ISOLATED SINGULARITIES OF THE SCHRÖDINGER EQUATION
WITH A GOOD POTENTIAL

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Abstract. We study the behaviour near an isolated singularity, say 0, of nonnegative solutions of the Schrödinger equation $-\Delta u + Vu = 0$ defined in a punctured ball $0 < |x| < R$. We prove that whenever the potential $V$ belongs to the Kato class $K_n$, the following alternative, well known in the case of harmonic functions, holds: either $|x|^{n-2}u(x)$ has a positive limit as $|x| \to 0$ or $u$ is continuous at 0. In the first case $u$ solves the equation $-\Delta u + Vu = a\delta$ in $\{|x| < R\}$. We discuss the optimality of the class $K_n$ and extend the result to solutions $u \not\geq 0$ of $-\Delta u + Vu = f$.

1. Introduction

In this paper we are concerned with the behaviour around an isolated singularity of the solutions of the equation

$$Lu \equiv -\Delta u + Vu = f,$$

where $V$ is a potential in the Kato class, $f$ belongs to a suitable $L^p$ space and the solution is assumed to be not too negative. The equation is posed in a domain $\Omega \subset \mathbb{R}^n$ with singularities at some isolated points. Without loss of generality we may assume that $\Omega$ is the punctured ball $B_R^* = B_R - \{0\}$, with $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, and we investigate the behaviour as $|x| \to 0$. We also assume for simplicity that $n \geq 3$, though the results have natural counterparts in two dimensions.

In the simplest case where $V$ and $f$ vanish identically this behaviour is well known. In fact, given any nonnegative harmonic function $u = u(x)$ defined in $B_R^*$, there exists the limit

$$\lim_{|x| \to 0} |x|^{n-2}u(x) = a \geq 0.$$

Moreover, if $a > 0$ then $u$ exhibits a singularity at 0 and the equation

$$-\Delta u = c_n a \delta$$

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is satisfied in $D'(B_R)$. Here $\delta$ denotes the Dirac mass at the origin and $c_n = (n - 2)s_n$, where $s_n$ is the $(n - 1)$-measure of the unit sphere $S_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Such a singularity is called a weak singularity. In case $a = 0$, $u$ can be extended to 0 as a harmonic function, i.e. the singularity is removable.

The fact that any isolated singularity is either weak or removable holds true for the nonnegative solutions of a wide class of elliptic equations, cf. [GS, S1, S2, SW]. In [S2] Serrin established the result for the Schrödinger equation

\[(4) \quad Lu \equiv -\Delta u + Vu = 0\]

with $V \in L^q_{\text{loc}}(B_R)$, $q > n/2$.

We consider here the same problem for the class of Kato potentials defined as follows. A function $V \in L^1_{\text{loc}}(B_R)$ belongs to $K_n = K_n(B_R)$ if and only if

\[(5) \quad \lim_{\alpha \to 0} \left\{ \sup_{|x| \in B_R} \int_{|x-y| \leq \alpha} |x-y|^{2-n} |V(y)| \, dy \right\} = 0.\]

In case we replace $x, y \in B_R$ by $x, y \in B_{R'}$ with $0 < R' < R$ we obtain the class $K_{n,\text{loc}} = K_{n,\text{loc}}(B_R)$. We may always pass from $K_{n,\text{loc}}$ to $K_n$ by reducing the size of the ball where we work. $K_n$ contains the space $L^q(B_R)$ if $q > n/2$, and it also contains other potentials as is easily seen in the radial case. Namely, if $V = V(|x|)$ then $V \in K_n$ if

\[(6) \quad \int_0^R r|V(r)| \, dr < \infty,\]

cf. [AS]. On the other hand $K_n$ does not contain $L^{n/2}(B_R)$, as the following example shows: take $V(y) = |y|^{-2} (\log |y|)^{-\alpha}$ with $1 \geq \alpha > 2/n$.

The class $K_n$ has received much attention recently in connection with the Schrödinger equation. Thus, Kato [K] proved the selfadjointness of the operator $L = -\Delta + V$ when $V \in K_n$. Aizenman and Simon [AS] have proved the Harnack inequality for nonnegative solutions of (3), cf. also [CFG]. Simon in his survey [Si] considers $K_n$ as the natural class of good potentials for different aspects in the theory of the Schrödinger equation.

Our results show that $K_n$ is a class of good potentials also from the point of view of isolated singularities.

Thus, our first result is

**Theorem 1.** Let $V \in K_n(B_R)$ and let $u \in L^1_{\text{loc}}(B_R^*)$ be such that

(i) $u \geq 0$ in $B_R$,

(ii) $\Delta u \in L^1_{\text{loc}}(B_R^*)$,

(iii) $\Delta u = V \cdot u$ in $B_R^*$.

Then there exists a constant $\alpha \geq 0$ such that

\[(7) \quad -\Delta u + Vu = \alpha \delta \quad \text{in} \quad D'(B_R),\]

\[(8) \quad \lim_{|x| \to 0} |x|^{n-2} u(x) = \alpha/c_n.\]
Moreover, if $\alpha = 0$ then $u$ is a continuous solution of (4) in $B_R$, i.e. $u$ satisfies (ii), (iii) in $B_R$.

Here $c_n$ is the constant appearing in (3). Equalities and inequalities between integrable functions are to be understood a.e. in the corresponding domain. The theorem has as a consequence the following removability result.

A nonnegative solution of $u(x) = o(|x|^{2-n})$ as $|x| \to 0$ or $u \in L^p_{\text{loc}}(B_R)$ for $p = n/(n - 2)$ is in fact a continuous solution in $B_R$.

In the work [VY2] (cf. also [VY1]) a study is made of the isolated singularities of the equation $Lu = 0$ when the potentials $V$ are radially symmetric$^1$. In particular, Theorem 1 is proved under this restriction. Moreover, it is shown that the class $K_n$ is optimal in the following sense. If $V \in C(0, R]$ is nonnegative and $V$ does not satisfy (6) then any nonnegative solution of (4) in $B_R^*$ which is singular satisfies as $|x| \to 0$

\[
\lim_{|x| \to 0} |x|^{n-2} u(x) = +\infty.
\]

The result actually holds for potentials with changing sign provided that $V^-(r) = \max(-V(r), 0) \in K_n$, cf. [VY2, Propositions 2.2, 2.3]. Moreover, if for such $V$ and $u$ we can assert that $Vu \in L^1_{\text{loc}}(B_R)$ then the singularity is automatically removable, cf. [VY2, Theorem B].

Theorem 1 can be extended to solutions of possibly changing sign for equation (1) with $f$ in some $L^p$-space by means of recent results in potential theory for Schrödinger operators with $V$ in $K_n$, taken from Cranston, Fabes and Zhao [CFZ]. In particular, we use the fact that for $r$ small enough there exists a fundamental solution for $L$ in $B_r$, i.e. a function $U \in C(B_r^*)$ which satisfies (7) and (8) with $\alpha = 1$. Then we obtain

**Theorem 2.** Let $V \in K_n(B_R)$ and let $u \in L^1_{\text{loc}}(B_R^*)$ be such that

(i) $u \geq O(|x|^{2-n})$ as $|x| \to 0$,

(ii) $u \in L^1_{\text{loc}}(B_R^*)$ and

(iii) $\Delta u - Vu + f$ in $D'(B_R^*)$ and $f \in L^p(B_R)$ with $p > n/2$.

Then (7) and (8) hold for some $\alpha \in \mathbb{R}$. Moreover, for $r$ small enough we may write

\[
u(x) = \alpha U(x) + g(x) \quad \text{in } B_r,
\]

where $U$ is the fundamental solution for $L$ in $B_r$ and $g$ is a continuous solution of (4) in $B_r$.

2. **Proof of Theorem 1**

We begin with some convenient notation and definitions. A solution of $Lu \equiv -\Delta u + Vu = f$ in a domain $\Omega \subset \mathbb{R}^n$ is a function $u \in L^1_{\text{loc}}(\Omega)$ such that

\[1\] They are also assumed to be continuous for $0 < |x| < R$ but this restriction is inessential for the results.
\(Vu \in L^1_{\text{loc}}(\Omega)\) and the equation holds in the sense of distributions in \(\Omega\). In this paper \(\Omega\) will be a ball or a punctured ball. We shall often use \textit{averages} over spheres: if \(u \in L^1_{\text{loc}}(B^*_R)\) we define its \textit{angular average}, \(\bar{u}(0, R) \to \mathbb{R}\) by

\[
\bar{u}(r) = s^{-1} \int_{S^{n-1}} u(r, \sigma) \, d\sigma,
\]

where \((r, \sigma)\) are spherical coordinates in \(B_R\). Clearly \(\bar{u} \in L^1_{\text{loc}}(0, R)\) and \(\overline{\Delta u} = \Delta(\bar{u})\).

The proof of Theorem 1 proceeds in three steps. First, we use a sharp version of the continuity results of [AS]. In fact, a careful scrutiny of their proofs of the Harnack inequality plus a scaling argument gives

\textbf{Lemma 3.} Let \(u\) be a nonnegative solution of \(Lu = 0\) in \(B_R^*\) with \(V \in K_n\). Then there exists a constant \(c\), depending only on \(n\) and the local norms of \(V\) in \(K_n\), such that for \(0 < r < R/2\)

\[
\max_{|x|=r} u(x) \leq c \min_{|x|=r} u(x).
\]

\textit{Proof.} Let us prove first that there exists \(a \in (0, 1)\) such that for every \(x_0\), \(0 < |x_0| < R/2\), we have

\[
\sup \{u(x) : x \in B_{ap}(x_0)\} \leq C \inf \{u(x) : x \in B_{ap}(x_0)\},
\]

where \(\rho = |x_0|/2\), \(B_{ap}(x_0)\) is the open ball of center \(x_0\) and radius \(ap\), and \(C\) depends on \(n\), \(a\), the norm of \(V\) in \(K_n\), but not on \(x_0\). To do this we rescale the problem and consider the function \(v = v_{x_0}\) defined by

\[
v(x) = u(\rho(x + x_0)).
\]

\(v\) is a solution of \(-\Delta v + W(x)v = 0\) in \(B_1\), with \(W(x) = \rho^2 V(\rho(x + x_0))\). By Theorem 3.10 of [AS] we get

\[
\sup_{B_a} v(x) \leq C \inf_{B_a} v(x).
\]

Moreover, (cf. Theorems 1.2, 1.6, 2.4, 3.7, 3.8 and Proposition 2.2 of [AS]) \(C\) may be taken as \(C = K/(1 - \beta)\), where \(K = K(n, a)\) and \(\beta\) is any number less than 1 such that

\[
\beta \geq K_1(n) \cdot \sup_{x \in B_1} \int_{|x-y| \leq 2a} |x-y|^{2-n} |W(y)| \, dy.
\]

This choice of \(\beta\) is possible for \(a\) small enough since

\[
\sup_{x \in B_1} \int_{|x-y| \leq 2a} |x-y|^{2-n} |W(y)| \, dy = \sup_{x \in B_{ap}(x_0)} \int_{|x-y| \leq 2ap} |x-y|^{2-n} V(y) \, dy \leq \sup_{x \in B_{3R/2}} \int_{|x-y| \leq aR/2} |x-y|^{2-n} |V(y)| \, dy
\]
and this integral tends to 0 as \(a \to 0\). This proves (15) and with it (13).

To obtain (12) from (13) we only have to cover the sphere \(|x| = |x_0|\) with a finite number of balls \(B_{a \rho}(x_i)\) with \(|x_i| = |x_0|\). Their number depends only on \(a\). □

The second step begins by writing \(Lu = 0\) in the form

\[
(16) \quad \Delta u = V^+ \cdot u - V^- \cdot u
\]

where \(V^+\) and \(V^-\) are the positive and negative parts of \(V\), i.e. \(V^+ = \frac{1}{2}(V + |V|), V^- = \frac{1}{2}(-V + |V|)\). Since \(V^- \cdot u \geq 0\) we have \(\Delta u \leq V^+ \cdot u\) and we are reduced to study the behaviour at 0 of nonnegative supersolutions of \(Lu = 0\) with a nonnegative potential in \(K_n\). This is done in the following result, which may have an independent interest and is therefore labeled as

**Theorem 4.** Let \(V\) be a nonnegative potential in \(K_n(B_R)\) and let \(u \in L^1_{loc}(B_R^*)\) satisfy: \(u \geq 0, \Delta u \in L^1_{loc}(B_R^*), V \cdot u \in L^1_{loc}(B_R^*)\) and

\[
(17) \quad \Delta u \leq V \cdot u + f \quad \text{in} \quad B_R^*
\]

for some \(f \in L^1_{loc}(B_R)\). If either \(V\) is radially symmetric or \(u\) satisfies (12), then \(u\) is a solution in \(B_R\) of the equation

\[
(18) \quad -\Delta u + Vu = \alpha \delta + \phi
\]

for some \(\alpha \in [0, \infty)\) and \(\phi \in L^1_{loc}(B_R)\). In particular \(u, Vu \in L^1_{loc}(B_R)\) and

\[
(19) \quad r^{n-2} \bar{u}(r) \to \alpha/c_n, \quad r^{n-1} \bar{u}_r(r) \to -\alpha/s_n.
\]

This theorem extends a result by Brezis and Lions [BL], which consider the case \(V = \text{constant}\). As in Theorem 1, it happens that \(V \in K_n\) is the necessary and sufficient condition for the result to hold in the class of nonnegative and radially symmetric potentials.

**Proof of Theorem 4.** The proof of (18) in [BL] begins by obtaining that \(Vu \in L^1_{loc}(B_R)\) and \(\bar{u}(r) = O(r^{2-n})\) as \(r \to 0\). Then it is concluded that the function \(\phi\) defined a.e. in \(B_R\) as \(\phi(x) = -\Delta u(x)\) belongs to \(L^1_{loc}(B_R)\). Finally they prove that the distribution \(T = \Delta u + \phi\) must be a Dirac mass at 0.

Our proof consists in establishing \(Vu \in L^1_{loc}(B_R)\). We then write (17) as

\[
\Delta u \leq g \equiv Vu + f \in L^1_{loc}(B_R)
\]

and apply [BL] with \(V \equiv 0\) to get the conclusion. We proceed as follows. Averaging (17) on the sets \(|x| = r\) for \(0 < r < R/2\), we get

\[
(20) \quad -(r^{n-1} \bar{u}_r), \leq r^{n-1}(\overline{Vu}(r) + \bar{f}(r)).
\]

It is at this stage that we need an extra assumption on \(V\) or \(u\) in order to relate \(\overline{Vu}\) to \(\overline{V \cdot u}\). If \(V\) is radially symmetric we get \(\overline{Vu} = \overline{V} \cdot \bar{u}\); on the other hand if (12) holds then \(\overline{Vu} \leq c \overline{V} \cdot \bar{u}\). Since \(c > 1\) we get after substituting into (20)

\[
(21) \quad -(r^{n-1} \bar{u}_r) \leq cr^{n-1} \overline{V}(r) \bar{u}(r) + r^{n-1} \bar{f}(r).
\]
Now we remark that \( V \in K_n \) implies \( \overline{V} \in K_n \), i.e. (cf. (6)) \( \int_0^{R/2} V(r) r \, dr < \infty \).

On the other hand \( \bar{u} \in C^1(0, R) \). Take \( 0 < R_1 < R/2 \) and integrate (21) over \((r, R_1)\) to obtain

\[
(22) \quad r^{n-1} \bar{u}_r(r) \leq c\psi(r) + C, \quad 0 < r < R_1,
\]

where \( \psi \) is defined by \( \psi(r) = \int_r^{R_1} \overline{V}(r) \bar{u}(r) r^{n-1} \, dr \) and \( C \) denotes a constant that depends on \( R_1 \) but not on \( r \). Dividing (22) by \( r^{n-1} \) and integrating again we have

\[
(23) \quad \bar{u}(r) \leq c \int_r^{R_1} \psi(s) s^{1-n} \, ds + C r^{2-n} + C.
\]

Now multiply by \( r^{n-1} \overline{V}(r) \), integrate once again and use the fact that \( \psi \) is nonincreasing to arrive at

\[
(24) \quad \psi(r) \leq \left( \frac{c}{n-2} \psi(r) + C \right) \int_r^{R_1} s \overline{V}(s) \, ds, \quad 0 < r < R_1.
\]

Since \( r \overline{V}(r) \in L^1(0, R/2) \), if we choose \( R_1 \) small enough we conclude that \( \psi \) is bounded in \((0, R_1)\). This implies that \( V u \in L^1_{\text{loc}}(B_R) \) since \( \overline{V} u \leq c \overline{V} u \) for \( 0 < r < R/2 \). From (23) it follows then that \( \bar{u}(r) = O(r^{2-n}) \) as \( r \to 0 \). As explained above, we obtain in this way (18).

Estimate (19) is easy to obtain from (18) by Gauss' theorem. Indeed for \( 0 < r < R \)

\[
\frac{1}{r} \int_{|x|<r} u(r) \, dx = \frac{1}{r} \int_{|x|<r} \overline{V}(r) u(r) r^{n-1} \, dr = \int_{|x|<r} \Delta u \, dx
\]

\[
= -\alpha + \int_{|x|<r} (V u - \varphi) \, dx \to -\alpha \quad \text{as} \quad r \to 0. \quad \square
\]

One of the good points of Theorem 4 is that the inequality in (17) allows to replace a potential \( V \) by any upper bound. In particular, the negative part of \( V \) plays no role in the result. This has for instance the following consequence.

**Proposition 5.** Let \( V \) be a radially symmetric potential such that \( V^+ \in K_n \) but \( V^- \not\in K_n \) and let \( u \) be a nonnegative solution of (1) in \( B^*_R \) with \(-\Delta u + V u \in L^1_{\text{loc}}(B_R)\). Then the singularity at 0 is removable. (In other words, all singularities are of oscillatory type.)

**Proof.** We apply Theorem 4 with potential \( V^+ \) to conclude that \( \bar{u}(r) \approx (\alpha/c_n) r^{2-n} \) near the origin and \( u, \Delta u - \alpha \delta \) and \( V^+ u \in L^1_{\text{loc}}(B_R) \). On the other hand, if \( \alpha > 0 \) we get

\[
\int_{|x|<R/2} V^-(x) u(x) \, dx = C \int_0^{R/2} V^-(r) \bar{u}(r) r^{n-1} \, dr
\]

\[
\geq C \int_0^{R/2} r V^-(r) \, dr = \infty.
\]
Since $V^- \cdot u = V^+ u - \Delta u$ we arrive at a contradiction. Hence $\alpha = 0$ and the singularity is removable. □

The last step in the proof of Theorem 1 starts from the application of Theorem 4 to $\Delta u \leq V^+ \cdot u$. By (12) and (19) we have

\begin{equation}
    u(x) \leq C|\alpha|^{2-n}
\end{equation}

if $|x|$ is small. Here in the sequel $C$ will denote different positive constants depending on $n$, $u$, $V$ and $R$ but not on $|x|$. Since $V \in K^n$ (25) implies that $Vu \in L^1_{\text{loc}}(B_R)$. Now the function $\varphi$ in (18) equals $V^- \cdot u$ a.e. in $B_R$, therefore we have

\begin{equation}
    -\Delta u + Vu = \alpha \delta \quad \text{in } D'(B_R)
\end{equation}

and the proof ends once we establish the asymptotic behaviour (8). For that we let $v$ be the solution of the problem

\[-\Delta v = \alpha \delta \quad \text{in } D'(B_{R/2}), \quad v = u \quad \text{on } |x| = R/2.\]

It is clear that $v \in C(B_{R/2})$ and $v(x)|x|^{n-2} \to \alpha/c_n$ as $|x| \to 0$. Therefore, we have to prove that $w \equiv u - v$ is $o(|x|^{2-n})$ as $|x| \to 0$. Notice that $w$ solves

\[\Delta w = Vu \quad \text{in } D'(B_{R/2}), \quad w = 0 \quad \text{on } |x| = R/2,\]

hence it can be represented in the form

\begin{equation}
    w(x) = -\int_{B_{R/2}} G(x, y)V(y)u(y) \, dy,
\end{equation}

where $G$ is the Green function for the Laplace operator $-\Delta$ in $B_{R/2}$. It is well known that $G(x, y)$ is a $C^\infty$-function for $x, y \in \overline{B}_{R/2}$, $x \neq y$. Moreover

\[0 \leq G(x, y) \leq c_n|x - y|^{2-n}\]

and $G$ is positive for $x, y \in B_{R/2}$. We shall estimate $w(x)$ for $|x|$ small using all these data. In fact, let $R_1 = \min(4, R/2, R^2/4)$. For $|x| \leq R_1$ we have

\[
w(x) \leq c_n \int_{|y| \leq R/2} |x - y|^{2-n} |V(y)|u(y) \, dy \leq \int_{|y| \leq |x|/2} |x - y|^{2-n} |V(y)|u(y) \, dy + \int_{|x|/2 < |y| \leq |x|/2} |x - y|^{2-n} |V(y)|u(y) \, dy + \int_{|x|/2 < |y| \leq R/2} |x - y|^{2-n} |V(y)|u(y) \, dy = I_1 + I_2 + I_3.
\]
A bound for $I_1$, for $|x| \approx 0$ is obtained as follows. Since $|y| \leq |x|/2$, we have $|x - y|^{2-n} \leq 2^{n-2}|x|^{2-n}$, so that

$$I_1 \leq 2^{n-2}|x|^{2-n} \int_{|y| \leq |x|/2} |V(y)| u(y) \, dy$$

$$= C|x|^{2-n} \int_{|y| \leq |x|/2} |y|^{2-n} V(y) \, dy = o(|x|^{2-n})$$

since $V \in K_n$. Likewise

$$I_3 \leq (|x|^{1/2} - |x|)^{2-n} \int_{|y| \leq R} u(y)|V(y)| \, dy = O(|x|^{1-n/2}) .$$

Finally let $D = \{ y \in B_R : |x|/2 < |y| < |x|/2 \}$. We have

$$I_2 \leq C \int_D |x - y|^{2-n} |y|^{2-n} V(y) \, dy$$

$$= C|x|^{2-n} \int_D |x - y|^{2-n} V(y) \, dy .$$

Since $V \in K_n$ the last integral vanishes as $|x| \to 0$, hence $I_2 = o(|x|^{2-n})$. We conclude that $w = o(|x|^{2-n})$, which ends the proof of Theorem 1. \( \square \)

3. Proof of Theorem 2

The proof relies on some properties of the Green function for the operator $L = \Delta + V$ in a small enough ball $B_R$. In [CFZ] (see also [CFZ2]) Cranston, Fabes and Zhao study the conditional gauge $F$ associated to the diffusion process generated by $L$ in a bounded Lipschitz domain $D$ of $\mathbb{R}^n$ and its relation with the Green function, $G_L$, of $L$ in $D$. We take the following results from [CFZ].

(a) Either $F$ is infinite everywhere or there exist positive constants $c_1, c_2$ such that

$$0 < c_1 \leq F(x, y) \leq c_2$$

for all $x, y \in D$ (cf. Theorem 2.2).

(b) In the latter case $F$ is continuous in $D \times D$ (Theorem 4.9) and

$$G_L(x, y) = F(x, y)G(x, y)$$

in $D \times D$, $G$ being the Green function for $-\Delta$ (Theorem 4.7).

(c) $F$ is finite if and only if $\inf(\text{spec}(L_{|D})) > 0$.

Now, the infimum of the spectrum of $L$ in $D$ can be made positive by choosing $D$ to be a sufficiently small ball. This is a consequence of the following embedding result for Kato potentials proved by Fabes [F], cf. also [CFG].

Lemma 6. For every $\varepsilon > 0$ there exists $r > 0$ such that

$$\int u^2 V \, dx \leq \varepsilon \int |\nabla u|^2$$

for every $u \in H_0^1(B_r)$.
In the sequel we fix $D = B_r$ with such an $r$. We also put $U(x) = G_L(x, 0)$. Then $U \in C(B_r^*)$, $U$ vanishes for $|x| = r$, $VU \in L^1(B_r)$ and (7), (8) hold with $\alpha = 1$.

Assume that $u$ is a function as in Theorem 2 and let $f = \Delta u - Vu \in L^p(B_r)$. We define

$$u_1(x) = \int_{B_r} G_L(x, y)f(y)\,dy.$$ 

It is clear that $u_1 \in C(B_r)$ and $Lu_1 = f$. Put $v = u - u_1$. It satisfies $Lv = 0$ in $B_r^*$ and

$$v(x) \geq -c|x|^{2-n}$$

for some constant $c$ and $|x| \leq r' < r$. Therefore the function $w = v + (c/c_n)U$ is a nonnegative solution of $Lu = 0$ in a small punctured neighbourhood of 0. By Theorem 1 there exists $\alpha_1 \geq 0$ such that

$$Lw = \alpha_1 \delta \quad \text{in } D'(B_{r'})$$

and

$$w(x)|x|^{n-2} \to \alpha_1/c_n \quad \text{as } |x| \to 0.$$ 

Put now $\alpha = \alpha_1 - (c/c_n)$ and let $g = u - \alpha U = w - \alpha_1 U + u_1$. The proof will be complete if we show that $g$ is continuous for $x \approx 0$. For this observe that

$$L(g - u_1) = L(w - \alpha_1 U) = 0 \quad \text{in } D'(B_{r'})$$

and $g - u_1 = o(|x|^{2-n})$ as $|x| \to 0$, hence $g - u_1 \in L^1(B_{r'})$. In this situation Theorem 15 of [AS] implies that $g - u_1$ is a continuous function in $B_{r'}$ and we are done. \qed

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