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RELATIONS BETWEEN H_u^p AND L_u^p IN A PRODUCT SPACE

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ABSTRACT. Relations between L_u^p and H_u^p are studied for the product space $\mathbf{R}^1 \times \mathbf{R}^1$ in the case $1 < p < \infty$ and $u(x_1, x_2) = |Q_1(x_1)|^p |Q_2(x_2)|^p w(x_1, x_2)$, where Q_1 and Q_2 are polynomials and w satisfies the A_p condition for rectangles. A description of the distributions in H_u^p is given. Questions about boundary values and about the existence of dense subsets of smooth functions satisfying appropriate moment conditions are also considered.

1. INTRODUCTION

In this paper we study in a product space setting the problem of identifying a weighted Hardy space H_u^p with the corresponding weighted Lebesgue space L_u^p, $1 < p < \infty$. We restrict our attention to the special product $\mathbf{R}^1 \times \mathbf{R}^1$ and consider weight functions u of the form

$$u(x_1, x_2) = |Q_1(x_1)|^p |Q_2(x_2)|^p w(x_1, x_2), \qquad x_1, x_2 \in \mathbf{R}^1,$$

where Q_1 and Q_2 are polynomials in x_1 and x_2, respectively, and w satisfies the A_p condition for rectangles, i.e., $w \geq 0$ and there is a finite constant c such that

$$\left(\frac{1}{|R|}\iint_R w(x_1, x_2)\, dx_1\, dx_2\right)\left(\frac{1}{|R|}\iint_R w(x_1, x_2)^{-1/(p-1)}\, dx_1\, dx_2\right)^{p-1} \leq c$$

for all rectangles R with sides parallel to the coordinate axes.

Our main results are that, for such u, H_u^p and L_u^p can be identified with equivalence of norms provided all the roots of Q_1 and Q_2 are real, while if there are complex roots, H_u^p can be identified with a subspace of L_u^p consisting of functions for which certain moments vanish. Similar results in the non-product case are given in [7 and 1]. These have had applications to fractional integrals and Sobolev embedding theorems [2], as well as to Fourier transform norm inequalities [5, 8]. Weights u of the form above are also of interest since they do not satisfy the A_p condition for rectangles; in fact, $u^{-1/(p-1)}$ is generally not locally integrable in either variable.

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By L_u^p, we mean the space

$$L_u^p = \left\{ f : \|f\|_{L_u^p} = \left(\iint_{\mathbf{R}^2} |f(x_1, x_2)|^p u(x_1, x_2)\, dx_1 dx_2 \right)^{1/p} < \infty \right\}.$$

We shall often write $\iint$ for $\iint_{\mathbf{R}^2}$.

To define H_u^p, let $\mathscr{S} = \mathscr{S}(\mathbf{R}^2)$ denote the Schwartz class of rapidly decreasing functions on $\mathbf{R}^2$, and let $\mathscr{S}' = \mathscr{S}'(\mathbf{R}^2)$ be the dual class of tempered distributions. For $l \in \mathscr{S}'$ and $\varphi \in \mathscr{S}$, let $N(l) = N_\varphi(l)$ denote the (product) nontangential maximal function defined by

$$N(l)(x_1, x_2) = \sup_{\substack{(\xi_1, t_1) \in \Gamma_1(x_1) \\ (\xi_2, t_2) \in \Gamma_2(x_2)}} |\langle l, \varphi_{t_1, t_2}(\xi_1 - \cdot, \xi_2 - \cdot) \rangle|,$$

where for $i = 1, 2$, $\Gamma_i(x_i)$ denotes the "cone" in $\mathbf{R}_+^2$ of points (ξ_i, t_i) with $|\xi_i - x_i| < \gamma_i t_i$, $\gamma_i > 0$, $\varphi_{t_1, t_2}(x_1, x_2) = t_1^{-1} t_2^{-1} \varphi(x_1/t_1, x_2/t_2)$, and $\langle l, \psi \rangle$ denotes the action of l on ψ. Then, by definition, H_u^p is the collection of all $l \in \mathscr{S}'$ such that $N(l) \in L_u^p$ for some $\gamma_1, \gamma_2 > 0$ and some φ with $\iint \varphi \neq 0$. All the weights u which we will consider belong to the class $A_\infty = \bigcup_{p>1} A_p$. The condition that $l \in H_u^p$ is then independent of the particular choice of γ_1, γ_2 and φ; in fact, as we shall see in Lemma (2.6), this is true if u merely satisfies the doubling condition in each variable uniformly in the other variable. We then set

$$\|l\|_{H_u^p} = \|N(l)\|_{L_u^p}$$

for some particular choice of $\gamma_1, \gamma_2 > 0$ and some $\varphi \in \mathscr{S}$ with $\iint \varphi \neq 0$.

To describe how H_u^p and L_u^p are related, we consider a polynomial $Q(x)$, $x \in \mathbf{R}^1$, normalized so that

$$Q(x) = \prod_{k=1}^{n} (x - a_k)^{\mu_k},$$

where $\{a_k\}$ are the distinct roots of Q, μ_k is the multiplicity of a_k, and Q has degree $N = \sum \mu_k$. Assuming that all the roots are real, we consider the partial fraction decomposition of $1/Q$ given by

$$\frac{1}{Q(x)} = \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(x - a_k)^l}.$$

With this decomposition, we associate the distribution

$$\mathscr{D}^Q = \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(l-1)!} \delta_{a_k}^{(l-1)},$$

where $\delta_a^{(j)}$ denotes the jth derivative of the delta function at a. (See [7].) If $F(x, y)$ is a function of two variables, $\mathscr{D}_y^Q F(x, y)$ will denote the action of $\mathscr{D}^Q$ on F as a function of y; the result is a function of x.

It is easy to check that

$$\mathscr{D}_y^Q\left(\frac{1}{x-y}\right)=\frac{1}{Q(x)}, \qquad \text{i.e.,} \quad Q(x)\mathscr{D}_y^Q\left(\frac{1}{x-y}\right)=1.$$

If $\varphi(x)\in\mathscr{S}(\mathbf{R}^1)$, let $\mathscr{P}_\varphi^Q(x)$ be the interpolating polynomial defined by

$$\mathscr{P}_\varphi^Q(x)=Q(x)\mathscr{D}_y^Q\left(\frac{\varphi(y)}{x-y}\right), \qquad x\in\mathbf{R}^1.$$

(See [7].) The degree of $\mathscr{P}_\varphi^Q$ is $N-1$, and the derivatives of φ and $\mathscr{P}_\varphi^Q$ up to order μ_k-1 at a_k are the same for $k=1,\dots,n$. One of the main results of [7] is that if $1<p<\infty$ and $u(x)=|Q(x)|^p w(x)$ with $w\in A_p(\mathbf{R}^1)$, then $H_u^p\equiv L_u^p$; in fact, there is a unique correspondence between distributions $l\in H_u^p$ and functions $f\in L_u^p$ given by

$$\langle l,\varphi\rangle=\int_{-\infty}^{\infty}f(x)[\varphi(x)-\mathscr{P}_\varphi^Q(x)]\,dx, \qquad \varphi\in\mathscr{S}(\mathbf{R}^1),$$

and $\|l\|_{H_u^p}\approx\|f\|_{L_u^p}$.

Now let $Q_1(x_1)$ and $Q_2(x_2)$ by polynomials of degree N_1 and N_2 all of whose roots are real. If $\varphi(x_1,x_2)\in\mathscr{S}(\mathbf{R}^2)$, let

$$(1.1)\quad \mathscr{P}_\varphi^{Q_1,Q_2}(x_1,x_2) = Q_1(x_1)Q_2(x_2)\mathscr{D}_{y_1}^{Q_1}\mathscr{D}_{y_2}^{Q_2}\left[\frac{\varphi(y_1,x_2)+\varphi(x_1,y_2)-\varphi(y_1,y_2)}{(x_1-y_1)(x_2-y_2)}\right].$$

To see how this function arises naturally in the product space setting, suppose that φ is the product of one-dimensional functions: $\varphi(x_1,x_2)=\varphi_1(x_1)\varphi_2(x_2)$. A simple computation then gives

$$(1.2)\quad \varphi(x_1,x_2)-\mathscr{P}_\varphi^{Q_1,Q_2}(x_1,x_2)=[\varphi_1(x_1)-\mathscr{P}_{\varphi_1}^{Q_1}(x_1)][\varphi_2(x_2)-\mathscr{P}_{\varphi_2}^{Q_2}(x_2)].$$

Although this formula holds only when φ is a product, the form of the right-hand side together with the result mentioned above from [7] motivate the definition of $\mathscr{P}_\varphi^{Q_1,Q_2}$ for general φ. Note that $\mathscr{P}_\varphi^{Q_1,Q_2}$ is not a polynomial even if φ is a product.

We can now state our main results. To simplify notation, it will be convenient to adopt a few conventions. We write

$$x=(x_1,x_2),\ z=(z_1,z_2),\ \text{etc.}, \qquad x_i,z_i\in\mathbf{R},\ i=1,2;$$
$$t=(t_1,t_2), \qquad t_1,t_2>0;$$
$$(x,t)=(x_1,t_1;x_2,t_2)\in\mathbf{R}_+^2\times\mathbf{R}_+^2;$$
$$\Gamma(x)=\Gamma_1(x_1)\times\Gamma_2(x_2)=\{(\xi,t)\in\mathbf{R}_+^2\times\mathbf{R}_+^2:(\xi_i,t_i)\in\Gamma_i(x_i),\ i=1,2\};$$
$$f(x)=f(x_1,x_2), \qquad \int f(x)\,dx=\iint f(x_1,x_2)\,dx_1dx_2;$$
$$\varphi_t(x)=t_1^{-1}t_2^{-1}\varphi(x_1/t_1,x_2/t_2); \qquad \mathscr{P}_\varphi^Q(x)=\mathscr{P}_\varphi^{Q_1,Q_2}(x_1,x_2);$$
$$|Q(x)|^p=|Q_1(x_1)Q_2(x_2)|^p.$$

We will prove the following results.

Theorem 1. *Let* $1 < p < \infty$ *and* $u = |Q|^p w$ *where* $Q_1(x_1)$ *and* $Q_2(x_2)$ *are polynomials of degrees* N_1 *and* N_2, *respectively, with all real roots, and* $w \in A_p$ *for rectangles. Let*

$$f(x,t) = \int f(z)[\varphi_t(x-z) - \mathscr{P}^Q_{\varphi_t(x-\cdot)}(z)]\,dz$$

where φ *has the property that*

$$(1+|x_1|)^{j_1+M}(1+|x_2|)^{j_2+M}\left|\left(\frac{\partial}{\partial x_1}\right)^{j_1}\left(\frac{\partial}{\partial x_2}\right)^{j_2}\varphi(x_1,x_2)\right|$$

is bounded for $0 \le j_1 \le N_1$, $0 \le j_2 \le N_2$, *and some* $M > 1$. *If*

$$N(f)(x) = \sup_{(\xi,t)\in\Gamma(x)} |f(\xi,t)|,$$

then there is a constant c *independent of* f *such that* $\|N(f)\|_{L^p_u} \le c\|f\|_{L^p_u}$.

Theorem 2. *Let* $1 < p < \infty$ *and* $u = |Q|^p w$ *where* Q_1 *and* Q_2 *are polynomials with all real roots and* $w \in A_p$ *for rectangles. Then* H^p_u *and* L^p_u *can be identified in the following sense: there is a unique correspondence between distributions* $l \in H^p_u$ *and functions* $f \in L^p_u$ *given by*

$$\langle l,\varphi\rangle = \int f(z)[\varphi(z) - \mathscr{P}^Q_\varphi(z)]\,dz, \qquad \varphi \in \mathscr{S}(\mathbf{R}^2).$$

Moreover, in this correspondence, $\|l\|_{H^p_u}$ *and* $\|f\|_{L^p_u}$ *are equivalent in the sense that* $c_1\|f\|_{L^p_u} \le \|l\|_{H^p_u} \le c_2\|f\|_{L^p_u}$ *for positive constants* c_1 *and* c_2 *which are independent of* f *and* l.

In defining $\mathscr{P}^Q_\varphi$ and stating Theorems 1 and 2, we have assumed tacitly that the degrees of both Q_1 and Q_2 are positive, i.e., that neither Q_1 nor Q_2 is identically 1. If, e.g., $Q_1 \equiv 1$ and $Q_2 \not\equiv 1$, we simply set

$$\mathscr{P}^Q_\varphi(x) = Q_2(x_2)\mathscr{D}^{Q_2}_{y_2}\left(\frac{\varphi(x_1,y_2)}{x_2-y_2}\right).$$

Similarly, if $Q_1 \equiv Q_2 \equiv 1$, we set $\mathscr{P}^Q_\varphi \equiv 0$.

Before stating a result which deals with the situation when Q_1, Q_2 have complex roots, we mention two facts which can be derived from the estimates used to prove Theorem 1.

Theorem 3. *Let* p, u, φ, f *and* $f(x,t)$ *be as in Theorem* 1. *Then for a.e.* x,

$$f(\xi,t) \to f(x)\int\varphi$$

as $(\xi,t) \to x$ *nontangentially, i.e., as* $(\xi_i,t_i) \to (x_i,0)$ *in such a way that* $(\xi_i,t_i) \in \Gamma_i(x_i)$, $i = 1,2$.

Theorem 4. *Let* $1 < p < \infty$ *and* $u = |Q|^p w$ *where* Q_1 *and* Q_2 *have only real roots and* $w \in A_p$ *for rectangles. Then the class of Schwartz functions whose*

Fourier transforms have compact support not containing either axis is dense in L_u^p. *In particular, the class of Schwartz functions* f *satisfying*

$$\int_{-\infty}^{\infty} f(x_1, x_2)x_1^{k_1}\,dx_1 = \int_{-\infty}^{\infty} f(x_1, x_2)x_2^{k_2}\,dx_2$$
$$= \iint f(x_1, x_2)x_1^{k_1}x_2^{k_2}\,dx_1dx_2 = 0$$

for $k_1, k_2 = 0, 1, 2, \ldots$ *is dense in* L_u^p.

For the case of complex roots, note that if $q_1(x_1)$ is a polynomial which has a total number d of complex roots, then

$$|q_1(x_1)| \approx (1 + |x_1|)^d |Q_1(x_1)|$$

where Q_1 is a polynomial with only real roots.

Theorem 5. *Let* $1 < p < \infty$, d_1 *and* d_2 *be positive integers and*

$$u = (1 + |x_1|)^{d_1 p}(1 + |x_2|)^{d_2 p}|Q|^p w$$

where $Q = Q_1(x_1)Q_2(x_2)$, Q_1 *and* Q_2 *are polynomials with only real roots, and* $w \in A_p$ *for rectangles. Then* H_u^p *can be identified with the subspace of* L_u^p *which consists of those* $f \in L_u^p$ *satisfying*

$$\int_{-\infty}^{\infty} f(x_1, x_2)Q_1(x_1)x_1^{k_1}\,dx_1 = \int_{-\infty}^{\infty} f(x_1, x_2)Q_2(x_2)x_2^{k_2}\,dx_2 = 0$$

for a.e. x_2 *and a.e.* x_1, *respectively, and for* $k_1 = 0, 1, \ldots, d_1 - 1$ *and* $k_2 = 0, 1, \ldots, d_2 - 1$. *The identification is given by* $l \equiv f$, $l \in H_u^p$, $f \in L_u^p$, *where*

$$\langle l, \varphi\rangle = \int f(z)[\varphi(z) - \mathscr{P}_\varphi^Q(z)]\,dz, \qquad \varphi \in \mathscr{S}(\mathbf{R}^2),$$

and $\|l\|_{H_u^p} \approx \|f\|_{L_u^p}$.

In case, e.g., $d_1 = 0$, the result remains true for the space of functions $f \in L_u^p$ with

$$\int_{-\infty}^{\infty} f(x_1, x_2)Q_2(x_2)x_2^{k_2}\,dx_2 = 0, \qquad k_2 = 0, 1, \ldots, d_2 - 1.$$

The proofs of Theorems 1–5 are given in §§4–8, respectively. In §2, we list as lemmas a few facts which will be needed in the proofs. We also show why the definition of H_u^p is independent of the cone apertures and of the convolution function φ. In §3, we derive the basic kernel estimates which are needed to prove Theorem 1.

2. Preliminaries

We shall use a few known facts about A_p weights. For example, it is a familiar fact that $w(x_1, x_2) \in A_p$ for rectangles (with sides parallel to the axes) if and only if $w \in A_p(\mathbf{R}^1)$ in each variable uniformly in the other—i.e., iff for a.e. x_1 and every interval $[a, b] \subset \mathbf{R}^1$,

$$\left(\frac{1}{b-a}\int_a^b w(x_1, x_2)\,dx_2\right)\left(\frac{1}{b-a}\int_a^b w(x_1, x_2)^{-1/(p-1)}\,dx_2\right)^{p-1} \le C$$

for a finite constant C which is independent of both x_1 and $[a, b]$, as well as a similar statement with the roles of x_1 and x_2 interchanged. The least constant C for which such an inequality holds is called the A_p constant of w.

The following two lemmas are special cases of the one-dimensional Hardy's inequality derived in [3].

Lemma (2.1). *If* $1 < p < \infty$, $v \in A_p(\mathbf{R}^1)$ *and* a *is real, then*

$$\int_{-\infty}^{\infty}\left(\int_{|r-a|\ge|s-a|}|f(r)|\frac{dr}{|r-a|}\right)^p v(s)\,ds \le c\int_{-\infty}^{\infty}|f(s)|^p v(s)\,ds$$

with c *equal to a constant multiple independent of* a, f *and* v *of the* A_p *constant of* v.

Lemma (2.2). *If* $1 < p < \infty$, $v \in A_p(\mathbf{R}^1)$ *and* a *is real, then*

$$\int_{-\infty}^{\infty}\left(\frac{1}{|s-a|}\int_{|r-a|\le|s-a|}|f(r)|\,dr\right)^p v(s)\,ds \le c\int_{-\infty}^{\infty}|f(s)|^p v(s)\,ds$$

with c *equal to a constant multiple independent of* a, f *and* v *of the* A_p *constant of* v.

In this form, Lemma (2.2) also follows immediately from [4].

We shall use the following known fact (see Lemma (5.4) of [7]) about distributions on $\mathbf{R}^1$.

Lemma (2.3). *Let* $l \in \mathscr{S}'(\mathbf{R}^1)$ *be a distribution with compact support such that* $l \in H^p_{|Q|^p w}$ *for some* p, $1 < p < \infty$, *where* Q *is a polynomial on* $\mathbf{R}^1$ *of degree* N *with all real roots, and* $w \in A_p(\mathbf{R}^1)$. *Then* $\langle l, x^k\rangle = 0$ *for* $k = 0, 1, \dots, N-1$. *In particular, if* $\langle l, \varphi\rangle = \langle l, \mathscr{P}^Q_\varphi\rangle$ *for all* $\varphi \in \mathscr{S}(\mathbf{R}^1)$, *then* $l \equiv 0$.

We say that $u(x_1, x_2)$ has doubling order ν_1 in the x_1-variable (uniformly in x_2) if for any intervals $I \subset J \subset \mathbf{R}^1$ and any x_2,

$$\int_J u(x_1, x_2)\,dx_1 \le c\left(\frac{|J|}{|I|}\right)^{\nu_1}\int_I u(x_1, x_2)\,dx_1$$

with c independent of I, J and x_2. A similar terminology applies to the other variable.

The purpose of the next three lemmas is to show that the definition of (product) H^p_u is independent of the cone apertures γ_1, γ_2 and of the particular convolution function φ, provided that u is doubling in each variable uniformly in the other variable. We shall use the notation

$$N_{\gamma_1,\gamma_2,\varphi}(l)(x_1,x_2) = \sup_{\substack{|\xi_1-x_1|<\gamma_1 t_1\\ |\xi_2-x_2|<\gamma_2 t_2}} |(l*\varphi_{t_1,t_2})(\xi_1,\xi_2)|.$$

Lemma (2.4). *Let $u(x_1,x_2)$ satisfy the doubling condition in each variable uniformly in the other, and let ν_1 and ν_2 be the doubling orders of u in the x_1 and x_2 variables, respectively. Then for $k_1,k_2=0,1,2,\dots$ and $0<p<\infty$,*

$$\|N_{2^{k_1}\gamma_1,2^{k_2}\gamma_2,\varphi}(l)\|_{L_u^p} \le c2^{(k_1\nu_1+k_2\nu_2)}\|N_{\gamma_1,\gamma_2,\varphi}(l)\|_{L_u^p}$$

with c independent of $k_1,k_2,\gamma_1,\gamma_2,l$ and φ.

Proof. For fixed $\alpha>0$, let

$$E=\{x\colon N_{2^{k_1}\gamma_1,2^{k_2}\gamma_2,\varphi}(l)(x)>\alpha\},$$
$$F=\{x\colon N_{2^{k_1}\gamma_1,\gamma_2,\varphi}(l)(x)>\alpha\},$$
$$G=\{x\colon N_{\gamma_1,\gamma_2,\varphi}(l)(x)>\alpha\}.$$

We shall use the notation $u(E)$ for $\int_E u\,dx$. It is enough to show that $u(E)\le c2^{k_1\nu_1+k_2\nu_2}u(G)$ with c independent of $\alpha,k_1,k_2,\gamma_1,\gamma_2,l$ and φ. We shall do this by showing that $u(E)\le c2^{k_2\nu_2}u(F)$ and $u(F)\le c2^{k_1\nu_1}u(G)$ for such c.

If $(x_1^0,x_2^0)\in E$, there is a rectangle

$$R=(|\xi_1-x_1^0|<2^{k_1}\gamma_1 t_1)\times(|\xi_2-x_2^0|<2^{k_2}\gamma_2 t_2)\equiv R_1\times R_2$$

such that $|(l*\varphi_{t_1,t_2})(\xi_1^0,\xi_2^0)|>\alpha$ at some point $(\xi_1^0,\xi_2^0)\in R$. By definition of $N_{2^{k_1}\gamma_1,\gamma_2,\varphi}(l)$, it follows that $N_{2^{k_1}\gamma_1,\gamma_2,\varphi}(l)>\alpha$ on the entire rectangle

$$S=(|x_1-\xi_1^0|<2^{k_1}\gamma_1 t_1)\times(|x_2-\xi_2^0|<\gamma_2 t_2)\equiv S_1\times S_2,$$

i.e., $S\subset F$. Therefore, since $x_1^0\in S_1$, we see that $(x_1^0,x_2)\in F$ if $x_2\in S_2$. Note also that $|S_2|=2^{-k_2}|R_2|$ and consequently, for some $c>0$,

$$\int_{S_2} u(x_1,x_2)\,dx_2 \ge c2^{-k_2\nu_2}\int_{R_2} u(x_1,x_2)\,dx_2$$

uniformly in x_1 by the doubling property of u. Moreover, $S_2\subset 2R_2$. Thus,

$$\begin{aligned}
&\int_{2R_2}\chi_F(x_1^0,x_2)u(x_1^0,x_2)\,dx_2\Big/\int_{2R_2}u(x_1^0,x_2)\,dx_2\\
&\ge \int_{S_2}\chi_F(x_1^0,x_2)u(x_1^0,x_2)\,dx_2\Big/\int_{2R_2}u(x_1^0,x_2)\,dx_2\\
&= \int_{S_2}u(x_1^0,x_2)\,dx_2\Big/\int_{2R_2}u(x_1^0,x_2)\,dx_2 \ge c2^{-k_2\nu_2}.
\end{aligned}$$

If we let $M_u^{(2)}$ denote the one-dimensional Hardy-Littlewood maximal function with respect to u in the second variable, i.e.,

$$M_u^{(2)}(g)(x_1,x_2)=\sup_{I_2\ni x_2}\int_{I_2}|g(x_1,x_2)|u(x_1,x_2)\,dx_2\Big/\int_{I_2}u(x_1,x_2)\,dx_2,$$

it follows since $x_2^0 \in 2R_2$ that

$$M_u^{(2)}(\chi_F)(x_1^0, x_2^0) \geq c2^{-k_2\nu_2}.$$

Therefore, $E \subset \{(x_1, x_2): M_u^{(2)}(\chi_F)(x_1, x_2) \geq c2^{-k_2\nu_2}\}$, and

$$\begin{aligned} u(E) &\leq \iint_{\{x_2:\ M_u^{(2)}(\chi_F)(x_1,x_2)\geq c2^{-k_2\nu_2}\}} u(x_1, x_2)\, dx_2 dx_1 \\ &\leq c2^{k_2\nu_2} \iint \chi_F(x_1, x_2) u(x_1, x_2)\, dx_2 dx_1 \\ &= c2^{k_2\nu_2} u(F) \end{aligned}$$

by weak-type $(1, 1)$ for the one-dimensional maximal function. This proves the first of the desired inequalities. The argument showing that $F \subset \{(x_1, x_2): M_u^{(1)}(\chi_G)(x_1, x_2) > c2^{-k_1\nu_1}\}$, and consequently that $u(F) \leq c2^{k_1\nu_1} u(G)$, is similar. This proves the lemma.

Lemma (2.5). *Let $\varphi, \psi \in \mathscr{S}(\mathbf{R}^2)$, $\int \varphi \neq 0$, and let λ be a positive constant. Then*

$$N_{1,1,\psi}(l)(x) \leq c \sup_{\substack{(\xi_i, t_i)\in\mathbf{R}_+^2 \\ i=1,2}} |(l * \varphi_{t_1,t_2}(\xi_1, \xi_2))| \left(1 + \frac{|x_1 - \xi_1|}{t_1}\right)^{-\lambda} \left(1 + \frac{|x_2 - \xi_2|}{t_2}\right)^{-\lambda}$$

with c independent of l and x.

Proof. If we add the restriction $t_1 = t_2$ to both sides above, the resulting inequality follows from the known inequality (see, e.g., [6])

$$\sup_{(\xi, t):\ |\xi - x| < 2t} |(l * \psi_{t,t})(\xi)| \leq c \sup_{(\xi, t)\in\mathbf{R}_+^3} |(l * \varphi_{t,t})(\xi)| \left(1 + \frac{|x - \xi|}{t}\right)^{-2\lambda},$$

$x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$, since $|x_1 - \xi_1| < t$ and $|x_2 - \xi_2| < t$ clearly implies that $|x - \xi| < 2t$, and for all x, ξ and $\lambda > 0$,

$$\begin{aligned} \left(1 + \frac{|x - \xi|}{t}\right)^{-2\lambda} &\leq \left(1 + \frac{|x_1 - \xi_1|}{t}\right)^{-\lambda} \left(1 + \frac{|x_2 - \xi_2|}{t}\right)^{-\lambda} \\ &\leq \left(1 + \frac{|x - \xi|}{t}\right)^{-\lambda}. \end{aligned}$$

The constant c above is independent of l and x.

Next, for $a > 0$, if we apply this special case at the point $(x_1/a, x_2)$ to the dilated distribution l_a defined by

$$\langle l_2, \theta \rangle = \left\langle l, \frac{1}{a}\theta\left(\frac{\cdot}{a}, \cdot\right)\right\rangle, \qquad \theta \in \mathscr{S}(\mathbf{R}^2),$$

and note that $(l_a * \theta_{t,t})(\xi_1, \xi_2) = (l * \theta_{at,t})(a\xi_1, \xi_2)$, we easily obtain

$$\sup_{\substack{\xi_1, \xi_2, t \\ |\xi_1 - x_1| < at \\ |\xi_2 - x_2| < t}} |(l * \psi_{at,t})(\xi_1, \xi_2)|$$

$$\leq c \sup_{\mathbf{R}_+^3} |(l * \varphi_{at,t})(\xi_1, \xi_2)| \left(1 + \frac{|x_1 - \xi_1|}{at}\right)^{-\lambda} \left(1 + \frac{|x_2 - \xi_2|}{t}\right)^{-\lambda}.$$

Since c is independent of a, this is equivalent to the desired inequality.

Lemma (2.6). *Let $u(x_1, x_2)$ satisfy the doubling condition in each variable uniformly in the other. Let $\varphi, \psi \in \mathscr{S}(\mathbf{R}^2)$, $\int \varphi \neq 0$, and let $\gamma_1, \gamma_2, \alpha_1, \alpha_2 > 0$. Then*

$$\|N_{\gamma_1, \gamma_2, \psi}(l)\|_{L_u^p} \leq c \|N_{\alpha_1, \alpha_2, \varphi}(l)\|_{L_u^p}, \qquad 0 < p < \infty,$$

with c independent of l.

Proof. By Lemma (2.4), we may assume $\gamma_1 = \gamma_2 = \alpha_1 = \alpha_2 = 1$. By Lemma (2.5), for $\lambda > 0$,

$$N_{1,1,\psi}(l)(x) \leq c \sum_{k_1, k_2 = 0}^{\infty} N_{2^{k_1}, 2^{k_2}, \varphi}(l)(x) \cdot 2^{-k_1\lambda - k_2\lambda}.$$

Thus, by Lemma (2.4),

$$\begin{aligned} \|N_{1,1,\psi}(l)\|_{L_u^p} &\leq c \sum_{k_1, k_2 = 0}^{\infty} 2^{(k_1\nu_1 + k_2\nu_2)} 2^{-(k_1 + k_2)\lambda} \|N_{1,1,\varphi}(l)\|_{L_u^p} \\ &= c \|N_{1,1,\varphi}(l)\|_{L_u^p} \end{aligned}$$

provided we choose λ sufficiently large. This completes the proof.

Lemma (2.7). *If d_1 and d_2 are nonnegative integers, then*

$$\begin{aligned} \mathscr{P}_\varphi^{x_1^{d_1} Q_1, x_2^{d_2} Q_2}(x_1, x_2) = \mathscr{P}_\varphi^{Q_1, Q_2}(x_1, x_2) &+ Q_1(x_1) Q_2(x_2) \\ &\times \{P_{d_1 - 1}(x_1) G_{d_2 - 1}(x_2) + P_{d_2 - 1}(x_2) G_{d_1 - 1}(x_1)\} \end{aligned}$$

where $P_{d_1 - 1}$, $P_{d_2 - 1}$ are polynomials of degrees $d_1 - 1$, $d_2 - 1$, respectively, and $G_{d_1 - 1}$, $G_{d_2 - 1}$ are functions satisfying

$$|G_{d_1 - 1}(x_1)| \leq c(1 + |x_1|^{d_1 - 1}), \qquad |G_{d_2 - 1}(x_2)| \leq c(1 + |x_2|^{d_2 - 1}).$$

The polynomials $P_{d_1 - 1}$, $P_{d_2 - 1}$ as well as the functions $G_{d_1 - 1}$, $G_{d_2 - 1}$ may depend on φ, Q_1 and Q_2. In case either d_1 or d_2 is 0, we interpret the corresponding polynomial to be 0.

Proof. We shall use the following formula (see Lemma (2.7) of [7]) for polynomials of a single variable $x \in \mathbf{R}^1$:

$$\mathscr{P}_\varphi^{xQ} = \mathscr{P}_\varphi^{Q} + \mathscr{D}^{xQ}(x) \cdot Q. \tag{2.8}$$

Consider first the case $d_1 = 1$, $d_2 = 0$. We shall assume that $\deg Q_1 > 0$. By definition,

$$\begin{aligned}
&\mathscr{P}_\varphi^{x_1 Q_1, Q_2}(x_1, x_2) \\
&\quad = x_1 Q_1(x_1) Q_2(x_2) \mathscr{D}_{y_1}^{x_1 Q_1} \mathscr{D}_{y_2}^{Q_2} \left\{ \frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \right\} \\
&\quad = Q_2(x_2) \mathscr{D}_{y_2}^{Q_2} \left[x_1 Q_1(x_1) \mathscr{D}_{y_1}^{x_1 Q_1} \left\{ \frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \right\} \right].
\end{aligned}$$

Now applying (2.8) to the term in square brackets for the variable x_1 and the function $\psi(z) = (\varphi(x_1, y_2) + \varphi(z, x_2) - \varphi(z, y_2))/(x_2 - y_2)$, we obtain

$$\begin{aligned}
&\mathscr{P}_\varphi^{x_1 Q_1, Q_2}(x_1, x_2) \\
&\quad = Q_2(x_2) \mathscr{D}_{y_2}^{Q_2} \left[Q_1(x_1) \mathscr{D}_{y_1}^{Q_1} \left\{ \frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \right\} \right. \\
&\qquad\qquad \left. + \mathscr{D}_{y_1}^{x_1 Q_1} \left(\frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{(x_2 - y_2)} \right) \cdot Q_1(x_1) \right] \\
&\quad = \mathscr{P}_\varphi^{Q_1, Q_2}(x_1, x_2) + Q_1(x_1) Q_2(x_2) \mathscr{D}_{y_2}^{Q_2} \mathscr{D}_{y_1}^{x_1 Q_1} \\
&\quad\quad \times \left(\frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{x_2 - y_2} \right).
\end{aligned}$$

In the last term on the right above we may drop the first of the three terms in the numerator since

$$\mathscr{D}_{y_1}^{x_1 Q_1} \left(\frac{\varphi(x_1, y_2)}{x_2 - y_2} \right) = 0;$$

in fact, $\varphi(x_1, y_2)/(x_2 - y_2)$ is independent of y_1, while $\mathscr{D}_{y_1}^{x_1 Q_1}$ is a linear combination of derivatives with respect to y_1 of order at least one (since $x_1 Q_1$ has degree at least 2). Moreover, for the remaining two terms we can write

$$\mathscr{D}_{y_2}^{Q_2} \mathscr{D}_{y_1}^{x_1 Q_1} \left(\frac{\varphi(y_1, x_2) - \varphi(y_1, y_2)}{x_2 - y_2} \right) = \mathscr{D}_{y_2} \left(\frac{\psi(x_2) - \psi(y_2)}{x_2 - y_2} \right) \tag{2.9}$$

where $\psi(z) = \mathscr{D}_{y_1}^{x_1 Q_1}(\varphi(y_1, z))$. It is easy to see that (2.9) is a function of x_2 alone, and that it is bounded in absolute value by a multiple of $(1 + |x_2|)^{-1}$. Thus, we have shown that

$$\mathscr{P}_\varphi^{x_1 Q_1, Q_2}(x_1, x_2) = \mathscr{P}_\varphi^{Q_1, Q_2}(x_1, x_2) + Q_1(x_1) Q_2(x_2) H_{-1}(x_2), \tag{2.10}$$

with $|H_{-1}(x_2)| \le c(1 + |x_2|)^{-1}$.

This formula was derived in case $\deg Q_1 > 0$.

If $\deg Q_1 = 0$, then $Q_1 \equiv 1$ and

$$\begin{aligned}
\mathscr{P}_\varphi^{x_1, Q_2}(x_1, x_2) &= Q_2(x_2) \mathscr{D}_{y_2}^{Q_2} \mathscr{D}_{y_1}^{x_1} \left(\frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{x_2 - y_2} \right) \\
&= Q_2(x_2) \mathscr{D}_{y_2}^{Q_2} \left(\frac{\varphi(x_1, y_2)}{x_2 - y_2} \right) + Q_2(x_2) \mathscr{D}_{y_2}^{Q_2} \mathscr{D}_{y_1}^{x_1} \left(\frac{\varphi(y_1, x_2) + \varphi(y_1, y_2)}{x_2 - y_2} \right).
\end{aligned}$$

The first term on the right equals $\mathscr{P}^{Q_2}_{\varphi(x_1,\cdot)}(x_2)$ and the second term can be treated as above.

Analogous results hold for $\mathscr{P}^{Q_1,x_2Q_2}_{\varphi}$.

For the general case, we argue by induction on d_1, d_2. Write

$$\begin{aligned}\mathscr{P}^{x_1^{d_1}Q_1,x_2^{d_2}Q_2}_{\varphi} &= \mathscr{P}^{x_1(x_1^{d_1-1}Q_1),x_2^{d_2}Q_2}_{\varphi} \\ &= \mathscr{P}^{x_1^{d_1-1}Q_1,x_2^{d_2}Q_2}_{\varphi} + (x_1^{d_1-1}Q_1)(x_2^{d_2}Q_2)H_{-1}(x_2) \qquad \text{(by 2.10)} \\ &= \mathscr{P}^{Q_1,Q_2}_{\varphi} + Q_1Q_2\{P_{d_1-2}(x_1)G_{d_2-1}(x_2) + P_{d_2-1}(x_2)G_{d_1-2}(x_1)\} \\ &\quad + (x_1^{d_1-1}Q_1)(x_2^{d_2}Q_2)H_{-1}(x_2)\end{aligned}$$

by applying the induction assumption to $\mathscr{P}^{x_1^{d_1-1}Q_1,x_2^{d_2}Q_2}$. We may rewrite the third term on the right as

$$(x_1^{d_1-1}Q_1)(x_2^{d_2}Q_2)H_{-1}(x_2) = Q_1Q_2x_1^{d_1-1}F_{d_2-1}(x_2)$$

where

$$F_{d_2-1}(x_2) = x_2^{d_2}H_{-1}(x_2) = O((1+|x_2|)^{d_2-1}),$$

and then combine this term with the other terms above to complete the proof of the lemma.

3. Kernel estimates

In this section, we derive the kernel estimates on which our proofs are based. As usual, we use the notation $Q(x) = Q(x_1, x_2) = Q_1(x_1)Q_2(x_2)$ where Q_1 and Q_2 are polynomials of degree N_1 and N_2, respectively, with all real zeros. We also write $D^{j_1}_{x_1} = \partial^{j_1}/\partial x_1^{j_1}$, etc.

Lemma (3.1). *Let $\varphi(x)$ be a function on $\mathbf{R}^2$ for which $D^{j_1}_{x_1}D^{j_2}_{x_2}\varphi$ is bounded for $j_1 \le N_1$ and $j_2 \le N_2$, and let*

$$c_\varphi = \sup_{\substack{(x_1,x_2)\in\mathbf{R}^2 \\ j_1\le N_1,\, j_2\le N_2}} |D^{j_1}_{x_1}D^{j_2}_{x_2}\varphi(x_1,x_2)|.$$

Then

$$|\varphi(x) - \mathscr{P}^Q_\varphi(x)| \le cc_\varphi\frac{|Q(x)|}{(1+|x_1|)(1+|x_2|)}, \qquad x = (x_1, x_2),$$

with c independent of φ and x.

Proof. Let R be a positive number which is larger than the absolute value of every root of Q_1 and every root of Q_2. We consider four cases. In case both $|x_1| > 2R$ and $|x_2| > 2R$, write $\varphi(x) - \mathscr{P}^Q_\varphi(x)$ as

$$Q(x)\mathscr{D}^{Q_1}_{y_1}\mathscr{D}^{Q_2}_{y_2}\left\{\frac{\varphi(x_1,x_2) - \varphi(y_1,x_2) - \varphi(x_1,y_2) + \varphi(y_1,y_2)}{(x_1-y_1)(x_2-y_2)}\right\}.$$

By simply applying Leibniz's differentiation rule term-by-term, we see this is bounded in absolute value by $cc_\varphi|Q(x)|/|x_1x_2|$.

In case $|x_1| \le 2R$ and $|x_2| > 2R$, we write the expression in the form

$$Q(x)\mathscr{D}_{y_1}^{Q_1}\mathscr{D}_{y_2}^{Q_2}\left\{\frac{[(\varphi(x_1,x_2)-\varphi(y_1,x_2))/(x_1-y_1)]-[(\varphi(x_1,y_2)-\varphi(y_1,y_2))/(x_1-y_1)]}{x_2-y_2}\right\}$$
$$= Q(x)\mathscr{D}_{y_1}^{Q_1}\mathscr{D}_{y_2}^{Q_2}$$
$$\times\left\{\frac{\int_0^1(D_{x_1}\varphi)(y_1+s_1(x_1-y_1),x_2)\,ds_1-\int_0^1(D_{x_1}\varphi)(y_1+s_1(x_1-y_1),y_2)\,ds_1}{x_2-y_2}\right\}.$$

Now performing the indicated differentiations in y_1 and y_2 and using the fact that $|x_2|$ is large, we see this is bounded in absolute value by $cc_\varphi|Q(x)|/|x_2|$. Since $|x_1|$ is bounded, we obtain the desired estimate. The case when $|x_1| > 2R$ and $|x_2| \le 2R$ is similar.

Finally, in case both $|x_1| \le 2R$ and $|x_2| \le 2R$, we write the expression as

$$Q(x)\mathscr{D}_{y_1}^{Q_1}\mathscr{D}_{y_2}^{Q_2}\left\{\int_0^1\int_0^1(D_{x_1}D_{x_2}\varphi)(y_1+s_1(x_1-y_1),y_2+s_2(x_2-y_2))\,ds_1\,ds_2\right\}.$$

By Leibniz's rule, this is bounded in absolute value by $cc_\varphi|Q(x)|$, and the lemma follows.

By using Lemma (3.1), we can show that if $f \in L_u^p$ for $u = |Q|^p w$ with $w \in A_p$ for rectangles, $1 < p < \infty$, and if l_f is defined by

$$\langle l_f, \varphi\rangle = \int_{\mathbf{R}^2} f(z)[\varphi(z) - \mathscr{P}_\varphi^Q(z)]\,dz, \qquad \varphi \in \mathscr{S}(\mathbf{R}^2),$$

then l_f defines a tempered distribution. In fact, by Lemma (3.1),

$$|\langle l_f,\varphi\rangle| \le \int |f(z)||\varphi(z)-\mathscr{P}_\varphi^Q(z)|\,dz$$
$$\le \int |f(z)|cc_\varphi\frac{|Q(z)|}{(1+|z_1|)(1+|z_2|)}\,dz.$$

Thus, by Hölder's inequality, $|\langle l_f,\varphi\rangle| \le cc_\varphi\|f\|_{L_u^p}\mathscr{I}$ where

$$\text{(3.2)} \qquad \mathscr{I} = \left(\int_{\mathbf{R}^2}\frac{w(z)^{-1/(p-1)}}{(1+|z_1|)^{p'}(1+|z_2|)^{p'}}\,dz\right)^{1/p'}, \qquad 1/p + 1/p' = 1.$$

If we show that $\mathscr{I}$ is finite, it will follow easily from the definition of c_φ that l_f is a tempered distribution. To accomplish this, we first claim that if $R_{1,1}$ is a square of edgelength 1 and R_{r_1,r_2} denotes the rectangle with the same center as $R_{1,1}$ whose x_i-edgelength is r_i, $i = 1, 2$, then if $w \in A_s$ for rectangles we have

$$\text{(3.3)} \qquad w(R_{r_1,r_2}) \le cr_1^s r_2^s w(R_{1,1}), \qquad r_1, r_2 > 1.$$

In fact, if $w \in A_s$ then

$$\left(\frac{1}{r_1r_2}\int_{R_{r_1,r_2}} w\right)\left(\frac{1}{r_1r_2}\int_{R_{r_1,r_2}} w^{-1/(s-1)}\right)^{s-1} \le c,$$

and so since $R_{1,1} \subset R_{r_1,r_2}$,

$$\left(\frac{1}{r_1 r_2}\int_{R_{r_1,r_2}} w\right)\left(\frac{1}{r_1 r_2}\int_{R_{1,1}} w^{-1/(s-1)}\right)^{s-1} \le c.$$

Then, by Hölder's inequality,

$$\left(\frac{1}{r_1 r_2}\int_{R_{r_1,r_2}} w\right)\left(\frac{1}{r_1 r_2}\right)^{s-1}\left(\int_{R_{1,1}} w\right)^{-1} \le c,$$

which proves the claim.

Now let R_{r_1,r_2} be centered at the origin, and write

$$\begin{aligned}\mathcal{I}^{p'} &= \int (1+|z_1|)^{-p'}(1+|z_2|)^{-p'} w(z)^{-1/(p-1)}\,dz \\ &= \int_{R_{1,1}} + \sum_{k_1,k_2=1}^{\infty} \int_{\substack{2^{k_1-1}<|z_1|<2^{k_1} \\ 2^{k_2-1}<|z_2|<2^{k_2}}} \\ &\le \int_{R_{1,1}} w^{-1/(p-1)}\,dz + \sum_{k_1,k_2=1}^{\infty} 2^{-p'(k_1+k_2)} \int_{R_{2^{k_1},2^{k_2}}} w^{-1/(p-1)}\,dz\end{aligned}$$

Since $w \in A_p$ for rectangles, we have $w^{-1/(p-1)} \in A_{p'-\varepsilon}$ for rectangles for some $\varepsilon > 0$. Thus, by (3.3) applied to $w^{-1/(p-1)}$ with $s = p' - \varepsilon$,

$$\begin{aligned}\mathcal{I}^{p'} &\le \left(\int_{R_{1,1}} w^{-1/(p-1)}\,dz\right)\left(1 + c\sum_{k_1,k_2=1}^{\infty} 2^{-p'(k_1+k_2)} 2^{(p'-\varepsilon)(k_1+k_2)}\right) \\ &\le c\int_{R_{1,1}} w^{-1/(p-1)}\,dz < \infty.\end{aligned}$$

Lemma (3.4). *Let $\{a_1^{(i)}\}$ and $\{a_2^{(j)}\}$ denote the roots of Q_1 and Q_2, respectively, and let $E^{i,j}$ denote the set of points of $\mathbf{R}^2$ which are closer to $(a_1^{(i)}, a_2^{(j)})$ than to any other $(a_1^{(k)}, a_2^{(l)})$. Given a function φ and a number $M \ge 1$, let*

$$A_{\varphi,M} = \sup_{\substack{(z_1,z_2)\in\mathbf{R}^2 \\ j_1\le N_1,\, j_2\le N_2}} (1+|z_1|)^{j_1+M}(1+|z_2|)^{j_2+M}|D_{z_1}^{j_1} D_{z_2}^{j_2}\varphi(z_1,z_2)|.$$

Then for $z \in (z_1, z_2) \in E^{i,j}$,

$$\begin{aligned}&|\varphi_t(x-z) - \mathcal{P}^{Q}_{\varphi_t(x-\cdot)}(z)| \\ &\quad\le cA_{\varphi,M}\frac{|Q(z)|}{|Q(x)|}\left(\frac{1}{|x_1-a_1^{(i)}| + |z_1-a_1^{(i)}|} + \frac{1}{t_1(1+|x_1-z_1|/t_1)^M}\right) \\ &\quad\times\left(\frac{1}{|x_2-a_2^{(j)}| + |z_2-a_2^{(j)}| + |z_2-a_2^{(j)}|} + \frac{1}{t_2(1+|x_2-z_2|/t_2)^M}\right),\end{aligned}$$

$x = (x_1, x_2)$, $t = (t_1, t_2)$, $t_1, t_2 > 0$, *with* c *independent of* x, z, t, φ *and* M.

Proof. Note first that $E^{i,j} = E_1^i \times E_2^j$ where

$$E_1^i = \left\{ z_1 \colon |z_1 - a_1^{(i)}| = \min_k |z_1 - a_1^{(k)}| \right\}$$

and

$$E_2^j = \left\{ z_2 \colon |z_2 - a_2^{(j)}| = \min_l |z_2 - a_2^{(l)}| \right\}.$$

The lemma will be proved by using the following one-dimensional estimate (see Lemma (3.3) of [7]): for a function $\varphi(z)$, $z \in \mathbf{R}^1$, and a polynomial $Q(z)$, $z \in \mathbf{R}^1$, with all real roots,

$$\begin{aligned} |\varphi_t(x-z) - \mathscr{P}^Q_{\varphi_t(x-\cdot)}(z)| &= \left| Q(z) \mathscr{D}^Q_y \left\{ \frac{\varphi_t(x-z) - \varphi_t(y-z)}{z-y} \right\} \right| \\ &\le c \frac{|Q(z)|}{|Q(x)|} \left[\frac{n_\varphi}{|x-a| + |z-a|} + |\varphi_t(x-z)| \right], \qquad t > 0, \end{aligned} \tag{3.5}$$

where a is the root of Q which is closest to z, and

$$n_\varphi = \sup_{\substack{z \in \mathbf{R}^1 \\ j \le \deg Q}} (1 + |z|)^{j+1} |D_z^j \varphi(z)|.$$

To see how the lemma follows from this estimate, let $(z_1, z_2) \in E^{i,j}$, and define for fixed x_2, z_2 and t_2,

$$\begin{aligned} \psi(z_1) &= \psi(z_1; x_2, z_2, t_2) \\ &= Q_2(z_2) \mathscr{D}^{Q_2}_{y_2} \left\{ \frac{\frac{1}{t_2} \varphi\left(z_1, \frac{x_2 - z_2}{t_2}\right) - \frac{1}{t_2} \varphi\left(z_1, \frac{x_2 - y_2}{t_2}\right)}{z_2 - y_2} \right\}. \end{aligned}$$

A computation gives

$$\varphi_t(x-z) - \mathscr{P}^Q_{\varphi_t(x-\cdot)}(z) = Q_1(z_1) \mathscr{D}^{Q_1}_{y_1} \left\{ \frac{\psi_{t_1}(x_1 - z_1) - \psi_{t_1}(x_1 - y_1)}{z_1 - y_1} \right\}.$$

By applying (3.5) to the right-hand side, we obtain

$$\begin{aligned} &|\varphi_t(x-z) - \mathscr{P}^Q_{\varphi_t(x-\cdot)}(z)| \\ &\qquad \le c \frac{|Q_1(z_1)|}{|Q_1(x_1)|} \left(\frac{n_\psi}{|x_1 - a_1^{(i)}| + |z_1 - a_1^{(i)}|} + |\psi_{t_1}(x_1 - z_1)| \right) \end{aligned} \tag{3.6}$$

where

$$n_\psi = \sup_{\substack{z \in \mathbf{R}^1 \\ j_1 \le N_1}} (1 + |z|_1)^{j_1 + 1} |D_{z_1}^{j_1} \psi(z_1)|.$$

Of course, n_ψ depends on x_2, z_2 and t_2 since ψ does.

To estimate the right side of (3.6), first note from the definition of ψ and (3.5) that

$$|\psi(z_1)| \leq c\frac{|Q_2(z_2)|}{|Q_2(x_2)|}\left(\frac{\tilde{n}(z_1)}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|} + \left|\frac{1}{t_2}\varphi\left(z_1, \frac{x_2 - z_2}{t_2}\right)\right|\right),$$

$$\tilde{n}(z_1) = \sup_{\substack{z_2 \in \mathbf{R}^1 \\ j_2 \leq N_2}} (1 + |z_2|)^{j_2+1} |D_{z_2}^{j_2} \varphi(z_1, z_2)|.$$

Clearly, by definition of $A_{\varphi,M}$, we have $\tilde{n}(z_1) \leq A_{\varphi,M}(1 + |z_1|)^{-M}$, $M \geq 1$, uniformly in z_1. In particular, by dilation and translation,

(3.7) $|\psi_{t_1}(x_1 - z_1)|$

$$\leq c\frac{|Q_2(z_2)|}{|Q_2(x_2)|}\left(A_{\varphi,M}\frac{1}{t_1}\frac{1}{(1 + \frac{|x_1 - z_1|}{t_1})^M}\frac{1}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|} + |\varphi_{t_1,t_2}(x_1 - z_1, x_2 - z_2)|\right).$$

To estimate n_ψ, write $(1 + |z_1|)^{j_1+1}\psi^{(j_1)}(z_1)$ as

$$(1 + |z_1|)^{j_1+1} Q_2(z_2)\mathscr{D}_{y_2}^{Q_2}\left\{\frac{\frac{1}{t_2}D_{z_1}^{j_1}\varphi\left(z_1, \frac{x_2 - z_2}{t_2}\right) - \frac{1}{t_2}D_{z_1}^{j_1}\varphi\left(z_1, \frac{x_2 - z_2}{t_2}\right)}{z_2 - y_2}\right\}.$$

By (3.5), this is bounded in absolute value by

(3.8)

$$c\frac{|Q_2(z_2)|}{|Q_2(x_2)|}\left(\frac{\tilde{\tilde{n}}(z_1)}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|} + \frac{1}{t_2}(1 + |z_1|)^{j_1+1}\left|D_{z_1}^{j_1}\varphi\left(z_1, \frac{x_2 - z_2}{t_2}\right)\right|\right),$$

where

$$\tilde{\tilde{n}}(z_1) = \sup_{\substack{z_1 \in \mathbf{R}^1 \\ j_2 \leq N_2}} (1 + |z_2|)^{j_2+1}(1 + |z_1|)^{j_1+1}|D_{z_2}^{j_2}D_{z_1}^{j_1}\varphi(z_1, z_2)|.$$

Clearly, $\tilde{\tilde{n}}(z_1) \leq A_{\varphi,M}$ uniformly in z_1 by definition of $A_{\varphi,M}$. Similarly, the second term inside the square brackets in (3.8) is bounded by

$$A_{\varphi,M}\frac{1}{t_2}\frac{1}{\left(1 + \frac{|x_2 - z_2|}{t_2}\right)^M}.$$

Thus

$$\text{(3.9)} \qquad n_\psi \leq cA_{\varphi,M}\frac{|Q_2(z_2)|}{|Q_2(x_2)|}\left(\frac{1}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|} + \frac{1}{t_2}\frac{1}{\left(1 + \frac{|x_2 - z_2|}{t_2}\right)^M}\right).$$

The lemma now follows by combining (3.7), (3.8) and (3.9), and by using in (3.8) the fact that

$$|\varphi_{t_1,t_2}(x_1 - z_1, x_2 - z_2)| \leq A_{\varphi,M}\frac{1}{t_1}\frac{1}{\left(1 + \frac{|x_1 - z_1|}{t_1}\right)^M}\cdot\frac{1}{t_2}\frac{1}{\left(1 + \frac{|x_2 - z_2|}{t_2}\right)^M}.$$

4. Proof of Theorem 1

Let $(\xi, t) = (\xi_1, t_1; \xi_2, t_2) \in \Gamma(x) = \Gamma_1(x_1) \times \Gamma_2(x_2)$, and write

$$f(\xi, t) = \int f(z)[\varphi_t(\xi - z) - \mathscr{P}^Q_{\varphi_t(\xi-\cdot)}(z)]\,dz$$
$$= \sum_{i,j} \int_{E^{i,j}}$$

where $E^{i,j}$ is defined as in Lemma (3.4). We will consider each term of the sum separately. Fix i and j and write $a_1^{(i)} = a_1$, $a_2^{(j)} = a_2$ and $E^{i,j} = E$ for simplicity. Thus, for the rest of the argument, $z \in E$ and the estimates of Lemma (3.4) hold. With x_1, ξ_1, t_1 and x_2, ξ_2, t_2 fixed, let

$$\theta(z_1, z_2) = \varphi\left(\frac{\xi_1 - x_1}{t_1} + z_1, \frac{\xi_2 - x_2}{t_2} + z_2\right).$$

Note that $|\xi_1 - x_1|/t_1 < \gamma_1$ and $|\xi_2 - x_2|/t_2 < \gamma_2$, where γ_1 and γ_2 are the apertures of $\Gamma_1(x_1)$ and $\Gamma_2(x_2)$. Moreover, note that

$$\varphi_t(\xi - z) - \mathscr{P}^Q_{\varphi_t(\xi-\cdot)}(z) = \theta_t(x - z) - \mathscr{P}^Q_{\theta_t(x-\cdot)}(z).$$

Thus, by applying Lemma (3.4) to the right-hand side, we obtain

$$|\varphi_t(\xi - z) - \mathscr{P}^Q_{\varphi_t(\xi-\cdot)}(z)| \le cA_{\theta,M}\frac{|Q(z)|}{|Q(x)|}K(x, z, t)$$

where

$$K(x, z, t) = \left[\frac{1}{|x_1 - a_1| + |z_1 - a_1|} + \frac{1}{t_1}\frac{1}{\left(1 + \frac{|x_1 - z_1|}{t_1}\right)^M}\right]$$
$$\times \left[\frac{1}{|x_2 - a_2| + |z_2 - a_2|} + \frac{1}{t_2}\frac{1}{\left(1 + \frac{|x_2 - z_2|}{t_2}\right)^M}\right].$$

Note that in this estimate we have $A_{\theta,M}$ (which also depends on x, ξ, t) rather than $A_{\varphi,M}$. However, $A_{\theta,M} \le cA_{\varphi,M}$ uniformly in x, ξ, t if $(\xi, t) \in \Gamma(x)$; in fact,

$$A_{\theta,M} = \sup_{\substack{z_1, z_2 \\ j_1 \le N_1, j_2 \le N_2}} (1 + |z_1|)^{j_1+M}(1 + |z_2|)^{j_2+M}$$
$$\times \left|D^{j_1}_{z_1}D^{j_2}_{z_2}\varphi\left(\frac{\xi_1 - x_1}{t_1} + z_1, \frac{\xi_2 - x_2}{t_2} + z_2\right)\right|,$$

and since if $|\xi_i - x_i|/t_i \le \gamma_i$, $i = 1, 2$, we have

$$\frac{1}{1 + \gamma_i} \le \frac{1 + |z_i|}{1 + \left|\frac{\xi_i - x_i}{t_i} + z_i\right|} \le 1 + \gamma_i,$$

it follows easily that $A_{\theta,M} \le cA_{\varphi,M}$ uniformly.

In particular

$$\sup_{(\xi,t)\in\Gamma(x)}\int_E |f(z)||\varphi_t(\xi-z)-\mathscr{P}^Q_{\varphi_t(\xi-\cdot)}(z)|\,dz \le cA_{\varphi,M}\frac{1}{|Q(x)|}\sup_{t_1,t_2>0}\int |f(z)Q(z)|K(x,z,t)\,dz. \tag{4.1}$$

To prove the theorem, we must show that the $L^p_{|Q|^p w}$ norm of the expression of the left of (4.1) is bounded by a constant times the same norm of f. By (4.1), it is enough to show that (with $g=fQ$) the L^p_w norm of

$$\sup_{t_1,t_2>0}\int |g(z)|K(x,z,t)\,dz \tag{4.2}$$

is at most $c\|g\|_{L^p_w}$. For fixed $x=(x_1,x_2)$, we divide the domain of integration in (4.2) into the following four regions:

$$\begin{aligned}
&\text{I: } \{(z_1,z_2): |z_1-a_1|>|x_1-a_1|, |z_2-a_2|>|x_2-a_2|\},\\
&\text{II: } \{(z_1,z_2): |z_1-a_1|>|x_1-a_1|, |z_2-a_2|<|x_2-a_2|\},\\
&\text{III: } \{(z_1,z_2): |z_1-a_1|<|x_1-a_1|, |z_2-a_2|>|x_2-a_2|\},\\
&\text{IV: } \{(z_1,z_2): |z_1-a_1|<|x_1-a_1|, |z_2-a_2|<|x_2-a_2|\}.
\end{aligned}$$

For the part of (4.2) with integration extended over region I, we have the bound

$$\begin{aligned}
\sup_{t_1,t_2>0}&\iint_{\substack{|z_1-a_1|>|x_1-a_1|\\|z_2-a_2|>|x_2-a_2|}} |g(z_1,z_2)|\left[\frac{1}{|z_1-a_1|}+\frac{1}{t_1}\frac{1}{\left(1+\frac{|x_1-z_1|}{t_1}\right)^M}\right]\\
&\qquad\times\left[\frac{1}{|z_2-a_2|}+\frac{1}{t_2}\frac{1}{\left(1+\frac{|x_2-z_2|}{t_2}\right)^M}\right]dz_1\,dz_2\\
\le&\int_{|z_1-a_1|>|x_1-a_1|}\left(\int_{|z_2-a_2|>|x_2-a_2|}|g(z_1,z_2)|\frac{dz_2}{|z_2-a_2|}\right)\frac{dz_1}{|z_1-a_1|}\\
&+c\int_{|z_1-a_1|>|x_1-a_1|}M^{(2)}g(z_1,x_2)\frac{dz_1}{|z_1-a_1|}\\
&+c\int_{|z_2-a_2|>|x_2-a_2|}M^{(1)}g(x_1,z_2)\frac{dz_2}{|z_2-a_2|}+cM^{(1)}M^{(2)}g(x_1,x_2),
\end{aligned}$$

where $M^{(1)}, M^{(2)}$ and $M^{(1)}M^{(2)}$ are respectively the classical one-dimensional Hardy-Littlewood maximal operator in the first variable, the same operator in the second variable, and the iterated Hardy-Littlewood maximal operator. To obtain the last inequality, we have chosen the constant $M>1$ and used the standard majorization of an approximation of the identity by the Hardy-Littlewood maximal function. Each of the last four terms has L^p_w norm bounded by $c\|g\|_{L^p_w}$, $1<p<\infty$, by repeated use of Lemma (2.1), the principal result of [4] in the one-dimensional case, and the previously mentioned fact that since $w \in A_p$ for rectangles, $w \in A_p(\mathbf{R}^1)$ on almost every line parallel to either axis uniformly in the other variable.

The arguments for the parts of (4.2) with the integration extended over regions II, III and IV are similar. For II, the bound is

$$\sup_{t_1,t_2>0} \iint_{\substack{|z_1-a_1|>|x_1-a_1| \\ |z_2-a_2|<|x_2-a_2|}} |g(z_1,z_2)| \times \left[\frac{1}{|z_1-a_1|} + \frac{1}{t_1}\frac{1}{\left(1+\frac{|x_1-z_1|}{t_1}\right)^M}\right] \times \left[\frac{1}{|x_2-a_2|} + \frac{1}{t_2}\frac{1}{\left(1+\frac{|x_2-z_2|}{t_2}\right)^M}\right] dz_1\,dz_2$$

$$\begin{aligned} &\leq \frac{1}{|x_2-a_2|}\int_{|z_2-a_2|<|x_2-a_2|}\left[\int_{|z_1-a_1|>|x_1-a_1|} |g(z_1,z_2)|\frac{dz_1}{|z_1-a_1|}\right] dz_2 \\ &\quad + c\int_{|z_1-a_1|>|x_1-a_1|} M^{(2)}g(z_1,x_2)\frac{dz_1}{|z_1-a_1|} \\ &\quad + c\frac{1}{|x_2-a_2|}\int_{|z_2-a_2|<|x_2-a_2|} M^{(1)}g(x_1,z_2)\,dz_2 + cM^{(1)}M^{(2)}g(x_1,x_2). \end{aligned}$$

Note that the third term is actually less than the fourth. At any rate, each of the terms can again be treated by Hardy's inequalities (2.1) and (2.2), and [3].

Region III is similar to II by symmetry. Finally, for IV, the bound is

$$\begin{aligned} &\frac{1}{|x_1-a_1||x_2-a_2|}\iint_{\substack{|z_1-a_1|<|x_1-a_1| \\ |z_2-a_2|<|x_2-a_2|}} |g(z_1,z_2)|\,dz_1\,dz_2 \\ &\quad + c\frac{1}{|x_1-a_1|}\int_{|z_1-a_1|<|x_1-a_1|} M^{(2)}g(z_1,x_2)\,dz_1 \\ &\quad + c\frac{1}{|x_2-a_2|}\int_{|z_2-a_2|<|x_2-a_2|} M^{(1)}g(x_1,z_2)\,dz_2 + cM^{(1)}M^{(2)}g(x_1,x_2). \end{aligned}$$

Each of these is majorized by the last, and Theorem 1 follows.

5. Proof of Theorem 2

Let Q_1 and Q_2 be polynomials on $\mathbf{R}^1$ with all real roots, let $Q(x) = Q_1(x_1)Q_2(x_2)$, $x = (x_1,x_2)$, and $u(x) = |Q(x)|^p w(x)$ with $w \in A_p$ for rectangles. We begin the proof of Theorem 2 by noting that L^p_u is continuously embedded in H^p_u, $1 < p < \infty$, in the sense that if $f \in L^p_u$ and l_f is defined by

$$\langle l_f, \varphi\rangle = \int_{\mathbf{R}^2} f(x)[\varphi(x) - \mathscr{P}^Q_\varphi(x)]\,dx, \qquad \varphi \in \mathscr{S}(\mathbf{R}^2),$$

then l_f is a tempered distribution in H^p_u and $\|l_f\|_{H^p_u} \leq c\|f\|_{L^p_u}$. In fact, this is a corollary of the discussion following Lemma (3.1) and of Theorem 1.

To prove the other half of Theorem 2, we will show that if $l \in H^p_u$ there exists $f \in L^p_u$ such that $l = l_f$ and $\|f\|_{L^p_u} \leq c\|l\|_{H^p_u}$. For fixed $\psi \in \mathscr{S}(\mathbf{R}^2)$

with $\int \psi = 1$, let

$$l(x_1, s_1; x_2, s_2) = (l * \psi_{s_1, s_2})(x_1, x_2).$$

By hypothesis, $\sup_{s_1, s_2>0} |l(x_1 s_1; x_2, s_2)| \in L^p_u$. Thus, taking $s_1 = s_2 = s$, we see that $\|l(\cdot, s; \cdot, s)\|_{L^p_u}$ is uniformly bounded in $s > 0$ by $c\|l\|_{H^p_u}$. In particular, there is a sequence $s^{(k)} \to 0$ and a function $f \in L^p_u$ such that $l(\cdot, s^{(k)}; \cdot, s^{(k)})$ converges weakly in L^p_u to f, and $\|f\|_{L^p_u} \le c\|l\|_{H^p_u}$. Let l_f be the distribution induced by f, i.e.,

$$\langle l_f, \varphi\rangle = \int f(x)[\varphi(x) - \mathscr{P}^Q_\varphi(x)]\,dx, \qquad \varphi \in \mathscr{S}(\mathbf{R}^2),$$

and let $l^{(1)}(s_1, s_2)$ be the distribution induced by $l(x_1, s_1; x_2, s_2)$, i.e.,

$$\langle l^{(1)}(s_1, s_2), \varphi\rangle = \int l(x_1, s_1; x_2, s_2)[\varphi(x) - \mathscr{P}^Q_\varphi(x)]\,dx.$$

As shown by the discussion following Lemma (3.1), $\varphi - \mathscr{P}^Q_\varphi \in L^{p'}_{u^{-1/(p-1)}}$ (the dual space of L^p_u). It then follows from the weak convergence mentioned above that $l^{(1)}(s^{(k)}, s^{(k)}) \to l_f$ as distributions.

Now define $l^{(2)}(s_1, s_2) \in \mathscr{S}'(\mathbf{R}^2)$ by

$$\langle l^{(2)}(s_1, s_2), \varphi\rangle = \int l(x_1, s_1; x_2, s_2)\varphi(x)\,dx, \qquad \varphi \in \mathscr{S}(\mathbf{R}^2). \tag{5.1}$$

This is well-defined since $l(x_1, s_1; x_2, s_2)$ is a locally bounded function of (x_1, x_2) which is also in L^p_u (see also (5.3)). By well-known facts about distributions, $l^{(2)}(s_1, s_2) \to l$ as distributions when $s_1, s_2 \to 0$. Hence, in order to show that $l = l_f$, and thus complete the proof, it is enough to show that $l^{(1)}(s_1, s_2) = l^{(2)}(s_1, s_2)$. For $\varphi \in \mathscr{S}(\mathbf{R}^2)$, we obtain by subtracting the two formulas above that

$$\langle l^{(2)}(s_1, s_2) - l^{(1)}(s_1, s_2), \varphi\rangle = \int l(x_1, s_1; x_2, s_2)\mathscr{P}^Q_\varphi(x)\,dx. \tag{5.2}$$

Note that by definition

$$\begin{aligned}\mathscr{P}^Q_\varphi(x) &= Q_1(x_1)Q_2(x_2)\mathscr{D}^{Q_1}_{y_1}\mathscr{D}^{Q_2}_{y_2}\left[\frac{\varphi(y_1, x_2) + \varphi(x_1, y_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}\right] \\ &= P_{1,x_2}(x_1) + P_{2,x_1}(x_2) + P(x_1, x_2),\end{aligned}$$

where $P_{1,x_2}(x_1)$ is a polynomial in x_1 of degree $N_1 - 1$ whose coefficients are bounded functions of x_2, $P_{2,x_1}(x_2)$ is a polynomial in x_2 of degree $N_2 - 1$ whose coefficients are bounded functions of x_1, and $P(x_1, x_2)$ is a polynomial in x_1, x_2 of degree $(N_1 - 1)(N_2 - 1)$. We claim that

(i) the resulting integrals in (5.2) converge absolutely;

(ii) for $s_1, s_2 > 0$, $l(x_1, s_1; x_2, s_2)$ has one dimensional x_i-moments of order $\le N_i - 1$ equal to 0 for $i = 1, 2$.

Once these claims are established, it will follow immediately from Fubini's theorem and (5.2) that $l^{(2)}(s_1, s_2) = l^{(1)}(s_1, s_2)$, as desired, and the proof of Theorem 2 will be complete.

To prove claim (i), we will show that

$$\int |l(x_1, s_1; x_2, s_2)|(1+|x_1|)^{N_1-1}(1+|x_2|)^{N_2-1}\,dx < \infty. \tag{5.3}$$

By Hölder's inequality, the integral in (5.3) is at most

$$\left(\int |l(x_1, s_2; x_2, s_2)|^p(1+|x_1|)^{N_1 p}(1+|x_2|)^{N_2 p} w(x)\,dx\right)^{1/p} \times \left(\int \frac{w(x)^{-1/(p-1)}}{(1+|x_1|)^{p'}(1+|x_2|)^{p'}}\,dx\right)^{1/p'}. \tag{5.4}$$

The second factor is finite by the argument given after (3.2). To show that the first factor is finite, given $s_1, s_1 > 0$, let Ω_{s_1, s_2} be the complement of

$$(x_1, x_2):\begin{cases} |x_1 - a_1^{(i)}| < s_1/4 & \text{for some } i, \text{ or} \\ |x_2 - a_2^{(j)}| < s_2/4 & \text{for some } j; \end{cases}$$

i.e., Ω_{s_1, s_2} is the points not belonging to any of the strips of width $\frac{1}{2}s_1$ centered around the lines $x_1 = a_1^{(i)}$, or to any of the strips of width $\frac{1}{2}s_2$ centered around $x_2 = a_2^{(j)}$, where $\{a_1^{(i)}\}$ and $\{a_2^{(j)}\}$ are the zeros of Q_1 and Q_2, respectively. Let

$$R_{s_1, s_2}(x_1, x_2) = \{(z_1, z_2): |z_i - x_i| < s_i,\ i = 1, 2\}.$$

If s_1 and s_2 are small, there is a constant $c > 0$ such that for any (x_1, x_2),

$$|R_{s_1, s_2}(x_1, x_2) \cap \Omega_{s_1, s_2}| > c|R_{s_1, s_2}(x_1, x_2)|.$$

Also, since $(x_1, s_1; x_2, s_2)$ lies in the product cone at any point of $R_{s_1, s_2}(x_1, x_2)$, we have

$$|l(x_1, s_1; x_2, s_2)| \le N(l)(z_1, z_2) \quad \text{if } (z_1, z_2) \in R_{s_1, s_2}(x_1, x_2).$$

Moreover, for such (z_1, z_2),

$$\begin{aligned} 1 + |x_i| &\le 1 + |z_i| + |x_i - z_i| \le 1 + |z_i| + s_i \\ &\le c(1 + |z_i|), \qquad i = 1, 2. \end{aligned}$$

Hence,

$$|l(x_1, s_1; x_2, s_2)|(1+|x_1|)^{N_1 p}(1+|x_2|)^{N_2 p} \le c\frac{1}{w(R_{s_1, s_2}(x_1, x_2))} \times \iint_{R_{s_1, s_2}(x_1, x_2)\cap\Omega_{s_1, s_2}} N(l)(z_1, z_2)^p(1+|z_1|)^{N_1 p}(1+|z_2|)^{N_2 p} w(z_1, z_2)\,dz_1\,dz_2.$$

It follows by integration that the pth power of the first factor in (5.4) is at most

$$c\iint_{\Omega_{s_1,s_2}} N(l)(z)^p(1+|z_1|)^{N_1p}(1+|z_2|)^{N_2p}w(z) \times\left(\iint_{R_{s_1,s_2}(z_1,z_2)}\frac{w(x)}{w(R_{s_1,s_2}(x_1,x_2))}\,dx\right)dz$$
$$\le c\iint_{\Omega_{s_1,s_2}} N(l)(z)^p(1+|z_1|)^{N_1p}(1+|z_2|)^{N_2p}w(z)\,dz,$$

since for $(x_1,x_2)\in R_{s_1,s_2}(z_1,z_2)$, we have $w(R_{s_1,s_2}(x_1,x_2)\approx w(R_{s_1,s_2}(z_1,z_2))$ and therefore

$$\iint_{R_{s_1,s_2}(z_1,z_2)}\frac{w(x)}{w(R_{s_1,s_2}(x_1,x_2))}\,dx\approx 1.$$

Finally, if $(z_1,z_2)\in\Omega_{s_1,s_2}$, then

$$(1+|z_1|)^{N_1p}(1+|z_2|)^{N_2p}\le c_{s_1,s_2}|Q_1(z_1)|^p|Q_2(z_2)|^p = c_{s_1,s_2}|Q(z)|^p,$$

and consequently the integral above is at most

$$c_{s_1,s_2}\iint_{\Omega_{s_1,s_2}} N(l)(z)^p|Q(z)|^p w(z)\,dz\le c_{s_1,s_2}\|l\|^p_{H^p_{|Q|^pw}}.$$

In particular, (5.3) is true for s_1,s_2 small. The argument for $s_1,s_2>c>0$ is similar and simpler. This completes the proof of the first claim (i).

To prove the second claim (ii), let us first show that for fixed $s_1,s_2>0$, $l^{(2)}(s_1,s_2)\in H^p_{|Q|^pw}$ (see (5.1) for the definition of $l^{(2)}(s_1,s_2)$). Since $l\in H^p_{|Q|^pw}$ and

$$l^{(2)}(s_1,s_2)*\varphi_{t_1,t_2}=(l*\psi_{s_1,s_2})*\varphi_{t_1,t_2}=l*(\psi_{s_1,s_2}*\varphi_{t_1,t_2}),$$

it suffices to show that

$$\sup_{\substack{(\xi_1,t_1)\in\Gamma_1(x_1)\\(\xi_2,t_2)\in\Gamma_2(x_2)}}|l*(\psi_{s_1,s_2}*\varphi_{t_1,t_2})(\xi_1,\xi_2)|\le cl^*(x_1,x_2),$$

with c independent of (s_1,s_2), where l^* denotes the "grand" maximal function defined by

$$l^*(x_1,x_2)=\sup_{\substack{(\xi_i,r_i)\in\Gamma_i(x_i),\,i=1,2\\ \theta\in\mathscr{A}}}|(l*\theta_{r_1,r_2})(\xi_1,\xi_2)|,$$

$\mathscr{A}$ being the collection of Schwartz functions θ for which a sufficiently large number of Schwartz seminorms are bounded by a fixed constant A.

We will show that

$$\psi_{s_1,s_2}*\varphi_{t_1,t_2}=\theta_{u_1,u_2},\quad u_i=\max\{s_i,t_i\},\qquad i=1,2,$$

where $\theta\in\mathscr{A}$ (although θ may depend on t_i,s_i). This will suffice since $(y_i,t_i)\in\Gamma_i(x_i)$ implies that $(y_i,u_i)\in\Gamma_i(x_i)$. By checking Fourier transforms, it is easy to see that

$$\psi_{s_1,s_2}*\varphi_{t_1,t_2}=(\psi_{s_1/u_1,s_2/u_2}*\varphi_{t_1/u_1,t_2/u_2})_{u_1,u_2}=\theta_{u_1,u_2},$$

where

$$\theta = \psi_{\bar{s}_1,\bar{s}_2} * \varphi_{\bar{t}_1,\bar{t}_2}, \quad \bar{s}_i = s_i/u_i, \quad \bar{t}_i = t_i/u_i, \qquad i = 1, 2.$$

Note that $\max\{\bar{s}_i, \bar{t}_i\} = 1$ for $i = 1, 2$. Thus, letting $|||\cdot|||_m$ denote the mth Schwartz norm, i.e,

$$|||\theta|||_m = \sup_{\substack{(x_1,x_2) \\ 0\le j,k\le m}} (1+|x_1|)^m(1+|x_2|)^m \left| \left(\frac{\partial}{\partial x_1}\right)^j \left(\frac{\partial}{\partial x_2}\right)^k \theta(x_1, x_2)\right|,$$

it is enough to show that

$$|||\psi_{s_1,s_2} * \varphi_{t_1,t_2}|||_m \le A \quad \text{when } \max\{s_i, t_i\} = 1, \; i = 1, 2,$$

with A independent of s_i, t_i.

Letting $r_i = \min\{s_i, t_i\}$, $i = 1, 2$, we see that there are four cases:
(1) $r_1 = t_1 \le s_1 = 1$ and $r_2 = t_2 \le s_2 = 1$;
(2) $r_1 = t_1 \le s_1 = 1$ and $r_2 = s_2 \le t_2 = 1$;
(3) $r_1 = s_1 \le t_1 = 1$ and $r_2 = t_2 \le s_2 = 1$;
(4) $r_1 = s_1 \le t_1 = 1$ and $r_2 = s_2 \le t_2 = 1$.
Since (1) and (4) are similar and (2) and (3) are similar, we shall consider only (1) and (2). In case (1),

$$(\psi * \varphi_{r_1,r_2})(x_1, x_2) = \int \psi(x_1 - \xi_1, x_2 - \xi_2)\varphi_{r_1,r_2}(\xi_1, \xi_2)\, d\xi_1 d\xi_2,$$

and since $|||\psi(\cdot - \xi_1, \cdot - \xi_2)|||_m \le |||\psi|||_m (1+|\xi_1|)^m(1+|\xi_2|)^m$, we have

$$|||\psi * \varphi_{r_1,r_2}|||_m \le |||\psi|||_m \int (1+|\xi_1|)^m(1+|\xi_2|)^m |\varphi_{r_1,r_2}(\xi_1, \xi_2)|\, d\xi_1 d\xi_2.$$

Changing variables in the integral and using $0 < r_1, r_2 \le 1$ we obtain $|||\psi * \varphi_{r_1,r_2}|||_m \le c|||\psi|||_m$ with $c = c_{\varphi,m}$ independent of r_1, r_2. In case 2, write

$$(\psi_{1,r_2} * \varphi_{r_1,1})(x_1, x_2) = \int \psi_{1,r_2}(x_1 - \xi_1, \xi_2)\varphi_{r_1,1}(\xi_1, x_2 - \xi_2)\, d\xi_1 d\xi_2.$$

Thus,

$$|||\psi_{1,r_2} * \varphi_{r_1,1}|||_m \le \int |||\psi_{1,r_2}(\cdot - \xi_1, \xi_2)|||_{m,\mathbf{R}^1} |||\varphi_{r_1,1}(\xi_1, \cdot - \xi_2)|||_{m,\mathbf{R}^1}\, d\xi_1 d\xi_2,$$

where $|||\cdot|||_{m,\mathbf{R}^1}$ denotes the mth Schwartz seminorm of a function of one variable. In particular, for any N, there is a constant c independent of r_1 and r_2 such that the last integral is bounded by

$$\begin{aligned} &c|||\psi|||_m|||\varphi|||_m \\ &\quad\times \int \left[(1+|\xi_1|)^m \frac{1}{r_2}\left(1 + \frac{|\xi_2|}{r_2}\right)^{-N}\right) \left[(1+|\xi_2|)^m \frac{1}{r_1}\left(1 + \frac{|\xi_1|}{r_1}\right)^{-N}\right] d\xi_1 d\xi_2 \\ &\le c|||\psi|||_m|||\varphi|||_m \end{aligned}$$

by changing variables, using $0 < r_1, r_2 \le 1$ and choosing $N \ge m + 2$. This shows that $l^{(2)}(s_1, s_2) \in H^p_{|Q|^p w}$ for each $s_1, s_2 > 0$.

It follows that the one-dimensional distribution defined by

$$\langle l_1, \varphi\rangle = \int l(x_1, s_1; x_2, s_2)\varphi(x_1)\,dx_1, \qquad \varphi \in \mathscr{S}(\mathbf{R}^1), \tag{5.5}$$

belongs to $H^p_{|Q_1|^p w(\cdot, x_2)}(\mathbf{R}^1)$ for each $s_1, s_2 > 0$ and a.e. x_2, as we now show. Pick $\varphi \in \mathscr{S}(\mathbf{R}^1)$ with $\varphi \geq 0$ and let $\overline{\varphi}(x_1, x_2) = \varphi(x_1)\varphi(x_2)$. Then $\overline{\varphi} \in \mathscr{S}(\mathbf{R}^2)$ and since $l^{(2)}(s_1, s_2) \in H^p_{|Q|^p w}$, we have $N_{\overline{\varphi}}(l^{(2)}(s_1, s_2)) \in L^p_{|Q|^p w}$, i.e.,

$$\sup_{(\xi_1, t_1)\in\Gamma_1(x_1)} \sup_{(\xi_2, t_2)\in\Gamma_2(x_2)} \times \left|\int \left(\int l(z_1, s_1; z_2, s_2)\varphi_{t_1}(\xi_1 - z_1)\,dz_1\right)\varphi_{t_2}(\xi_2 - z_2)\,dz_2\right| \in L^p_{|Q_1 Q_1|^p w}(\mathbf{R}^2).$$

However, this double supremum exceeds a positive constant times

$$\sup_{(\xi_1, t_1)\in\Gamma_1(x_1)} \left|\int l(z_1, s_1; x_2, s_2)\varphi_{t_1}(\xi_1 - z_1)\,dz_1\right|.$$

Thus, by Fubini's theorem, the last expression belongs to $L^p_{|Q_1|^p w(\cdot, x_2)}(\mathbf{R}^1)$ for a.e. x_2, as desired. A similar fact holds for the other variable.

We can now complete the proof of the second claim (ii) made earlier. By above, the distribution l_1 defined in (5.5) belongs to $H^p_{|Q_1|^p w(\cdot, x_2)}(\mathbf{R}^1)$ for a.e. x_2. Also, since $l(\cdot, s_1; x_2, s_2) \in L^p_{|Q_1|^p w(\cdot, x_2)}(\mathbf{R}^1)$ for a.e. x_2, we have from the one-dimensional theory in [7] that the distribution defined by

$$\langle l_2, \varphi\rangle = \int l(x_1, s_1; x_2, s_2)[\varphi(x_1) - \mathscr{P}^{Q_1}_{\varphi}(x_1)]\,dx_1, \qquad \varphi \in \mathscr{S}(\mathbf{R}^1),$$

belongs to $H^p_{|Q_1|^p w(\cdot, x_2)}(\mathbf{R}^1)$, and therefore, so does $l_1 - l_2$. Since

$$\langle l_1 - l_2, \varphi\rangle = \int l(x_1, s_1; x_2, s_2)\mathscr{P}^{Q_1}_{\varphi}(x_1)\,dx_1,$$

we see that $l_1 - l_2$ has compact support, and it follows from Lemma (2.3) that $l_1 - l_2 = 0$. Now choosing $\varphi_k \in \mathscr{S}(\mathbf{R}^1)$ such that $\mathscr{P}^{Q_1}_{\varphi_k}(x_1) = x_1^k$ for each $k = 0, \ldots, N_1 - 1$ (e.g., let $\varphi_k(x_1) = x_1^k \rho(x_1)$ where $\rho \in \mathscr{S}(\mathbf{R}^1)$ and $\rho = 1$ on a set containing all the roots of $Q_1(x_1)$), we obtain

$$\int l(x_1, s_1; x_2, s_2)x_1^k\,dx_1 = 0, \qquad k = 0, 1, \ldots, N_1 - 1,$$

for a.e. x_2, as desired. A similar fact holds for the variable x_2. This completes the proof of claim (ii), and so also of Theorem 2.

6. Proof of Theorem 3

Let f and φ satisfy the hypothesis of Theorem 3, and let

$$f(\xi, t) = \int f(z)[\varphi_t(\xi - z) - \mathscr{P}^{Q}_{\varphi_t(\xi - \cdot)}(z)]\,dz,$$

$\xi = (\xi_1, \xi_2)$, $t = (t_1, t_2)$. Suppose first that φ has compact support. Note that if $z = (z_1, z_2)$ then

$$\mathscr{P}^{Q}_{\varphi_t(\xi-\cdot)}(z) = Q_1(z_1)Q_2(z_2)\mathscr{D}^{Q_1}_{y_1}\mathscr{D}^{Q_2}_{y_2} \times \left\{ \frac{\varphi_{t_1,t_2}(\xi_1 - y_1, \xi_2 - z_2) - \varphi_{t_1,t_2}(\xi_1 - z_1, \xi_2 - y_2) - \varphi_{t_1,t_2}(\xi_1 - y_1, \xi_2 - y_2)}{(z_1 - y_1)(z_2 - y_2)} \right\}.$$

If $x = (x_1, x_2)$ does not lie on any line $x_1 = a_1$ or $x_2 = a_2$, where a_1 and a_2 denote roots of Q_1 and Q_2 respectively, then $|\xi_i - a_i|/t_i \to \infty$ as $(\xi_i, t_i) \to x_i, i = 1, 2$, for each root a_i. Therefore, since φ has compact support, we see that $\mathscr{P}^{Q}_{\varphi_t(\xi-\cdot)}(z) = 0$ for (ξ_i, t_i) close to x_i, $i = 1, 2$, for such x (in fact, each of the three terms of $\mathscr{P}$ is zero), and consequently,

$$f(\xi, t) = \int f(z)\varphi_t(\xi - z)\,dz$$

for such x and (ξ, t). This expression converges nontangentially in the product sense to $f(x)\int\varphi$ a.e. by standard facts about "strong" differentiability since $\exists q > 1$ so that f is locally in L^q away from the lines $x_i = a_i$: in fact, away from these lines $|Q_1|$ and $|Q_2|$ are bounded below away from zero, so that $f \in L^p_w$ there; thus, f is locally in L^q there for some $q > 1$ since $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$.

In case φ does not have compact support, let $r > 0$ and write

$$\varphi(x) = \rho(rx)\varphi(x) + (1 - \rho(rx))\varphi(x) = \tilde{\varphi}(x) + \tilde{\tilde{\varphi}}(x),$$

where ρ is a smooth truncation with $\rho(x) = 1$ for $|x| < 1$, $\rho(x) = 0$ for $|x| > 2$, and $\rho \in C^\infty$. For fixed $r, \tilde{\varphi}$ has compact support and the corresponding extension $\tilde{f}(\xi, t)$ converges nontangentially to $f(x)\int\tilde{\varphi}$ a.e. by the previous case. Moreover, as $r \to 0$ we see that $\int\tilde{\varphi} \to \int\varphi$ and that the constant $A_{\tilde{\tilde{\varphi}},M}$ defined in Lemma (3.4) tends to zero. Hence, to prove the theorem, it is enough to show that for $M > 1$ and a.e. x, there is a finite number $c_{x,f,M}$ such that $N(f)(x) \le c_{x,f,M}A_{\varphi,M}$. However, from the proof of Theorem 1, $N(f)(x)$ is bounded by a sum of terms of the type in (4.1), one term for each pair a_1, a_2 of roots. Note that $A_{\varphi,M}$ is a factor on the right in (4.1). Moreover, as shown in the argument following (4.1), the remaining factor in (4.1) belongs to L^p_u if $M > 1$; in particular, this factor is finite a.e., and the proof is complete.

7. Proof of Theorem 4

Let $f \in L^p_u$ and let $f(x, t)$ be the extension formed by using a convolver φ with $\int\varphi = 1$. By the proof of Theorem 2 (see (ii) in §5), $f(x, t)$ has one dimensional x_i-moments of order $\le N_i - 1$ equal to zero for $i = 1, 2$. Moreover, $f(x, t) \in L^1$ as a function of x by (5.3). Also, by Theorem 3, $f(x, t)$ converges pointwise a.e. to $f(x)$ as $t \to 0$, and since $\sup_{t>0}|f(x, t)| \in L^p_u$ by Theorem 1, we see from the dominated convergence theorem that $f(x, t) \to f(x)$ in L^p_u

as $t \to 0$. Thus, we may assume from the start that $f \in L_u^p \cap L^1$ and that f satisfies

$$\int_{-\infty}^{\infty} f(x_1, x_2) x_1^{k_1}\, dx_1 = \int_{-\infty}^{\infty} f(x_1, x_2) x_2^{k_2}\, dx_2 = \iint_{R^2} f(x_1, x_2) x_1^{k_1} x_2^{k_2}\, dx_1\, dx_2 = 0$$

for $k_1 = 0, 1, \dots, N_1$ and $k_2 = 0, 1, \dots, N_2$.

Let us show that if $t = (t_1, t_2)$ and if either t_1 or $t_2 \to \infty$, then $f(x, t) \to 0$ in L_u^p. By definition of $N(f)$, we have

$$|f(x,t)| \le \left(\frac{1}{\iint_{\substack{|z_1 - x_1| \le t_1 \\ |z_2 - x_2| < t_2}} u\, dz_1 dz_2} \iint_{\substack{|z_1 - x_1| < t_1 \\ |z_2 - x_2| < t_2}} N(f)^p u\, dz \right)^{1/p} \le \frac{c}{\left(\iint_{\substack{|z_1 - x_1| < t_1 \\ |z_2 - x_2| < t_2}} u\, dz_1 dz_2 \right)^{1/p}} \|f\|_{L_u^p} \to 0$$

as either t_1 or $t_2 \to \infty$ since u satisfies the doubling condition in each variable. This shows pointwise convergence; norm convergence follows from the dominated convergence theorem since $\sup_{t_1, t_2 > 0} |f(x, t)| \le N(f)(x)$.

Now suppose in addition that the convolution function φ is a product $\varphi_1(x_1)\varphi_2(x_2)$ of one-dimensional functions with $\hat{\varphi}_i(x_i)$ compactly supported and equal to 1 near $x_i = 0$. Let us write $f(x, t) = F_{t_1, t_2}(x_1, x_2)$. Note that due to the moment conditions on f,

$$F_{t_1, t_2}(x_1, x_2) = \iint f(z_1, z_2) \varphi_{t_1, t_2}(x_1 - z_1, x_2 - z_2)\, dz_1 dz_2,$$

and therefore, since $f \in L^1$,

$$\widehat{F}_{t_1, t_2}(x_1, x_2) = \hat{f}(x_1, x_2) \hat{\varphi}_1(t_1 x_1) \hat{\varphi}_2(t_2 x_2).$$

Now, for t_1, t_2 small and T_1, T_2 large, let

$$G = F_{t_1, t_2} - F_{t_1, T_2} - F_{T_1, t_2} + F_{T_1, T_2}.$$

Note that

$$\widehat{G}(x_1, x_2) = \hat{f}(x_1, x_2)[\hat{\varphi}_1(t_1 x_1)\hat{\varphi}_2(t_2 x_2) - \hat{\varphi}_1(t_1 x_1)\hat{\varphi}_2(T_2 x_2) - \hat{\varphi}_1(T_1 x_1)\hat{\varphi}_2(t_2 x_2) + \hat{\varphi}_1(T_1 x_1)\hat{\varphi}_2(T_2 x_2)].$$

Since $\hat{\varphi}_i = 1$ near $x_i = 0$ and $\hat{\varphi}_i$ has compact support, it follows that $\widehat{G}$ vanishes near both axes and that $\widehat{G}$ has compact support. Also,

$$\|G - f\|_{L_u^p} \le \|F_{t_1, t_2} - f\|_{L_u^p} + \|F_{t_1, T_2}\|_{L_u^p} + \|F_{T_1, t_2}\|_{L_u^p} + \|F_{T_1, T_2}\|_{L_u^p}.$$

The first term on the right is small if t_1, t_2 are small. With t_1, t_2 fixed, each of the remaining three terms on the right is small if T_1, T_2 are large. Thus, G

has all the required properties except that G and its derivatives may not decay at ∞. Note that $G \in C^\infty$. Let $\hat{\psi} \in C_0^\infty$ and $\psi(0,0) = 1$. Define H_t by

$$\widehat{H}_t = \widehat{G} * t^{-2}\hat{\psi}\left(\frac{\cdot}{t}, \frac{\cdot}{t}\right).$$

Then $\widehat{H}_t$ has compact support and vanishes near the axes when t is small. Also, $\widehat{H}_t \in C^\infty$, so that $\widehat{H}_t \in \mathscr{S}$ and, consequently, $H_t \in \mathscr{S}$. Finally, since

$$H_t(x_1, x_2) = G(x_1, x_2)\psi(tx_1, tx_2)$$

and ψ is bounded and $\psi(tx_1, tx_2) \to \psi(0,0) = 1$ as $t \to 0$, it follows that $H_t \to G$ in L_u^p. This completes the proof of Theorem 4.

8. Proof of Theorem 5

To prove the first half of Theorem 5, suppose that $f \in L_u^p$ with $u = (1+|x_1|)^{d_1 p}(1+|x_2|)^{d_2 p}|Q|^p w$, $Q = Q_1(x_1)Q_2(x_2)$ and $w \in A_p$ for rectangles. Suppose also that f satisfies

$$\int_{-\infty}^{\infty} f(x_1, x_2)Q_1(x_1)x_1^{k_1}\,dx_1 = \int_{-\infty}^{\infty} f(x_1, x_2)Q_2(x_2)x_2^{k_2}\,dx_2 = 0 \tag{8.1}$$

for a.e. x_2 and a.e. x_1, respectively, and for $k_1 = 0, 1, \dots, d_1 - 1$ and $k_2 = 0, 1, \dots, d_2 - 1$. We wish to show that the distribution l defined by

$$\langle l, \varphi\rangle = \int f(z)[\varphi(z) - \mathscr{P}_\varphi^Q(z)]\,dz, \qquad \varphi \in \mathscr{S}(\mathbf{R}^2),$$

satisfies $l \in H_u^p$ with $\|l\|_{H_u^p} \le c\|f\|_{L_u^p}$.

Let us first show that the integrals in (8.1) are well-defined a.e. It is easy to see that

$$\iint |f(x_1, x_2)||Q_1(x_1)||Q_2(x_2)|(1+|x_1|)^{d_1-1}(1+|x_2|)^{d_2-1}\,dx_1dx_2 < \infty \tag{8.2}$$

if $f \in L_u^p$ since by Hölder's inequality the integral is at most

$$\|f\|_{L_u^p}\left(\iint \frac{w^{-1/(p-1)}}{(1+|x_1|)^{p'}(1+|x_2|)^{p'}}\,dx_1dx_2\right)^{1/p'},$$

which is finite by (3.2). It then follows from Fubini's theorem that the integrals in (8.1) converge absolutely a.e.

Since

$$(1+|x_1|)^{d_1 p}(1+|x_2|)^{d_2 p} \approx 1 + |x_1|^{d_1 p} + |x_2|^{d_2 p} + |x_1^{d_1}x_2^{d_2}|^p,$$

the fact that $f \in L_u^p$ implies that f also belongs to each of $L_{u_j}^p$, $j = 1, 2, 3, 4$, where $u_1 = |Q_1Q_2|^p w$, $u_2 = |x_1^{d_1}Q_1|^p|Q_2|^p w$, $u_3 = |Q_1|^p|x_2^{d_2}Q_2|^p w$ and $u_4 = |x_1^{d_1}Q_1|^p|x_2^{d_2}Q_2|^p w$. Moreover,

$$\|f\|_{L_u^p} \approx \sum_{j=1}^4 \|f\|_{L_{u_j}^p}.$$

From Theorem 2, f defines four distributions l_j, $j = 1, 2, 3, 4$, by

$$\begin{aligned}\langle l_1, \varphi\rangle &= \int f[\varphi - \mathscr{P}_\varphi^{Q_1, Q_2}]\,dx,\\ \langle l_2, \varphi\rangle &= \int f[\varphi - \mathscr{P}_\varphi^{x_1^{d_1}Q_1, Q_2}]\,dx,\\ \langle l_3, \varphi\rangle &= \int f[\varphi - \mathscr{P}_\varphi^{Q_1, x_2^{d_2}Q_2}]\,dx,\\ \langle l_4, \varphi\rangle &= \int f[\varphi - \mathscr{P}_\varphi^{x_1^{d_1}Q_1, x_2^{d_2}Q_2}]\,dx,\end{aligned}$$

and $\|l_j\|_{H_{u_j}^p} \le c\|f\|_{L_{u_j}^p}$ for each j. In particular, $\|l_j\|_{H_{u_j}^p} \le c\|f\|_{L_u^p}$ for each j. We claim that $l_1 = l_2 = l_3 = l_4$. Taking this momentarily for granted and calling the common value l, it follows that $l \in H_u^p$ and $\|l\|_{H_u^p} \approx \sum \|l\|_{H_{u_j}^p} \le c\|f\|_{L_u^p}$, as desired.

To verify the claim, we will show that $l_1 = l_4$; the proofs that $l_1 = l_2$ and $l_1 = l_3$ are similar. We have

$$\begin{aligned}\langle l_1, \varphi\rangle - \langle l_4, \varphi\rangle &= \int f[\mathscr{P}_\varphi^{x_1^{d_1}Q_1, x_2^{d_2}Q_2} - \mathscr{P}_\varphi^{Q_1, Q_2}]\,dx\\ &= \iint fQ_1Q_2\{P_{d_1-1}(x_1)G_{d_2-1}(x_2) + P_{d_2-1}(x_2)G_{d_1-1}(x_1)\}\,dx_1\,dx_2\end{aligned}$$

by Lemma (2.7), where P_{d_1-1} and P_{d_2-1} are polynomials of degrees $d_1 - 1$ and $d_2 - 1$, and

$$\begin{aligned}|G_{d_1-1}(x_1)| &\le c(1 + |x_1|)^{d_1-1},\\ |G_{d_2-1}(x_2)| &\le c(1 + |x_2|)^{d_2-1}.\end{aligned}$$

The last integral converges absolutely by (8.2), and we may rewrite the integral by using Fubini's theorem as

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x_1, x_2)Q_1(x_1)P_{d_1-1}(x_1)\,dx_1\right)Q_2(x_2)G_{d_2-1}(x_2)\,dx_2\\ &\quad + \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x_1, x_2)Q_2(x_2)P_{d_2-1}(x_2)\,dx_2\right)Q_1(x_1)G_{d_1-1}(x_1)\,dx_1.\end{aligned}$$

The inner integrals in both these terms vanish for a.e. x_2 and for a.e. x_1, respectively, by (8.1), and the claim follows. This completes half of the proof of Theorem 5.

To prove the other half, let l be any element of H_u^p. Then, with u_j defined as above, $l \in \bigcap_{j=1}^4 H_{u_j}^p$. By Theorem 2, there exist $f_j \in L_{u_j}^p$ for each j such that

$$\begin{aligned}\langle l,\varphi\rangle &= \int f_1[\varphi-\mathscr{P}_\varphi^{Q_1,Q_2}]\,dx\\ &= \int f_2[\varphi-\mathscr{P}_\varphi^{x_1^{d_1}Q_1,Q_2}]\,dx\\ &= \int f_3[\varphi-\mathscr{P}_\varphi^{Q_1,x_2^{d_2}Q_2}]\,dx\\ &= \int f_4[\varphi-\mathscr{P}_\varphi^{x_1^{d_1}Q_1,x_2^{d_2}Q_2}]\,dx\end{aligned}$$

and $\|f_j\|_{L^p_{u_j}} \le c\|l\|_{H^p_{u_j}} \le c\|l\|_{H^p_u}$ for each j. Choosing φ to be supported away from the lines corresponding to zeros of $x_i^{d_i}Q_i$, $i=1,2$, we see that

$$\mathscr{P}_\varphi^{Q_1,Q_2} = \mathscr{P}_\varphi^{x_1^{d_1}Q_1,Q_2} = \mathscr{P}_\varphi^{Q_1,x_2^{d_2}Q_2} = \mathscr{P}_\varphi^{x_1^{d_1}Q_1,x_2^{d_2}Q_2} = 0$$

and that

$$\int f_1\varphi = \int f_2\varphi = \int f_3\varphi = \int f_4\varphi.$$

Therefore, $f_1=f_2=f_3=f_4$ a.e. If we call this common value f, we see that

$$\|f\|_{L^p_u} \le c\sum_1^4 \|f\|_{L^p_{u_j}} \le c\|l\|_{H^p_u}.$$

It remains to show that f satisfies the moment conditions (8.1). Pick $\varphi(x_1,x_2)$ to be a product of one-dimensional functions:

$$\varphi(x_1,x_2)=\varphi_1(x_1)\varphi_2(x_2).$$

In this case, by (1.2),

$$\varphi-\mathscr{P}_\varphi^{Q_1,Q_2} = [\varphi_1(x_1)-\mathscr{P}_{\varphi_1}^{Q_1}(x_1)][\varphi_2(x_2)-\mathscr{P}_{\varphi_2}^{Q_2}(x_2)],$$

and similar formulas hold for $\varphi-\mathscr{P}_\varphi^{x_1^{d_1}Q_1,Q_2}$, etc. Thus, by subtracting the first and second representations of $\langle l,\varphi\rangle$ above, and also the first and third representations, we obtain both

$$\iint f(x_1,x_2)[\mathscr{P}_{\varphi_1}^{x_1^{d_1}Q_1}(x_1)-\mathscr{P}_{\varphi_1}^{Q_1}(x_1)][\varphi_2(x_2)-\mathscr{P}_{\varphi_2}^{Q_2}(x_2)]\,dx_1\,dx_2=0$$

and

$$\iint f(x_1,x_2)[\varphi_1(x_1)-\mathscr{P}_{\varphi_1}^{Q_1}(x_1)][\mathscr{P}_{\varphi_2}^{x_2^{d_2}Q_2}(x_2)-\mathscr{P}_{\varphi_2}^{Q_2}(x_2)]\,dx_1\,dx_2=0.$$

Consider the first of these with $\varphi_1(x_1)=x_1^{k_1}Q_1(x_1)\rho(x_1)$ for $k_1=0,1,\dots,d_1-1$ and $\rho\in C_0^\infty$ with $\rho=1$ near the zeros of $x_1^{k_1}Q_1(x_1)$. Then, by Lemma (2.5) of [7],

$$\mathscr{P}_{\varphi_1}^{x_1^{d_1}Q_1} = \mathscr{P}_{x_1^{k_1}Q_1}^{x_1^{d_1}Q_1} = x_1^{k_1}Q_1 \quad\text{and}\quad \mathscr{P}_{\varphi_1}^{Q_1} = \mathscr{P}_{x_1^{k_1}Q_1}^{Q_1}=0.$$

Consequently,

$$\iint f(x_1, x_2) x_1^{k_1} Q_1(x_1)[\varphi_2(x_2) - \mathscr{P}_{\varphi_2}^{Q_2}(x_2)]\, dx_1\, dx_2 = 0.$$

Since φ_2 is arbitrary and $\mathscr{P}_{\varphi_2}^{Q_2}(x_2) = 0$ if φ_2 is supported away from the zeros of Q_2, it follows that

$$\int_{-\infty}^{\infty} f(x_1, x_2) x_1^{k_1} Q_1(x_1)\, dx_1 = 0 \quad \text{a.e. } x_2.$$

Similarly,

$$\int_{-\infty}^{\infty} f(x_1, x_2) x_2^{k_2} Q_2(x_2)\, dx_2 = 0 \quad \text{a.e. } x_1$$

if $k_2 = 0, 1, \dots, d_2 - 1$. This completes the proof of Theorem 5.

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