

INJECTIVITY OF OPERATOR SPACES

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ABSTRACT. We study the structure of injective operator spaces and the existence and uniqueness of the injective envelopes of operator spaces. We give an easy example of an injective operator space which is not completely isometric to any C^* -algebra. This answers a question of Wittstock [23]. Furthermore, we show that an operator space E is injective if and only if there exists an injective C^* -algebra A and two projections p and q in A such that E is completely isometric to pAq .

1. INTRODUCTION

Let E be a vector space over the complex numbers C and $M_n(E)$ the space of $n \times n$ matrices with entries in E . We write

$$x \oplus y = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in M_{n+m}(E)$$

and

$$\alpha x = \left[\sum_{j=1}^n \alpha_{ij} x_{jk} \right] \quad \text{and} \quad x \alpha = \left[\sum_{j=1}^n x_{ij} \alpha_{jk} \right] \in M_n(E)$$

for $x = [x_{ij}] \in M_n(E)$, $y = [y_{ij}] \in M_m(E)$ and $\alpha = [\alpha_{ij}] \in M_n(C)$.

An *m.n. space* (matricially normed space) is a vector space E together with a norm $\| \cdot \|_n$ on each $M_n(E)$, which satisfies:

- (i) $\|x \oplus 0\|_{n+m} = \|x\|_n$,
- (ii) $\|\alpha x\|_n \leq \|\alpha\| \|x\|_n$ and $\|x \alpha\|_n \leq \|x\|_n \|\alpha\|$

for all $\alpha \in M_n(C)$, $x \in M_n(E)$ and the zero element $0 \in M_m(E)$. We use the notation $(E, \{\| \cdot \|_n\})$, or simply E . An *m.n. space* is L^∞ if it satisfies

$$(L^\infty) \quad \|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}.$$

An *m.n. space* is L^p ($1 \leq p < \infty$) if it satisfies

$$(L^p) \quad \|x \oplus y\|_{n+m} = (\|x\|_n^p + \|y\|_m^p)^{1/p}.$$

Remark. Notice that our definition of m.n. spaces is different from that in Effros [5] and that in Wittstock [23]. According to our notation, the matricially normed

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space defined in [5] is a vector space with a norm $\|\cdot\|_n$ on each $M_n(E)$, and the matricial normed space defined in [23] is an L^∞ -m.n. space.

Let E and F be m.n. spaces and $\phi: E \rightarrow F$ a linear map. We can define a linear map $\phi_n: M_n(E) \rightarrow M_n(F)$ by letting

$$\phi_n([x_{ij}]) = [\phi(x_{ij})]$$

for $[x_{ij}] \in M_n(E)$. Write

$$\|\phi\|_{cb} = \sup\{\|\phi_n\|; n \in \mathbb{N}\}.$$

We call ϕ a *completely bounded map* if $\|\phi\|_{cb} < \infty$, a *complete contraction* if $\|\phi\|_{cb} \leq 1$ and a *complete isometry* if each ϕ_n is an isometry.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H with the operator norm defined by

$$\|x\| = \sup\{\|x\xi\|; \xi \in H, \|\xi\| \leq 1\}.$$

Identifying $M_n(B(H))$ with $B(H^n)$ where $H^n = H \oplus \cdots \oplus H$, we obtain a natural operator norm on each $M_n(B(H))$. This family of norms $\{\|\cdot\|_n\}$ over $B(H)$ is called an *operator matricial norm on $B(H)$* . A linear subspace of $B(H)$ with the above operator matricial norm is called a *concrete operator space*. An m.n. space E is called an *abstract operator space*, or simply an *operator space* if E is completely isometric to a concrete operator space.

It is obvious that operator spaces are L^∞ -m.n. spaces. Ruan [18] showed that an m.n. space E is an operator space iff E is an L^∞ -m.n. space. This gives a matricial norm characterization of operator spaces. For our convenience, we will regard the words “operator spaces” and “ L^∞ -m.n. space” as the same notation.

In classical functional analysis, we are interested in studying the properties of normed (Banach) spaces and bounded linear maps between these spaces. It is known that every normed (Banach) space can be canonically embedded into $C(\Omega)$, where Ω is a compact Hausdorff space and $C(\Omega)$ is the space of all continuous functions on Ω . If Ω is a stonian space (cf. [2, §7]), then $C(\Omega)$ is an injective Banach space. Furthermore, Nachbin [14], Goodner [7] and Kelley [12] (Hasumi [10]) have proved that a real (complex) Banach space is injective if and only if it is linearly isometric to $C(\Omega)$ for some stonian space Ω . The existence and uniqueness of the injective envelopes of Banach spaces were studied by Cohen [4].

In quantized functional analysis, we consider operator spaces and completely bounded maps between these spaces. The Arveson-Wittstock Hahn-Banach Theorem has shown that $B(H)$ is an injective object in the category of operator spaces and complete contractions, i.e. $B(H)$ is an injective operator space. Thus every injective C^* -algebra is an injective operator space. Choi and Effros [3] showed that an operator system is injective in the category of operator systems and completely positive maps if and only if it is completely order isomorphic to an injective C^* -algebra. In [23], Wittstock asked a question: “Does there

exist an injective operator space (i.e. an injective L^∞ -m.n. space), which is not completely isometric to an injective C^* -algebra?"

In this paper, we study the structure of injective operator spaces and the unique existence of the injective envelopes of operator spaces. We give an example of an injective operator space which is not completely isometric to any C^* -algebra (Theorem 4.3). This gives a negative answer to Wittstock's question. In general, an injective operator space is a triple subsystem of a C^* -algebra (cf. Youngson [24]). Indeed, we show in Theorem 4.5 that an operator space E is injective if and only if there exists an injective C^* -algebra A and two projections p and q such that E is completely isometric to pAq . Finally we show in §5 that for every operator space E , there exists an essentially unique injective envelope of E . Our proof of this result is inspired by Hamana [9].

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2. MATRICIALLY SEMINORMED SPACES

Definition 2.1. Let E be a vector space and p_n a seminorm on $M_n(E)$ ($n \in N$). $(E, \{p_n\})$ is called an *m.s.n. space* (*matricially seminormed space*) if it satisfies the conditions (i) and (ii) in §1, i.e.

$$(i) \quad p_{n+m}(x \oplus 0) = p_n(x),$$

$$(ii) \quad p_n(\alpha x) \leq \|\alpha\|p_n(x) \quad \text{and} \quad p_n(x\alpha) \leq \|\alpha\|p_n(x)$$

for all $\alpha \in M_n(C)$, $x \in M_n(E)$ and 0 the zero element in $M_m(E)$. The family of seminorms $\{p_n\}$ is called a *matricial seminorm on E* . An m.s.n. space is called an *L^p -m.s.n. space* ($1 \leq p \leq \infty$) if it satisfies the L^p -condition.

Proposition 2.2. Let $(E, \{p_n\})$ be an m.s.n. space and $K_n = \ker p_n$, the kernel of p_n . Then $K_n = M_n(K_1)$ for each $n \in N$.

Proof. Let $[x_{ij}] \in K_n$. Then

$$p_1(x_{ij}) = p_1 \left(\begin{array}{c} [0 \cdots 1 \cdots 0][x_{ij}] \\ \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{array} \right] \end{array} \right) \leq p_n([x_{ij}]) = 0.$$

Hence $p_1(x_{ij}) = 0$ and $x_{ij} \in K_1$.

Conversely if $x_{ij} \in K_1$, we have $p_n([x_{ij}]) \leq \sum_{i,j=1}^n p_1(x_{ij}) = 0$ and thus $[x_{ij}] \in K_n$. Therefore $K_n = M_n(K_1)$. \square

Identifying $M_n(E/K_1)$ with $M_n(E)/K_n$, we may define a norm $\|\cdot\|_{p_n}$ on each $M_n(E/K_1)$ by letting

$$\|[\bar{x}_{ij}]\|_{p_n} = p_n([x_{ij}])$$

where \bar{x}_{ij} is the corresponding equivalence class in E/K_1 . It is easy to see that the quotient space E/K_1 with the matricial norm defined above is an m.n. space. Furthermore if $(E, \{p_n\})$ is an L^p -m.s.n. space, then $(E/K_1, \{\|\cdot\|_{p_n}\})$ is an L^p -m.n. space.

Suppose that E and F are two m.n. spaces and $\phi: E \rightarrow F$ is a complete contraction.

Example 2.3. For each $n \in N$, we define a seminorm p_n^ϕ on $M_n(E)$ by

$$p_n^\phi(x) = \|\phi_n(x)\|_n$$

for $x \in M_n(E)$. Then $(E, \{p_n^\phi\})$ is an m.s.n. space with $p_n^\phi(x) \leq \|x\|_n$ for all $x \in M_n(E)$. If F is an L^p -m.n. space, then $(E, \{p_n^\phi\})$ is an L^p -m.s.n. space.

Proof. It follows easily from the definition.

Example 2.4. Suppose that F is an m.n. subspace of E . For each $n \in N$, we define a seminorm q_n^ϕ on $M_n(E)$ by

$$q_n^\phi(x) = \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_n + \cdots + \phi_n^k}{k} \right) (x) \right\|_n$$

for $x \in M_n(E)$. Then $(E, \{q_n^\phi\})$ is an m.s.n. space with $q_n^\phi(x) \leq \|x\|_n$ for all $x \in M_n(E)$. If E is an L^∞ -m.n. space, then $(E, \{q_n^\phi\})$ is an L^∞ -m.s.n. space.

Proof. Given $x \in M_n(E)$ and 0 the zero element in $M_m(E)$, we have

$$\begin{aligned} q_{n+m}^\phi(x \oplus 0) &= \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_{n+m} + \cdots + \phi_{n+m}^k}{k} \right) (x \oplus 0) \right\|_{n+m} \\ &= \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_n + \cdots + \phi_n^k}{k} \right) (x) \oplus \left(\frac{\phi_m + \cdots + \phi_m^k}{k} \right) (0) \right\|_{n+m} \\ &= \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_n + \cdots + \phi_n^k}{k} \right) (x) \right\|_n = q_n^\phi(x). \end{aligned}$$

For $x \in M_n(E)$ and $\alpha \in M_n(C)$, we have

$$\begin{aligned} q_n^\phi(\alpha x) &= \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_n + \cdots + \phi_n^k}{k} \right) (\alpha x) \right\|_n \\ &\leq \|\alpha\| \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_n + \cdots + \phi_n^k}{k} \right) (x) \right\|_n \\ &= \|\alpha\| q_n^\phi(x). \end{aligned}$$

Similarly, we have $q_n^\phi(x\alpha) \leq \|\alpha\| q_n^\phi(x)$. Hence $(E, \{q_n^\phi\})$ is an m.s.n. space. If E is an L^∞ -m.n. space, then F is also an L^∞ -m.n. space. For $x \in M_n(E)$

and $y \in M_m(E)$, we have

$$\begin{aligned}
 q_{n+m}^\phi(x \oplus y) &= \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_{n+m} + \cdots + \phi_{n+m}^k}{k} \right) (x \oplus y) \right\|_{n+m} \\
 &= \limsup_{k \rightarrow \infty} \left\{ \max \left\{ \left\| \left(\frac{\phi_n + \cdots + \phi_n^k}{k} \right) (x) \right\|_n, \right. \right. \\
 &\quad \left. \left. \left\| \left(\frac{\phi_m + \cdots + \phi_m^k}{k} \right) (y) \right\|_m \right\} \right\} \\
 &= \max \left\{ \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_n + \cdots + \phi_n^k}{k} \right) (x) \right\|_n, \right. \\
 &\quad \left. \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi_m + \cdots + \phi_m^k}{k} \right) (y) \right\|_m \right\} \\
 &= \max \{ q_n^\phi(x), q_m^\phi(y) \}.
 \end{aligned}$$

Hence $(E, \{q_n^\phi\})$ is an L^∞ -m.s.n. space. \square

Now we consider the Arveson-Wittstock Hahn-Banach Theorem for L^∞ -m.s.n. spaces.

Theorem 2.5. *Let $(E, \{p_n\})$ be an L^∞ -m.s.n. space, $(F, \{p_n\})$ an L^∞ -m.s.n. subspace of E and $\phi: F \rightarrow B(H)$ a linear map such that $\|\phi_n(x)\|_n \leq p_n(x)$ for all $x \in M_n(F)$ and $n \in N$. Then there exists a linear map $\bar{\phi}: E \rightarrow B(H)$ which extends ϕ and $\|\bar{\phi}_n(x)\|_n \leq p_n(x)$ for all $x \in M_n(E)$.*

Proof. Let $K_1 = \ker p_1$. Then $(E/K_1, \{\|\cdot\|_{p_n}\})$ is an L^∞ -m.n. space. By linear algebra, we may identify $F/(F \cap K_1)$ with $(F + K_1)/K_1$. Hence $(F/(F \cap K_1), \{\|\cdot\|_{p_n}\})$ can be regarded as an L^∞ -m.n. subspace of E/K_1 . We define a map $\psi: F/(F \cap K_1) \rightarrow B(H)$ by $\psi(\bar{x}) = \phi(x)$ for $\bar{x} \in F/(F \cap K_1)$. The map ψ is well-defined since the kernel of ϕ contains $F \cap K_1$. For each $[\bar{x}_{ij}] \in M_n(F/(F \cap K_1))$, we have

$$\|\psi_n([\bar{x}_{ij}])\|_n = \|\phi_n([x_{ij}])\|_n \leq p_n([x_{ij}]) = \|[\bar{x}_{ij}]\|_{p_n}.$$

Hence $\psi: F/(F \cap K_1) \rightarrow B(H)$ is a complete contraction. By the Arveson-Wittstock Hahn-Banach Theorem (cf. Theorem 4.2), there exists a completely contractive map $\bar{\psi}: E/K_1 \rightarrow B(H)$, which extends ψ and preserves the cb-norm. Let $\pi: E \rightarrow E/K_1$ be the natural quotient map and let $\bar{\phi} = \bar{\psi} \circ \pi: E \rightarrow B(H)$. Then $\bar{\phi}$ extends ϕ and we have

$$\|\bar{\phi}_n([x_{ij}])\|_n = \|(\bar{\psi}_n \circ \pi_n)([x_{ij}])\|_n \leq \|[\bar{x}_{ij}]\|_{p_n} = p_n([x_{ij}])$$

for all $[x_{ij}] \in M_n(E)$. \square

3. MINIMAL E -PROJECTION OF $B(H)$

The main result (Theorem 3.4) of this section is to show that “For every operator space E contained in $B(H)$ and a minimal L^∞ -matricial E -seminorm $\{p_n\}$ on $B(H)$, there exists a minimal E -projection ϕ of $B(H)$ with $\{p_n^\phi\} = \{p_n\}$ ”. This result is motivated by Hamana [9] and will play a key role in the study of the injective envelopes of operator spaces.

Definition 3.1. Let $B(H)$ be the space of all bounded operators on a Hilbert space H . A linear map $\phi: B(H) \rightarrow B(H)$ is called a *completely contractive projection* if $\|\phi\|_{cb} \leq 1$ and $\phi^2 = \phi$. Let E be an operator space contained in $B(H)$. An E -projection of $B(H)$ is a completely contractive projection $\phi: B(H) \rightarrow B(H)$ such that $\phi(x) = x$ for all $x \in E$.

Let $\Xi_{B(H)}^E$ be the set of all E -projections of $B(H)$. Then $\Xi_{B(H)}^E$ is nonempty. Define a partial ordering on $\Xi_{B(H)}^E$ by saying

$$\psi \preceq \phi \quad \text{if and only if} \quad \psi \circ \phi = \phi \circ \psi = \psi.$$

A *minimal E -projection of $B(H)$* is an E -projection of $B(H)$ which is minimal under this partial ordering.

Definition 3.2. Let E be an L^∞ -m.n. space and F an L^∞ -m.n. subspace of E . An L^∞ -matricial seminorm $\{p_n\}$ on E is called an *L^∞ -matricial F -seminorm* if it satisfies:

$$(I) \quad p_n(x) \leq \|x\|_n \quad \text{for all } x \in M_n(E),$$

$$(II) \quad p_n(x) = \|x\|_n \quad \text{for all } x \in M_n(F).$$

Let Γ_E^F be the set of all L^∞ -matricial F -seminorms on E . Then Γ_E^F is nonempty. Define a partial ordering on Γ_E^F by saying

$$\{p_n\} \preceq \{q_n\} \quad \text{if and only if} \quad p_n(x) \leq q_n(x)$$

for all $x \in M_n(E)$ and $n \in \mathbb{N}$.

Let E be an operator space contained in $B(H)$ and $\phi: B(H) \rightarrow B(H)$ a complete contraction such that $\phi(x) = x$ for all $x \in E$. Then the matricial seminorms $\{p_n^\phi\}$ and $\{q_n^\phi\}$ in Example 2.3 and Example 2.4 are L^∞ -matricial E -seminorms on $B(H)$ and it is easy to see that $\{q_n^\phi\} \preceq \{p_n^\phi\}$.

Proposition 3.3. *The partially ordered set Γ_E^F must have at least one minimal element.*

Proof. Suppose that $\{\{p_n^\gamma\}_{n=1}^\infty\}_{\gamma \in \Lambda}$ is an arbitrary chain in Γ_E^F . Consider a function p_n on $M_n(E)$ defined by

$$p_n(x) = \inf\{p_n^\gamma(x); \gamma \in \Lambda\}$$

for $x \in M_n(E)$. It is easy to see that for each $n \in N$, p_n is a seminorm on $M_n(E)$ which satisfies (I) and (II) in Definition 3.2. For $x \in M_n(E)$ and $y \in M_m(E)$, we have

$$\begin{aligned} p_{n+m}(x \oplus y) &= \inf\{p_{n+m}^\gamma(x \oplus y); \gamma \in \Lambda\} \\ &= \inf\{\max\{p_n^\gamma(x), p_m^\gamma(y)\}; \gamma \in \Lambda\} \\ &= \max\{\inf\{p_n^\gamma(x); \gamma \in \Lambda\}, \inf\{p_m^\gamma(y); \gamma \in \Lambda\}\} \\ &= \max\{p_n(x), p_m(y)\}. \end{aligned}$$

For $\alpha \in M_n$ and $x \in M_n(E)$, we have

$$\begin{aligned} p_n(\alpha x) &= \inf\{p_n^\gamma(\alpha x); \gamma \in \Lambda\} \\ &\leq \|\alpha\| \inf\{p_n^\gamma(x); \gamma \in \Lambda\} \\ &= \|\alpha\| p_n(x). \end{aligned}$$

Similarly, we have $p_n(x\alpha) \leq \|\alpha\| p_n(x)$. Hence $\{p_n\}$ is an L^∞ -matricial F -seminorm on E such that $\{p_n\} \lesssim \{p_n^\gamma\}$ for all $\gamma \in \Lambda$. By Zorn's lemma, the partially ordered set Γ_E^F must have a minimal element. \square

A minimal element in the partially ordered set Γ_E^F is called a *minimal L^∞ -matricial F -seminorm on E* .

Theorem 3.4. *Let E be an operator space contained in $B(H)$. For every minimal L^∞ -matricial E -seminorm $\{p_n\}$ on $B(H)$, there is a minimal E -projection ϕ of $B(H)$ with $\{p_n^\phi\} = \{p_n\}$.*

Proof. Let $\text{id}: E \rightarrow B(H)$ be the inclusion map. Then we have

$$\|\text{id}_n(x)\|_n = \|x\|_n = p_n(x)$$

for all $x \in M_n(E)$ and $n \in N$. By Theorem 2.5, there is a linear map $\phi: B(H) \rightarrow B(H)$ which extends id such that

$$\|\phi_n(x)\|_n \leq p_n(x) \leq \|x\|_n$$

for all $x \in M_n(B(H))$ and $n \in N$. Then $\{p_n^\phi\}$ and $\{q_n^\phi\}$ are L^∞ -matricial E -seminorms on $B(H)$ such that

$$\{q_n^\phi\} \lesssim \{p_n^\phi\} \lesssim \{p_n\}.$$

This implies that $\{q_n^\phi\} = \{p_n^\phi\} = \{p_n\}$ since $\{p_n\}$ is a minimal element in $\Gamma_{B(H)}^E$.

Next we want to show that ϕ is a minimal E -projection of $B(H)$. Since

$$\begin{aligned} \|\phi(x) - \phi^2(x)\|_1 &= \|\phi(x - \phi(x))\|_1 \\ &= p_1^\phi(x - \phi(x)) = q_1^\phi(x - \phi(x)) \\ &= \limsup_{k \rightarrow \infty} \left\| \left(\frac{\phi + \cdots + \phi^k}{k} \right) (x - \phi(x)) \right\|_1 \\ &= \lim_{k \rightarrow \infty} \left\| \frac{\phi(x) - \phi^{k+1}(x)}{k} \right\|_1 = 0 \end{aligned}$$

for all $x \in B(H)$ we get $\phi^2 = \phi$ and ϕ is an E -projection of $B(H)$. Let ψ be an arbitrary E -projection of $B(H)$ with $\psi \lesssim \phi$. We have $\{q_n^\psi\} = \{p_n^\phi\} = \{p_n\}$. Then

$$\begin{aligned} \|\phi(x) - \psi(x)\|_1 &= \|\phi(x) - \phi \circ \psi(x)\|_1 = \|\phi(x - \psi(x))\|_1 \\ &= p_1^\phi(x - \psi(x)) = q_1^\psi(x - \psi(x)) = 0 \end{aligned}$$

for all $x \in B(H)$. Therefore $\psi = \phi$ and ϕ is a minimal E -projection of $B(H)$ with $\{p_n^\phi\} = \{p_n\}$. \square

4. INJECTIVE OPERATOR SPACES

Definition 4.1. Let E be an operator space. E is called an *injective operator space* if for every operator space F , every operator subspace F_0 of F and every completely bounded map $\phi: F_0 \rightarrow E$, there exists a linear map $\bar{\phi}: F \rightarrow E$ such that $\bar{\phi}|_{F_0} = \phi$ and $\|\bar{\phi}\|_{cb} = \|\phi\|_{cb}$.

The following theorem is due to Wittstock [22 and 23]. It is called the Arveson-Wittstock Hahn-Banach Theorem. Haagerup [8], Paulsen [15 and 16], and recently Effros-Ruan [6] have given different proofs of this result.

Theorem 4.2. Let E be an operator space contained in a unital C^* -algebra A and $\phi: E \rightarrow B(H)$ a complete contraction. Then there exists a complete contraction $\bar{\phi}: A \rightarrow B(H)$ such that $\bar{\phi}|_E = \phi$ and $\|\bar{\phi}\|_{cb} = \|\phi\|_{cb}$.

Remark. The Arveson-Wittstock Hahn-Banach Theorem fails for general m.n. spaces. We have shown in [6] that there exist L^1 -m.n. spaces $F \subseteq E$ and a completely bounded map $\phi: F \rightarrow M_2(C)$ so that ϕ cannot be extended to a completely bounded map $\hat{\phi}: E \rightarrow M_2(C)$ with $\|\hat{\phi}\|_{cb} = \|\phi\|_{cb}$.

From the Arveson-Wittstock Hahn-Banach Theorem, we know that $B(H)$ is an injective operator space for arbitrary Hilbert space H . Then an operator space E contained in $B(H)$ is injective if and only if there exists a completely contractive projection ϕ from $B(H)$ onto E . This implies that injective operator spaces must be norm closed. Obviously, every injective C^* -algebra is an injective operator space. Furthermore if A is an injective C^* -algebra and p, q are projections in A , then pAq is again an injective operator space. In

particular, we know that $B(H, K)$ is an injective operator space, where H and K are Hilbert spaces. Our next theorem gives a negative answer to Wittstock's question [23, Problem 5.1].

Theorem 4.3. *There exists an injective operator space E which is not completely isometric to any C^* -algebra.*

Proof. Let $E = \{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, a, b \in C \}$. It is easy to see that E is an injective operator space contained in $M_2(C)$. Consider $F = \{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, a, b \in C \}$. Then F is also an operator space. Let $\theta: F \rightarrow E$ be the transpose map defined by

$$\theta \left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \right) = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Then θ is an isometry. On the other hand, for

$$\begin{bmatrix} E_{11} & E_{21} \\ 0 & 0 \end{bmatrix} \in M_2(M_2(C)),$$

we have

$$\left\| \begin{bmatrix} E_{11} & E_{21} \\ 0 & 0 \end{bmatrix} \right\|_2 = 1$$

but

$$\left\| \theta_2 \left(\begin{bmatrix} E_{11} & E_{21} \\ 0 & 0 \end{bmatrix} \right) \right\|_2 = \left\| \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \right\|_2 = 2^{1/2}.$$

Therefore we get $\|\theta\|_{cb} = \|\theta_2\| \geq 2^{1/2}$ (cf. Smith [19]).

Suppose that E is completely isometric to a C^* -algebra A . Since $\dim A = 2$, we must have that A is $*$ -isomorphic to $C \oplus C$. Thus A is a commutative C^* -algebra and we have $\|\theta\|_{cb} = \|\theta\| = 1$ (cf. Loeb [13, Lemma 1]), a contradiction. Thus E cannot be completely isometric to any C^* -algebra. \square

If E is an operator space contained in $B(H)$, then there exists a minimal E -projection of $B(H)$ (Theorem 3.4). The following lemma shows that if we choose such a minimal E -projection carefully, we can find two other unital completely positive projections $\psi_i: B(H) \rightarrow B(H)$ ($i = 1, 2$) so that the map

$$\Phi = \begin{bmatrix} \psi_1 & \phi \\ \phi^* & \psi_2 \end{bmatrix}: M_2(B(H)) \rightarrow M_2(B(H))$$

is a unital completely positive projection.

Lemma 4.4. *Let E be an operator space contained in $B(H)$. Then there exists a minimal E -projection ϕ of $B(H)$ and two unital completely positive projections ψ_i ($i = 1, 2$) so that the map*

$$\Phi = \begin{bmatrix} \psi_1 & \phi \\ \phi^* & \psi_2 \end{bmatrix}$$

is a unital completely positive projection.

Proof. Let $L^E = \{ \begin{pmatrix} \lambda & x \\ y^* & \mu \end{pmatrix}; \lambda, \mu \in C, x, y \in E \}$. Then L^E is an operator system contained in $M_2(B(H))$. We have an L^∞ -matricial L^E -seminorm on

$M_2(B(H))$ which is minimal in the partially ordered set $\Gamma_{M_2(B(H))}^{L^E}$ (Proposition 3.3). Fixing such a minimal L^∞ -matricial L^E -seminorm $\{p_n\}$ on $M_2(B(H))$, there exists a minimal L^E -projection Φ of $M_2(B(H))$ such that $\{p_n^\Phi\} = \{p_n\}$ (Theorem 3.4). We know from Arveson [1] that Φ must be completely positive since Φ is a complete contraction with $\Phi(1_2) = 1_2$. By Stinespring's Theorem [20], there exists a Hilbert space \bar{K} , a $*$ -representation $\bar{\pi}: M_2(B(H)) \rightarrow B(\bar{K})$ with $\bar{\pi}(1_2) = 1_{\bar{K}}$ and an isometry $V: H \oplus H \rightarrow \bar{K}$ such that

$$\Phi([x_{ij}]) = V^* \bar{\pi}([x_{ij}]) V$$

for all $[x_{ij}] \in M_2(B(H))$.

Let $\{E_{ij}\}$ be the usual matrix units in $M_2(B(H))$. Then $\{\bar{\pi}(E_{ij})\}$ are matrix units in $B(\bar{K})$ which give a decomposition of \bar{K} into $K \oplus K$. If we write $\varepsilon_{ij} = \bar{\pi}(E_{ij})$, then ε_{11} is the projection onto $K \oplus 0$. Consider

$$\pi(x) = \bar{\pi} \left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right) = \varepsilon_{11} \bar{\pi}(x \oplus x) \varepsilon_{11}$$

for all $x \in B(H)$. Then $\pi: B(H) \rightarrow B(K)$ is a $*$ -representation with $\pi(1) = 1_K$ and we have $\bar{\pi} = \pi \otimes \text{id}_2$. Following the proof in [16, Theorem 2.7], we have $V(\xi_1 \oplus \xi_2) = V_1 \xi_1 \oplus V_2 \xi_2$ for some isometries $V_i: H \rightarrow K$ ($i = 1, 2$) and

$$\begin{aligned} \Phi([x_{ij}]) &= V^* (\pi \otimes \text{id}_2)([x_{ij}]) V \\ &= \begin{bmatrix} V_1^* & 0 \\ 0 & V_2^* \end{bmatrix} \begin{bmatrix} \pi(x_{11}) & \pi(x_{12}) \\ \pi(x_{21}) & \pi(x_{22}) \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \\ &= \begin{bmatrix} V_1^* \pi(x_{11}) V_1 & V_1^* \pi(x_{12}) V_2 \\ V_2^* \pi(x_{21}) V_1 & V_2^* \pi(x_{22}) V_2 \end{bmatrix} \end{aligned}$$

for all $[x_{ij}] \in M_2(B(H))$. Therefore, we have

$$\Phi = \begin{bmatrix} \psi_1 & \phi \\ \phi^* & \psi_2 \end{bmatrix}$$

where $\phi = V_1^* \pi V_2$ is an E -projection of $B(H)$ and $\psi_i = V_i^* \pi V_i$ ($i = 1, 2$) are unital completely positive projections from $B(H)$ into $B(H)$.

We need to show that ϕ is a minimal E -projection of $B(H)$. Let $E^\phi = \phi(B(H))$. Then E^ϕ is an injective operator space contained in $B(H)$. Suppose that ϕ_0 is another E -projection of $B(H)$ such that $\phi_0 \circ \phi = \phi \circ \phi_0 = \phi_0$. Then the injective operator space E^{ϕ_0} is contained in E^ϕ . Assume that $\phi_0 \neq \phi$. Then E^{ϕ_0} is a proper subspace of E^ϕ . The identity map $\text{id}: E^{\phi_0} \rightarrow E^{\phi_0}$ can be extended to a complete contraction $T: E^\phi \rightarrow E^{\phi_0}$. Hence there exists an element $x_0 \neq 0 \in \ker T$.

Let $L^{E^\phi} = \{ \begin{pmatrix} \lambda & x \\ y^* & \mu \end{pmatrix}; \lambda, \mu \in C, x, y \in E^\phi \}$. Then L^{E^ϕ} is an operator system contained in $M_2(B(H))$ and we consider a map $\Psi: L^{E^\phi} \rightarrow M_2(B(H))$ defined by

$$\Psi \left(\begin{pmatrix} \lambda & x \\ y^* & \mu \end{pmatrix} \right) = \begin{bmatrix} \lambda & T(x) \\ (T(y))^* & \mu \end{bmatrix}.$$

Then Ψ is a unital completely positive map since T is a complete contraction (Paulsen [17, Lemma 7.1]). Therefore we have

$$\|\Psi_n([u_{ij}])\|_n \leq \| [u_{ij}] \|_n = \|\Phi_n([u_{ij}])\|_n = p_n([u_{ij}])$$

for all $[u_{ij}] \in M_n(L^{E^\phi})$. We can extend Ψ to $\bar{\Psi}: M_2(B(H)) \rightarrow M_2(B(H))$ such that

$$\|\bar{\Psi}_n([u_{ij}])\|_n \leq p_n([u_{ij}])$$

for all $[u_{ij}] \in M_n(M_2(B(H)))$ and $n \in N$. Then $\{p_n^{\bar{\Psi}}\}$ is an L^∞ -matricial L^E -seminorm on $M_2(B(H))$ such that $\{p_n^{\bar{\Psi}}\} \lesssim \{p_n\}$. It follows from the assumption that

$$p_1^{\bar{\Psi}} \left(\begin{bmatrix} 0 & x_0 \\ x_0^* & 0 \end{bmatrix} \right) = 0$$

and

$$\begin{aligned} p_1 \left(\begin{bmatrix} 0 & x_0 \\ x_0^* & 0 \end{bmatrix} \right) &= \left\| \Phi \left(\begin{bmatrix} 0 & x_0 \\ x_0^* & 0 \end{bmatrix} \right) \right\|_1 \\ &= \left\| \begin{bmatrix} 0 & x_0 \\ x_0^* & 0 \end{bmatrix} \right\|_1 = \|x_0\| \neq 0. \end{aligned}$$

Then $\{p_n^{\bar{\Psi}}\} \neq \{p_n\}$. This contradicts the minimality of $\{p_n\}$. Therefore we must have $\phi_0 = \phi$ and ϕ is a minimal E -projection. \square

In the following theorem, we study the structures of injective operator spaces.

Theorem 4.5. *Let E be an operator space. Then TFAE :*

- (1) E is an injective operator space;
- (2) there exists an injective C^* -algebra A and two projections $p, q \in A$ such that E is completely isometric to pAq .

Proof. (2) \Rightarrow (1) is trivial.

(1) \Rightarrow (2) We may assume that E is an injective operator space contained in $B(H)$. Using the same notations as in Lemma 4.4, we have a minimal E -projection ϕ of $B(H)$ and two unital completely contractive projections ψ_i ($i = 1, 2$) from $B(H)$ into $B(H)$ such that the map

$$\Phi = \begin{bmatrix} \psi_1 & \phi \\ \phi^* & \psi_2 \end{bmatrix} : M_2(B(H)) \rightarrow M_2(B(H))$$

is a unital completely positive projection. Then

$$A = \Phi(M_2(B(H))) \subseteq M_2(B(H))$$

with the multiplication defined by $x \cdot y = \Phi(xy)$ for all $x, y \in A$ is an injective C^* -algebra (cf. Choi and Effros [4]).

Consider $p = \Phi(E_{11})$ and $q = \Phi(E_{22})$. Obviously, p and q are two projections in A . Since E is an injective operator space and ϕ is a minimal E -projection, we must have $E = E^\phi$. Define a map $T: E \rightarrow pAq$ by

$$T(x) = p \cdot \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \cdot q = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

for all $x \in E \subseteq B(H)$. Thus T is a well-defined complete isometry. Notice that

$$p \cdot A \cdot q = \Phi(E_{11}M_2(B(H))E_{22}) = \Phi\left(\begin{bmatrix} 0 & B(H) \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & E^\phi \\ 0 & 0 \end{bmatrix}.$$

This implies that T is onto. Deleting the dot notation for the product in A , we are done. \square

5. INJECTIVE ENVELOPES OF OPERATOR SPACES

Definition 5.1. Let E be an operator space. An extension of E is a pair (Z, κ) of an operator space Z and a completely isometric embedding $\kappa: E \rightarrow Z$. An extension (Z, κ) of E is called injective if Z is an injective operator space. An extension (Z, κ) of E is called an injective envelope of E if (Z, κ) is an injective extension of E and id_Z is the only complete contraction (from Z into Z) which extends $\text{id}_{\kappa(E)}: \kappa(E) \rightarrow Z$ from $\kappa(E)$ to Z .

Remark. In the category of Banach spaces and contractive linear maps, the existence and uniqueness of injective envelopes of Banach spaces were studied by Cohen [4]. The original definition of injective envelopes of Banach spaces is as follows:

Let E be a Banach space. An injective envelope of E is a pair (Z, κ) of an injective Banach space Z and an isometric embedding $\kappa: E \rightarrow Z$ such that the only injective subspace of Z containing $\kappa(E)$ is Z itself.

Isbell [10] pointed out that this definition is equivalent to the following one:

Let E be a Banach space. An injective envelope of E is a pair (Z, κ) of an injective Banach space Z and an isometric embedding $\kappa: E \rightarrow Z$ such that id_Z is the only contractive extension of $\text{id}_{\kappa(E)}: \kappa(E) \rightarrow Z$ from $\kappa(E)$ to Z .

Our definition of injective envelopes of operator spaces is an analogue of the second one (see, however, Theorem 5.6). Most of this section is inspired by Hamana [8], where he studied the injective envelopes of unital C^* -algebras.

Theorem 5.2. Every operator space has an injective envelope.

Proof. Assume that E is an operator space contained in $B(H)$. Then there exists a minimal L^∞ -matricial E -seminorm $\{p_n\}$ on $B(H)$ and a minimal E -projection ϕ of $B(H)$ such that $\{p_n^\phi\} = \{p_n\}$. Then $E^\phi = \phi(B(H))$ is an injective operator space containing E as a subspace. The pair (E^ϕ, id_E) is an injective extension of E , where $\text{id}_E: E \rightarrow E^\phi$ is the natural embedding map. Let $\psi: E^\phi \rightarrow E^\phi$ be a completely contractive extension of id_E . Then

$$\|\psi_n(x)\|_n \leq \|x\|_n = \|\phi_n(x)\|_n = p_n(x)$$

for all $x \in M_n(E^\phi)$. We can extend ψ again to a complete contraction $\bar{\psi}: B(H) \rightarrow E^\phi$ such that $\|\bar{\psi}_n(x)\| \leq p_n(x)$ for all $x \in M_n(B(H))$. Then $\{q_n^{\bar{\psi}}\}$ is an L^∞ -matricial E -seminorm on $B(H)$ such that

$$\{q_n^{\bar{\psi}}\} \lesssim \{p_n\}.$$

Hence we must have $\{q_n^{\bar{\psi}}\} = \{p_n\}$ since $\{p_n\}$ is a minimal L^∞ -matricial E -projection. Now for every $x \in E^\phi$, we have

$$\|x - \psi(x)\|_1 = p_1(x - \bar{\psi}(x)) = q_1^{\bar{\psi}}(x - \bar{\psi}(x)) = 0.$$

Therefore $\psi = \text{id}_{E^\phi}$ and (E^ϕ, κ) is an injective envelope of E . \square

Proposition 5.3. *Let E_1, E_2 be two operator spaces and (Z_1, κ_1) and (Z_2, κ_2) injective envelopes of E_1, E_2 , respectively. Then for every complete isometry λ from E_1 onto E_2 , there exists a unique complete isometry $\hat{\lambda}$ from Z_1 onto Z_2 such that*

$$\hat{\lambda} \circ \kappa_1 = \kappa_2 \circ \lambda.$$

Proof. Let $\lambda: E_1 \rightarrow E_2$ be a complete isometry from E_1 onto E_2 . Then

$$\mu = \kappa_2 \circ \lambda \circ \kappa_1^{-1}: \kappa_1(E_1) \rightarrow \kappa_2(E_2)$$

is a complete isometry from $\kappa_1(E_1)$ onto $\kappa_2(E_2)$. μ can be extended to a complete contraction $\hat{\lambda}: Z_1 \rightarrow Z_2$. Since $\mu^{-1}: \kappa_2(E_2) \rightarrow \kappa_1(E_1)$ is also a complete isometry, μ^{-1} can be extended to a complete contraction $\hat{\mu}: Z_2 \rightarrow Z_1$.

Then $\hat{\mu} \circ \hat{\lambda}: Z_1 \rightarrow Z_1$ is a complete contraction such that $\hat{\mu} \circ \hat{\lambda}|_{\kappa_1(E_1)} = \text{id}_{\kappa_1(E_1)}$. Hence $\hat{\mu} \circ \hat{\lambda} = \text{id}_{Z_1}$. Similarly we have $\hat{\lambda} \circ \hat{\mu} = \text{id}_{Z_2}$. This implies that $\hat{\lambda}$ is a complete isometry from Z_1 onto Z_2 and $\hat{\lambda} \circ \kappa_1 = \kappa_2 \circ \lambda$.

If we have another complete isometry λ^* from Z_1 onto Z_2 such that $\lambda^* \circ \kappa_1 = \kappa_2 \circ \lambda$, it follows easily from the above arguments that $\lambda^* = \hat{\lambda}$. \square

Theorem 5.4. *Let E be an operator space. If (Z_1, κ_1) and (Z_2, κ_2) are two injective envelopes of E , then there exists a unique complete isometry $\hat{\lambda}$ from Z_1 onto Z_2 such that $\hat{\lambda} \circ \kappa_1 = \kappa_2$. Hence every operator space E has an essentially unique injective envelope.*

Proof. Let $\lambda = \kappa_2 \circ \kappa_1^{-1}: \kappa_1(E) \rightarrow \kappa_2(E)$. Then λ is a complete isometry from $\kappa_1(E)$ onto $\kappa_2(E)$ and λ can be extended uniquely to a complete isometry $\hat{\lambda}$ from Z_1 onto Z_2 . \square

Lemma 5.5. *Let $E \subseteq Z \subseteq B(H)$ and (Z, id_E) an injective extension of E . Then (Z, id_E) is an injective envelope of E if and only if there exists a minimal E -projection ϕ of $B(H)$ such that $Z = E^\phi = \phi(B(H))$.*

Proof. \Rightarrow Since Z is an injective subspace of $B(H)$, there exists a Z -projection ϕ of $B(H)$ such that $E^\phi = \phi(B(H)) = Z$. Then ϕ is also an E -projection of $B(H)$.

Let ψ be a minimal E -projection of $B(H)$ such that $\psi \preceq \phi$, i.e. $\phi \circ \psi = \psi \circ \phi = \psi$. Then $E^\psi \subseteq E^\phi$ and E^ψ is an injective operator space containing E as a subspace. The identity map $\text{id}_{E^\psi}: E^\psi \rightarrow E^\psi$ can be extended to a complete

contraction $\lambda: E^\phi \rightarrow E^\psi$. Thus $\lambda: E^\phi \rightarrow E^\psi \subseteq E^\phi$ is a complete contraction such that $\lambda|_E = \text{id}_E$. Hence we have $\lambda = \text{id}_{E^\phi}$ and $E^\psi = E^\phi$. This implies that $\psi = \phi$ and ϕ is a minimal E -projection of $B(H)$.

\Leftarrow Suppose that ϕ is a minimal E -projection of $B(H)$ such that $Z = \phi(B(H))$. Let $\lambda: Z \rightarrow Z$ be a complete contraction such that $\lambda|_E = \text{id}_E$. We want to show that $\lambda = \text{id}_Z$. Obviously, $\{q_n^\lambda\}$ is an L^∞ -matricial E -seminorm on Z such that

$$\{q_n^\lambda\} \lesssim \{p_n^\phi\} = \{\| \cdot \|_n\}.$$

Following the proofs of Proposition 3.3 and Theorem 3.4, we can show that there exists a minimal L^∞ -matricial E -seminorm $\{p_n\}$ on Z such that $\{p_n\} \lesssim \{q_n^\lambda\}$ and a minimal E -projection ψ of Z such that $\{p_n^\psi\} = \{p_n\}$.

Let $\bar{\psi} = \psi \circ \phi$. Then $\bar{\psi}$ is an E -projection of $B(H)$ and $\bar{\psi} \lesssim \phi$. Thus we have $\bar{\psi} = \phi$ since ϕ is a minimal E -projection of $B(H)$. This implies that $p_1(x) = p_1^\psi(x) = \|x\|_1$ and then $\|x\|_1 = q_1^\lambda(x)$ for all $x \in Z$. Therefore we have $\lambda = \text{id}_Z$ since $\|x - \lambda(x)\|_1 = q_1^\lambda(x - \lambda(x)) = 0$ for all $x \in Z$. \square

Remark. Let E be an operator space containing 1 in $B(H)$ and (Z, κ) an injective envelope of E . Then Z is completely isometric to a injective C^* -algebra. To see this, we assume that $1 \in E \subseteq B(H)$ and $\phi: B(H) \rightarrow B(H)$ is a minimal E -projection of $B(H)$. Then (E^ϕ, id_E) is completely isometric to (Z, κ) and E^ϕ is an injective operator system, which is completely order isomorphic to a unital injective C^* -algebra (cf. Choi and Effros [4]). In particular, if A is a unital C^* -algebra and (B, κ) is an injective envelope of A with $\kappa(1) = 1$, then (B, κ) is also an injective envelope of A in the category of unital C^* -algebras and unital completely positive maps (cf. Hamana [8]).

The following theorem will show that the injective envelopes of operator spaces are the smallest injective extensions of E .

Theorem 5.6. *Let (Z, κ) be an injective extension of E . Then (Z, κ) is an injective envelope of E if and only if the only injective subspace of Z containing $\kappa(E)$ is Z itself, i.e. if there exists an injective subspace Z_1 of Z containing $\kappa(E)$ then $Z_1 = Z$.*

Proof. \Rightarrow Without loss of generality, we may assume that E is a subspace of Z and $\kappa = \text{id}_E$ is the inclusion map from E into Z . If Z_1 is an injective operator subspace of Z and E is contained in Z_1 , then the identity map $\text{id}_{Z_1}: Z_1 \rightarrow Z_1$ can be extended to a complete contraction $\lambda: Z \rightarrow Z_1 \subseteq Z$ such that $\lambda|_E = \text{id}_E$. Hence we must have that $\lambda = \text{id}_Z$ and $Z_1 = Z$.

\Leftarrow Assume that $E \subseteq Z \subseteq B(H)$. The identity map $\text{id}_Z: Z \rightarrow Z$ can be extended to a complete contraction $\phi: B(H) \rightarrow Z \subseteq B(H)$. Then ϕ is an E -projection of $B(H)$ such that $Z = E^\phi$. If we have another E -projection ψ of $B(H)$ such that $\psi \circ \phi = \phi \circ \psi = \psi$, we have $E \subseteq E^\psi \subseteq E^\phi$ and E^ψ is injective. Then we have $E^\psi = E^\phi$ and $\psi = \phi$. Hence ϕ is a minimal E -projection of $B(H)$ and (Z, κ) is an injective envelope of E by Lemma 5.5. \square

Definition 5.7. Let E be an operator space. An extension (Z, κ) of E is called an *essential extension* if for any operator space F and a complete contraction $\phi: Z \rightarrow F$, ϕ is a complete isometry if $\phi \circ \kappa$ is a complete isometry.

Lemma 5.8. Let (Z, κ) be an injective envelope of an operator space E . Then (Z, κ) is an essential extension of E .

Proof. Assume that E is contained in Z and $\kappa = \text{id}_E$. Suppose that F is an operator space and $\phi: Z \rightarrow F$ is a complete contraction such that $\phi \circ \text{id}_E$ is a complete isometry. Then $\phi^{-1}: \phi(E) \rightarrow Z$ is a complete isometry which can be extended to a complete contraction $\psi: F \rightarrow Z$. Then $\psi \circ \phi: Z \rightarrow Z$ is a complete contraction such that $\psi \circ \phi|_E = \text{id}_E$. Hence $\psi \circ \phi = \text{id}_Z$ since Z is an injective envelope of E . This implies that ϕ must be a complete isometry. \square

Theorem 5.9. Let (Z, κ) be an extension of an operator space E . Then (Z, κ) is an injective envelope of E if and only if it is an injective and essential extension of E .

Proof. \Rightarrow follows from Lemma 5.8.

\Leftarrow Let $(\hat{Z}, \hat{\kappa})$ be an injective envelope of E and $\lambda: Z \rightarrow \hat{Z}$ a completely contractive extension of $\hat{\kappa} \circ \kappa^{-1}: \kappa(E) \rightarrow \hat{Z}$. Then λ is a complete isometry since Z is an essential extension of E with $\lambda \circ \kappa = \hat{\kappa}$. This implies that $(\lambda(Z), \hat{\kappa})$ is an injective extension of E which is contained in the injective envelope of $(\hat{Z}, \hat{\kappa})$. From Theorem 5.6, we have that $\lambda(Z) = \hat{Z}$ and then (Z, κ) is an injective envelope of E . \square

Corollary 5.10. Let $(\hat{Z}, \hat{\kappa})$ be an injective envelope of an operator space E . Then for every essential extension (Z, κ) of E , there exists a complete isometry $\lambda: Z \rightarrow \hat{Z}$ such that $\lambda \circ \kappa = \hat{\kappa}$. Hence the injective envelope of an operator space E is a "maximal" essential extension of E .

Proof. The first statement of this corollary follows easily from the proof of Theorem 5.9. The proof of the second statement is trivial. \square

Remark. Masamichi Hamana has kindly informed the author that he has independently obtained Theorem 4.5 and Theorem 5.4 of this paper in connection to his earlier work: *Injective envelopes of operator systems*, Publ. Res. Inst. Math. Sci. **15** (1979), 773–785.

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