

SCATTERING FOR THE YANG-MILLS EQUATIONS

JOHN C. BAEZ

ABSTRACT. We construct wave and scattering operators for the Yang-Mills equations on Minkowski space, $M_0 \cong \mathbf{R}^4$. Sufficiently regular solutions of the Yang-Mills equations on M_0 are known to extend uniquely to solutions of the corresponding equations on the universal cover of its conformal compactification, $\widetilde{M} \cong \mathbf{R} \times S^3$. Moreover, the boundary of M_0 as embedded in \widetilde{M} is the union of "lightcones at future and past infinity", C_{\pm} . We construct wave operators W_{\pm} as continuous maps from a space X of time-zero Cauchy data for the Yang-Mills equations to Hilbert spaces $H(C_{\pm})$ of Goursat data on C_{\pm} . The scattering operator is then a homeomorphism $S: X \rightarrow X$ such that $UW_+ = W_-S$, where $U: H(C_+) \rightarrow H(C_-)$ is the linear isomorphism arising from a conformal transformation of \widetilde{M} mapping C_- onto C_+ . The maps W_{\pm} and S are shown to be smooth in a certain sense.

1. INTRODUCTION

The conformal invariance of the Yang-Mills equations in four dimensions greatly facilitates the study of the temporal asymptotic behavior of their solutions. There is a natural conformal embedding ι of Minkowski space, $M_0 \cong \mathbf{R}^4$, into the universal cover of its conformal compactification, $\widetilde{M} \cong \mathbf{R} \times S^3$. Conformally invariant wave equations on M_0 may thereby be extended to corresponding equations on \widetilde{M} [1, 2]. In particular, if A is a connection satisfying the Yang-Mills equations on \widetilde{M} , ι^*A satisfies the Yang-Mills equations on M_0 . Techniques used to prove global existence for the Yang-Mills Cauchy problem on M_0 extend to prove global existence on \widetilde{M} , allowing the derivation of precise asymptotics for Yang-Mills fields on M_0 [3, 4]. Here we construct wave and scattering operators for the Yang-Mills equations on M_0 in terms of the embedding $\iota: M_0 \rightarrow \widetilde{M}$.

The boundary of M_0 in \widetilde{M} is the union of two characteristic cones C_{\pm} , the "lightcones at past and future infinity". Points of C_{\pm} correspond to limits of null lines in M_0 as the time coordinate in M_0 approaches $\pm\infty$. The scattering theory of conformally invariant wave equations on M_0 is thus closely related to the Goursat problem on \widetilde{M} , in which solutions are determined by data on a characteristic cone [5]. Moreover the surface where the time coordinate of M_0

Received by the editors April 28, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 81E13; Secondary 35Q20, 58G16, 81D25.

is zero is just the Cauchy surface $\{0\} \times S^3 \subset \widetilde{M}$ with a single point removed. For the equation $\square f + \lambda f^3 = 0$ this allowed the formulation of wave operators W_{\pm} as nonlinear maps from a Hilbert space of Cauchy data at time zero to Hilbert spaces of Goursat data on the cones C_{\pm} [6]. These maps can be correlated with the traditional wave operators mapping sufficiently regular solutions of $\square f + \lambda f^3 = 0$ to temporally asymptotic solutions of the free wave equation. Moreover, the maps W_{\pm} are smooth with smooth inverses, so the scattering operator $W_+(W_-)^{-1}$ exists and is a diffeomorphism [7].

A difficulty in extending this approach to the Yang-Mills equations is that global existence has not been proven for finite-energy Cauchy data, but only for data having more derivatives. The techniques used in [6] to solve the Goursat problem and invert the operators W_{\pm} rely heavily upon the fact that the spaces of Goursat data have energy-type norms. Thus while we construct wave operators for the Yang-Mills equations and show they are smooth in a certain sense, we do not prove them invertible.

Nonetheless, a scattering operator of a different sort can be constructed. The group $SO^{\sim}(2, 4)$ of conformal transformations of \widetilde{M} has a unique central element ζ mapping C_- onto C_+ , and the action of this element ζ on solutions of conformally invariant equations on \widetilde{M} corresponds to scattering [5]. We show that for the Yang-Mills equations this map is smooth, and describe its relationship to the wave operators.

2. THE SCATTERING OPERATOR

First we recall the basic global existence theorem of [4]. We shall identify the universal cover of conformally compactified Minkowski space, \widetilde{M} , with $\mathbf{R} \times S^3$ given the metric $dt^2 - ds^2$, where dt^2 and ds^2 are the standard Riemannian metrics on \mathbf{R} and S^3 , respectively. Let \mathfrak{g} be the Lie algebra of a compact Lie group. Given a smooth manifold M , possibly with boundary, let $\Omega^p(M, \mathfrak{g})$ denote the \mathfrak{g} -valued differential p -forms over M . (We use this notation in informal contexts when no particular degree of differentiability need be specified.) Following the notation of [8], the Yang-Mills equations for $A \in \Omega^1(\widetilde{M}, \mathfrak{g})$ may be written as:

$$(1) \quad F = dA + \frac{1}{2}[A, A]; \quad d * F + [A, * F] = 0.$$

In temporal gauge, the dt component of A is assumed to vanish, where t is the \mathbf{R} -valued coordinate on \widetilde{M} . We shall identify elements $A \in \Omega^1(\widetilde{M}, \mathfrak{g})$ with vanishing dt component with functions $A: \mathbf{R} \rightarrow \Omega^1(S^3, \mathfrak{g})$. Let d_s denote exterior differentiation of \mathfrak{g} -valued forms on S^3 , and let $*_s$ denote the Hodge $*$ -operator on \mathfrak{g} -valued forms on S^3 with respect to the metric ds^2 . Given $A, B \in \Omega^1(S^3, \mathfrak{g})$, we make the following definitions:

$$\begin{aligned} A \times B &= *_s [A, B], & [A; B] &= *_s [A, *_s B], \\ \nabla \cdot A &= *_s d_s *_s A, & \nabla \times A &= *_s d_s A, \end{aligned}$$

and given $f \in \Omega^0(S^3, \mathfrak{g})$ we define ∇f to be $d_s f$. Note that if Δ denotes the Laplace-Beltrami operator on $\Omega^1(S^3, \mathfrak{g})$, we have $\Delta A = \nabla \times (\nabla \times A) - \nabla(\nabla \cdot A)$. Using $'$ to denote ∂_t , and working in temporal gauge, the equations (1) are equivalent to the evolution equation

$$(2) \quad A'' + \nabla \times (\nabla \times A) + A \times (\nabla \times A) + \frac{1}{2} \nabla \times (A \times A) + \frac{1}{2} A \times (A \times A) = 0$$

together with the constraint

$$(3) \quad \nabla \cdot A' + [A; A'] = 0.$$

Identifying S^3 with $SU(2)$, let $V_i, 1 \leq i \leq 3$, be an orthonormal basis of left-invariant vector fields on S^3 . Let $\Omega_k^1(S^3, \mathfrak{g})$ denote the space of \mathfrak{g} -valued one-forms on S^3 with all components lying in the Sobolev space $H^k(S^3)$, given the structure of a real Hilbert space with the following norm:

$$\|A\|_k^2 = \sum_{1 \leq i \leq 3} \int_{S^3} |(\Delta + 1)^{k/2} A(V_i)|^2$$

where here Δ denotes the Laplacian as a selfadjoint operator on $L^2(S^3, \mathfrak{g})$, and $|\cdot|$ denotes any Hilbert norm on \mathfrak{g} .

The evolution equation (2) may be desingularized by differentiating it with respect to t and using the constraint (3) to rewrite the resulting term $\nabla \times (\nabla \times A')$ as $\Delta A' - \nabla[A; A']$. Let \mathbf{H} denote the real Hilbert space $\Omega_3^1(S^3, \mathfrak{g}) \oplus \Omega_2^1(S^3, \mathfrak{g}) \oplus \Omega_1^1(S^3, \mathfrak{g})$; \mathbf{H} will be used as a space of Cauchy data (A, A', A'') . Let \mathbf{X} denote the set of Cauchy data in \mathbf{H} satisfying (2) and (3), that is, those $(A_1, A_2, A_3) \in \mathbf{H}$ such that $\nabla \cdot A_2 + [A_1; A_2] = 0$ and

$$A_3 + \nabla \times (\nabla \times A_1) + A_1 \times (\nabla \times A_1) + \frac{1}{2} \nabla \times (A_1 \times A_1) + \frac{1}{2} A_1 \times (A_1 \times A_1) = 0.$$

Note that \mathbf{X} is a closed subset of \mathbf{H} ; we give \mathbf{X} the metric arising from the norm on \mathbf{H} . Let L be the unbounded operator on \mathbf{H} with domain $\Omega_4^1(S^3, \mathfrak{g}) \oplus \Omega_3^1(S^3, \mathfrak{g}) \oplus \Omega_2^1(S^3, \mathfrak{g})$ given by

$$L(A_1, A_2, A_3) = (A_2, A_3, -(\Delta + 1)A_2),$$

and let $N: \mathbf{H} \rightarrow \mathbf{H}$ be the function given by

$$N(A_1, A_2, A_3) = (0, 0, k(A_1, A_2))$$

where

$$k(A_1, A_2) = \nabla[A_1; A_2] - A_1 \times (\nabla \times A_2) - A_2 \times (\nabla \times A_1) - \nabla \times (A_1 \times A_2) - A_1 \times (A_1 \times A_2) - \frac{1}{2} A_2 \times (A_1 \times A_1) + A_2.$$

It follows from results in [4] that L generates a strongly continuous one-parameter semigroup on \mathbf{H} , and that the function N is C^∞ from \mathbf{H} to \mathbf{H} with all derivatives bounded on bounded sets.

The global existence result may be stated as follows:

Lemma 1. *For any Cauchy datum $u_0 \in \mathbf{X}$ there is a unique continuous $u: \mathbf{R} \rightarrow \mathbf{X}$ satisfying*

$$(4) \quad u(t) = e^{Lt} u_0 + \int_0^t e^{L(t-s)} N(u(s)) ds.$$

Let $(A_1(t), A_2(t), A_3(t)) = u(t)$, and let $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$ denote the \mathfrak{g} -valued one-form corresponding to the continuous function $A_1: \mathbf{R} \rightarrow \Omega^1(S^3, \mathfrak{g})$. Then A satisfies (2) and (3) in the distributional sense.

Proof. This follows from Theorems 3 and 4 of [4]. \square

Define the map $T: \mathbf{R} \times \mathbf{X} \rightarrow \mathbf{X}$ as follows: given $u_0 \in \mathbf{X}$, let $u: \mathbf{R} \rightarrow \mathbf{X}$ be the unique continuous solution of (4), and let $T(t)u_0 = u(t)$. T is clearly a group action of \mathbf{R} on the space \mathbf{X} . This group action extends to an action of $\mathbf{R} \times SO(4)$, which is the identity component of the group of isometries of $\widetilde{\mathbf{M}}$:

Lemma 2. *The group $SO(4)$ has a strongly continuous orthogonal representation on \mathbf{H} given by:*

$$g(A_1, A_2, A_3) = ((g^{-1})^* A_1, (g^{-1})^* A_2, (g^{-1})^* A_3)$$

where $g^*: \Omega^1(S^3, \mathfrak{g}) \rightarrow \Omega^1(S^3, \mathfrak{g})$ denotes the pullback map induced by the diffeomorphism $g: S^3 \rightarrow S^3$ corresponding to $g \in SO(4)$. If $g \in SO(4)$ and $u \in \mathbf{X}$, then $gu \in \mathbf{X}$. Moreover, there is a group action $\Gamma: \mathbf{R} \times SO(4)\mathbf{X} \rightarrow \mathbf{X}$ given by:

$$\Gamma(t, g)u = T(t)(gu), \quad (t, g) \in \mathbf{R} \times SO(4), \quad u \in \mathbf{H}.$$

Proof. Since the action of $SO(4)$ on S^3 is isometric the linear transformation of \mathbf{H} given by $u \mapsto gu$ is an isometry. The action of $SO(4)$ on any Sobolev space $H^k(S^3)$ is strongly continuous, so $(g, u) \mapsto gu$ is a continuous orthogonal representation. Moreover, if $u = (A_1, A_2, A_3) \in \mathbf{X}$ and $gu = (B_1, B_2, B_3)$, then simple computations show that $\nabla \cdot B_2 + [B_1; B_2] = 0$ and

$$B_3 + \nabla \times (\nabla \times B_1) + B_1 \times (\nabla \times B_1) + \frac{1}{2} \nabla \times (B_1 \times B_1) + \frac{1}{2} B_1 \times (B_1 \times B_1) = 0,$$

hence $gu \in \mathbf{X}$. That Γ is a group action follows from the fact that $T(t)gu = gT(t)u$, a consequence of the $SO(4)$ -invariance of equation (4). \square

As mentioned above, the action of the central element $\zeta \in SO^\sim(2, 4)$ on the space of solutions of the Yang-Mills equation may be interpreted as the scattering operator [5]. In its action as a conformal transformation of $\widetilde{\mathbf{M}}$ it corresponds to the isometry $(\pi, -I) \in \mathbf{R} \times SO(4)$. Thus we define the scattering operator $S: \mathbf{X} \rightarrow \mathbf{X}$ by $S = \Gamma(\pi, -I)$. In a certain sense S is smooth, but since \mathbf{X} is not a submanifold of \mathbf{H} [9], some care is required to make this precise. We will show that the action Γ of $\mathbf{R} \times SO(4)$ on \mathbf{X} extends to a local action $\bar{\Gamma}$ of $\mathbf{R} \times SO(4)$ as C^∞ maps defined on neighborhoods of \mathbf{X} .

More precisely:

Theorem 3. For any $(t, g) \in \mathbf{R} \times SO(4)$ there is an open set $U_t \subseteq \mathbf{H}$ containing \mathbf{X} and a function $\bar{\Gamma}(t, g): U_t \rightarrow \mathbf{H}$ such that:

- (a) The restriction of $\bar{\Gamma}(t, g)$ to \mathbf{X} is $\Gamma(t, g)$.
- (b) When both sides are defined, $\bar{\Gamma}(t, g)\bar{\Gamma}(s, h)v = \bar{\Gamma}(t + s, gh)v$.
- (c) If $|s| \leq |t|$ then $U_t \subseteq U_s$. If $t \geq 0$ and $v \in U_t$, $\bar{\Gamma}(s, I)v$ is continuous as a function of s from $[-t, t]$ to \mathbf{H} .
- (d) For all $n \geq 1$ the function $\bar{\Gamma}(t, g): U_t \rightarrow \mathbf{H}$ has a continuous n th Fréchet derivative $D^n \bar{\Gamma}(t, g): \mathbf{H} \times \mathbf{H}^n \rightarrow \mathbf{H}$, where \mathbf{H}^n denotes the n -fold product of \mathbf{H} . For all $v \in U_t$ the multilinear map $D^n \bar{\Gamma}(s, g)v: \mathbf{H}^n \rightarrow \mathbf{H}$ satisfies

$$\sup\{\|D^n \bar{\Gamma}(s, g)v\|: g \in SO(4), |s| \leq |t|\} < \infty .$$

Proof. Suppose that $u_0 \in \mathbf{X}$ and $t \geq 0$. We shall show that for some $\varepsilon, M > 0$, if $\|v_0 - u_0\| < \varepsilon$, $0 \leq t_0 \leq t$, and $v: [-t_0, t_0] \rightarrow \mathbf{H}$ is a continuous functions such that

$$(5) \quad v(s) = e^{Ls}v_0 + \int_0^s e^{L(s-s')}N(v(s'))ds',$$

then $\sup_{s \in [-t_0, t_0]} \|v(s)\| \leq M$. It then follows from the theory of nonlinear semigroups [10] that for any $v_0 \in \mathbf{H}$ with $\|v_0 - u_0\| < \varepsilon$ there is a unique continuous function $v: [-t, t] \rightarrow \mathbf{H}$ satisfying (5).

Let $F: \mathbf{R}^2 \rightarrow \mathbf{R}$ be a continuous function, increasing in each variable, such that

$$\|N(x) - N(y)\| \leq F(\|x\|, \|y\|)\|x - y\|, \quad x, y \in \mathbf{H} .$$

Let $u: \mathbf{R} \rightarrow \mathbf{X}$ be as in Lemma 1, and let

$$M = \sup_{s \in [-t, t]} \|u(s)\| + 1 \quad \text{and} \quad C = \sup_{s \in [-t, t]} \|e^{Ls}\| .$$

Choose $\varepsilon > 0$ such that $\varepsilon C e^{CF(M, M)t} < 1$. The value of ε depends on u_0 and t ; let us write it as $\varepsilon(u_0, t)$. By the definition of ε and the $SO(4)$ -invariance of the equations (4) and (5), we may choose this function so that $\varepsilon(u_0, t) = \varepsilon(gu_0, t)$ for all $g \in SO(4)$. Moreover we may choose this function so that $0 \leq s \leq t$ implies $\varepsilon(u_0, t) \leq \varepsilon(u_0, s)$.

Let $v_0 \in \mathbf{H}$ have $\|v_0 - u_0\| < \varepsilon(u_0, t)$. Let $0 \leq t_0 \leq t$, and let $v: [-t_0, t_0] \rightarrow \mathbf{H}$ be continuous, satisfying (5). The proof now proceeds by contradiction: assume that for some $s \in [-t_0, t_0]$ we have $\|v(s) - u(s)\| > 1$. Let

$$\tau = \inf\{|s|: s \in [-t_0, t_0], \|v(s) - u(s)\| > 1\} .$$

By continuity, either $\|v(\tau) - u(\tau)\| = 1$ or $\|v(-\tau) - u(-\tau)\| = 1$. We may assume that $\|v(\tau) - u(\tau)\| = 1$, as the other case is analogous. By (4) and (5), for any $s \in [0, \tau]$ we have

$$\begin{aligned} \|v(s) - u(s)\| &\leq C\varepsilon(u_0, t) + C \int_0^s F(\|v(s')\|, \|u(s')\|)\|v(s') - u(s')\| ds' \\ &\leq C\varepsilon(u_0, t) + CF(M, M) \int_0^s \|v(s') - u(s')\| ds' . \end{aligned}$$

Thus by Gronwall’s inequality and our choice of $\varepsilon(u_0, t)$,

$$\|v(\tau) - u(\tau)\| \leq \varepsilon(u_0, t) C e^{CF(M, M)\tau} < 1,$$

a contradiction. We conclude that for all $s \in [-t_0, t_0]$, $\|v(s) - u(s)\| \leq 1$; hence $\sup_{s \in [-t_0, t_0]} \|v(s)\| \leq M$ as was to be shown.

Now for any $t \in \mathbf{R}$ define the open set $U_t \subseteq \mathbf{H}$ as follows:

$$U_t = \bigcup_{u \in \mathbf{X}} \{v : \|v - u\| \leq \varepsilon(u, |t|)\}.$$

Given $v_0 \in U_t$, let $v : [-t, t] \rightarrow \mathbf{H}$ be the unique continuous function satisfying (5). Given s with $|s| \leq t$ let $\bar{T}(s)v_0 = v(s)$, and for $g \in SO(4)$ let

$$(7) \quad \bar{\Gamma}(s, g)v_0 = g\bar{T}(s)v_0.$$

Statements (a) and (c) of the theorem now follow directly from the definitions. Statement (b) follows from the same arguments used in Lemma 2. To prove the differentiability claimed in (d) it suffices by (7) and Lemma 2 to prove the existence of continuous Fréchet derivatives $D^n \bar{T}(t) : \mathbf{H} \times \mathbf{H}^n \rightarrow \mathbf{H}$. This follows by induction on n using Theorem B’ of [11] and the fact that N is C^∞ with bounded derivatives on bounded sets. This theorem also establishes that for $v \in U_t$ and $|s| \leq |t|$, the Frechet derivative $D^n \bar{T}(s)v : \mathbf{H}^n \rightarrow \mathbf{H}$ is a strongly continuous function of s . By the uniform boundedness theorem we conclude that

$$\sup_{|s| \leq |t|} \|D^n \bar{T}(s)v\| < \infty.$$

Since $SO(4)$ acts as orthogonal transformations of \mathbf{H} and $\bar{\Gamma}(s, g) = g\bar{T}(s)$, we have $\|D^n \bar{\Gamma}(s, g)v\| = \|D^n \bar{T}(s)v\|$, hence

$$\sup\{\|D^n \bar{\Gamma}(s, g)v\| : g \in SO(4), |s| \leq |t|\} < \infty \quad \square$$

Corollary 4. *For some open sets $U, V \subseteq \mathbf{H}$ containing \mathbf{X} , the scattering operator $S : \mathbf{X} \rightarrow \mathbf{X}$ extends to a diffeomorphism $\bar{S} : U \rightarrow V$.*

Proof. Take $U = U_\pi \cap \bar{\Gamma}(-\pi, -I)U_\pi$, let \bar{S} be the restriction of $\bar{\Gamma}(\pi, -I)$ to U , and let $V = \bar{S}U$. Then $\bar{S} : U \rightarrow V$ is C^∞ by part (c) of the theorem, and if $v \in U$, $\bar{\Gamma}(-\pi, -I)\bar{S}v = v$ by part (b), so $\bar{\Gamma}(-\pi, -I)|V$ is a C^∞ inverse of \bar{S} . \square

3. THE WAVE OPERATORS

In this section we describe wave operators mapping a Cauchy datum in \mathbf{X} to restrictions of the corresponding solution of the Yang-Mills equation to the “lightcones at infinity”, $C_\pm \subset \tilde{\mathbf{M}}$. Let ρ denote the arclength from the point in S^3 corresponding to the identity of $SU(2)$. Let C_\pm be the subsets of $\tilde{\mathbf{M}}$ given by the equations $t = \pm(\pi - \rho)$. We define the Sobolev spaces $H^k(C_\pm)$ through the identification of C_\pm with S^3 by means of the one-to-one maps

$$(t, x) \mapsto x, \quad (t, x) \in C_\pm \subset \mathbf{R} \times S^3.$$

The Sobolev space $H^1(C_{\pm})$ is of particular significance, being the space of “finite-energy Goursat data” for the conformally invariant scalar wave equation on $\widetilde{\mathbf{M}}$, that is, the space of restrictions to C_{\pm} of finite-energy solutions of $(\square + 1)\varphi = 0$, where \square denotes the D’Alembertian on functions $\varphi: \widetilde{\mathbf{M}} \rightarrow \mathbf{R}$ [6]. Let E_{\pm} denote the pullback of the bundle of \mathfrak{g} -valued one-forms on $\widetilde{\mathbf{M}}$ to the subspace C_{\pm} , and let $\mathbf{H}(C_{\pm})$ denote the space of sections of E_{\pm} with all components lying in the Sobolev space $H^1(C_{\pm})$.

We shall formulate the wave operators for the Yang-Mills equations in terms of the spaces $\mathbf{H}(C_{\pm})$, as follows. As Cauchy datum $u \in \mathbf{X}$ determines a solution of the Yang-Mills equations in temporal gauge, $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$. Letting X_i denote the lifts to $\widetilde{\mathbf{M}}$ of the previously described vector fields V_i on S^3 , and letting $X_0 = \partial_t$, the vector fields X_i , $0 \leq i \leq 3$, form an orthonormal basis of vector fields on $\widetilde{\mathbf{M}}$. We shall show that the restrictions $A|_{C_{\pm}}$, $(X_i A)|_{C_{\pm}}$, and $(X_0 X_i A)|_{C_{\pm}}$ are well-defined as elements of $\mathbf{H}(C_{\pm})$. Moreover, the functions $u \mapsto A|_{C_{\pm}}$, $u \mapsto (X_i A)|_{C_{\pm}}$, and $u \mapsto (X_0 X_i A)|_{C_{\pm}}$ extend to smooth functions defined on an open neighborhood of \mathbf{X} .

We make use of a lemma on the inhomogeneous wave equation for \mathfrak{g} -valued one-forms. Let R be the region of $\widetilde{\mathbf{M}}$ defined by $\{-\pi \leq t \leq \pi\}$, and let \mathbf{V} be the space of \mathfrak{g} -valued one-forms on R with all components in the Sobolev space $L^2(R)$, given the structure of a real Hilbert space.

Lemma 5. *If $A_1, A_2 \in \Omega^1(S^3, \mathfrak{g})$ and $h \in \Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$ are C^∞ sections, then there is a unique C^∞ section $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$ such that*

$$(6) \quad (\square + 1)A = h, \quad (A, X_0 A)|_{t=0} = (A_1, A_2),$$

where \square denotes the D’Alembertian on $\Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$. The restrictions $A|_{C_{\pm}}$ are functions of $h|_R, A_1$, and A_2 , and the functions $(h|_R, A_1, A_2) \mapsto A|_{C_{\pm}}$ thus defined extend uniquely to continuous maps

$$T_{\pm}: \mathbf{V} \oplus \Omega^1(S^3, \mathfrak{g}) \oplus \Omega^1_0(S^3, \mathfrak{g}) \rightarrow \mathbf{H}(C_{\pm}).$$

Proof. This is a straightforward consequence of Lemma 6 of [7], which states an analogous result for the inhomogeneous scalar wave equation. \square

Note that by the theory of the Cauchy problem for the inhomogeneous wave equation, if $(A_1, A_2) \in \Omega^1_1(S^3, \mathfrak{g}) \oplus \Omega^1_0(S^3, \mathfrak{g})$ and $h \in \mathbf{V}$ then there exists a unique $A \in \Omega^1_1(R, \mathfrak{g})$ satisfying (6) in the distributional sense. The above lemma then allows us to define $A|_{C_{\pm}}$ as an element of $\mathbf{H}(C_{\pm})$.

Theorem 6. *Given $u \in \mathbf{X}$, let $(A_1(t), A_2(t), A_3(t)) = T(t)u$, and let A denote the element of $\Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$ corresponding to $A_1: \mathbf{R} \rightarrow \Omega^1_3(S^3, \mathfrak{g})$. Then $A|_{C_{\pm}}$, $(X_i A)|_{C_{\pm}}$, and $(X_0 X_i A)|_{C_{\pm}}$, where $0 \leq i \leq 3$, are well-defined elements of $\mathbf{H}(C_{\pm})$ as in Lemma 5. Moreover, for some open set $U \subseteq \mathbf{H}$ containing \mathbf{X} the functions $u \mapsto A|_{C_{\pm}}$, $u \mapsto (X_i A)|_{C_{\pm}}$, and $u \mapsto (X_0 X_i A)|_{C_{\pm}}$ extend to C^∞ maps from U to $\mathbf{H}(C_{\pm})$.*

Proof. By symmetry it suffices to consider the restrictions to C_+ . Suppose that $u \in U_\pi$, and for $|t| \leq \pi$ let $(A_1(t), A_2(t), A_3(t)) = \bar{\Gamma}(t, I)u$, where the open set $U_\pi \subseteq \mathbf{H}$ and $\bar{\Gamma}(t, I)$ are as in Theorem 3. Let A denote the element of $\Omega^1(R, \mathbf{g})$ corresponding to $A_1: [-\pi, \pi] \rightarrow \Omega_3^1(S^3, \mathbf{g})$, and let $h(u)$ denote the element of \mathbf{V} corresponding to

$$A_3 + (\Delta + 1)A_1: [-\pi, \pi] \rightarrow \Omega_1^1(S^3, \mathbf{g}).$$

By equation (5) and the definition of L and N , we have $(\square + 1)A = h(u)$ in the distributional sense. Note that by statements (c), (d) of Theorem 3, the functions $u \mapsto h(u)$ is C^∞ from U_π to \mathbf{V} . Let us write $u = (B_1, B_2, B_3)$. By Lemma 5 and the remarks following, we may define $A|C_+ = T_+(h(u), B_1, B_2)$, and conclude that $u \mapsto A|C_+$ is a C^∞ function from U_π to $\mathbf{H}(C_+)$.

Next we treat $X_i A|C_+$. Since the Lie derivations X_i commute with the differential operator \square , we have

$$(\square + 1)X_i A = X_i h(u)$$

in the distributional sense. For $1 \leq i \leq 3$ the map $u \mapsto X_i h(u)$ is C^∞ from U_π to \mathbf{V} , by Theorem 3. Moreover, by (5) we have

$$(7) \quad X_0 h(u) = X_0(A_3 + (\Delta + 1)A_1) = k(A_1, A_2)$$

in the distributional sense. Since N is C^∞ with bounded derivatives on bounded sets, k is C^∞ from $\Omega_3^1(S^3, \mathbf{g}) \oplus \Omega_2^1(S^3, \mathbf{g})$ to $\Omega_1^1(S^3, \mathbf{g})$, with bounded derivatives on bounded sets. It follows from Theorem 3 that the map $u \mapsto X_0 h(u)$ is C^∞ from U_π to \mathbf{V} .

For each i , $X_i A|_{t=0}$ and $X_0 X_i A|_{t=0}$ are well-defined distributions by the theory of the Cauchy problem. Then for $1 \leq i \leq 3$ we have

$$(X_i A, X_0 X_i A)|_{t=0} = (V_i B_1, V_i B_2),$$

while for $i = 0$ we have

$$(X_0 A, X_0^2 A)|_{t=0} = (B_2, B_3).$$

For each i it is evident that the map $u \mapsto (X_i A, X_0 X_i A)|_{t=0}$ is C^∞ from \mathbf{H} to $\Omega_1^1(S^3, \mathbf{g}) \oplus \Omega_0^1(S^3, \mathbf{g})$. Thus by Lemma 5 and the remarks following, we may define

$$(X_i A)|C_+ = T_+(X_i h(u), (X_i A, X_0 X_i A)|_{t=0}),$$

and conclude that $u \mapsto (X_i A)|C_+$ is a C^∞ function from U_π to $\mathbf{H}(C_+)$.

We use a similar argument for $(X_0 X_i A)|C_+$. Given $u \in \mathbf{X}$ we have

$$(\square + 1)X_0 X_i A = X_0 X_i h(u)$$

in the distributional sense. By (7) we have

$$X_0 X_i h(u) = X_i k(A_1, A_2).$$

For $1 \leq i \leq 3$, since $X_i k$ is C^∞ from $\Omega_3^1(S^3, \mathfrak{g}) \oplus \Omega_2^1(S^3, \mathfrak{g})$ to $\Omega_0^1(S^3, \mathfrak{g})$, it follows from Theorem 3 that the map $u \mapsto X_0 X_i h(u)$ is C^∞ from U_π to \mathbf{V} . For $i = 0$ we obtain the same conclusion as follows. By the chain rule we have

$$X_0^2 h(u) = X_0 k(A_1, A_2) = k_1(A_1, A_2)A_2 + k_2(A_1, A_2)A_3$$

where k_i denotes the Fréchet derivative with respect to A_i of the expression for $k(A_1, A_2)$. It follows from the Sobolev inequalities that the right side is C^∞ from \mathbf{H} to $\Omega_0^1(S^3, \mathfrak{g})$, with bounded derivatives on bounded sets (see also Lemma 2.2 of [4]). Thus by Theorem 3, the map $u \mapsto X_0^2 h(u)$ is C^∞ from U_π to \mathbf{V} .

For each i , $X_0 X_i A|_{t=0}$ and $X_0^2 X_i A|_{t=0}$ are well-defined distributions, by the theory of the Cauchy problem. For $1 \leq i \leq 3$ we have

$$(X_0 X_i A, X_0^2 X_i A)|_{t=0} = (V_i B_2, V_i B_3),$$

and for $i = 0$ we have

$$(X_0^2 A, X_0^3 A)|_{t=0} = (B_3, -(\Delta + 1)B_2 + k(B_1, B_2)).$$

For each i , it is evident that the map $u \mapsto (X_0 X_i A, X_0^2 X_i A)|_{t=0}$ is C^∞ from U_π to $\Omega_1^1(S^3, \mathfrak{g}) \oplus \Omega_0^1(S^3, \mathfrak{g})$. Thus by Lemma 5 we may define

$$(X_0 X_i A)|_{C_+} = T_+(X_0 X_i h(u), (X_0 X_i A, X_0^2 X_i A)|_{t=0}),$$

and note that $u \mapsto (X_0 X_i A)|_{C_+}$ is C^∞ from U_π to $\mathbf{H}(C_+)$. \square

By the above theorem there exist continuous “wave operators” $W_\pm: \mathbf{X} \rightarrow \mathbf{H}(C_\pm)$ such that $W_\pm(u) = A|_{C_\pm}$ where $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$ is as in the theorem. The relation between these wave operators and the scattering operator S is given as follows:

Theorem 6. *Let $U: \mathbf{H}(C_+) \rightarrow \mathbf{H}(C_-)$ be the isomorphism of real Hilbert spaces given by*

$$(UA)(t, x) = A(t + \pi, -x) \quad (t, x) \in C_+ \subset \mathbf{R} \times S^3.$$

Then we have $W_- S = U W_+$.

Proof. Suppose $u \in \mathbf{X}$, and let $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$ be the corresponding solution of the Yang-Mills equation in temporal gauge, as in the statement of Theorem 5. Then if $B \in \Omega^1(\widetilde{\mathbf{M}}, \mathfrak{g})$ denotes the solution of the Yang-Mills equation in temporal gauge corresponding to $Su \in \mathbf{X}$, by the definition of S we have $B(t, x) = A(t + \pi, -x)$. Thus

$$W_- Su = B|_{C_-} = U(A|_{C_+}) = U W_+ u. \quad \square$$

Acknowledgments. The author wishes to thank Irving Segal and Zheng-fang Zhou for many discussions concerning the Goursat problem and the Yang-Mills equations, without which this work could not have been done.

REFERENCES

1. S. M. Paneitz and I. E. Segal, *Analysis in space-time bundles. I*, J. Funct. Anal. **47** (1982), 78–142.
2. Y. Choquet-Bruhat and D. Christodoulou, *Existence of global solutions the Yang-Mills, Higgs, and spinor field equations in 3+1 dimensions*, Ann. Ecole. Norm. Sup. (4) **14** (1981), 481–506.
3. D. Eardley and V. Moncrief, *The global existence of Yang-Mills-Higgs fields on 4-dimensional Minkowski space*, Comm. Math. Phys. **83** (1982), 121–191, 193–212.
4. Y. Choquet-Bruhat, S. M. Paneitz and I. E. Segal, *The Yang-Mills equations on the universal cosmos*, J. Funct. Anal. **53** (1983), 112–150.
5. I. E. Segal, *Reduction of scattering to an invariant finite displacement in an ambient space-time*, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), 7266–7268.
6. J. Baez, I. E. Segal and Z. Zhou, *The global Goursat problem and scattering for nonlinear wave equations* (to appear).
7. J. Baez, *Scattering and the geometry of the solution manifold of $\square f + \lambda f^3 = 0$* , J. Funct. Anal. (to appear).
8. D. Bleeker, *Gauge theory and variational principles*, Addison-Wesley, Reading, Mass. 1981.
9. V. Moncrief, *Reduction of the Yang-Mills equations*, Differential Geometrical Methods in Mathematical Physics, Lecture Notes in Math., vol. 836, Springer, New York 1980.
10. I. E. Segal, *Non-linear semi-groups*, Ann. of Math. **78** (1964), 339–364.
11. ———, *Differential operators in the manifold of solutions of a non-linear differential equation*, J. Math. Pures Appl. **44** (1965), 71–132.

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520

Current address: Department of Mathematics, University of California, Riverside, California 95121