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PIXLEY-ROY HYPERSPACES OF ω-GRAPHS

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ABSTRACT. The techniques developed by Wage and Norden are used to show that the Pixley-Roy hyperspaces of any two ω-graphs are homeomorphic. The Pixley-Roy hyperspaces of several subsets of $\mathbf{R}^n$ are also shown to be homeomorphic.

I. INTRODUCTION

Since it was introduced in 1969, the Pixley-Roy hyperspace, $\mathrm{PR}[X]$, of a topological space X has been intensely studied with the hope of establishing how the properties of X affect those of $\mathrm{PR}[X]$. This study has met with some success, especially in the area of cardinal functions. However, there is a class of questions which, until recently, eluded investigators: For which spaces X and Y will $\mathrm{PR}[X]$ be homeomorphic to $\mathrm{PR}[Y]$? For several years the only results in this area were some embedding results obtained by van Douwen [vD] and Lutzer [L]. In 1985 Wage [W] achieved a breakthrough by developing a technique for breaking up neighborhoods around points in certain spaces which allowed him to define homeomorphisms between those neighborhoods. Using this technique he was able to show that Pixley-Roy hyperspaces of spaces like R or $[0,1]$ are homogeneous. In 1986 Norden [N] extended Wage's technique to one which broke up an entire space. With this he was able to show that the Pixley-Roy hyperspaces of any two P-graphs (one-dimensional polyhedra with a finite number of points removed) are homeomorphic. It follows that the Pixley-Roy hyperspaces of spaces like R, $[0,1]$, and the circle are all homeomorphic. It is the purpose of this paper to use Norden's technique to show that Pixley-Roy hyperspaces of infinite, as well as finite, graphs are all the same.

Definition. A T_2 space X with no isolated points is an ω-graph if there is a countable discrete subset D of X and a countable collection $\mathbf{I}$ of pairwise disjoint copies of $(0,1)$ such that $X\backslash D = \bigcup \mathbf{I}$, $\mathbf{I}$ is locally finite on X, and for every $x \in D$, $\{x\} \cup (\bigcup\{I \in \mathbf{I} \colon x \in \overline{I}\})$ is a neighborhood of x which can be embedded in $\mathbf{R}^2$. The set D is called a dividing set for X.

The main result of this paper can be stated as follows.

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Theorem 1. *If X and Y are ω-graphs then* $\mathrm{PR}[X]$ *is homeomorphic to* $\mathrm{PR}[Y]$.

§II will consist of preliminary definitions, notation, and observations necessary for the proof of the Theorem 1. Theorem 1 will be proved in §III, and §IV will contain some related results.

We will use $\mathrm{PR}[X]$ to denote the Pixley-Roy hyperspace of X. Our notation for the open subsets of $\mathrm{PR}[X]$ will be standard. We will use $F[A]$ to denote the set of nonempty finite subsets of a set A, and $F'[A]$ to denote the set of all finite subsets of A. The notation "$X \approx Y$" will mean that X is homeomorphic to Y.

II. Preliminary matters

Let X be an ω-graph and let X_0 be a dividing set for X. Enumerate X_0 as $\{x_n : n < \omega\}$. Let $\mathbf{I}_0$ be the countable collection of pairwise disjoint copies of $(0, 1)$ whose union makes up $X \backslash X_0$. We may assume that every element of I_0 has at least one endpoint in X_0. For each $n < \omega$ let $\mu(n)$ be the number of elements of $X \backslash X_0$ having x_n as an endpoint. For each $I \in \mathbf{I}_0$, fix a linear structure and orientation for I. Let Q_0 be the set of all midpoints of elements of $\mathbf{I}_0$ and, for each $p \in X_0$, let O_p be the component of $X \backslash Q_0$ containing p. Then Q_0 is a discrete subset of X and $O_p \cap O_q = \varnothing$ if $p \neq q$.

For each $p \in X_0$ and each $I \in \mathbf{I}_0$ having p as an endpoint, choose a sequence of points in $I \cap O_p$ converging monotonically to p. This can be done because each element of X_0 is the endpoint of at least one element of $\mathbf{I}_0$. Let Q_1 be the set of all points of X which are elements either of Q_0 or of the sequences just chosen. Call Q_1 the 1st cut-set of X. Set $\widehat{Q}_1 = Q_1$. Let $\mathbf{I}_1$ be the countable collection of pairwise disjoint copies of $(0, 1)$ whose union makes up $X \backslash (\widehat{Q}_1 \cup X_0)$. Call I_1 the set of intervals in X derived from $\widehat{Q}_1$.

Assume that $n < \omega$, that Q_n is a discrete subset of $X \setminus X_0$, and that $\mathbf{I}_n$ is a countable collection of pairwise disjoint intervals in X. Let Q_{n+1}, the $(n+1)$th cut-set of X, be the set of midpoints of elements of $\mathbf{I}_n$ and let $\widehat{Q}_{n+1} = \widehat{Q}_n \cup Q_{n+1}$. Let $\mathbf{I}_{n+1}$, the set of intervals in X derived from $\widehat{Q}_{n+1}$, be the countable collection of pairwise disjoint copies of $(0, 1)$ whose union makes up $X \backslash (\widehat{Q}_{n+1} \cup X_0)$. Set $Q = \bigcup_{n<\omega} Q_n$.

For every $1 \leq m < \omega$ and every $n < \omega$, let $\mathbf{I}_{m,n} = \{I \in \mathbf{I}_m : I \subset O_{x_n}\}$. This is the set of those elements of $\mathbf{I}_m$ which "cluster" around x_n.

For every $1 \leq n < \omega$ let $\Sigma(n)$ be the set of sequences, σ, defined on $n+1$ such that $\sigma(0), \sigma(1) \in \omega$ and $\sigma(m) \in \{0, 1\}$ for all $1 < m \leq n$. Let $m < \omega$. Since $\mathbf{I}_{1,m}$ is countable, it can be enumerated as $\{I_{\langle m,n\rangle} : n < \omega\}$. In this way the set $\mathbf{I}_1$ is indexed by $\Sigma(1)$. Assume that the elements of $\Sigma(n)$ have been used to index the elements of $\mathbf{I}_n$. Let $I \in \mathbf{I}_{n+1}$. There is a unique $\sigma \in \Sigma(n)$ such that $I \subset I_\sigma$. If I is the left-hand half of I_σ, then let τ be the element of $\Sigma(n+1)$ such that $\tau \restriction n+1 = \sigma$ and $\tau(n+1) = 0$ and set $I_\tau = I$. If I is the right-hand half of I_σ, then let τ be the element of $\Sigma(n+1)$ such that $\tau \restriction n+1 = \sigma$ and $\tau(n+1) = 1$ and set $I_\tau = I$. Let $\Sigma = \bigcup_{1 \leq n < \omega} \Sigma(n)$.

The following lemma consists of observations which are immediate consequences of the previous definitions and its proof is omitted.

Lemma 2. *Let* $1 \le m \le n < \omega$.

1. *If* $I \in \mathbf{I}_n$ *then* $I \cap Q_m \ne \varnothing$.
2. *If* $p \in Q_m$ *then there are exactly two elements,* I_1 *and* I_2, *of* $\mathbf{I}_n$ *such that* p *is an endpoint of both* I_1 *and* I_2. *Furthermore,* $I_1 \cup I_2 \cup \{p\}$ *is open in* X.
3. *If* $I \in \mathbf{I}_m$ *then there are exactly two elements of* $\mathbf{I}_{m+1}$ *that are subintervals of* I.
4. *If* $I_\sigma \in \mathbf{I}_n$ *then there is exactly one element,* $I_{\sigma\restriction m+1}$, *of* $\mathbf{I}_m$ *that contains* I_σ.
5. *If* $\sigma \in \Sigma(1)$, $\sigma(0) = k$, *and* $\sigma(1) = 1$, *then* I_σ *is the* l*th element of* $\mathbf{I}_{1,k}$.
6. *If* $J_\sigma \in \mathbf{I}_{n,k}$ *then* $\sigma \in \Sigma(n)$ *and* $\sigma(0) = k$.
7. *For any* $n, k < \omega$, $\{\mathrm{Int}[\mathrm{Cl}(\bigcup\{\mathbf{I}_\sigma \in I_{n,k}: \sigma(1) > a\})]: a < \omega\}$ *forms a local base for* x_k.

For each $p \in X$ and each $1 \le n < \omega$ let $\mathbf{A}_n(p) = \{I \in \mathbf{I}_n : p \in \overline{I}\}$ and let $\mathbf{A}_n^*(p) = \bigcup \mathbf{A}_n(p)$. If $p \in Q_n$ then $\mathbf{A}(p)$ and $\mathbf{A}^*(p)$ will denote $\mathbf{A}_{n+1}(p)$ and $\mathbf{A}_{n+1}^*(p)$ respectively. If $B \in \mathrm{PR}[X]$ then set $\mathbf{A}_n(B) = \bigcup_{p\in B} \mathbf{A}_n(p)$ and $\mathbf{A}_n^*(p) = \bigcup_{p \in B} \mathbf{A}_n^*(p)$. If $B \in F[Q_n]$ then set $\mathbf{A}(B) = \bigcup_{p\in B}\mathbf{A}(p)$ and $\mathbf{A}^*(B) = \bigcup_{p\in B}\mathbf{A}^*(p)$.

Set $M_0 = \{\varnothing\}$ and, for each $1 \le n < \omega$, let $M_n = \{E \in F(\widehat{Q}_n): E \cap Q_m \ne \varnothing$ for all $1 \le m \le n\}$. For $1 \le n < \omega$ call M_n the set of elements of $\mathrm{PR}[X]$ compatible with $\widehat{Q}_n$. Note that if $m > n$ and $E \in M_n$ then $E \cap Q_m = \varnothing$. Also, if $k \ne l$ then $M_k \cap M_l = \varnothing$. For each $n < \omega$ and each $E \in M_n$, let $S_E = \{A \in \mathrm{PR}[X]: A \cap \widehat{Q}_{n+1} = E\}$. Thus, if $A \in S_E$ and $E \in M_n$, then $A \cap Q_{n+1} = \varnothing$. The set $\{S_E : E \in M\}$ where $M = \bigcup_{n<\omega} M_n$ is a partition of $\mathrm{PR}[X]$ and is called the fundamental partition of $\mathrm{PR}[X]$ based on M. If $E \in M_n$ then S_E can be written as $\{A \cup B \cup E: A \in F'[X_0]$ and $B \in F'[X \backslash (\widehat{Q}_{n+1} \cup X_0)]\}$. Recall that $X \backslash (\widehat{Q}_{n+1} \cup X_0) = \bigcup \mathbf{I}_{n+1}$.

For each $E \in M_n$ let $\widehat{F}_E = \{I \in \mathbf{I}_{n+1} : I \subset A^*(E)\}$. If $n \ge 2$, let $E' = E \backslash Q_n = E \cap \widehat{Q}_{n-1}$. If $n \ge 3$ then E'' is $E \cap \widehat{Q}_{n-2}$. If $n = 2$ then set $E'' = \varnothing$.

Now let Y be another ω-graph and let Y_0 be a dividing set for Y. Enumerate Y_0 as $\{y_n : n < \omega\}$. Then the function $\lambda: X_0 \to Y_0$ given by $\lambda(x_n) = y_n$ is a bijection. Let $\mathbf{J}_0$ be a countable collection of pairwise disjoint copies of $(0, 1)$ whose union is $Y \backslash Y_0$. We may again assume that every element of $\mathbf{J}_0$ has at least one endpoint in Y_0. Let R_0 be the set of midpoints of elements of $\mathbf{J}_0$. Let $\{R_n : 1 \le n < \omega\}$ be the collection of cut-sets for Y and set $R = \bigcup_{n>\omega} R_n$. Let P_n be the component of $Y \backslash R_0$ that contains y_n. For each $0 < n < \omega$ let $\mathbf{J}_n$ be the set of intervals of $\mathrm{PR}[Y]$ derived from R_n, each indexed as before by the elements of Σ. Let $\{N_k : k < \omega\}$ be the collection of sets of elements of PR$[Y]$ compatible with $\{\widehat{R}_k : k < \omega\}$ and let $\{T_E : E \in N\}$ be the fundamental partition of PR$[Y]$ based on $N = \bigcup_{k<\omega} N_k$. If $E \subset Q$ and $f: E \to R$, then f is level preserving if $f(E \cap Q_n) \subset R_n$ for all $n < \omega$.

For each $I \in \mathbf{I}_n$ and $J \in \mathbf{J}_n$ there is a unique linear homeomorphism between I and J that preserves orientation. Denote this homeomorphism by $\eta_{I,J}$. If $\sigma, \tau \in \Sigma(n)$, $I = I_{\sigma\restriction m+1}$, and $J = J_{\tau\restriction m+1}$ for some $m < n$, then $\eta_{I,J}(I_\sigma) = J_\tau$ if and only if $\sigma(k) = \tau(k)$ for all $m < k \le n$. If $\Gamma: \mathbf{I}_n \to \mathbf{J}_n$ is a bijection, then $\Gamma^*: \bigcup \mathbf{I}_n \to \bigcup \mathbf{J}_n$ is the function $\bigcup_{I \in I_n} \eta_{I,\Gamma(I)}$. Γ^* is a homeomorphism that is linear and orientation preserving on each element of $\mathbf{I}_n$.

Now order each $\mathbf{I}_n$ and $\mathbf{J}_n$ lexicographically using the indices of their elements. These collections then have order-type ω^2. Let $\mathbf{F} \subset \mathbf{I}_n$ and $\mathbf{G} \subset \mathbf{J}_n$ be equipotent finite sets and let $\gamma: F \to G$ be a bijection. Then $\mathbf{I}_n \backslash \mathbf{F}$ and $\mathbf{J}_n \backslash \mathbf{G}$ still have order-type ω^2, so there is a unique order isomorphism $\Delta_F: \mathbf{I}_n \backslash F \to \mathbf{J}_n \backslash G$. Define $\Gamma: \mathbf{I}_n \to \mathbf{J}_n$ by $\Gamma = \gamma \cup \Delta_F$. Then Γ is a bijection.

In those situations where more than one F is being considered and subscripts are used to distinguish the various set, the same subscripts will be used to distinguish the corresponding γ, Δ, and Γ functions. For example, the functions associated with $\mathbf{F}_1$ will be γ_1, Δ_1, and Γ_1.

It will be necessary in what follows to compare the index of I_σ with that of $\gamma(I_\sigma)$ or $\Gamma(I_\sigma)$. In order to facilitate this, we will use $\gamma(\sigma)$ and $\Gamma(\sigma)$ to denote the indices of $\gamma(I_\sigma)$ and $\Gamma(I_\sigma)$ respectively.

The next lemma is obvious and its proof is omitted.

Lemma 3. *Let $m \le n < \omega$ and let $\mathbf{F}_1 \subset \mathbf{I}_m$ and $\mathbf{F}_2 \subset \mathbf{I}_n$ with $\{I \in \mathbf{I}_n : I \subset \mathbf{F}_1\} \subset \bigcup \mathbf{F}_2$. If $\gamma_1: F_1 \to \mathbf{J}_m$ is a one-to-one function and $\gamma_2: \mathbf{F}_2 \to \mathbf{J}_n$ is defined by $\gamma_2(I) = \Gamma_1^*(I)$, then $\Gamma_1^*(I) = \Gamma_2^*(I)$ for all $I \in \mathbf{I}_n$.*

Lemma 4. *Let $\mathbf{F} \subset \mathbf{I}_k$ be finite and let $\gamma: \mathbf{F} \to \mathbf{J}_k$ be a one-to-one function. Assume that there are $b, c, m < \omega$ such that*

1. *$c - m > b$;*
2. *if $I_\sigma \in \mathbf{F}$ then either $\sigma(1) \le b$ or $\sigma(1) > c$;*
3. *if $I_\sigma \in \mathbf{F} \cap \mathbf{I}_{k,n}$ and $m \le \sigma(1) \le b$ then $\gamma(I_\sigma) \in \mathbf{J}_{k,n}$ and $\gamma(\sigma)(1) \le b$; and*
4. *if $I_\sigma \in \mathbf{F} \cap \mathbf{I}_{k,n}$ and $\sigma(1) > c$ then $\gamma(I_\sigma) \in \mathbf{J}_{k,n}$ and $\gamma(\sigma)(1) > b$.*

Then $\Gamma(I_\sigma) \in \mathbf{J}_{k,n}$ and $\Gamma(\sigma)(1) > b$ for all $I_\sigma \in \mathbf{I}_{k,n}$ with $\sigma(1) > c$.

Proof. Let $n < \omega$. The elements of $\mathbf{J}_{k,n} \backslash \gamma(\mathbf{F})$ are the images under Δ_F of $\mathbf{I}_{k,n} \backslash \mathbf{F}$. By conditions 2 and 3,

$$\begin{aligned} |\mathbf{F} \cap \{I_\sigma \in \mathbf{I}_{k,n} : m \le \sigma(1) \le c\}| &= |\mathbf{F} \cap \{I_\sigma \in \mathbf{I}_{k,n} : m \le \sigma(1) \le b\}| \\ &= |\{\gamma(I_\sigma) : I_\sigma \in \mathbf{I}_{k,n} \text{ and } m \le \sigma(1) \le b\}| \\ &\le |\{J_\sigma \in \mathbf{J}_{k,n} : J_\sigma \in \gamma(F) \text{ and } \sigma(1) \le b\}| \\ &= |\gamma(\mathbf{F}) \cap \{J_\sigma \in \mathbf{J}_{k,n} : \sigma(1) \le b\}|. \end{aligned}$$

Also, $|\{I_\sigma \in \mathbf{I}_{k,n}: m \leq \sigma(1) \leq c\}| \geq |\{J_\sigma \in \mathbf{J}_{k,n}: \sigma(1) \leq b\}|$ because $c - m > b$. Therefore,

$$\begin{aligned}
&|\{I_\sigma \in \mathbf{I}_{k,n}: m \leq \sigma(1) \leq c\} \backslash \mathbf{F}| \\
&\quad = |\{I_\sigma \in \mathbf{I}_{k,n}: m \leq \sigma(1) \leq c\} \backslash (\mathbf{F} \cap \{I_\sigma \in \mathbf{I}_{k,n}: m \leq \sigma(1) \leq c\})| \\
&\quad \geq |\{J_\sigma \in \mathbf{J}_{k,n}: \sigma(1) \leq b\} \backslash (\gamma(\mathbf{F}) \cap \{J_\sigma \in \mathbf{J}_{k,n}: \sigma(1) \leq b\})| \\
&\quad = |\{J_\sigma \in \mathbf{J}_{k,n}: \sigma(1) \leq b\} \backslash \gamma(\mathbf{F})|.
\end{aligned}$$

Thus, if $J_\tau \in \mathbf{J}_{k,n}$ and $\tau(1) \leq b$ then there is $I_\sigma \in \mathbf{I}_k$ such that either $I_\sigma \in \mathbf{F}$ or $I_\sigma \in \mathbf{I}_{k,n}$ and $\sigma(1) \leq c$, and $\Gamma(I_\sigma) = \mathbf{J}_\tau$. It follows from this and condition 4 that if $I_\sigma \in \mathbf{I}_{k,n}$ and $\sigma(1) > c$, then $\Gamma(I_\sigma) \in \mathbf{J}_{k,n}$ and $\Gamma(\sigma)(1) > b$.

Lemma 5. *Let $\mathbf{F}_1, \mathbf{F}_2 \subset I_k$ be finite and let $\gamma_1: \mathbf{F}_1 \to \mathbf{J}_k$ and $\gamma_2: \mathbf{F}_2 \to \mathbf{J}_k$ be one-to-one functions. Let $a, b, m < \omega$ such that*

1. $b - a > m$;
2. $\{I_\sigma \in \mathbf{F}_1: \sigma(1) \leq a\} = \{I_\sigma \in \mathbf{F}_2: \sigma(1) \leq a\} = \mathbf{G}$; *and*
3. $\gamma_1(I_\sigma) = \gamma_2(I_\sigma)$ *for all* $I_\sigma \in \mathbf{G}$;

and that for $i = 1$ or 2,

4. *if* $J_\sigma \in \gamma_i(\mathbf{F}_i)$ *then either* $\sigma(1) \leq a$ *or* $\sigma(1) > b$;
5. *if* $I_\sigma \in \mathbf{F}_i$ *and* $\sigma(1) > b$ *then* $\gamma_i(\sigma)(1) > a$; *and*
6. *for all* $n < \omega$, *if* $J_\sigma \in \gamma_i(\mathbf{F}_i) \cap \mathbf{J}_{k,n}$ *and* $\gamma_i^{-1}(J_\sigma) \notin \mathbf{I}_{k,n}$ *then* $\sigma(1) < m$.

Then $\Gamma_1(I_\sigma) = \Gamma_2(I_\sigma)$ for all $I_\sigma \in \mathbf{I}_n$ with $\sigma(1) \leq a$.

Proof. Let $n < \omega$. By condition 2,

$$\{I_\sigma \in \mathbf{I}_{k,n}: \sigma(1) \leq a\} \cap \mathbf{F}_1 = \mathbf{I}_{k,n} \cap \mathbf{G} = \{I_\sigma \in \mathbf{I}_{k,n}: \sigma(1) \leq a\} \cap \mathbf{F}_2$$

and

$$\begin{aligned}
\{I_\sigma \in \mathbf{I}_{k,n}: \sigma(1) \leq a\} \backslash \mathbf{F}_1 &= \{I_\sigma \in \mathbf{I}_{k,n}: \sigma(1) \leq a\} \backslash \mathbf{G} \\
&= \{I_\sigma \in \mathbf{I}_{k,n}: \sigma(1) \leq a\} \backslash \mathbf{F}_2.
\end{aligned}$$

By conditions 2, 3, and 4,

$$\begin{aligned}
\{J_\sigma \in \mathbf{J}_{k,n}: \sigma(1) \leq b\} \backslash \gamma_1(\mathbf{F}_1) &= \{J_\sigma \in \mathbf{J}_{k,n}: \sigma(1) \leq b\} \backslash \gamma_1(\mathbf{G}) \\
&= \{J_\sigma \in \mathbf{J}_{k,n}: \sigma(1) \leq b\} \backslash \gamma_2(\mathbf{F}_2).
\end{aligned}$$

If $I_\sigma \in \mathbf{I}_{k,n} \cap \mathbf{G}$ then $\Gamma_1(I_\sigma) = \gamma_1(I_\sigma) = \gamma_2(I_\sigma) = \Gamma_2(I_\sigma)$. The values of Γ_1 and Γ_2 on $\{I_\sigma \in \mathbf{I}_{k,n}: \sigma(1) \leq a\} \backslash \mathbf{G}$ are determined by Δ_1 and Δ_2 respectively. We can establish the equality of Γ_1 and Γ_2 on $\{I_\sigma \in \mathbf{I}_{k,n}: \sigma(1) \leq a\} \backslash \mathbf{G}$ by showing that this set is no larger than $\{J_\sigma \in \mathbf{J}_{k,n}: \sigma(1) \leq b\} \backslash \gamma_1(\mathbf{G})$. Then, since both Δ_1 and Δ_2 take the α th element of $\{I_\sigma \in \mathbf{I}_{k,n}: \sigma(1) \leq a\} \backslash \mathbf{G}$ to the

αth element of $\{J_\sigma \in \mathbf{G}J_{k,n}\colon \sigma(1) \le b\}\backslash\gamma_1(\mathbf{G})$, they must be equal.

$$\begin{aligned}
&|\{J_\sigma \in \mathbf{J}_{k,n}\colon \sigma(1) \le b\}\backslash\gamma_1(\mathbf{G})| \\
&\qquad = |\{J_\sigma \in \mathbf{J}_{k,n}\colon \sigma(1) \le a\}\backslash\gamma_1(\mathbf{G})| + |\{J_\sigma \in \mathbf{J}_{k,n}\colon a < \sigma(1) \le b\}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by condition 4)} \\
&\qquad = |\{J_\sigma \in \mathbf{J}_{k,n}\colon \sigma(1) \le a\}\backslash(\{\gamma_1(I_\sigma) \in \mathbf{J}_{k,n}\colon I_\sigma \in \mathbf{G}\backslash\mathbf{I}_{k,n}\} \\
&\qquad\quad \cup \{\gamma_1(I_\sigma) \in \mathbf{J}_{k,n}\colon I_\sigma \in \mathbf{G}\cap\mathbf{I}_{k,n}\})| \\
&\qquad\quad + |\{J_\sigma \in \mathbf{J}_{k,n}\colon a < \sigma(1) \le b\}| \\
&\qquad \ge |\{J_\sigma \in \mathbf{J}_{k,n}\colon \sigma(1) \le a\}\backslash\{\gamma_1(I_\sigma) \in \mathbf{J}_{k,n}\colon I_\sigma \in \mathbf{G}\cap\mathbf{I}_{k,n}\}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by conditions 1 and 6)} \\
&\qquad = |\{I_\sigma \in \mathbf{I}_{k,n}\colon \sigma(1) \le a\}\backslash\{I_\sigma \in \mathbf{G}\cap\mathbf{I}_{k,n}\colon \gamma_1(I_\sigma) \in \mathbf{J}_{k,n}\}| \\
&\qquad \ge |\{I_\sigma \in \mathbf{I}_{k,n}\colon \sigma(1) \le a\}\backslash\mathbf{G}|.
\end{aligned}$$

III. Proof of Theorem 1

Let X and Y be ω-graphs with dividing sets X_0 and Y_0. We will use the structures and definitions developed in §II. Let $g\colon Q_1 \to R_1$ be a bijection such that $g(Q_1 \cap O_n) = R_1 \cap P_n$ for all $n < \omega$. Then $g(Q_0) = R_0$. For our convenience later in the proof, we will assume that the first $\mu(n)$ elements of any $\mathbf{I}_{m,n}$ are those elements of $\mathbf{I}_{m,n}$ having an element of Q_0 as an endpoint.

The homeomorphism we will define is essentially that defined by Norden in [N].

Define $\Gamma_\phi\colon \mathbf{I}_1 \to \mathbf{J}_1$ by $\Gamma_\phi(I_\sigma) = J_\sigma$, and $h_\phi\colon \bigcup\mathbf{I}_1 \to \bigcup\mathbf{J}_1$ by $h_\phi = \Gamma_\phi^*$. Then h_ϕ is a homeomorphism. Set $\theta(\phi) = \phi$.

Let $E \in M_1$. Set $f_E = g \restriction E$ and $\theta(E) = f_E(E)$. Let $\mathbf{F}_E = \widehat{\mathbf{F}}_E$ and $\mathbf{F}_{\theta(E)} = \widehat{\mathbf{F}}_{\theta(E)}$. Each $I \in \mathbf{F}_E$ is adjacent to exactly one element of E and each element of E is the endpoint of exactly two elements of F_E. Similarly, each element of $F_{\theta(E)}$ is adjacent to exactly one element of $\theta(E)$ and each element of $\theta(E)$ is the endpoint of exactly two elements of $F_{\theta(E)}$. Define $\gamma_E\colon \mathbf{F}_E \to \mathbf{F}_{\theta(E)}$ as follows. Let $I \in \mathbf{F}_E$ and let $p \in E$ be an endpoint of I. If p is the right-hand endpoint of I, then set $\gamma_E(I)$ equal to the element of $\mathbf{F}_{\theta(E)}$ which has $g(p)$ for its right-hand endpoint. If p is the left-hand endpoint of I, then set $\gamma_E(I)$ equal to the element of $\mathbf{F}_{\theta(E)}$ which has $g(p)$ for its left-hand endpoint. Then γ_E is a bijection. Define $h_E\colon (\bigcup\mathbf{I}_2) \cup E \to (\bigcup\mathbf{J}_2) \cup \theta(E)$ by $h_E = \Gamma_E^* \cup f_E$. Both Γ_E^* and f_E are bijections so h_E is a bijection. It is also a homeomorphism on $\bigcup\mathbf{I}_2$ because Γ_E^* is. Let $x \in E$ and let V be a neighborhood of $f_E(x)$ in Y. By the definition of γ_E there is a neighborhood U of x in $A^*(x) \cup \{x\}$ such that $h_E(U) \subset V$. Thus h_E is continuous at x. A similar argument shows that h_E^{-1} is continuous at $h_E(x)$, so h_E is a homemorphism.

Let $2 \leq 1 < \omega$ and assume that for all $k < 1$ and all $E \in M_k$,

1. $f_E \colon E \to \widehat{R}_k$ is a level preserving one-to-one function and $\theta(E) = f_E(E)$;
2. $\mathbf{F}_E \subset \mathbf{I}_{k+1}$ and $\mathbf{F}_{\theta(E)} \subset \mathbf{J}_{k+1}$ are finite and $\gamma_E \colon \mathbf{F}_E \to \mathbf{F}_{\theta(E)}$ is a bijection; and
3. the function $h_E \colon (\bigcup \mathbf{I}_{k+1}) \cup E \to (\bigcup \mathbf{J}_{k+1}) \cup \theta(E)$ given by $h_E = \Gamma_E^* \cup f_E$ is a homeomorphism.

Fix $E \in M_l$. Each element of $E \cap Q_l$ is the midpoint of some element of I_{l-1} and $h_{E''}$, which is defined on $\bigcup I_{l-1}$, takes midpoints to midpoints. Thus $h_{E''}(p) \in R_l$ for all $p \in E \cap Q_l$. Define $f_E \colon E \to \widehat{R}_l$ by

$$f_E(p) = \begin{cases} h_{E'}(p) & \text{if } p \in E \cap \widehat{Q}_{l-1}, \\ h_{E''}(p) & \text{if } p \in E \cap Q_l. \end{cases}$$

Then f_E is a one-to-one level preserving function. Note that if $p \in E \cap \widehat{Q}_{l-1}$ then $f_E(p) = h_{E'}(p) = f_{E'}(p)$. Extending this backward, we can see that if $1 \leq k < l$ and $p \in E \cap \widehat{Q}_k$ then $f_E(p) = f_{E \cap \widehat{Q}_k}(p)$.

Let $\mathbf{F}_{E1} = \mathbf{A}(E \cap Q_l)$ and $\mathbf{F}_{\theta(E)1} = \mathbf{A}(\theta(E) \cap R_l)$. Let $I \in \mathbf{F}_{E1}$ and let $p \in E \cap Q_l$ be and endpoint of I. Then $f_E(p) = h_{E''}(p) \in R_l$ and $h_{E''}(p)$ is an endpoint of $h_{E''}(I)$ because $h_{E''}$ is continuous. Thus $h_{E''}(I) \in \mathbf{F}_{\theta(E)1}$. A similar argument shows that if $h_{E''}(I) \in \mathbf{F}_{\theta(E)1}$ then $I \in \mathbf{F}_{E1}$.

Let $\mathbf{F}_{E2} = \{I \in \widehat{\mathbf{F}}_E \backslash \mathbf{F}_{E1} \colon h_{E'}(I) \in \widehat{\mathbf{F}}_{\theta(E)} \backslash \mathbf{F}_{\theta(E)1}\}$ and let $\mathbf{F}_{\theta(E)2} = \{J \in \widehat{\mathbf{F}}_{\theta(E)} \backslash \mathbf{F}_{\theta(E)1} \colon h_{E'}^{-1}(J) \in \widehat{\mathbf{F}}_E \backslash \mathbf{F}_{E1}\}$. Clearly $I \in \mathbf{F}_{E2}$ if and only if $h_{E'}(\mathbf{I}) \in \mathbf{F}_{\theta(E)2}$. Set $\mathbf{F}_E = \mathbf{F}_{E1} \cup \mathbf{F}_{E2}$ and $\mathbf{F}_{\theta(E)} = \mathbf{F}_{\theta(E)1} \cup \mathbf{F}_{\theta(E)2}$. Define $\gamma_E \colon \mathbf{F}_E \to \mathbf{F}_{\theta(E)}$ by

$$\gamma_E(I) = \begin{cases} h_{E''}(I) & \text{if } I \in \mathbf{F}_{E1}, \\ h_{E'}(I) & \text{if } I \in \mathbf{F}_{E2}. \end{cases}$$

Then γ_E is a bijection.

Define $h_E \colon (\bigcup \mathbf{I}_{l+1}) \cup E \to (\bigcup \mathbf{J}_{l+1}) \cup \theta(E)$ by $h_E = \Gamma_E^* \cup f_E$. The function h_E is a bijection because Γ_E^* and f_E are bijections and is a homemorphism on $\bigcup \mathbf{I}_{l+1}$ because Γ_E^* is. If $p \in E \cap Q_l$ then $\mathbf{A}(p) \subset \mathbf{F}_{E1}$ and $h_E(\mathbf{A}^*(p) \cup \{p\}) = h_{E''}(\mathbf{A}^*(p) \cup \{p\})$. Now let $p \in E'$. If $I \in \mathbf{A}_{l+1}(p)$ then $I \in \widehat{\mathbf{F}}_E$. Since p is an endpoint of I and $p \in \widehat{Q}_{l-1}$, the other endpoint of I must be an element of Q_{l+1}. Hence $I \notin \mathbf{F}_{E1}$. To show that $h_{E'}(I) \in \widehat{\mathbf{F}}_{\theta(E)} \backslash \mathbf{F}_{\theta(E)1}$, note that $p \in E'$ and $h_{E'}$ is continuous on $(\bigcup I_l) \cup E'$. So $f_E(p) = \mathbf{F}_{E'}(p)$ is an endpoint of $h_{E'}(I)$. But $f_{E'}$ is level preserving, so $f_{E'}(p) \in \widehat{R}_{l+1}$. Again, the other endpoint of $h_{E'}(I)$ must be an element of R_{l+1}. Hence $h_{E'}(I) \in \widehat{\mathbf{F}}_{\theta(E)} \backslash \mathbf{F}_{\theta(E)1}$. It follows that $\mathbf{A}_{l+1}(p) \subset \mathbf{F}_{E2}$ and $h_E(\mathbf{A}_{l+1}^*(p) \cup \{p\}) = h_{E'}(\mathbf{A}_{l+1}^*(p) \cup \{p\})$. But $h_{E'}$ is a homeomorphism on $(\bigcup \mathbf{I}_l) \cup E'$ and $h_{E''}$ is a homeomorphism on $\bigcup \mathbf{I}_{l-1}$, so h_E is a homeomorphism on $(\bigcup \mathbf{I}_{l+1}) \cup E$.

Notice that for any $k < \omega$, $E \in M_k$, $x_n \in X_0$, and $I_\sigma \in \mathbf{I}_{k,n}$, if $\Gamma_E(I_\sigma) \notin \mathbf{J}_{k,n}$ then $\sigma(1) < \mu(n)$ because only the first $\mu(n)$ elements of $\mathbf{I}_{1,n}$ have endpoints in Q_0.

For all $n < \omega$ and all $E \in M_n$, define $H_E \colon S_E \to T_{\theta(E)}$ by $H_E(A) = \lambda(A \cap X_0) \cup h_E(A \backslash X_0)$. Finally, define $H\colon \mathrm{PR}[X] \to \mathrm{PR}[Y]$ by $H = \bigcup_{E \in M} H_E$. To show that H is a bijection it is sufficient to show that θ is a bijection. Let $E, D \in M$ and $E \neq D$. Then $\theta(E) = f_E(E)$ and $\theta(D) = f_D(D)$. Both f_E and f_D are level-preserving one-to-one functions, so $\theta(E) \neq \theta(D)$ if $E \in M_k$ and $D \in M_l$ and $k \neq l$. Assume that $E, D \in M_1$. Then $\theta(E) = g(E) \neq g(D) = \theta(D)$ since g is a bijection. Assume that $E, D \in M_k$ for some $k > 1$. Either $E \cap Q_k \neq D \cap Q_k$ or $E' \neq D'$. But the functions $h_{E'}$, $h_{E''}$, $h_{D'}$, and $h_{D''}$ are all one-to-one, so either $h_{E''}(E \cap Q_k) \neq h_{D''}(D \cap Q_k)$ or $h_{E'}(E') \neq h_{D'}(D')$. In either case, $\theta(E) \neq \theta(D)$.

Let $A \in S_E$ where $E \in M_k$ and let V be a neighborhood of $H(A)$ in Y. Pick $a < \omega$ such that if $I_\sigma \in \mathbf{A}_1(A)$ then $\sigma(1) \leq a$ and if $J_\sigma \in \mathbf{A}_1(H(A))$ then $\sigma(1) \leq a$. Let $m = \max\{\mu(n)\colon \mathbf{A}_1(A) \cap \mathbf{I}_{1,n} \neq \varnothing \text{ or } \mathbf{A}_1(H(A)) \cap \mathbf{J}_{1,n} \neq \varnothing\} + 1$. Pick $b \in \omega$ such that $b - m > a$ and

$$\mathrm{Int}\left[\mathrm{Cl}\left(\bigcup\{J_\sigma \in \mathbf{J}_{1,n}\colon \sigma(1) > b\}\right)\right] \subset V$$

for all $y_n \in H(A) \cap Y_0$. Set

$$V_{y_n} = \mathrm{Int}\left[\mathrm{Cl}\left(\bigcup\{J_\sigma \in \mathbf{J}_{1,n}\colon \sigma(1) > b\}\right)\right]$$

and set $V_0 = \bigcup_{p \in H(A) \cap Y_0} V_p$. Pick $c \in \omega$ such that $c - m > b$ and if $x_n \in A \cap X_0$ and $p \in Q_1 \cap \mathrm{Int}[\mathrm{Cl}(\bigcup\{I_\sigma \in \mathbf{I}_{1,n}\colon \sigma(1) > c\})]$, then $g(p) \in V_{y_n}$. For each $x_n \in A \cap X_0$ set $U_{x_n} = \mathrm{Int}[\mathrm{Cl}(\bigcup\{I_\sigma \in \mathbf{I}_{1,n}\colon \sigma(1) > c\})]$. Let $U_0 = \bigcup_{p \in A \cap X_0} U_p$. If $A \cap X_0 = \varnothing$ then set $U_0 = \varnothing$. Pick $r \geq k+1$ such that $h_E(\mathbf{A}_r^*(p)) \subset V$ for all $p \in A \backslash X_0$. Set $U_p = \mathbf{A}_r^*(p) \cup \{p\}$ for $p \in A \backslash X_0$ and set $U_1 = \bigcup_{p \in A \backslash X_0} U_p$. Let $U = U_0 \cup U_1$. Note that:

1. if $I_\sigma \cap U_1 \neq \varnothing$ then $\sigma(1) \leq a$;
2. if $\overline{J}_\sigma \cap (H(A) \backslash Y_0) \neq \varnothing$ then $\sigma(1) \leq a$;
3. if $I_\sigma \cap U_{x_n} \neq \varnothing$ for some $x_n \in A \cap X_0$ then $I_{\sigma \restriction 1} \in \mathbf{I}_{1,n}$ and $\sigma(1) > c$;
4. if $J_\sigma \cap V_{y_n} \neq \varnothing$ for some $y_n \in H(A) \cap Y_0$ then $J_{\sigma \restriction 1} \in \mathbf{J}_{1,n}$ and $\sigma(1) > b$;
5. if $p \in A \backslash X_0$ then $U_p \cap \widehat{Q}_{k+1} \subset \{p\}$.
6. a, b, c and m satisfy condition 1 in Lemmas 4 and 5; and
7. if $I_\sigma \in \mathbf{I}_{1,n}$ and $m \leq \sigma(1)$ then $H_D(I_\sigma) \subset \bigcup \mathbf{J}_{1,n}$ for any $0 < 1 < \omega, n < \omega$, and $D \in M$.

The heart of the proof that $H([A, U]) \subset [H(A), V]$ is contained in Lemmas 6 and 7.

Lemma 6. *Let $D \in M_j$ where $1 \leq j \leq k$, $D \subset U$, and $D \cap U_1 = E \cap \widehat{Q}_j$. Let $C = E \cap \widehat{Q}_j$. Then*

1. *if $p \in D \cap U_q$ for some $q \in A \cap X_0$ then $f_D(p) \in V_{\lambda(q)}$;*

2. *if* $p \in D \cap U_q$ *for some* $q \in E$ *then* $p = q$ *and* $f_D(p) = F_E(p)$;
3. *if* $I_\sigma \in \mathbf{I}_{j+1}$ *and* $\sigma(1) \le a$ *then* $\Gamma_C(I_\sigma) = \Gamma_D(I_\sigma)$; *and*
4. *if* $I_\sigma \in \mathbf{I}_{j+1,n}$, $x_n \in A \cap X_0$, *and* $\sigma(1) > c$, *then* $\Gamma_D(I_\sigma) \in \mathbf{J}_{j+1,n}$ *and* $\Gamma_D(\sigma)(1) > b$.

Proof. To begin with, let us take note of three useful facts. First, since $\Gamma_\phi(I_\sigma) = J_\sigma$ for all $I_\sigma \in I_1$, if $I_\sigma \in I_{1,n}$ and $\sigma(1) > c$, then $\Gamma_\phi(I_\sigma) = J_\sigma \in \mathbf{J}_{1,n}$ and $\Gamma_\phi(\sigma)(1) = \sigma(1) > c > b$. Also, for any j, if $p \in C$ then $f_C(p) = f_E(p)$. Furthermore, if $I_\sigma \in F_D$ then either $\sigma(1) \le a < b$ or $\sigma(1) > c$.

Let $j = 1$. Then $D \subset Q_1$ and $D \cap U_1 = E \cap Q_1$. Let $p \in D$. If $p \in U_q$ for some $q \in A \cap X_0$, then $f_D(p) = g(p) \in V_{\lambda(q)}$. If $p \in U_q$ for some $q \in A \backslash X_0$, then $q \in E, p = q$, and $f_D(p) = g(p) = f_C(p)$.

Let $n < \omega$ and let $I_\sigma \in \mathbf{I}_{2,n} \cap \mathbf{F}_D$ with $\sigma(1) > c$. Let $p \in D$ be an endpoint of I_σ. Since $\sigma(1) > c$, p must be in U_{x_n}. Then $f_D(p)$, which is an endpoint of $\gamma_D(I_\sigma)$, is in V_{y_n}. Thus $\gamma_D(I_\sigma) \in \mathbf{J}_{2,n}$ and $\gamma_D(\sigma)(1) > b > a$.

It follows from $D \cap U_1 = C$ that $F_C = \{I_\sigma \in \mathbf{F}_D : \sigma(1) \le a\}$. Let $I_\sigma \in \mathbf{F}_C$. Let $p \in D$ be an endpoint of I_σ. Then p must be an element of U_1, so $f_D(p) = f_E(p) = f_C(p)$. Thus $f_E(p)$ is an endpoint for both $\gamma_C(I_\sigma)$ and $\gamma_D(I_\sigma)$. Since both γ_C and γ_D preserve orientation, it must be true that $\gamma_C(I_\sigma) = \gamma_D(I_\sigma)$. Also, $\gamma_D(\sigma)(1) \le a < b$ because $f_D(p) \in H(A) \backslash Y_0$.

By Lemma 4, if $I_\sigma \in \mathbf{I}_{2,n}$ and $\sigma(1) > c$, then $\Gamma_D(I_\sigma) \in \mathbf{J}_{2,n}$ and $\Gamma(\sigma(1)) > b$. By Lemma 5, if $I_\sigma \in \mathbf{I}_2$ and $\sigma(1) \le a$, then $\Gamma_D(I_\sigma) = \Gamma_C(I_\sigma)$.

Let $2 \le j \le k$ and assume that the lemma is valid for all $1 \le i < j$ and all $D \in M_i$ with $D \subset U$ and $D \cap U_1 = E \cap \widehat{Q}_i$. Let $D \in M_j$ with $D \subset U$ and $D \cap U_1 = E \cap \widehat{Q}_j$. Then $D' \in M_{j-1}, D' \subset U$, and $D' \cap U_1 = E \cap \widehat{Q}_{j-1} = C'$, so the lemma is valid for D'. If $j = 2$, then $D'' = C'' = \varnothing$. If $j > 2$, then $D'' \in M_{j-2}, D'' \subset U$, and $D'' \cap U_1 = E \cap \widehat{Q}_{j-2} = C''$. Thus the lemma is valid for D''.

Let $p \in D \cap U_{x_n}$ for some $x_n \in A \cap X_0$. If $p \in \widehat{Q}_{j-1}$ then $f_D(p) = f_{D'}(p) \in V_{y_n}$. If $p \in Q_j$ then $f_D(p) = h_{D''}(p)$. Now p is the midpoint of some element I_σ of $I_{j-1,n}$ where $\sigma(1) > c$. But $\Gamma_{D''}(I_\sigma) \in J_{j-1,n}, \Gamma_{D''}(\sigma)(1) > b$, and $h_{D''}(p)$ is the midpoint of $\Gamma_{D''}(I_\sigma)$. Hence $f_D(p) \in V_{y_n}$.

Let $p \in D \cap U_q$ for some $q \in A \backslash X_0$. Then $q \in E$ and $q = p$. If $p \in \widehat{Q}_{j-1}$ then $f_D(p) = f_{d'}(p) = f_E(p)$. If $p \in Q_j$ then

$$f_D(p) = h_{D''}(p) = \Gamma^*_{D''}(p) = \Gamma^*_{C''}(p) = h_{C''}(p) = f_C(p) = f_E(p).$$

Let $n < \omega$ and let $I_\sigma \in F_D \cap \mathbf{I}_{j+1,n}$ with $\sigma(1) > c$. Either $\gamma_D(I_\sigma) = \Gamma^*_{D'}(I_\sigma)$ or $\gamma_D(I_\sigma) = \Gamma^*_{D''}(I_\sigma)$. In either case, $\gamma_D(I_\sigma) \in \mathbf{J}_{j+1,n}$ and $\gamma_D(\sigma)(1) > b > a$.

It follows from the inductive hypotheses that $\mathbf{F}_{C1} = \{I_\sigma \in \mathbf{F}_{D1} : \sigma(1) \le a\}$ and $\mathbf{F}_{C2} = \{I_\sigma \in \mathbf{F}_{D2} : \sigma(1) \le a\}$. Thus $\mathbf{F}_C = \{I_\sigma \in \mathbf{F}_D : \sigma(1) \le a\}$. Let $I_\sigma \in \mathbf{F}_C$. If $I_\sigma \in \mathbf{F}_{D1}$ then $\gamma_D(I_\sigma) = \Gamma^*_{D''}(I_\sigma)$. But $\Gamma^*_{D''}(I_\sigma) = \Gamma^*_{C''}(I_\sigma)$ so $\gamma_D(I_\sigma) = \gamma_C(I_\sigma)$. If $I_\sigma \in \mathbf{F}_{D2}$ then $\gamma_D(I_\sigma) = \Gamma^*_{D'}(I_\sigma)$. But $\Gamma^*_{D'}(I_\sigma) = \Gamma^*_{C'}(I_\sigma)$ so $\gamma_D(I_\sigma) = \gamma_C(I_\sigma)$. In either case, $\gamma_D(\sigma)(1) \le a < b$.

By Lemma 4, if $I_\sigma \in \mathbf{I}_{j+1,n}$ and $\sigma(1) > c$, then $\Gamma_D(I_\sigma) \in \mathbf{J}_{j+1,n}$ and $\Gamma_D(\sigma)(1) > b$. By Lemma 5, if $I_\sigma \in \mathbf{I}_{j+1}$ and $\sigma(1) \le a$, then $\Gamma_D(I_\sigma) = \Gamma_C(I_\sigma)$.

Lemma 7. *If $k \le l, D \in M_l$, and $E \subset D \subset U$, then*

1. *if $p \in D \cap U_q$ for some $q \in A \cap X_0$ then $f_D(p) \in V_{\lambda(q)}$;*
2. *if $p \in D \cap U_q$ for some $q \in A\backslash X_0$ then $f_D(p) \in V$;*
3. *if $I_\sigma \in \mathbf{I}_{l+1,n}$ for some $x_n \in A \cap X_0$ and $\sigma(1) > c$, then $\Gamma_D(I_\sigma) \in \mathbf{J}_{l+1,n}$ and $\Gamma_D(\sigma)(1) > b$; and*
4. *if $I_\sigma \in \mathbf{I}_{l+1}$ and $\sigma(1) \le a$ then $\Gamma_D(I_\sigma) = \Gamma^*_E(I_\sigma)$.*

Note that condition 4 implies that $\gamma_D(\sigma)(1) \le a$ for all $I_\sigma \in \mathbf{F}_D$ with $\sigma(1) \le a$.

Proof. The case $k = 1$ is given by Lemma 6.

Assume that $l = k + 1$. Then $D' \in M_k, D' \subset U$, and $D' \cap U_1 = E$. Also, $D'' \in M_{k-1}$, $D'' \subset U$, and $D'' \cap U_1 = E'$. So Lemma 6 holds for D' and D''.

Let $p \in D \cap U_{x_n}$ for some $x_n \in A \cap X_0$. if $p \in \widehat{Q}_k$ then $f_D(p) = f_{D'}(p) \in U_{x_n}$. Let $p \in Q_1$. Then p is the midpoint of some element I_σ of $\mathbf{I}_{k,n}$ where $\sigma(1) > c$. Also, $f_D(p) = h_{D''}(p)$ and $h_{D''}(p)$ is the midpoint of $\Gamma_{D''}(I_\sigma)$. But $\Gamma_{D''}(I_\sigma) \in \mathbf{J}_{k,n}$ and $\Gamma_{D''}(\sigma)(1) > b$. Thus $f_D(p) \in U_{x_n}$.

Let $p \in D \cap U_q$ for some $q \in A\backslash X_0$. Now $U_q \cap \widehat{Q}_1 \subset \{q\}$ so $p = q$ and $p \in \widehat{Q}_k$. Thus $f_D(p) = f_{D'}(p) = f_E(p) \in V$.

Let $I_\sigma \in \mathbf{F}_D \cap \mathbf{I}_{l+1,n}$ for some $x_n \in A \cap X_0$ and let $\sigma(1) > c$. Either $\gamma_D(I_\sigma) = \Gamma^*_{D'}(I_\sigma)$ or $\gamma_D(I_\sigma) = \Gamma^*_{D''}(I_\sigma)$. In either case, $\gamma_D(I_\sigma) \in \mathbf{J}_{l+1,n}$ and $\gamma_D(\sigma)(1) > b > a$.

To show that conditions 3 and 4 hold, consider the sets $\mathbf{F} = \{\mathbf{I} \in I_{l+1} : I \subset \bigcup \mathbf{F}_E\}$ and $\mathbf{G} = \{I_\sigma \in \mathbf{F}_D : \sigma(1) \le a\}$. Define γ on $\mathbf{G}$ by $\gamma(I) = \Gamma^*_E(I)$. We will show that $\mathbf{F} \subset \mathbf{G}$. Let $I_\sigma \in \mathbf{F}$. Then $\sigma(1) \le a$ and $I_{\sigma\restriction k+1} \in \mathbf{F}_E$. Now $A(E) \subset A(D)$ because $E \subset D$. Also, $A(\theta(E)) \subset A(\theta(D))$. Thus $I_\sigma \in \widehat{\mathbf{F}}_D$ and $h_{D'}(I_\sigma) = h_E(I_\sigma) \in \widehat{\mathbf{F}}_{\theta(D)}$. If $I_\sigma \in \mathbf{F}_{D1}$ then there is $p \in D \cap Q_l$ such that p is an endpoint of I_σ. Then, since $\sigma(1) \le a$, $p \in U_1$. But $U_1 \cap Q_l = \varnothing$, so $I_\sigma \notin \mathbf{F}_{D1}$. If $p \in D \cap Q_l$ then $p \in U_0$ and $f_D(p) \in V_0$. But $\Gamma_{D'}(\sigma)(1) \le a$ so $h_{D'}(I_\sigma)$ cannot have an endpoint in $\theta(D) \cap R_l$. Therefore $h_{D'}(I_\sigma) \in \widehat{\mathbf{F}}_{\theta(D)} \backslash \mathbf{F}_{\theta(D)1}$, and $I_\sigma \in \mathbf{G}$. By Lemma 3, $\Gamma(I) = \Gamma^*_E(I)$ for all $I \in \mathbf{I}_{l+1}$. If $I \in \mathbf{G}$ then $I \in \mathbf{F}_{D2}$ so $\gamma_D(I) = \Gamma^*_{D'}(I) = \Gamma^*_E(I) = \gamma(I)$. Thus $\gamma_D(I_\sigma) \in \mathbf{J}_{l+1,n}$ and $\gamma_D(\sigma)(1) \le a < b$ for all $I_\sigma \in \mathbf{F}_D \cap \mathbf{I}_{l+1,n}$ with $m \le \sigma(1) \le b$. By Lemma 4, if $I_\sigma \in \mathbf{I}_{l+1,n}$ for some $x_n \in A\backslash X_0$ and $\sigma(1) > c$, then $\Gamma_D(I_\sigma) \in \mathbf{J}_{l+1,n}$ and $\Gamma_D(\sigma)(1) > b$. By Lemma 5, $\Gamma_D(I_\sigma) = \Gamma(I_\sigma) = \Gamma^*_E(I_\sigma)$ for all $I_\sigma \in \mathbf{I}_{l+1}$ with $\sigma(1) \le a$.

Let $l \ge k + 2$ and assume that if $j = l - 1$ or $j = l - 2$, $C \in M_j$, and $E \subset C \subset U$, then the lemma holds for C. Let $D \in M_l$ with $E \subset D \subset U$. Then $D \cap U_1 \cap \widehat{Q}_{k+1} = E$. Furthermore $D' \in M_{l-1}$, $E \subset D' \subset U$, $D'' \in M_{l-2}$, and $E \subset D'' \subset U$. Thus the lemma holds for D' and D''.

Let $p \in D \cap U_{x_n}$ for some $x_n \in A \cap X_0$. If $p \in \widehat{Q}_{l-1}$ then $f_D(p) \subset f_{D'}(p) \in V_{y_n}$. If $p \in Q_l$ then p is the midpoint of some $I_\sigma \in \mathbf{I}_{l-1,n}$ with $\sigma(1) > c$. But $f_D(p) = h_{D''}(p)$ is the midpoint of $\Gamma_{D''}(I_\sigma)$ and $\Gamma_{D''}(I_\sigma) \in \mathbf{J}_{l-1,n}$ with $\Gamma_{D''}(\sigma)(1) > b$. Hence $f_D(p) \in V_{y_n}$.

Let $p \in D \cap U_q$ for some $q \in A \backslash X_0$. If $p \in \widehat{Q}_{l-1}$ then $f_D(p) = f_{D'}(p) \in V$. If $p \in Q_l$ then $f_D(p) = h_{D''}(p) = \Gamma^*_{D''}(p) = \Gamma^*_E(p) \in V$ because $h_E(U_q) \subset V$.

Let $I_\sigma \in \mathbf{F}_D \cap \mathbf{I}_{l+1,n}$ for some $x_n \in A \cap X_0$ and let $\sigma(1) > c$. Either $\gamma_D(I_\sigma) = \Gamma^*_{D'}(I_\sigma)$ or $\gamma_D(I_\sigma) = \Gamma^*_{D''}(I_\sigma)$. In either case, $\gamma_D(I_\sigma) \in \mathbf{J}_{l+1,n}$ and $\gamma_D(\sigma)(1) > b > a$.

To show that conditions 3 and 4 hold, consider the sets $\mathbf{F} = \{I \in \mathbf{I}_{l+1} : I \subset \bigcup \mathbf{F}_E\}$ and $\mathbf{G} = \{I \in \mathbf{F}_D : \sigma(1) \le a\}$. Define γ on $\mathbf{G}$ by $\gamma(I) = \Gamma^*_E(I)$. Let $I_\sigma \in \mathbf{F}$. Then $I_\sigma \in \widehat{\mathbf{F}}_D$ because $E \subset D$ and $h_{D'}(I_\sigma) = h_E(I_\sigma) \in \widehat{\mathbf{F}}_{\theta(D)}$ because $\theta(E) \subset \theta(D)$. Assume that $I_\sigma \notin \mathbf{F}_{D1}$. Let $p \subset D \cap Q_1$. We will show that $f_D(p)$ cannot be an endpoint of $h_{D'}(I_\sigma)$. If $p \in U_0$, then $f_D(p) \in V_0$. But $\Gamma_{D'}(\sigma)(1) \le a$ so $f_D(p)$ is not an endpoint of $h_{D'}(I_\sigma)$. If $p \in U_1$ then $p \in I_\tau$ for some $I_\tau \in \mathbf{I}_{k+2}$ with $\tau(1) \le a$. By the induction hypotheses, $f_D(p) = h_{D''}(p) = h_E(p) \in h_E(I_\tau)$. If $\sigma \restriction k+2 \neq \tau$ then $I_{\sigma \restriction k+2} \cap I_\tau = \varnothing$ so p cannot be an endpoint of any subinterval of $I_{\sigma \restriction k+2}$. If $\sigma \restriction k+2 = \tau$ then p is not an endpoint of I_σ because $I_\sigma \notin \mathbf{F}_{D1}$. The assumption that $I_\sigma \notin \mathbf{F}_{D1}$ also implies that $h_D(I_\sigma) = h_{D'}(I_\sigma) = h_E(I_\sigma)$. But h_E^{-1} is continuous at $h_E(p)$, so $h_E(p)$ cannot be an endpoint of $h_E(I_\sigma)$. Therefore $h_D(I_\sigma) \in \widehat{\mathbf{F}}_{\theta(D)} \backslash \mathbf{F}_{\theta(D)1}$ and $I_\sigma \in \mathbf{F}_{D2}$. By Lemma 3, $\gamma(I) = \Gamma^*_E(I)$ for all $I \in \mathbf{I}_{l+1}$. If $I \in \mathbf{G}$ then either $\gamma_D(I) = \Gamma^*_{D'}(I)$ or $\gamma_D(I) = \Gamma^*_{D''}(I)$. In either case, $\gamma_D(I) = \Gamma^*_E(I) = \gamma(I)$. Thus $\gamma_D(I_\sigma) \in \mathbf{J}_{l+1,n}$ and $\gamma_D(\sigma)(1) \le a < b$ for all $I_\sigma \in \mathbf{F}_D \cap \mathbf{I}_{l+1,n}$ with $m \le \sigma(1) \le b$. By Lemma 4, if $I_\sigma \in \mathbf{I}_{l+1,n}$ for some $x_n \in A \cap X_0$ and $\sigma(1) > c$, then $\Gamma_D(I_\sigma) \in \mathbf{J}_{l+1,n}$ and $\Gamma_D(\sigma)(1) > b$. By Lemma 5, if $I_\sigma \in \mathbf{I}_{l+1}$ and $\sigma(1) \le a$, then $\Gamma_D(I_\sigma) = \Gamma^*_E(I_\sigma)$.

Now let $B \in [A, U]$ and let $B \in S_D$. Then $D \in M_l$ for some $l \ge k$ and $E \subset D \subset U$. Also, $B \cap X_0 = A \cap X_0$ so $\lambda(B \cap X_0) = \lambda(A \cap X_0) \subset V$. Let $p \in B \backslash X_0$. If $p \in D$ then $f_D(p) \in V$ by Lemma 7. Assume that $p \notin D$. There is $I_\sigma \in \mathbf{I}_{l+1}$ such that $p \in I_\sigma$. If $p \in U_{x_n}$ for some $x_n \in A \cap X_0$ then $I_\sigma \in \mathbf{I}_{l+1,n}$ and $\sigma(1) > c$. By Lemma 7, $h_D(I_\sigma) = \Gamma^*_D(I_\sigma) \in \mathbf{J}_{l+1,n}$ and $\Gamma^*_D(\sigma)(1) > b$. Thus $h_D(p) \in V$. If $p \in U_q$ for some $q \in A \backslash X_0$ then $\sigma(1) \le a$. By Lemma 7, $h_D(I_\sigma) = \Gamma^*_D(I_\sigma) = \Gamma^*_E(I_\sigma)$. Thus $h_D(p) \in V$ because $h_E(U_q) \subset V$. Therefore $H(B) \in [H(A), V]$ and H is continuous. A similar argument shows that H^{-1} is continuous.

IV. Related results

Corollary 8. *If X and Y are ω-graphs and D and E are equipotent discrete subsets of X and Y respectively, then $\bigcup_{p \in D}[p, X]$ is homeomorphic to $\bigcup_{p \in E}[p, Y]$.*

Proof. Extend D and E to dividing sets X_0 and Y_0 of X and Y. Order the sets X_0 and Y_0 so that $\lambda(D) = E$. Then the homeomorphism defined in the proof of Theorem 1 takes $\bigcup_{p\in D}[p, X]$ to $\bigcup_{p\in E}[p, Y]$, so these two sets are homeomorphic.

The finally results are about spaces other than graphs or ω-graphs. Theorem 2 of [N] shows that points may be removed from certain T_1 spaces without affecting its Pixley-Roy hyperspace. The next three lemmas generalize this result. Theorem 12 applies this procedure to $\mathbf{R}^n$.

Lemma 9. *If* $\langle Z_n : n < \omega\rangle$ *is a sequence of disjoint homeomorphic open and closed subsets of* $\mathrm{PR}[X]$ *such that* $\bigcup_{n<\omega} Z_n$ *is open and closed in* $\mathrm{PR}[X]$, *then* $\mathrm{PR}[X]\backslash Z_0 \approx \mathrm{PR}[X]$.

Proof. For each $n < \omega$ let $H_n : Z_n \to Z_{n+1}$ be a homeomorphism. Define $H: \mathrm{PR}[X] \to \mathrm{PR}[X]\backslash Z_0$ by

$$H(A) = \begin{cases} A & \text{if } A \notin \bigcup_{n<\omega} Z_n, \\ H_n(A) & \text{if } A \in Z_n. \end{cases}$$

Then H is a homeomorphism.

Lemma 10. *If* U *is an open subset of space* X *and* C *is closed in* U *then* $\bigcup_{p\in C}[p, U]$ *is open and closed in* $\mathrm{PR}[X]$.

Proof. Clearly $\bigcup_{p\in C}[p, U]$ is an open subset of $\mathrm{PR}[X]$. Let

$$A \in U\backslash \bigcup_{p\in C}[p, U].$$

If $A \not\subset U$ then $[A, X]$ is a neighborhood of A that misses $\bigcup_{p\in C}[p, U]$. If $A \subset U$ then $A\cap C = \varnothing$, so $[A, U\backslash C]$ is a neighborhood of A in $\mathrm{PR}[X]$ that misses $\bigcup_{p\in C}[p, U]$.

Lemma 11. *Let* $\langle U_n : n < \omega\rangle$ *be a sequence of disjoint open subsets of a space* X *and let* $\langle C_n : n < \omega\rangle$ *be a sequence of subsets of* X *such that* $C_n \subset U_n$ *and* C_n *is closed in* U_n *for all* $n < \omega$. *Then* $\bigcup_{n<\omega}\bigcup_{p\in C_n}[p, U_n]$ *is open and closed in* $\mathrm{PR}[X]$.

Proof. It is clear that $\bigcup_{n<\omega}\bigcup_{p\in C_n}[p, U_n]$ is open in $\mathrm{PR}[X]$. By Lemma 10, each $\bigcup_{p\in C_n}[p, U_n]$ is closed in $\mathrm{PR}[X]$. Let $A \in \mathrm{PR}[X]$. Since A is finite and the U_n's are disjoint, there is a finite subset B of ω such that $A\cap U_n \neq \varnothing$ if and only if $n \in B$. Then $(\bigcup_{m\in B}[A, U_m]) \cap (\bigcup_{p\in U_n}[p, U_n]) \neq \varnothing$ only if $n \in B$. Thus $\{\bigcup_{p\in C_n}[p, U_n] : n < \omega\}$ is locally finite, and $\bigcup_{n<\omega}\bigcup_{p\in C_n}[p, U_n]$ is closed.

Theorem 12. *Let* $0 < n < \omega$ *and let* $X = \{\overline{x} \in R^n : 0 < |\overline{x}| < 1\}$ *where* $|\overline{x}|$ *denotes the Euclidean norm. For any* $0 < m < \omega$,

$$\mathrm{PR}[\mathbf{R}^n] \approx \mathrm{PR}[m \times \mathbf{R}^n] \approx \mathrm{PR}[\omega \times \mathbf{R}^n] \approx \mathrm{PR}[m \times X] \approx \mathrm{PR}[\omega \times X].$$

Proof. We will show that each of these spaces is homeomorphic to $\mathrm{PR}[\mathbf{R}^n]$. Let D be a discrete subset of $\{x \in \mathbf{R} : x \geq 0\}$ which contains 0 and let $\pi : \mathbf{R}^n \to \mathbf{R}$

be the projection onto the first coordinate. Let $L = \{\overline{x} \in R^n : \pi(\overline{x}) \in D\}$ and let $C = \{\overline{x} \in \mathbf{R}^n : |\overline{x}| \in D\}$. If D is finite then $\mathbf{R}^n \backslash L = (|D| + 1) \times \mathbf{R}^n$ and $\mathbf{R}^n \backslash C = |D| \times X$. If D is infinite then $\mathbf{R}^n \backslash L \approx \omega \times \mathbf{R}^n$ and $\mathbf{R}^n \backslash C \approx \omega \times X$. Let $U_0 = \mathbf{R}^n$ and let $\langle U_k : 0 < k < \omega \rangle$ be a sequence of disjoint open balls in $\mathbf{R}^n$, each of which has empty intersection with L and C.

Set $C_0 = L$. For every $0 < k < \omega$ let C_k be a subset of U_k which is homeomorphic to L. Then C_k is closed in U_k for all $k < \omega$. For each $k < \omega$ set $Z_k = \bigcup_{p \in C_k} [p, U_k]$. By Lemma 10, each Z_k is open and closed in $\mathrm{PR}[\mathbf{R}^n]$. By Lemma 11, $\bigcup_{0<k<\omega} Z_k$ is open and closed in $\mathrm{PR}[\mathbf{R}^n]$, so $\bigcup_{k<\omega} Z_k$ is open and closed in $\mathrm{PR}[\mathbf{R}^n]$. Clearly each Z_k is homeomorphic to every other Z_k, so $\mathrm{PR}[\mathbf{R}^n] \approx \mathrm{PR}[\mathbf{R}^n] \backslash Z_0 \approx \mathrm{P}R[\mathbf{R}^n \backslash L]$. If D is finite then $\mathrm{PR}[\mathbf{R}^n] \approx \mathrm{PR}[(|D| + 1) \times \mathbf{R}^n]$. If D is infinite then $\mathrm{PR}[\mathbf{R}^n] \approx \mathrm{PR}[\omega \times \mathbf{R}^n]$.

Now let $C_0 = C$ and for every $k < \omega$ let C_k be a subset of U_k homeomorphic to C. Set $Z_k = \bigcup_{p \in C_k} [p, U_k]$ for all $k < \omega$. Again, $\langle Z_k : k < \omega \rangle$ is a sequence of disjoint homeomorphic open and closed subsets of $\mathrm{PR}[\mathbf{R}^n]$ so $\mathrm{PR}[\mathbf{R}^n] \approx \mathrm{PR}[\mathbf{R}^n] \backslash Z_0 \approx \mathrm{PR}[\mathbf{R}^n \backslash C]$. If D is finite then $\mathrm{PR}[\mathbf{R}^n] \approx \mathrm{PR}[|D| \times X]$. If D is infinite then $\mathrm{PR}[\mathbf{R}^n] \approx \mathrm{PR}[\omega \times X]$.

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