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ISOLATED SINGULARITIES OF THE SCHRÖDINGER EQUATION WITH A GOOD POTENTIAL

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ABSTRACT. We study the behaviour near an isolated singularity, say 0, of nonnegative solutions of the Schrödinger equation $-\Delta u + Vu = 0$ defined in a punctured ball $0 < |x| < R$. We prove that whenever the potential V belongs to the Kato class K_n the following alternative, well known in the case of harmonic functions, holds: either $|x|^{n-2}u(x)$ has a positive limit as $|x| \to 0$ or u is continuous at 0. In the first case u solves the equation $-\Delta u + Vu = a\delta$ in $\{|x| < R\}$. We discuss the optimality of the class K_n and extend the result to solutions $u \not\geq 0$ of $-\Delta u + Vu = f$.

1. INTRODUCTION

In this paper we are concerned with the behaviour around an isolated singularity of the solutions of the equation

$$Lu \equiv -\Delta u + Vu = f, \tag{1}$$

where V is a potential in the Kato class, f belongs to a suitable L^p space and the solution is assumed to be not too negative. The equation is posed in a domain $\Omega \subset \mathbf{R}^n$ with singularities at some isolated points. Without loss of generality we may assume that Ω is the punctured ball $B_R^* = B_R - \{0\}$, with $B_R = \{x \in \mathbf{R}^n : |x| < R\}$, and we investigate the behaviour as $|x| \to 0$. We also assume for simplicity that $n \geq 3$, though the results have natural counterparts in two dimensions.

In the simplest case where V and f vanish identically this behaviour is well known. In fact, given any nonnegative harmonic function $u = u(x)$ defined in B_R^*, there exists the limit

$$\lim_{|x|\to 0} |x|^{n-2}u(x) = a \geq 0. \tag{2}$$

Moreover, if $a > 0$ then u exhibits a singularity at 0 and the equation

$$-\Delta u = c_n a\delta \tag{3}$$

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is satisfied in $D'(B_R)$. Here δ denotes the Dirac mass at the origin and $c_n = (n-2)s_n$, where s_n is the $(n-1)$-measure of the unit sphere $S_{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$. Such a singularity is called a *weak* singularity. In case $a = 0$, u can be extended to 0 as a harmonic function, i.e. the singularity is *removable.*

The fact that any isolated singularity is either weak or removable holds true for the nonnegative solutions of a wide class of elliptic equations, cf. [GS, S1, S2, SW]. In [S2] Serrin established the result for the Schrödinger equation

$$Lu \equiv -\Delta u + Vu = 0 \tag{4}$$

with $V \in L^q_{\text{loc}}(B_R)$, $q > n/2$.

We consider here the same problem for the class of *Kato* potentials defined as follows. A function $V \in L^1_{\text{loc}}(B_R)$ belongs to $K_n = K_n(B_R)$ if and only if

$$\lim_{\alpha \downarrow 0} \left\{ \sup_{|x| \in B_R} \int_{\substack{|x-y| \le \alpha \\ y \in B_R}} |x-y|^{2-n} |V(y)|\, dy \right\} = 0. \tag{5}$$

In case we replace $x, y \in B_R$ by $x, y \in B_{R'}$ with $0 < R' < R$ we obtain the class $K_{n,\text{loc}} = K_{n,\text{loc}}(B_R)$. We may always pass from $K_{n,\text{loc}}$ to K_n by reducing the size of the ball where we work. K_n contains the space $L^q(B_R)$ if $q > n/2$, and it also contains other potentials as is easily seen in the radial case. Namely, if $V = V(|x|)$ then $V \in K_n$ if

$$\int_0^R r|V(r)|\, dr < \infty, \tag{6}$$

cf. [AS]. On the other hand K_n does not contain $L^{n/2}(B_R)$, as the following example shows: take $V(y) = |y|^{-2}(\log|y|)^{-\alpha}$ with $1 \ge \alpha > 2/n$.

The class K_n has received much attention recently in connection with the Schrödinger equation. Thus, Kato [K] proved the selfadjointness of the operator $L = -\Delta + V$ when $V \in K_n$. Aizenman and Simon [AS] have proved the Harnack inequality for nonnegative solutions of (3), cf. also [CFG]. Simon in his survey [Si] considers K_n as the natural class of good potentials for different aspects in the theory of the Schrödinger equation.

Our results show that K_n is a class of good potentials also from the point of view of isolated singularities.

Thus, our first result is

Theorem 1. *Let* $V \in K_n(B_R)$ *and let* $u \in L^1_{\text{loc}}(B_R^*)$ *be such that*

(i) $u \ge 0$ *in* B_R,

(ii) $\Delta u \in L^1_{\text{loc}}(B_R^*)$,

(iii) $\Delta u = V \cdot u$ *in* B_R^*.

Then there exists a constant $\alpha \ge 0$ *such that*

$$-\Delta u + Vu = \alpha\delta \quad \text{in } D'(B_R), \tag{7}$$

$$\lim_{|x| \to 0} |x|^{n-2} u(x) = \alpha / c_n. \tag{8}$$

Moreover, if $\alpha = 0$ *then* u *is a continuous solution of* (4) *in* B_R, *i.e.* u *satisfies* (ii), (iii) *in* B_R.

Here c_n is the constant appearing in (3). Equalities and inequalities between integrable functions are to be understood a.e. in the corresponding domain. The theorem has as a consequence the following removability result.

A nonnegative solution of (4) *in* B_R^* *such that either* $u(x) = o(|x|^{2-n})$ *as* $|x| \to 0$ *or* $u \in L^p_{\mathrm{loc}}(B_R)$ *for* $p = n/(n-2)$ *is in fact a continuous solution in* B_R.

In the work [VY2] (cf. also [VY1]) a study is made of the isolated singularities of the equation $Lu = 0$ when the potentials V are radially symmetric [1]. In particular, Theorem 1 is proved under this restriction. Moreover, it is shown that the class K_n is *optimal* in the following sense. If $V \in C(0, R]$ is nonnegative and V does not satisfy (6) then any nonnegative solution of (4) in B_R^* which is singular satisfies as $|x| \to 0$

$$\lim_{|x|\to 0} |x|^{n-2} u(x) = +\infty. \tag{9}$$

The result actually holds for potentials with changing sign provided that $V^-(r) = \max(-V(r), 0) \in K_n$, cf. [VY2, Propositions 2.2, 2.3]. Moreover, if for such V and u we can assert that $Vu \in L^1_{\mathrm{loc}}(B_R)$ then the singularity is automatically removable, cf. [VY2, Theorem B].

Theorem 1 can be extended to solutions of possibly changing sign for equation (1) with f in some L^p-space by means of recent results in potential theory for Schrödinger operators with V in K_n, taken from Cranston, Fabes and Zhao [CFZ]. In particular, we use the fact that for r small enough there exists a *fundamental solution* for L in B_r, i.e. a function $U \in C(B_r^*)$ which satisfies (7) and (8) with $\alpha = 1$. Then we obtain

Theorem 2. *Let* $V \in K_n(B_R)$ *and let* $u \in L^1_{\mathrm{loc}}(B_R^*)$ *be such that*

(i) $u \geq O(|x|^{2-n})$ *as* $|x| \to 0$,

(ii) $u \in L^1_{\mathrm{loc}}(B_R^*)$ *and*

(iii) $\Delta u = Vu + f$ *in* $D'(B_R^*)$ *and* $f \in L^p(B_R)$ *with* $p > n/2$.

Then (7) *and* (8) *hold for some* $\alpha \in \mathbf{R}$. *Moreover, for* r *small enough we may write*

$$u(x) = \alpha U(x) + g(x) \quad \text{in } B_r, \tag{10}$$

where U *is the fundamental solution for* L *in* B_r *and* g *is a continuous solution of* (4) *in* B_r.

2. Proof of Theorem 1

We begin with some convenient notation and definitions. A solution of $Lu \equiv -\Delta u + Vu = f$ in a domain $\Omega \subset \mathbf{R}^n$ is a function $u \in L^1_{\mathrm{loc}}(\Omega)$ such that

[1] They are also assumed to be continuous for $0 < |x| < R$ but this restriction is inessential for the results.

$Vu \in L^1_{\mathrm{loc}}(\Omega)$ and the equation holds in the sense of distributions in Ω. In this paper Ω will be a ball or a punctured ball. We shall often use *averages* over spheres: if $u \in L^1_{\mathrm{loc}}(B_R^*)$ we define its *angular average*, $\bar{u}\colon (0, R) \to \mathbf{R}$ by

$$\bar{u}(r) = s_n^{-1} \int_{S_{n-1}} u(r, \sigma)\, d\sigma, \tag{11}$$

where (r, σ) are spherical coordinates in B_R. Clearly $\bar{u} \in L^1_{\mathrm{loc}}(0, R)$ and $\overline{\Delta u} = \Delta(\bar{u})$.

The proof of Theorem 1 proceeds in three steps. First, we use a sharp version of the continuity results of [AS]. In fact, a careful scrutiny of their proofs of the Harnack inequality plus a scaling argument gives

Lemma 3. *Let u be a nonnegative solution of $Lu = 0$ in B_R^* with $V \in K_n$. Then there exists a constant c, depending only on n and the local norms of V in K_n, such that for $0 < r < R/2$*

$$\max_{|x|=r} u(x) \le c \min_{|x|=r} u(x). \tag{12}$$

Proof. Let us prove first that there exists $a \in (0, 1)$ such that for every x_0, $0 < |x_0| < R/2$, we have

$$\sup\{u(x)\colon x \in B_{a\rho}(x_0)\} \le C \inf\{u(x)\colon x \in B_{a\rho}(x_0)\}, \tag{13}$$

where $\rho = |x_0|/2$, $B_{a\rho}(x_0)$ is the open ball of center x_0 and radius $a\rho$, and C depends on n, a, the norm of V in K_n, but not on x_0. To do this we rescale the problem and consider the function $v = v_{x_0}$ defined by

$$v(x) = u(\rho(x + x_0)). \tag{14}$$

v is a solution of $-\Delta v + W(x)v = 0$ in B_1, with $W(x) = \rho^2 V(\rho(x + x_0))$. By Theorem 3.10 of [AS] we get

$$\sup_{B_a} v(x) \le C \inf_{B_a} v(x). \tag{15}$$

Moreover, (cf. Theorems 1.2, 1.6, 2.4, 3.7, 3.8 and Proposition 2.2 of [AS]) C may be taken as $C = K/(1 - \beta)$, where $K = K(n, a)$ and β is any number less than 1 such that

$$\beta \ge K_1(n) \cdot \sup_{x \in B_1} \int_{|x-y| \le 2a} |x - y|^{2-n} |W(y)|\, dy.$$

This choice of β is possible for a small enough since

$$\begin{aligned} &\sup_{x \in B_1} \int_{|x-y| \le 2a} |x - y|^{2-n} |W(y)|\, dy \\ &\quad = \sup_{x \in B_\rho(x_0)} \int_{|x-y| \le 2a\rho} |x - y|^{2-n} V(y)|\, dy \\ &\quad \le \sup_{x \in B_{3R/2}} \int_{|x-y| \le aR/2} |x - y|^{2-n} |V(y)|\, dy \end{aligned}$$

and this integral tends to 0 as $a \to 0$. This proves (15) and with it (13).

To obtain (12) from (13) we only have to cover the sphere $|x| = |x_0|$ with a finite number of balls $B_{a\rho}(x_i)$ with $|x_i| = |x_0|$. Their number depends only on a. □

The second step begins by writing $Lu = 0$ in the form

$$\Delta u = V^+ \cdot u - V^- \cdot u \tag{16}$$

where V^+ and V^- are the positive and negative parts of V, i.e. $V^+ = \frac{1}{2}(V + |V|)$, $V^- = \frac{1}{2}(-V + |V|)$. Since $V^- \cdot u \geq 0$ we have $\Delta u \leq V^+ \cdot u$ and we are reduced to study the behaviour at 0 of nonnegative *supersolutions* of $Lu = 0$ with a nonnegative potential in K_n. This is done in the following result, which may have an independent interest and is therefore labeled as

Theorem 4. *Let V be a nonnegative potential in $K_n(B_R)$ and let $u \in L^1_{\text{loc}}(B^*_R)$ satisfy: $u \geq 0$, $\Delta u \in L^1_{\text{loc}}(B^*_R)$, $V \cdot u \in L^1_{\text{loc}}(B^*_R)$ and*

$$\Delta u \leq V \cdot u + f \quad \text{in } B^*_R \tag{17}$$

for some $f \in L^1_{\text{loc}}(B_R)$. If either V is radially symmetric or u satisfies (12), *then u is a solution in B_R of the equation*

$$-\Delta u + Vu = \alpha\delta + \varphi \tag{18}$$

for some $\alpha \in [0, \infty)$ and $\varphi \in L^1_{\text{loc}}(B_R)$. In particular u, $Vu \in L^1_{\text{loc}}(B_R)$ and

$$r^{n-2}\bar{u}(r) \to \alpha/c_n, \qquad r^{n-1}\bar{u}_r(r) \to -\alpha/s_n. \tag{19}$$

This theorem extends a result by Brezis and Lions [BL], which consider the case $V =$ constant. As in Theorem 1, it happens that $V \in K_n$ is the necessary and sufficient condition for the result to hold in the class of nonnegative and radially symmetric potentials.

Proof of Theorem 4. The proof of (18) in [BL] begins by obtaining that $Vu \in L^1_{\text{loc}}(B_R)$ and $\bar{u}(r) = O(r^{2-n})$ as $r \to 0$. Then it is concluded that the function φ defined a.e. in B_R as $\varphi(x) = -\Delta u(x)$ belongs to $L^1_{\text{loc}}(B_R)$. Finally they prove that the distribution $T = \Delta u + \varphi$ must be a Dirac mass at 0.

Our proof consists in establishing $Vu \in L^1_{\text{loc}}(B_R)$. We then write (17) as

$$\Delta u \leq g \equiv Vu + f \in L^1_{\text{loc}}(B_R)$$

and apply [BL] with $V \equiv 0$ to get the conclusion. We proceed as follows. Averaging (17) on the sets $|x| = r$ for $0 < r < R/2$, we get

$$-(r^{n-1}\bar{u}_r)_r \leq r^{n-1}(\overline{Vu}(r) + \bar{f}(r)). \tag{20}$$

It is at this stage that we need an extra assumption on V or u in order to relate $\overline{Vu}$ to $\overline{V} \cdot \bar{u}$. If V is radially symmetric we get $\overline{Vu} = \overline{V} \cdot \bar{u}$; on the other hand if (12) holds then $\overline{Vu} \leq c\overline{V} \cdot \bar{u}$. Since $c > 1$ we get after substituting into (20)

$$-(r^{n-1}\bar{u}_r)_r \leq cr^{n-1}\overline{V}(r)\bar{u}(r) + r^{n-1}\bar{f}(r). \tag{21}$$

Now we remark that $V \in K_n$ implies $\overline{V} \in K_n$, i.e. (cf. (6)) $\int_0^{R/2} \overline{V}(r) r\,dr < \infty$. On the other hand $\bar{u} \in C^1(0,R)$. Take $0 < R_1 < R/2$ and integrate (21) over (r, R_1) to obtain

$$r^{n-1}\bar{u}_r(r) \leq c\psi(r) + C, \qquad 0 < r < R_1, \tag{22}$$

where ψ is defined by $\psi(r) = \int_r^{R_1} \overline{V}(r)\bar{u}(r) r^{n-1}\,dr$ and C denotes a constant that depends on R_1 but not on r. Dividing (22) by r^{n-1} and integrating again we have

$$\bar{u}(r) \leq c\int_r^{R_1} \psi(s) s^{1-n}\,ds + Cr^{2-n} + C. \tag{23}$$

Now multiply by $r^{n-1}\overline{V}(r)$, integrate once again and use the fact that ψ is nonincreasing to arrive at

$$\psi(r) \leq \left(\frac{c}{n-2}\psi(r) + C\right)\int_r^{R_1} s\overline{V}(s)\,ds, \qquad 0 < r < R_1. \tag{24}$$

Since $r\overline{V}(r) \in L^1(0, R/2)$, if we choose R_1 small enough we conclude that ψ is bounded in $(0, R_1)$. This implies that $Vu \in L^1_{\mathrm{loc}}(B_R)$ since $\overline{Vu} \leq c\overline{V}\bar{u}$ for $0 < r < R/2$. From (23) it follows then that $\bar{u}(r) = O(r^{2-n})$ as $r \to 0$. As explained above, we obtain in this way (18).

Estimate (19) is easy to obtain from (18) by Gauss' theorem. Indeed for $0 < r < R$

$$s_n r^{n-1}\bar{u}_r(r) = \int_{|x|=r} \frac{\partial u}{\partial r}(r,\sigma) r^{n-1}\,d\sigma = \int_{|x|<r} \Delta u\,dx$$
$$= -\alpha + \int_{|x|<r} (Vu - \varphi)\,dx \to -\alpha \quad \text{as } r \to 0. \quad \square$$

One of the good points of Theorem 4 is that the inequality in (17) allows to replace a potential V by any upper bound. In particular, the negative part of V plays no role in the result. This has for instance the following consequence.

Proposition 5. *Let V be a radially symmetric potential such that $V^+ \in K_n$ but $V^- \notin K_n$ and let u be a nonnegative solution of (1) in B_R^* with $-\Delta u + Vu \in L^1_{\mathrm{loc}}(B_R)$. Then the singularity at 0 is removable. (In other words, all singularities are of oscillatory type.)*

Proof. We apply Theorem 4 with potential V^+ to conclude that $\bar{u}(r) \approx (\alpha/c_n) r^{2-n}$ near the origin and u, $\Delta u - \alpha\delta$ and $V^+u \in L^1_{\mathrm{loc}}(B_R)$. On the other hand, if $\alpha > 0$ we get

$$\int_{|x|<R/2} V^-(x)u(x)\,dx = C\int_0^{R/2} V^-(r)\bar{u}(r) r^{n-1}\,dr$$
$$\geq C\int_0^{R/2} rV^-(r)\,dr = \infty.$$

Since $V^- \cdot u = V^+ u - \Delta u$ we arrive at a contradiction. Hence $\alpha = 0$ and the singularity is removable. □

The last step in the proof of Theorem 1 starts from the application of Theorem 4 to $\Delta u \le V^+ \cdot u$. By (12) and (19) we have

$$u(x) \le C\alpha |x|^{2-n} \tag{25}$$

if $|x|$ is small. Here in the sequel C will denote different positive constants depending on n, u, V and R but not on $|x|$. Since $V \in K_n$ (25) implies that $Vu \in L^1_{\mathrm{loc}}(B_R)$. Now the function φ in (18) equals $V^- \cdot u$ a.e. in B_R, therefore we have

$$-\Delta u + Vu = \alpha\delta \quad \text{in } D'(B_R) \tag{26}$$

and the proof ends once we establish the asymptotic behaviour (8). For that we let v be the solution of the problem

$$-\Delta v = \alpha\delta \quad \text{in } D'(B_{R/2}), \qquad v = u \quad \text{on } |x| = R/2.$$

It is clear that $v \in C(B^*_{R/2})$ and $v(x)|x|^{n-2} \to \alpha/c_n$ as $|x| \to 0$. Therefore, we have to prove that $w \equiv u - v$ is $o(|x|^{2-n})$ as $|x| \to 0$. Notice that w solves

$$\Delta w = Vu \quad \text{in } D'(B_{R/2}), \qquad w = 0 \quad \text{on } |x| = R/2,$$

hence it can be represented in the form

$$w(x) = -\int_{B_{R/2}} G(x, y)V(y)u(y)\,dy, \tag{27}$$

where G is the Green function for the Laplace operator $-\Delta$ in $B_{R/2}$. It is well known that $G(x,y)$ is a C^∞-function for $x, y \in \overline{B}_{R/2}$, $x \ne y$. Moreover

$$0 \le G(x,y) \le c_n |x-y|^{2-n}$$

and G is positive for $x, y \in B_{R/2}$. We shall estimate $w(x)$ for $|x|$ small using all these data. In fact, let $R_1 = \min(4, R/2, R^2/4)$. For $|x| \le R_1$ we have

$$\begin{aligned} w(x) &\le c_n \int_{|y|\le R/2} |x-y|^{2-n}|V(y)|u(y)\,dy \\ &\le \int_{|y|\le |x|/2} |x-y|^{2-n}|V(y)|u(y)\,dy \\ &\quad + \int_{|x|/2<|y|\le |x|^{1/2}} |x-y|^{2-n}|V(y)|u(y)\,dy \\ &\quad + \int_{|x|^{1/2}<|y|\le R/2} |x-y|^{2-n}|V(y)|u(y)\,dy \\ &= I_1 + I_2 + I_3. \end{aligned}$$

A bound for I_1 for $|x| \approx 0$ is obtained as follows. Since $|y| \leq |x|/2$, we have $|x-y|^{2-n} \leq 2^{n-2}|x|^{2-n}$, so that

$$\begin{aligned} I_1 &\leq 2^{n-2}|x|^{2-n} \int_{|y|\leq|x|/2} |V(y)|u(y)\,dy \\ &= C|x|^{2-n} \int_{|y|\leq|x|/2} |y|^{2-n} V(y)\,dy = o(|x|^{2-n}) \end{aligned}$$

since $V \in K_n$. Likewise

$$I_3 \leq (|x|^{1/2} - |x|)^{2-n} \int_{|y|\leq R} u(y)|V(y)|\,dy = O(|x|^{1-n/2}).$$

Finally let $D = \{y \in B_R : |x|/2 \leq |y| \leq |x|^{1/2}\}$. We have

$$\begin{aligned} I_2 &\leq C \int_D |x-y|^{2-n}|y|^{2-n}|V(y)|\,dy \\ &C|x|^{2-n} \int_D |x-y|^{2-n}|V(y)|\,dy. \end{aligned}$$

Since $V \in K_n$ the last integral vanishes as $|x| \to 0$, hence $I_2 = o(|x|^{2-n})$. We conclude that $w = o(|x|^{2-n})$, which ends the proof of Theorem 1. □

3. Proof of Theorem 2

The proof relies on some properties of the Green function for the operator $L = -\Delta + V$ in a small enough ball B_R. In [CFZ] (see also [CFZ2]) Cranston, Fabes and Zhao study the *conditional gauge* F associated to the diffusion process generated by L in a bounded Lipschitz domain D of $\mathbf{R}^n$ and its relation with the Green function, G_L, of L in D. We take the following results from [CFZ].

(a) Either F is infinite everywhere or there exist positive constants c_1, c_2 such that

$$0 < c_1 \leq F(X, Y) \leq c_2 \tag{28}$$

for all $x, y \in D$ (cf. Theorem 2.2).

(b) In the latter case F is continuous in $\overline{D} \times \overline{D}$ (Theorem 4.9) and

$$G_L(x, y) = F(x, y)G(x, y) \tag{29}$$

in $D * D$, G being the Green function for $-\Delta$ (Theorem 4.7).

(c) F is finite if and only if $\inf(\operatorname{spec}(L|_D)) > 0$.

Now, the infimum of the spectrum of L in D can be made positive by choosing D to be a sufficiently small ball. This is a consequence of the following embedding result for Kato potentials proved by Fabes [F], cf. also [CFG].

Lemma 6. *For every $\varepsilon > 0$ there exists $r > 0$ such that*

$$\int u^2|V|\,dx \leq \varepsilon \int |\nabla u|^2 \tag{30}$$

for every $u \in H_0^1(B_r)$.

In the sequel we fix $D = B_r$ with such an r. We also put $U(x) = G_L(x, 0)$. Then $U \in C(B_r^*)$, U vanishes for $|x| = r$, $VU \in L^1(B_r)$ and (7), (8) hold with $\alpha = 1$.

Assume that u is a function as in Theorem 2 and let $f = \Delta u - Vu \in L^p(B_r)$. We define

$$u_1(x) = \int_{B_r} G_L(x, y) f(y) dy.$$

It is clear that $u_1 \in C(B_r)$ and $Lu_1 = f$. Put $v = u - u_1$. It satisfies $Lv = 0$ in B_r^* and

$$v(x) \geq -c|x|^{2-n}$$

for some constant c and $|x| \leq r' < r$. Therefore the function $w = v + (c/c_n)U$ is a nonnegative solution of $Lu = 0$ in a small punctured neighbourhood of 0. By Theorem 1 there exists $\alpha_1 \geq 0$ such that

$$Lw = \alpha_1 \delta \quad \text{in } D'(B_{r'})$$

and

$$w(x)|x|^{n-2} \to \alpha_1 / c_n \quad \text{as } |x| \to 0.$$

Put now $\alpha = \alpha_1 - (c/c_n)$ and let $g = u - \alpha U = w - \alpha_1 U + u_1$. The proof will be complete if we show that g is continuous for $x \approx 0$. For this observe that

$$L(g - u_1) = L(w - \alpha_1 U) = 0 \quad \text{in } D'(B_{r'})$$

and $g - u_1 = o(|x|^{2-n})$ as $|x| \to 0$, hence $g - u_1 \in L^1(B_{r'})$. In this situation Theorem 15 of [AS] implies that $g - u_1$ is a continuous function in $B_{r'}$ and we are done. □

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