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# A PICARD THEOREM WITH AN APPLICATION TO MINIMAL SURFACES

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ABSTRACT. We prove a Picard theorem for holomorphic maps from $\mathbf{C}$ to a quadric hypersurface. This implies a theorem on the number of directions in general position omitted by the normals to a minimal surface of the conformal type of $\mathbf{C}$.

The distribution of the normals to a two-dimensional minimal surface in $\mathbf{R}^n$ has been studied by Chern and Osserman [1]. This paper is concerned only with a special case of their theorem. For a minimal surface of the type of the entire plane $\mathbf{C}$, Chern and Osserman prove that, if the normals to the surface omit $n+1$ directions in general position, where $n$ is the dimension of the ambient space, the image of the Gauss map lies in a proper linear subspace of $\mathbf{CP}^{n-1}$. Theorem 1 of this paper improves on their result in two ways. First, it is only assumed that the normals omit $n$ directions in general position. Secondly, we prove that the image of the Gauss map lies in a linear subspace of codimension two; in consequence, the minimal surface "decomposes" into a holomorphic function and a minimal surface in $\mathbf{R}^{n-2}$, in a sense that will be made precise below. The method, which derives from a paper of M. L. Green [4], is to apply value-distribution theory to maps into a quadric hypersurface instead of maps into projective space.

In the definitions that follow we adopt the notation used by Hoffman and Osserman in their memoir [5], to which we refer for details.

Let $M$ be a Riemann surface and $f\colon M \to \mathbf{R}^n$, where $n \geq 2$, be a nonconstant smooth map, with components $(f_1, \dots, f_n)$. If $z = x + iy$ is a local coordinate on $M$, let

$$\varphi_k = \frac{\partial f_k}{\partial x} - i\frac{\partial f_k}{\partial y}.$$

The map $f$ is called a *minimal surface* if the $\varphi_k$ are holomorphic and satisfy the equation of conformality

$$\varphi_1^2 + \cdots + \varphi_n^2 = 0. \tag{1}$$

If the vector $(\varphi_1, \dots, \varphi_k)$ is nonzero then it gives the homogeneous coordinates of some point in the complex projective space $\mathbf{CP}^{n-1}$. Since the $\varphi_k$ are

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holomorphic, this defines a map $\Phi\colon M \to \mathbf{CP}^{n-1}$ except on a discrete subset of $M$, and $\Phi$ may be extended to the whole of $M$ by analytic continuation. Since the $\varphi_k$ satisfy equation (1) we see that the image of $\Phi$ lies on the quadric

$$Q_{n-2}\colon x_1^2 + \cdots + x_n^2 = 0, \tag{2}$$

where $(x_1, \ldots, x_n)$ are homogeneous coordinates on $\mathbf{CP}^{n-1}$.

Geometrically, $Q_{n-2}$ may be identified with the Grassmannian $G_{2,n}(\mathbf{R})$ of oriented two-planes through the origin in $\mathbf{R}^n$. We shall refer to $\Phi$ as the *Gauss map* of $f$, although strictly it is the complex conjugate of the Gauss map. The extension of $\Phi$ by analytic continuation across the points where $(\varphi_1, \ldots, \varphi_n)$ is zero defines the *tangent plane* $T_p f$ at a singular point $p$ of $f$ in such a way that $T_p f$ is a continuous function on $M$.

Suppose $g$ is a holomorphic lift

$$\begin{tikzcd} & \mathbf{C}^n - \{0\} \arrow[d] \\ M \arrow[ur, "g"] \arrow[r, "\Phi"] & \mathbf{CP}^{n-1} \end{tikzcd} \tag{3}$$

with components $(g_1, \ldots, g_n)$. (This is possible by the Behnke-Stein theorem [2, Theorem 26.5], since $M$ cannot be compact.) From (1) we have the identity

$$g_1^2 + \cdots + g_n^2 = 0. \tag{4}$$

A direction $v$ in $\mathbf{R}^n$ will be called *normal* to $f$ at $p$ if it is orthogonal to $T_p f$. The real and imaginary parts of $g(p)$ span $T_p f$, so that the equation for $v$ to be normal to $f$ at $p$ is

$$v_1 g_1(p) + \cdots + v_n g_n(p) = 0, \tag{5}$$

where $(v_1, \ldots, v_n)$ are the components of $v$, as on p. 118 of [7]. Equation (5) can be interpreted as saying that $\Phi(p)$ lies in the hyperplane of $\mathbf{CP}^{n-1}$ which is described by the equation

$$v_1 x_1 + \cdots + v_n x_n = 0. \tag{6}$$

We observe that (6) is an equation with real coefficients. The normals to $f$ omit the direction $v$ if and only if the image of $\Phi$ lies in the complement of the hyperplane (6).

In equation (1), it may happen that for some $k < n$ the equation

$$\varphi_1^2 + \cdots + \varphi_k^2 = 0 \tag{7}$$

holds identically on $M$. Then we must also have

$$\varphi_{k+1}^2 + \cdots + \varphi_n^2 = 0. \tag{8}$$

Consider the orthogonal decomposition $\mathbf{R}^n = \mathbf{R}^k \oplus \mathbf{R}^{n-k}$ and let $p_a$, $p_b$ be the projections onto the factors. Equations (7) and (8) mean that the maps

$$f_a = p_a \circ f\colon M \to \mathbf{R}^k \quad \text{and} \quad f_b = p_b \circ f\colon M \to \mathbf{R}^{n-k}$$

are minimal surfaces, unless $k = 1$ or $k = n - 1$, in which case one of $f_a$ and $f_b$ is a constant. (We comment that $f_a$ and $f_b$ may both have singularities even if $f$ is an immersion.) We are particularly interested in the case $k = 2$. A minimal surface in $\mathbf{R}^2$ is the same thing as a holomorphic or antiholomorphic function. If, after a rotation of $\mathbf{R}^n$, (7) and (8) hold with $k = 2$, we shall say that $f$ *has a holomorphic factor.* When $f$ lies fully in $\mathbf{R}^n$ this is what Hoffman and Osserman call 2-decomposable [5, p. 50]. If $n = 2m$ is even, it may happen that we can obtain an orthogonal decomposition $\mathbf{R}^{2m} = \mathbf{R}^2 \oplus \cdots \oplus \mathbf{R}^2$ such that the projection of $f$ into each $\mathbf{R}^2$ is holomorphic or antiholomorphic. Then, if we identify $\mathbf{R}^{2m}$ with $\mathbf{C}^m$ in the obvious manner, $f$ becomes a holomorphic curve. We say that $f$ is holomorphic in an *orthogonal complex structure* on $\mathbf{R}^{2m}$ [5, p. 54; 7, p. 164].

The property that $f$ be holomorphic in an orthogonal complex structure has been characterized by Lawson in terms of the Gauss map $\Phi$ [5, Proposition 4.3; 7, p. 165, Proposition 16]. He shows that $f$ is holomorphic in some orthogonal complex structure if and only if the image of $\Phi$ lies in a linear subspace of $Q_{n-2}$. In the same vein, Hoffman and Osserman have shown that $f$ has a holomorphic factor if and only if the image of $\Phi$ lies in a hyperplane tangent to $Q_{n-2}$ [5, Proposition 4.1].

We now state our theorem on the number of directions that do not occur as normals. The theorem is given here for minimal surfaces defined on $\mathbf{C}$ but could be proved by the same argument for a punctured disc if $\Phi$ had an essential singularity at the puncture.

**Theorem 1.** *Let $f\colon \mathbf{C} \to \mathbf{R}^n$ be a minimal surface and suppose that the normals to $f$ omit $n$ directions in general position. Then $f$ has a holomorphic factor.*

*Proof.* From the remarks above, we see that the hypothesis is equivalent to saying that the image of the Gauss map $\Phi$ lies in the complement of $n$ hyperplanes with real coefficients and in general position. The conclusion is equivalent to saying that the image of $\Phi$ lies in a hyperplane tangent to $Q_{n-2}$. Theorem 1 now follows from Theorem 2 below.

**Corollary 1.** *If $f\colon \mathbf{C} \to \mathbf{R}^4$ is minimal and the normals to $f$ omit four directions in general position, then $f$ is holomorphic in some orthogonal complex structure on $\mathbf{R}^4$.*

*Proof.* Theorem 1 gives an orthogonal decomposition $\mathbf{R}^4 = \mathbf{R}^2 \oplus \mathbf{R}^2$ such that the projection of $f$ into each factor is holomorphic or antiholomorphic.

**Corollary 2.** *If $f\colon \mathbf{C} \to \mathbf{R}^4$ is minimal and the normals to $f$ omit five directions in general position, then $f$ is planar.*

*Proof.* By Corollary 1, $f$ is holomorphic in an orthogonal complex structure and hence the image of $\Phi$ lies in a line $L$ on $Q_2$. Since the hyperplane (6) of $\mathbf{CP}^3$ corresponding to a direction in $\mathbf{R}^4$ has real coefficients, the intersection of three such hyperplanes in general position is a real point. The equation (2) of

$Q_2$ shows that $Q_2$ has no real points. Hence the five hyperplanes corresponding to the omitted directions intersect $L$ in at least three points. By the standard Little Picard Theorem, $\Phi$ is a constant, which implies that $f$ is planar.

We shall require a standard result of complex analysis.

**Lemma 1** (Borel Lemma). *Let*

$$h_1, \dots, h_m : \mathbf{C} \to \mathbf{C}\backslash\{0\}$$

*be nowhere-zero holomorphic functions that satisfy an identity*

$$\lambda_1 h_1 + \cdots + \lambda_m h_m = 0,$$

*where* $\lambda_1, \dots, \lambda_m$ *are nonzero constants. Then there is a partition of the indices* $1, \dots, m$ *such that*

(1) *each subset in the partition contains at least two indices; and*

(2) *if* $i$ *and* $j$ *are indices in the same subset of the partition then the identity* $h_i = a h_j$ *holds for some nonzero constant* $a$, *depending on* $i$ *and* $j$.

Proofs of the Borel Lemma may be found in works on value-distribution theory. A recent survey has been given by Lang [6]. Statements of the Borel Lemma in the form in which it is used here are given by Nevanlinna [8, article 57] and M. L. Green [3] (see Green's Remark 2 on p. 100).

We shall also require a lemma from projective geometry.

**Lemma 2.** *Let* $L$ *be a linear subspace of* $\mathbf{CP}^{n-1}$ *of codimension* $2$. *Then* $L$ *lies in a hyperplane tangent to the nonsingular quadric* $Q_{n-2}$.

*Proof.* We shall write $\operatorname{polar}(A)$ for the polar with respect to $Q_{n-2}$ of a linear subspace $A$ of $\mathbf{CP}^{n-1}$. The tangent at a point $p$ of $Q_{n-2}$ is $\operatorname{polar}(\{p\})$. We wish to find $p$ on $Q_{n-2}$ such that

$$L \subset \operatorname{polar}(\{p\}).$$

This is equivalent to

$$\{p\} \subset \operatorname{polar}(L).$$

Since $L$ has codimension 2, $\operatorname{polar}(L)$ is a line and hence intersects $Q_{n-2}$ in at least one point. Taking $p$ to be such a point, we obtain the result.

**Theorem 2.** *Let*

$$\Phi : \mathbf{C} \to Q_{n-2} \subset \mathbf{CP}^{n-1}$$

*be holomorphic. Suppose that* $\Phi$ *does not meet the union of* $n$ *hyperplanes* $\Pi_1, \dots, \Pi_n$ *with real coefficients and in general position. Then the image of* $\Phi$ *lies in a linear subspace of codimension* $2$, *and hence in a hyperplane tangent to* $Q_{n-2}$.

*Proof.* We continue to use the notation established above.

To illustrate the method we first prove a special case. Suppose that $\Pi_1, \dots, \Pi_n$ are the coordinate hyperplanes. Then the components $g_i$ of the lift $g$

defined in (3) are nowhere-zero holomorphic functions. Their squares $g_i^2$ are also nowhere-zero holomorphic functions and satisfy the linear identity (4). By Lemma 1, corresponding to each index $i = 1, \dots, n$ there is at least one other index $j \neq i$ such that

$$g_i^2 = a g_j^2 \tag{9}$$

for some nonzero constant $a$, depending on $i$ and $j$. From (9) we must have one or the other of the identities $g_i = \pm a^{1/2} g_j$. The result is trivial for $n = 2$, and for $n > 3$ there are at least two such identities, so that the image of $\Phi$ lies in a linear subspace $L$ of codimension 2. By Lemma 2, $L$ is contained in a tangent hyperplane of $Q_{n-2}$.

To prove Theorem 2 in general, it suffices by Lemma 2 to show that the image of $\Phi$ lies in a linear subspace $L$ of codimension 2.

Take new coordinates $(y_1, \dots, y_n)$ on $\mathbf{CP}^{n-1}$ so that $\Pi_1, \dots, \Pi_n$ are the coordinate hyperplanes. Since $\Pi_1, \dots, \Pi_n$ have real coefficients, the change of coordinates is given by a real matrix. Therefore it preserves the signature of real quadratic forms. The form in equation (2) for $Q_{n-2}$ is a positive definite real quadratic form, so is carried into some other real quadratic form which is also positive definite. Let the new equation of $Q_{n-2}$ be

$$\sum_{i,j=1}^{n} M_{ij} y_i y_j = 0. \tag{10}$$

Since $M_{ij}$ is a positive definite matrix, the diagonal entries are all nonzero.

Take a holomorphic lift $g$ of $\Phi$ as in (3) but now let $g_1, \dots, g_n$ be components with respect to the coordinate system $y_1, \dots, y_n$. In these coordinates $g_1, \dots, g_n$ are nowhere-zero holomorphic functions and so is the product $g_i g_j$ of any two of them. Since the image of $\Phi$ lies on $Q_{n-2}$ we have an identity

$$\sum_{i,j=1}^{n} M_{ij} g_i g_j = 0, \tag{11}$$

which may be regarded as a linear identity among the products $g_i g_j$. Not all the coefficients in (11) need be nonzero, but as we have seen the coefficients of the squares $g_1^2, \dots, g_n^2$ are nonzero.

Apply Lemma 1 to the identity (11). We obtain a system of identities among the $g_i g_j$ which includes $n$ identities of the form

$$g_i^2 = a_i g_j g_k, \qquad i = 1, \dots, n, \tag{12}$$

in which $j$ and $k$ are not both equal to $i$ and $a_i$ is a nonzero constant. We shall show that the identities (12) may be used to derive two independent linear identities—but not, in general, three.

Observe that an equation of the form

$$g_i^p = a g_j^p, \tag{13}$$

where $i \neq j$, $p \geq 1$ and $a$ is a nonzero constant, means that $\Phi$ lies in the union of $p$ hyperplanes and hence lies in one of them. Therefore it is sufficient to derive two identities such as (13).

There are now three cases to consider. First, it may be that one of the identities in (12) is already of the form (13). Secondly, there may be an identity such as

$$g_i^2 = a_i g_i g_j, \tag{14}$$

in which $g_i$ occurs on both sides. This is equivalent to

$$g_i(g_i - a_i g_j) = 0,$$

so that we may cancel $g_i$ from each side and obtain a linear identity. Thirdly, if neither of the first two cases occurs, we may use the last identity of (12) to eliminate $g_n$ from each of the other $n-1$ identities. For $i = 1, \ldots, n-1$, if $g_n$ appears on the right-hand side of identity $i$, we square both sides and substitute for $g_n^2$ using identity $n$. This leads to a system of $n-1$ identities of the form

$$g_i^4 = b_i g_j^2 g_k g_l, \qquad i = 1, \ldots, n-1, \tag{15}$$

in which $j \neq i$, $k \neq i$ but possibly $l = i$, and $b_i$ is a nonzero constant. If one of the identities (15) involves only two indices, we may obtain a linear identity as we did in the first two cases with the identities (12). Otherwise, we may now use identity $n-1$ to eliminate $g_{n-1}$ from the first $n-2$ identities. By iterating this procedure we must reach an identity of the form (13). This yields our first linear identity.

Renumber the indices so that the linear identity just obtained is

$$g_2 = \alpha g_1. \tag{16}$$

In the system of identities (12), delete the first two identities and substitute $\alpha g_1$ for $g_2$ in the others. This yields $n-2$ idetities in the $n-1$ functions $g_1, g_3, \ldots, g_n$. Eliminating successive $g_i$ as before leads to another identity

$$g_i^p = b g_j^p, \tag{17}$$

in which $i \geq 3$, $j \neq i$, $p \geq 1$ and $b$ is a nonzero constant. This yields a second linear identity.

*Remarks.* 1. It is not possible in general to obtain any more linear identities. Suppose that the linear identity obtained from (17) is

$$g_4 = \beta g_3. \tag{18}$$

Then we may use (18) to obtain $n-4$ quadratic identities in $g_1, g_3, g_5, \ldots, g_n$. The problem is that these identities may yield only relations involving at least

three indices, from which it is impossible to derive a linear identity. For example, we may have

$$\begin{aligned} g_1 &= ig_2 = e^z, \\ g_3 &= ig_4 = e^{2z}, \\ g_5^2 &= g_1 g_3, \\ g_6^2 &= g_1 g_5, \\ &\vdots \\ g_n^2 &= g_1 g_{n-1}. \end{aligned}$$

This may be represented as the Gauss map of an immersed minimal surface using Hoffman and Osserman's generalized Enneper-Weierstrass representation [5, Theorem 3.1].

2. The conclusion of Corollary 1 to Theorem 1 cannot be improved to say that $f$ is planar; if $f$ is the holomorphic curve defined by $f(z) = (e^z, e^{2z})$, then the normals to $f$ omit all four coordinate directions.

3. The proof of Theorem 2 is modelled on the proof of a Picard theorem for complex Grassmannians due to M. L. Green [4, Part 4]. Green is only able to conclude that the image of the map lies in a section by a quadric hypersurface, not that it lies in a linear subspace, as he shows by an example. The difficulty has to do with the case when the omitted hyperplanes or their intersections become tangential to the variety. The special conditions that arise in our application to minimal surfaces are sufficient to avoid that.

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