(X)\}$ and $c = \inf\{x : [x, \infty) \subset Z(X)\}$. Choose a sequence of positive real numbers α_n decreasing to 0 such that $\alpha_n \mathbb{Z}$ does not intersect the countable set $(b, c) \cap Z(X)$. For each n let Q_n denote the quotient map from $H^1(\mathbb{R})$ onto $H_0^1(\mathbb{T})$ determined by $\alpha_n Z$ and let S_n be

a right inverse. By the proof of Proposition 1.6 and the remark following it, we may assume that $\|S_n\| \, \|Q_n\| \le M$. (M does not depend on n.)

Let P be a projection with kernel X and for fixed n consider the sequence of operators on $H_0^1(\mathbf{T})$ defined by

$$T_{n,k} f = Q_n({}_k P(S_n f)), \quad \text{for } k \in \mathbf{N}.$$

It follows from Lemma 2.8 that ${}_k P^* e_\rho$ converges w^* to 0 or e_ρ for each $\rho \in \mathbf{R}$. It follows that $T_{n,k}$ converges (as k tends to ∞) pointwise in the w^* topology of $H^1(\mathbf{T})$ to an idempotent multiplier M_n on $H_0^1(\mathbf{T})$. Moreover because $Z(X) \cap (b,c)$ contains no limit points, $M_n^*(e_j) = 0$ for all j such that $\alpha_n j \in (b,c)$. Similarly, because $[0,b] \cup [c,\infty) \subset Z(X)$, $M_n^*(e_j) = e_j$ for all j such that $\alpha_n j \in [0,b] \cup [c,\infty)$.

Observe that if $b \ne 0$ or $c \ne \infty$ then both the zero set and the 1-set of $\hat{M}_n$ intersected with $\alpha_n \mathbf{Z}$ must get large with n. We will show that such a sequence of multipliers cannot be bounded uniformly.

Indeed, suppose that m is a multiplier on $H_0^1(\mathbf{T})$ such that $\hat{m}(n) = 1$ if $n \le N$ or $n \ge J > N$ and $\hat{m}(n) = 0$ otherwise. Suppose that $N > 1$. Let $r = \min(N/3, J - N)$. Let $f = e_{N-r}(2K_{2r} - K_r)$, and note that $f \in H_0^1(\mathbf{T})$. Then $\hat{m}(n)\hat{f}(n) = 0$ if $n \ge N$ and $\hat{m}(n)\hat{f}(n) = 1$ if $N - 2r \le n < N$. Applying Hardy's inequality to $e_N \overline{mf}$ we see that $\|mf\| \ge C \log 2r$. In the situation above $r \to \infty$ which is a contradiction of the uniform boundedness of the M_n.

In the case $J \ne \infty$ we take $r = J - N$, and $f = e_{J+r}(2K_{2r} - K_r)$ and proceed similarly. □

Corollary 2.10. *If X is a complemented translation-invariant subspace of $H^1(\mathbf{R})$, then $Z(X)$ has no limit points in $(0,\infty)$.*

Corollary 2.11. *If X is a complemented translation-invariant subspace of $H^1(\mathbf{R})$, then the intersection of any arithmetic progression with $Z(X)$ is in the ring of sets generated by subarithmetic progressions and lacunary subsets of the original arithmetic progression.*

Proof. The averaging argument used above shows that for each arithmetic progression there is a 0-1 valued multiplier on $H^1(\mathbf{T})$ with support which corresponds to the intersection with $Z(X)$. By Klemes' result [K] this intersection is of the required form. □

We know now that $Z(X) \cap (0,\infty)$ contains only isolated points but there still remains the possibility that $\inf\{|a-b| : a \ne b \in Z(X)\} = 0$, either near ∞ or 0. The proof that this is not the case is very similar to the proof of Lemma 2.3 of [AM] except that lacunary sequences are used in place of arithmetic progressions.

Lemma 2.12. *Suppose that A is a discrete subset of $(0,\infty)$ with no finite limit points except perhaps 0. Suppose that $(x_n)_{n\in M}$ is a generalized lacunary sequence contained in A, i.e., there is a $q > 1$ such that for all $n, n+1 \in M$, $qx_n < x_{n+1}$, where $M = \mathbb{N}$, $-\mathbb{N}$, or $\mathbb{Z}$. Then $\overline{[x_n]}^{w^*}$ is w^*-complemented in $\overline{[A]}^{w^*}$, where the w^* topology is the $\sigma([A], H^1(\mathbb{R})/I(A))$ topology. Moreover $\overline{[A]}^{w^*} = \overline{[x_n]}^{w^*} \oplus \overline{[A\setminus\{x_n\}]}^{w^*}$.*

Proof. We have already proved the first assertion for the case $M = \mathbb{N}$. (See Lemma 1.2.) We will assume that $M = \mathbb{Z}$ and leave the modifications for the remaining cases to the reader.

For each $n \in \mathbb{N}$ let $X_n = [x_k : -n \le k \le n]$. By Lemma 1.2 there is a w^*-continuous projection P_n from $H^1(\mathbb{R})^*$ onto X_n and the norm of P_n does not depend on n. By first convolving by a measure μ_n with transform equal to 1 on $\{x_k : -n \le k \le n\}$ and 0 off of a compact set not containing zero, [Ru1, p. 53], we may assume that the kernel of P_n contains all but finitely many points of A. Restricting these operators to $\overline{[A]}^{w^*}$ and averaging we get for each n a sequence $({}_rP_n)_{r=1}^{\infty}$ of projections from $\overline{[A]}^{w^*}$ onto X_n. Clearly ${}_rP_n e_a$ tends to 0 as r goes to ∞ for each $a \in A\setminus\{x_k : -n \le k \le n\}$ and thus a w^*-operator limit point Q_n is a norm continuous projection of $\overline{[A]}^{w^*}$ onto X_n with kernel $\overline{[A\setminus\{x_k : -n \le k \le n]}^{w^*}$. Because both the kernel and range are w^*-closed it follows that Q_n is w^*-continuous on $\overline{[A]}^{w^*}$.

Now let Q be a w^*-operator limit point of the sequence Q_n. Clearly range $Q \supset \bigcup_n$ range Q_n and Q is the identity on $\bigcup_n$ range Q_n. Because $[x_n]$ is isomorphic to l^2, a reflexive space, the norm closure of $\bigcup_n X_n$ is also the w^*-closure. Thus Q is the identity on $\overline{[x_n]}^{w^*}$. Also the kernel of Q contains $\bigcap_n \ker Q_n = \bigcap_n \overline{[A\setminus\{x_k : -n \le k \le n\}]}^{w^*} = \overline{[A\setminus\{x_n\}]}^{w^*}$ and the result is proved. □

We are ready to prove the main result of this section.

Theorem 2.13. *Suppose that X is a complemented translation-invariant subspace of $H^1(\mathbb{R})$. Then there is an $\varepsilon > 0$ such that $|a-b| > \varepsilon$ for all $a, b \in Z(X)$, $a \ne b$.*

Proof. We have already established that $Z(X)$ is a discrete set with no nonzero finite limit points. Suppose first that 0 is a limit point of $Z(X)$ and let (x_n) be a generalized lacunary sequence in $Z(X)$ with 0 as a limit point. Lemma 2.12 implies that $\overline{[x_n]}^{w^*}$ is complemented in $\overline{[Z(X)]}^{w^*}$ by a projection with kernel $\overline{[Z(X)\setminus\{x_n\}]}^{w^*}$. Because the kernel and range are w^*-closed the projection is w^*-continuous. Hence $\overline{[x_n]}^{w^*}$ is w^*-complemented in $H^1(\mathbb{R})^*$. Let P^* be the projection onto $\overline{[x_n]}^{w^*}$ and let Y be the range of P. It follows that Y is isomorphic to the predual of $\overline{[x_n]}^{w^*}$. Therefore the mapping $f \to (\hat{f}(x_n))$

is an isomorphism from Y onto l^2. Because the unit ball of Y is weakly compact, for every $\varepsilon > 0$ there is an interval $[-K, K]$ such that for all $f \in Y$, $\|f|_{[-K,K]}\|_1 > (1-\varepsilon)\|f\|_1$. However $x_n \to 0$ so $e_{x_n} \to 1$ uniformly on $[-K, K]$. Hence there is no bounded sequence (f_n) in Y such that $\hat{f}_n(x_k) = \delta_n^k$. This contradiction shows that 0 is not a limit point of $Z(X)$.

Next suppose that there is an increasing sequence $(x_n) \subset Z(X)$ such that $\inf\{|x_j - y| : j \in N \text{ and } y \in Z(X) \setminus \{x_n\}\} = 0$. Without loss of generality we may assume that (x_n) is lacunary. By Lemma 2.12 $\overline{[x_n]}^{w^*}$ is w^*-complemented in $\overline{[Z(X)]}^{w^*}$ with complement $\overline{[Z(X) \setminus (x_n)]}^{w^*}$. This implies that there is a subspace Y of $I(Z(X) \setminus (x_n))$ which is isomorphic to l^2 with the isomorphism given by $f \to (\hat{f}(x_n))$. (See Lemma 2.1 of [AM].) Suppose that (f_n) is a uniformly bounded sequence in Y such that $\hat{f}_n(x_k) = \delta_n^k$. Then $\hat{f}_n|_{Z(X)\setminus(x_n)} = 0$ but $\hat{f}_n(x_n) = 1$. This implies that the sequence (f_n) is not relatively weakly compact (not uniformly integrable or not essentially supported on a compact set), which contradicts the fact that Y was isomorphic to l^2. □

3. Some partial results

In the previous section we have shown that the hull of a complemented translation-invariant subspace must be ε-separated but we have not shown that the hull is in $\Omega_l(\mathbf{R})$. In this section we will show that if Klemes' result generalizes to $H^1(\mathbf{T}^n)$ then the hull is in $\Omega_l(\mathbf{R})$ and that in any case the hull must be somewhat like an element of $\Omega_l(\mathbf{R})$. The main difficulty is that unlike Klemes we are forced to deal with operators which are not 0-1 valued multipliers. First we will prove a result similar to Theorem 2 of [K].

Proposition 3.1. *Suppose that A is an ε-separated subset of $\mathbf{R}^+$ and that $I(A)$ is complemented in $H^1(\mathbf{R})$. If A has arbitrarily large gaps, then A is a finite union of lacunary sequences.*

The proof of this is very similar to Klemes' proof of Theorem 2 [K] so we will just show that we can set things up to use his ideas. Let $b\mathbf{R}$ denote the Bohr compactification of $\mathbf{R}$ and let $\mathscr{HM}(b\mathbf{R})$ be the set of all finite Radon measures on $b\mathbf{R}$ with transform equal to zero on the negative real axis.

Proposition 3.2. *Suppose that A is an ε-separated subset of $\mathbf{R}^+$ and that $I(A)$ is complemented in $\mathscr{HM}(b\mathbf{R})$. If A has arbitrarily large gaps, then A is a finite union of lacunary sequences.*

Before explaining how one proves this we will prove the next lemma which will allow us to deduce Proposition 3.1 from Proposition 3.2.

Lemma 3.3. *If T is a projection on $H^1(\mathbf{R})$ with translation-invariant kernel $I(A)$, then there is a translation-invariant projection S on $\mathscr{HM}(b\mathbf{R})$ with kernel $\{\mu \in \mathscr{HM}(b\mathbf{R}) : \hat{\mu}(a) = 0 \text{ for all } a \in A\}$.*

Proof. Consider the sequence of operators $({}_nT)$ and let S_0 be a w^* operator limit point of $({}_nT^*)$. S_0 is a translation-invariant operator on $L^\infty(\mathbb{R})/H^1(\mathbb{R})^\perp$, with range in $\overline{[A]}^{w^*}/H^1(\mathbb{R})^\perp$. Indeed $\langle S_0(g), f\rangle = \lim_n \langle g, {}_nTf\rangle = 0$ if $f \in I(A)$. Thus $S_0(g) \in \overline{[A]}^{w^*}/H^1(\mathbb{R})^\perp$. Also $\langle S_0(e_t), f_s\rangle = \langle S_0((e_t)_s), f\rangle$ for all $s \in \mathbb{R}$ and $f \in H^1(\mathbb{R})$, so that $S_0(e_t) = m(t)e_t$ modulo elements of $H^1(\mathbb{R})^\perp$.

Next we wish to show that the induced operator S_1 on $C(b\mathbb{R})/(C(b\mathbb{R}) \cap H^1(\mathbb{R})^\perp)$ defined by

$$S_1\Big(e_t + \big(C(b\mathbb{R}) \cap H^1(\mathbb{R})^\perp\big)\Big) = m(t)e_t + \big(C(b\mathbb{R}) \cap H^1(\mathbb{R})^\perp\big), \quad \text{for } t \in \mathbb{R}^+ ,$$

is bounded. It will follow then that S_1 is a well-defined operator on

$$C(b\mathbb{R})/\big(C(b\mathbb{R}) \cap H^1(\mathbb{R})^\perp\big)$$

and hence $S = S_1^*$ is an operator on $\mathscr{HM}(b\mathbb{R})$. Clearly S will be a multiplier.

Now suppose that p is a trigonometric polynomial, and let λ denote Lebesgue measure on the image of $\mathbb{R}$ in $b\mathbb{R}$. Because $\{f d\lambda : f \in H^1(\mathbb{R})\}$ is w^* dense in $\big(C(b\mathbb{R}) \cap H^1(\mathbb{R})^\perp\big)^\perp$,

$$\begin{aligned}
&\|p\|_{C(b\mathbb{R})/(C(b\mathbb{R})\cap H^1(\mathbb{R})^\perp)} \\
&\quad = \sup\{|\langle p, \mu\rangle| : \mu \in \big(C(b\mathbb{R}) \cap H^1(\mathbb{R})^\perp\big)^\perp = \mathscr{HM}(b\mathbb{R}) \ , \ \|\mu\| = 1\} \\
&\quad = \sup\{|\langle p, \mu\rangle| : \mu = f d\lambda \ , \ f \in H^1(\mathbb{R}) \ , \ \|\mu\| = 1\} \\
&\quad = \sup\{|\langle p, f\rangle| : f \in H^1(\mathbb{R}) \ , \ \|f\| = 1\} \\
&\quad = \|p\|_{L^\infty(\mathbb{R})/H^1(\mathbb{R})^\perp} .
\end{aligned}$$

This shows that the $L^\infty(\mathbb{R})/H^1(\mathbb{R})^\perp$ and $C(b\mathbb{R})/(C(b\mathbb{R}) \cap H^1(\mathbb{R})^\perp)$ norms are equal on trigonometric polynomials and thus the boundedness of S_0 implies the boundedness of S_1. Obviously the range of S_1 is contained in $(C(b\mathbb{R}) \cap [A])/(C(b\mathbb{R}) \cap H^1(\mathbb{R})^\perp)$ and thus S has the required kernel.

Finally suppose that T is a projection. Then ${}_nT^*$ is also a projection with range $\overline{[A]}^{w^*}/H^1(\mathbb{R})^\perp$. Hence S_0, S_1 and S are also projections and

$$(\text{range } S_1)^\perp = \big[[A]/H^1(\mathbb{R})^\perp\big]^\perp = \ker S . \quad \square$$

The two main ingredients in the proof of Theorem 3 of [K] are standard properties of the Fejér kernels and the solution of the Littlewood Conjecture [MPS]. In the case of the Bohr compactification of $\mathbb{R}$ the Fejér kernels correspond to the singular (with respect to the Haar measure) measures $K_\delta d\lambda$, where λ is Lebesgue measure on the image of $\mathbb{R}$ in $b\mathbb{R}$, and as noted in [MPS] the generalized Hardy's Inequality holds in any compact abelian group with ordered dual. It is now an easy exercise to show that Klemes' argument proves our Proposition 3.2. Proposition 3.1 follows then from Proposition 3.2 and Lemma 3.3.

We can use this result to reduce somewhat the number of possible zero sets for complemented translation-invariant subspaces of $H^1(\mathbf{R})$. In particular we can reduce the verification of the conjecture to subsets of finitely generated subgroups of $\mathbf{R}$.

Theorem 3.4. *Assume that it is true that whenever B is a subset of a finitely generated subgroup of $\mathbf{R}$ and $I(B)$ is complemented in $H^1(\mathbf{R})$, then B is in $\Omega_l(\mathbf{R})$. Suppose that X is a complemented translation-invariant subspace of $H^1(\mathbf{R})$ and $Z(X) = A$. Then there is a finitely generated subgroup G of $\mathbf{R}$ such that $A \setminus G$ is a finite union of lacunary sequences, i.e., $A \in \Omega_l(\mathbf{R})$.*

We need to prove some other results to prepare for the proof of this theorem. Our next result shows that the zero set of a complemented translation-invariant subspace of $H^1(\mathbf{R})$ must have a large part near an arithmetic progression.

Theorem 3.5. *Suppose that P is a projection on $H^1(\mathbf{R})$ with $\ker P = I(A)$ and A is an ε-separated subset of $[0,\infty)$. Then there exists a real number $\tau > 0$ and a finite set $F \subset [0,\tau)$ such that for every $\delta > 0$, $A \cap [F + \tau\{0,1,2,3,\dots\} + (-\delta,\delta)]^c$ has arbitrarily large gaps.*

The proof of this result requires some preliminaries. First we recall a result of Rosenthal [R1].

Lemma 3.6. *If S is an operator on $L^\infty(\mathbf{R})$ and $\operatorname{range}(S) = \overline{[A]}^{w^*}$ then there is a translation-invariant operator T on $L^\infty(\mathbf{R})$ with $\operatorname{range}(T) \subset \overline{[A]}^{w^*}$ such that if $a \in A$ and $Sa = a$ then $Ta = a$.*

The next lemma will allow us to imitate the proof of Theorem 2.1 of [K].

Lemma 3.7. *Suppose that (S_k) is a sequence of operators on $L^1(\mathbf{R})$ such that for any $f \in L^1(\mathbf{R})$ the sequence $(S_k f^\wedge)$ is equicontinuous, $\ker S_k = I(A_k)$, A_k is a finite ε-separated set for each k, and $A_k \to A$, i.e., for every $\delta > 0$ there is a $K \in N$ such that $A + (-\delta,\delta) \supset A_k$ for all $k > K$ and for any L there is a K such that for all $k > K$,*

$$A \cap [-L,L] \subset A_k + (-\delta,\delta).$$

If $S = w^*\text{-op-}\lim S_k^*$, then $\operatorname{range} S \subset \overline{[A]}^{w^*}$. If, in addition, for every $L > 0$ there is a K such that if $k > K$ then $S_k^* a = a$ for all $a \in A_k \cap [-L,L]$, then S is a projection onto $\overline{[A]}^{w^*}$.

Proof. The first assertion is easy. Indeed, if $f \in L^1(\mathbf{R})$ and for some $\delta > 0$, $\hat{f}|_{A+(-\delta,\delta)} = 0$, then $\langle g, f\rangle = 0$ for all $g \in [A_k]$ if k is sufficiently large. Thus for any $h \in L^\infty(\mathbf{R})$, $w^* \lim S_k^* h \in \overline{[A + (-\delta,\delta)]}^{w^*}$ for every $\delta > 0$. Hence $Sh \in \overline{[A]}^{w^*}$.

Now suppose that for every $L > 0$ there is a K such that if $k > K$ then $S_k^* a = a$ for all $a \in A_k \cap [-L,L]$. If $a \in A$, $\varepsilon > 0$ and $f \in L^1(\mathbf{R})$, then for

sufficiently large k we can find $a_k \in A_k \cap (-\varepsilon + a, \varepsilon + a)$. Now

$$\langle Sa, f\rangle = \lim\langle S_k^* a, f\rangle = \lim -\langle S_k^* a_k, f\rangle + \langle S_k^* a, f\rangle + \langle a_k, f\rangle,$$

because $S_k^* a_k = a_k$. Of course $\langle a_k, f\rangle = \hat{f}(a_k) \to \hat{f}(a)$; in fact the equicontinuity assumption implies that $\langle S_k^* a_k, f\rangle - \langle S_k^* a, f\rangle = S_k f\hat{}(a_k) - S_k f\hat{}(a) \to 0$. Hence $Sa = a$ for all $a \in A$. Finally to see that S is the identity on $\overline{[A]}^{w^*}$, suppose that $g \in \overline{[A]}^{w^*}$. Let $f \in L^1(\mathbb{R})$ and assume that supp $\hat{f} \subset [-L, L]$ for some $L < \infty$. If $g_n \in [A_n]$ and $g_n \xrightarrow{w^*} g$, then $\langle Sg, f\rangle = \lim\langle S_k^* g, f\rangle = \lim_k \lim_n \langle S_k^* g_n, f\rangle$. Because supp $\hat{f} \subset [-L, L]$ and $(S_k f\hat{})_k$ is equicontinuous, $\langle S_k^* g_k, f\rangle - \langle S_k^* g_n, f\rangle$ is small for all large n, given k large enough. But $\langle S_k^* g_k, f\rangle = \langle g_k, f\rangle$ and therefore $\langle Sg, f\rangle = \lim_k \langle g_k, f\rangle = \langle g, f\rangle$. Because L was arbitrary, $Sg = g$. □

Proof of Theorem 3.5. Define operators S_n on $L^1(\mathbb{R})$ by

$$S_n f = e_{-3\tilde{n}} \mu_n * [P(\mu_n * e_{3\tilde{n}} f)]$$

where

$$\hat{\mu}_n(x) = \begin{cases} 0 & \text{if } -\infty < x + 3(n - \tilde{n}) \le 0, \\ (x + 3(n - \tilde{n}))/2n & \text{if } 0 < x + 3(n - \tilde{n}) < 2n, \\ 1 & \text{if } 2n \le x + 3(n - \tilde{n}) \le 4n, \\ 3 - [(x + 3(n - \tilde{n}))/2n] & \text{if } 4n < x + 3(n - \tilde{n}) < 6n, \\ 0 & \text{if } 6n \le x + 3(n - \tilde{n}) < \infty, \end{cases}$$

and $\tilde{n} \ge n$ (chosen below). Because range $P^* \subset \overline{[A]}^{w^*}$,

$$\begin{aligned} \text{range } S_n^* &\subset \overline{\Big[\big[A \cap [3(\tilde{n} - n)\ ,\ 6n + 3(\tilde{n} - n)]\big] - 3\tilde{n}\Big]} \\ &= \Big[\big[A \cap [3(\tilde{n} - n)\ ,\ 6n + 3(\tilde{n} - n)]\big] - 3\tilde{n}\Big] \\ &= \Big[A - 3\tilde{n} \cap [-3n, 3n]\Big]. \end{aligned}$$

Let $A_n = (A - 3\tilde{n}) \cap [-3n, 3n]$. If $\nu + 3\tilde{n} \in A \cap [2n + 3(\tilde{n} - n),\ 4n + 3(\tilde{n} - n)]$ then $S_n^* \nu = \nu$. Because A is ε-separated, by choosing $\tilde{n}$ appropriately we may assume that $A_n \to A'$ for some set A'. Indeed, if $A_{n,k} = (A - 3k) \cap [-3n, 3n]$, then card $A_{n,k} \le \frac{6n+1}{2}$. Hence we may choose for each n a subsequence $\{k_j\}$ such that A_{n,k_j} converges. A standard diagonalization argument yields the sequence $\tilde{n}$.

We are almost ready to apply Lemma 2, but we have not yet checked the equicontinuity of $(S_k f\hat{})$. For n large $\mu_n * e_{3\tilde{n}} f$ is close in norm to $e_{3\tilde{n}} f$ and $e_{3\tilde{n}} f \to 0$ weakly. Hence $\mu_n * e_{3\tilde{n}} f \to 0$ weakly. Because every norm bounded operator is weakly continuous $P(\mu_n * e_{3\tilde{n}} f) \to 0$ weakly. Thus $P(\mu_n * e_{3\tilde{n}} f)$ is uniformly integrable and essentially supported on some fixed set of finite measure, [DS, p. 292]. Therefore $(P(\mu_n * e_{3\tilde{n}} f)\hat{})$ is equicontinuous, and hence

so is $(S_n f\hat{})$. Applying Lemma 3.7 we see that $S_n^* \to S$ a projection onto $\overline{[A']}^{w^*}$.

By Lemma 3.6 there is a translation-invariant projection onto the same space and thus by Rosenthal's (Cohen's) Theorem [R1], A' is a finite union of cosets of $\mathbf{Z}$ (up to a finite set). Because A was ε-separated so is A' and thus the periods must be rationally dependent. Let τ be the GCD of the periods. Now consider the final assertion of the theorem. Because $A_n \to A'$, given $\delta > 0$, $A_n \subset A' + (-\delta, \delta)$ for large n. Hence $A_n + 3\tilde{n} = A \cap [3(\tilde{n} - n), 6n + 3(\tilde{n} - n)] \subset A' + (-\delta, \delta) + 3\tilde{n}$. By passing to a subsequence we may assume that $3\tilde{n}$ converges to $\alpha \bmod \tau$. Hence for n in this subsequence and large, $A_n + 3\tilde{n} \subset A' + \alpha + (-\delta, \delta)$, and therefore for $F = (A' + \alpha) \cap [0, \tau)$ we have a gap of size $6n$ in $A \cap [F + \tau\{0, 1, 2, \dots\} + (-\delta, \delta)]^c$. □

We are now ready for the proof of Theorem 3.4.

Proof. We have by Theorem 2.13 that A is an ε-separated subset of $\mathbf{R}$ and that there is a translation-invariant projection S on $\mathscr{HM}(b\mathbf{R})$ with kernel $I(A)$. Let $A = \{a_n\}$ and let G_k be the group generated by $\{a_n : n \le k\}$. By the easy part of Cohen's Idempotent Theorem for each k there is a norm-one measure μ_k on $b\mathbf{R}$ such that $\hat{\mu}_k = 1_{G_k}$. Let $T_k f = Sf - \mu_k * Sf$ and $U_k f = \mu_k * Sf$. Observe that T_k is a projection with kernel $I(A \setminus G_k)$ and U_k is a projection with kernel $I(A \cap G_k)$. We claim that for some k, $A \setminus G_k$ has large gaps and thus by Proposition 3.2 is a finite union of lacunary sequences.

Suppose that this is false. We are assuming that for each k, $A \cap G_k \in \Omega_l(\mathbf{R})$ and thus there are real numbers α_n^k, β_n^k, $n = 1, 2, 3, \dots, N_k$ and finite unions of lacunary sequences C_n^k, B_k such that $A \cap G_k = \bigcup_n \alpha_n^k(\mathbf{Z} \setminus C_n^k) + \beta_n^k \cup B_k$. Theorem 3.5 implies that there is a $\tau > 0$ and a finite subset F of $[0, \tau)$, independent of k, such that $\alpha_n^k(\mathbf{Z} \setminus C_n) + \beta_n^k \subset \tau\mathbf{Z} + F$. Hence $A \cap G_k \setminus \tau\mathbf{Z} + F$ is a finite union of lacunary sequences for each k. For each k the operator $V_k f = \nu * U_k f$, where $\hat{\nu} = 1_{(\tau\mathbf{Z}+F)^c}$, is a projection with kernel $I(A \cap G_k \setminus \tau\mathbf{Z} + F)$, and $\|V_k\|$ is bounded independent of k. It remains to show that $(\bigcup B_k) \setminus \tau\mathbf{Z} + F$ is a finite union of lacunary sequences.

As noted by Klemes the proof of (2.2) [K] shows that if m is a 0-1 valued multiplier on $H^1(\mathbf{T})$ with large gaps in the support of $\hat{m}$ then $\sup_y \operatorname{card}[3y, 6y) \cap \operatorname{supp} \hat{m} \le \exp(\exp(c\|m\|))$. A similar estimate is valid for the proof of Proposition 3.2. Because this parameter determines the number of lacunary sequences needed in the union and a lower bound on the ratio of lacunarity q, we obtain that

$\sup_{k,y} \operatorname{card}[3y, 6y) \cap A \cap G_k \setminus \tau\mathbf{Z} + F = \sup_y \operatorname{card}[3y, 6y) \cap A \setminus \tau\mathbf{Z} + F$ is finite and thus that $A \setminus \tau\mathbf{Z} + F$ is a finite union of lacunary sequences. □

One way to think of a finitely generated subgroup of $\mathbf{R}$ is to find a slightly larger subgroup G of $\mathbf{R}$ which is isomorphic to $\mathbf{Z}^n$ and consider this as the dual group of the quotient $b\mathbf{R}/G_\perp$. In this way the multiplier acting on $\mathscr{HM}(b\mathbf{R})$

can be considered as acting on $H^1(\mathbf{T}^n)$. (By $H^1(\mathbf{T}^n)$ we mean the subspace of $L^1(\mathbf{T}^n)$ consisting of functions with transform zero on a negative half-space of $\mathbf{Z}^n$. See [Ru1, p. 197].) It might be possible to prove the analog of Klemes' result here and then deduce the result for $\mathbf{R}$ from Theorem 3.4.

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