PIXLEY-ROY HYPERSPACES OF $\omega$-GRAPHS

J. D. MASHBURN

Abstract. The techniques developed by Wage and Norden are used to show that the Pixley-Roy hyperspaces of any two $\omega$-graphs are homeomorphic. The Pixley-Roy hyperspaces of several subsets of $\mathbb{R}^n$ are also shown to be homeomorphic.

I. Introduction

Since it was introduced in 1969, the Pixley-Roy hyperspace, $\text{PR}[X]$, of a topological space $X$ has been intensely studied with the hope of establishing how the properties of $X$ affect those of $\text{PR}[X]$. This study has met with some success, especially in the area of cardinal functions. However, there is a class of questions which, until recently, eluded investigators: For which spaces $X$ and $Y$ will $\text{PR}[X]$ be homeomorphic to $\text{PR}[Y]$? For several years the only results in this area were some embedding results obtained by van Douwen [vD] and Lutzer [L]. In 1985 Wage [W] achieved a breakthrough by developing a technique for breaking up neighborhoods around points in certain spaces which allowed him to define homeomorphisms between those neighborhoods. Using this technique he was able to show that Pixley-Roy hyperspaces of spaces like $\mathbb{R}$ or $[0,1]$ are homogeneous. In 1986 Norden [N] extended Wage's technique to one which broke up an entire space. With this he was able to show that the Pixley-Roy hyperspaces of any two $P$-graphs (one-dimensional polyhedra with a finite number of points removed) are homeomorphic. It follows that the Pixley-Roy hyperspaces of spaces like $\mathbb{R}$, $[0,1]$, and the circle are all homeomorphic. It is the purpose of this paper to use Norden's technique to show that Pixley-Roy hyperspaces of infinite, as well as finite, graphs are all the same.

Definition. A $T_2$ space $X$ with no isolated points is an $\omega$-graph if there is a countable discrete subset $D$ of $X$ and a countable collection $I$ of pairwise disjoint copies of $(0,1)$ such that $X \setminus D = \bigcup I$, $I$ is locally finite on $X$, and for every $x \in D$, $\{x\} \cup (\bigcup \{I \in I : x \in I\})$ is a neighborhood of $x$ which can be embedded in $\mathbb{R}^2$. The set $D$ is called a dividing set for $X$.

The main result of this paper can be stated as follows.
Theorem 1. If $X$ and $Y$ are $\omega$-graphs then $\text{PR}[X]$ is homeomorphic to $\text{PR}[Y]$.

§II will consist of preliminary definitions, notation, and observations necessary for the proof of the Theorem 1. Theorem 1 will be proved in §III, and §IV will contain some related results.

We will use $\text{PR}[X]$ to denote the Pixley-Roy hyperspace of $X$. Our notation for the open subsets of $\text{PR}[X]$ will be standard. We will use $F[A]$ to denote the set of nonempty finite subsets of a set $A$, and $F'[A]$ to denote the set of all finite subsets of $A$. The notation "$X \approx Y$" will mean that $X$ is homeomorphic to $Y$.

II. Preliminary matters

Let $X$ be an $\omega$-graph and let $X_0$ be a dividing set for $X$. Enumerate $X_0$ as $\{x_n : n < \omega\}$. Let $I_0$ be the countable collection of pairwise disjoint copies of $(0,1)$ whose union makes up $X \setminus X_0$. We may assume that every element of $I_0$ has at least one endpoint in $X_0$. For each $n < \omega$ let $\mu(n)$ be the number of elements of $X \setminus X_0$ having $x_n$ as an endpoint. For each $I \in I_0$, fix a linear structure and orientation for $I$. Let $Q_0$ be the set of all midpoints of elements of $I_0$ and, for each $p \in X_0$, let $O_p$ be the component of $X \setminus Q_0$ containing $p$. Then $Q_0$ is a discrete subset of $X$ and $O_p \cap O_q = \emptyset$ if $p \neq q$.

For each $p \in X_0$ and each $I \in I_0$ having $p$ as an endpoint, choose a sequence of points in $I \cap Q_0$ converging monotonically to $p$. This can be done because each element of $X_0$ is the endpoint of at least one element of $I_0$. Let $Q_1$ be the set of all points of $X$ which are elements either of $Q_0$ or of the sequences just chosen. Call $Q_1$ the 1st cut-set of $X$. Set $\hat{Q}_1 = Q_1$. Let $I_1$ be the countable collection of pairwise disjoint copies of $(0,1)$ whose union makes up $X \setminus (\hat{Q}_1 \cup X_0)$. Call $I_1$ the set of intervals in $X$ derived from $\hat{Q}_1$.

Assume that $n < \omega$, that $Q_n$ is a discrete subset of $X \setminus X_0$, and that $I_n$ is a countable collection of pairwise disjoint intervals in $X$. Let $Q_{n+1}$, the $(n+1)$th cut-set of $X$, be the set of midpoints of elements of $I_n$ and let $\hat{Q}_{n+1} = \hat{Q}_n \cup Q_{n+1}$. Let $I_{n+1}$, the set of intervals in $X$ derived from $\hat{Q}_{n+1}$, be the countable collection of pairwise disjoint copies of $(0,1)$ whose union makes up $X \setminus (\hat{Q}_{n+1} \cup X_0)$. Set $Q = \bigcup_{n<\omega} Q_n$.

For every $1 < m < \omega$ and every $n < \omega$, let $I_{m,n} = \{I \in I_m : I \subset O_{x_n}\}$. This is the set of those elements of $I_m$ which "cluster" around $x_n$.

For every $1 < n < \omega$ let $\Sigma(n)$ be the set of sequences, $\sigma$, defined on $n+1$ such that $\sigma(0), \sigma(1) \in \omega$ and $\sigma(m) \in \{0,1\}$ for all $1 < m \leq n$. Let $m < \omega$. Since $I_{1,m}$ is countable, it can be enumerated as $\{I_{(m,n)} : n < \omega\}$. In this way the set $I_{1}$ is indexed by $\Sigma(1)$. Assume that the elements of $\Sigma(n)$ have been used to index the elements of $I_n$. Let $I \in I_{n+1}$. There is a unique $\sigma \in \Sigma(n)$ such that $I \subset I_\sigma$. If $I$ is the left-hand half of $I_\sigma$, then let $\tau$ be the element of $\Sigma(n+1)$ such that $\tau \upharpoonright n + 1 = \sigma$ and $\tau(n+1) = 0$ and set $I_\tau = I$. If $I$ is the right-hand half of $I_\sigma$, then let $\tau$ be the element of $\Sigma(n+1)$ such that $\tau \upharpoonright n + 1 = \sigma$ and $\tau(n+1) = 1$ and set $I_\tau = I$. Let $\Sigma = \bigcup_{1 \leq n < \omega} \Sigma(n)$.
The following lemma consists of observations which are immediate consequences of the previous definitions and its proof is omitted.

**Lemma 2.** Let \( 1 \leq m \leq n < \omega \).

1. If \( I \in I_n \) then \( I \cap Q_m \neq \emptyset \).
2. If \( p \in Q_m \) then there are exactly two elements, \( I_1 \) and \( I_2 \), of \( I_n \) such that \( p \) is an endpoint of both \( I_1 \) and \( I_2 \). Furthermore, \( I_1 \cup I_2 \cup \{p\} \) is open in \( X \).
3. If \( I \in I_m \) then there are exactly two elements of \( I_{m+1} \) that are subintervals of \( I \).
4. If \( I_\sigma \in I_n \) then there is exactly one element, \( I_{\sigma 1} \), of \( I_m \) that contains \( I_\sigma \).
5. If \( \sigma \in \Sigma(1) \), \( \sigma(0) = k \), and \( \sigma(1) = 1 \), then \( I_{\sigma 1} \) is the \( I \)th element of \( I_{1,k} \).
6. If \( J_\sigma \in I_{n,k} \) then \( \sigma \in \Sigma(n) \) and \( \sigma(0) = k \).
7. For any \( n, k < \omega \), \( \{\text{Int}(\text{Cl}(\bigcup\{I_\sigma \in I_{n,k} : \sigma(1) > a\})) : a < \omega\} \) forms a local base for \( x_k \).

For each \( p \in X \) and each \( 1 \leq n < \omega \) let \( A_n(p) = \{I \in I_n : p \in I\} \) and let \( A^*_n(p) = \bigcup A_n(p) \). If \( p \in Q_n \) then \( A(p) \) and \( A^*(p) \) will denote \( A_{n+1}(p) \) and \( A^*_{n+1}(p) \) respectively. If \( B \in \text{PR}[X] \) then set \( A_n(B) = \bigcup_{p \in B} A_n(p) \) and \( A^*_n(B) = \bigcup_{p \in B} A^*_n(p) \). If \( B \in \text{PR}[Q_n] \) then set \( A(B) = \bigcup_{p \in B} A(p) \) and \( A^*(B) = \bigcup_{p \in B} A^*(p) \).

Set \( M_0 = \{\emptyset\} \) and, for each \( 1 \leq n < \omega \), let \( M_n = \{E \in F(\hat{Q}_n) : E \cap Q_m \neq \emptyset \} \) for all \( 1 \leq m \leq n \}. \) For \( 1 \leq n < \omega \) call \( M_n \) the set of elements of \( \text{PR}[X] \) compatible with \( \hat{Q}_n \). Note that if \( m > n \) and \( E \in M_n \) then \( E \cap Q_m = \emptyset \). Also, if \( k \neq l \) then \( M_k \cap M_l = \emptyset \). For each \( n < \omega \) and each \( E \in M_n \), let \( S_E = \{A \in \text{PR}[X] : A \cap \hat{Q}_{n+1} = E\} \). Thus, if \( A \in S_E \) and \( E \in M_n \), then \( A \cap Q_{n+1} = \emptyset \). The set \( \{S_E : E \in M\} \) where \( M = \bigcup_{n<\omega} M_n \) is a partition of \( \text{PR}[X] \) and is called the fundamental partition of \( \text{PR}[X] \) based on \( M \). If \( E \in M_n \) then \( S_E \) can be written as \( \{A \cup B \cup E : A \in F'[X_0] \) and \( B \in F'[X \setminus (\hat{Q}_{n+1} \cup X_0)]\} \). Recall that \( X \setminus (\hat{Q}_{n+1} \cup X_0) = \bigcup I_{n+1} \).

For each \( E \in M_n \) let \( \hat{F}_E = \{I \in I_{n+1} : I \subset A^*(E)\} \). If \( n \geq 2 \), let \( E' = E \setminus Q_n = E \cap \hat{Q}_{n-1} \). If \( n \geq 3 \) then \( E'' = E \setminus \hat{Q}_{n-2} \). If \( n = 2 \) then set \( E'' = \emptyset \).

Now let \( Y \) be another \( \omega \)-graph and let \( Y_0 \) be a dividing set for \( Y \). Enumerate \( Y_0 \) as \( \{y_n : n < \omega\} \). Then the function \( \lambda : X_0 \to Y_0 \) given by \( \lambda(x_n) = y_n \) is a bijection. Let \( J_0 \) be a countable collection of pairwise disjoint copies of \((0, 1)\) whose union is \( Y \setminus Y_0 \). We may again assume that every element of \( J_0 \) has at least one endpoint in \( Y_0 \). Let \( R_0 \) be the set of midpoints of elements of \( J_0 \). Let \( \{R_n : 1 \leq n < \omega\} \) be the collection of cut-sets for \( Y \) and set \( R = \bigcup_{n<\omega} R_n \). Let \( P_n \) be the component of \( Y \setminus R_0 \) that contains \( y_n \). For each \( 0 < n < \omega \) let \( J_n \) be the set of intervals of \( \text{PR}[Y] \) derived from \( R_n \), each indexed as before by the elements of \( \Sigma \). Let \( \{N_k : k < \omega\} \) be the collection of
sets of elements of \( PR[Y] \) compatible with \( \{ R_k : k < \omega \} \) and let \( \{ T_E : E \in N \} \) be the fundamental partition of \( PR[Y] \) based on \( N = \bigcup_{k < \omega} N_k \). If \( E \in Q \) and \( f : E \to R \), then \( f \) is level preserving if \( f(E \cap Q_n) \subset R_n \) for all \( n < \omega \).

For each \( I \in I_n \) and \( J \in J_n \), there is a unique linear homeomorphism between \( I \) and \( J \) that preserves orientation. Denote this homeomorphism by \( \eta_{I,J} \). If \( \sigma, \tau \in \Sigma(n) \), \( I = I_{\sigma}|_{m+1} \), and \( J = J_{\tau}|_{m+1} \) for some \( m < n \), then \( \eta_{I,J}(I_\sigma) = J_\tau \) if and only if \( \sigma(k) = \tau(k) \) for all \( m < k \leq n \). If \( \Gamma : I_n \to J_n \) is a bijection, then \( \Gamma^* : \bigcup I_n \to \bigcup J_n \) is the function \( \bigcup_{I \in I_n} \eta_{I,J(I)} \). \( \Gamma^* \) is a homeomorphism that is linear and orientation preserving on each element of \( I_n \).

Now order each \( I_n \) and \( J_n \) lexicographically using the indices of their elements. These collections then have order-type \( \omega^2 \). Let \( F \subset I_n \) and \( G \subset J_n \) be equipotent finite sets and let \( \gamma : F \to G \) be a bijection. Then \( I_n \setminus F \) and \( J_n \setminus G \) still have order-type \( \omega^2 \), so there is a unique order isomorphism \( \Delta_F : I_n \setminus F \to J_n \setminus G \). Define \( \Gamma : I_n \to J_n \) by \( \Gamma = \gamma \cup \Delta_F \). Then \( \Gamma \) is a bijection.

In those situations where more than one \( F \) is being considered and subscripts are used to distinguish the various set, the same subscripts will be used to distinguish the corresponding \( \gamma \), \( \Delta \), and \( \Gamma \) functions. For example, the functions associated with \( F_1 \) will be \( \gamma_1 \), \( \Delta_1 \), and \( \Gamma_1 \).

It will be necessary in what follows to compare the index of \( I_\sigma \) with that of \( \gamma(I_\sigma) \) or \( \Gamma(I_\sigma) \). In order to facilitate this, we will use \( \gamma(\sigma) \) and \( \Gamma(\sigma) \) to denote the indices of \( \gamma(I_\sigma) \) and \( \Gamma(I_\sigma) \) respectively.

The next lemma is obvious and its proof is omitted.

**Lemma 3.** Let \( m < n < \omega \) and let \( F_1 \subset I_m \) and \( F_2 \subset I_n \) with \( \{ I \in I_n : I \subset F_1 \} \subset \bigcup F_2 \). If \( \gamma_1 : F_1 \to J_m \) is a one-to-one function and \( \gamma_2 : F_2 \to J_n \) is defined by \( \gamma_2(I) = \Gamma_1(I) \), then \( \Gamma_2(I) = \Gamma_2^*(I) \) for all \( I \in I_n \).

**Lemma 4.** Let \( F \subset I_k \) be finite and let \( \gamma : F \to J_k \) be a one-to-one function. Assume that there are \( b, c, m < \omega \) such that

1. \( c - m > b \);
2. if \( I_\sigma \in F \) then either \( \sigma(1) \leq b \) or \( \sigma(1) > c \);
3. if \( I_\sigma \in F \cap I_{k,n} \) and \( m \leq \sigma(1) \leq b \) then \( \gamma(I_\sigma) \in J_{k,n} \) and \( \gamma(\sigma)(1) \leq b \);
4. if \( I_\sigma \in F \cap I_{k,n} \) and \( \sigma(1) > c \) then \( \gamma(I_\sigma) \in J_{k,n} \) and \( \gamma(\sigma)(1) > b \).

Then \( \Gamma(I_\sigma) \in J_{k,n} \) and \( \Gamma(\sigma)(1) > b \) for all \( I_\sigma \in I_{k,n} \) with \( \sigma(1) > c \).

**Proof.** Let \( n < \omega \). The elements of \( J_{k,n} \setminus \gamma(F) \) are the images under \( \Delta_F \) of \( I_{k,n} \setminus F \). By conditions 2 and 3,

\[
|F \cap \{ I_\sigma : I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c \}| = |F \cap \{ I_\sigma : I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq b \}|
\]

\[
= |\{ \gamma(I_\sigma) : I_\sigma \in I_{k,n} \text{ and } m \leq \sigma(1) \leq b \}|
\]

\[
\leq |\{ J_\sigma \in J_{k,n} : J_\sigma \in \gamma(F) \text{ and } \sigma(1) \leq b \}|
\]

\[
= |\gamma(F) \cap \{ J_\sigma : J_\sigma \in J_{k,n} : \sigma(1) \leq b \}|.
\]
Also, $|\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}| \geq |\{I_\sigma \in I_{k,n} : \sigma(1) \leq c\}|$ because $c - m > b$. Therefore,

$$
|\{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c\}\setminus F|
= |\{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c\}\setminus (F \cap \{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c\})|
\geq |\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}\setminus (\gamma(F) \cap \{J_\sigma \in J_{k,n} : \sigma(1) \leq b\})|
= |\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}\setminus \gamma(F)|.
$$

Thus, if $J_\tau \in J_{k,n}$ and $\tau(1) \leq b$ then there is $I_\sigma \in I_k$ such that either $I_\sigma \in F$ or $I_\sigma \in I_{k,n}$ and $\sigma(1) \leq c$, and $\Gamma(I_\sigma) = J_\tau$. It follows from this and condition 4 that if $I_\sigma \in I_{k,n}$ and $\sigma(1) > c$, then $\Gamma(I_\sigma) \in J_{k,n}$ and $\Gamma(\sigma)(1) > b$.

**Lemma 5.** Let $F_1, F_2 \subseteq I_k$ be finite and let $\gamma_1 : F_1 \to J_k$ and $\gamma_2 : F_2 \to J_k$ be one-to-one functions. Let $a, b, m < \omega$ such that

1. $b - a > m$;
2. $\{I_\sigma \in F_1 : \sigma(1) \leq a\} = \{I_\sigma \in F_2 : \sigma(1) \leq a\} = G$; and
3. $\gamma_1(I_\sigma) = \gamma_2(I_\sigma)$ for all $I_\sigma \in G$;

and that for $i = 1$ or 2,

4. if $J_\sigma \in \gamma_i(F_i)$ then either $\sigma(1) \leq a$ or $\sigma(1) > b$;
5. if $I_\sigma \in F_i$ and $\sigma(1) > b$ then $\gamma_i(\sigma)(1) > a$; and
6. for all $n < \omega$, if $J_\sigma \in \gamma_i(F_i) \cap J_{k,n}$ and $\gamma_i^{-1}(J_\sigma) \notin I_{k,n}$ then $\sigma(1) < m$.

Then $\Gamma_1(I_\sigma) = \Gamma_2(I_\sigma)$ for all $I_\sigma \in I_n$ with $\sigma(1) \leq a$.

**Proof.** Let $n < \omega$. By condition 2,

$$
\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap F_1 = I_{k,n} \cap G = \{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap F_2
$$

and

$$
\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\}\setminus F_1 = \{I_\sigma \in I_{k,n} : \sigma(1) \leq a\}\setminus G
= \{I_\sigma \in I_{k,n} : \sigma(1) \leq a\}\setminus F_2.
$$

By conditions 2, 3, and 4,

$$
\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}\setminus \gamma_1(F_1) = \{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}\setminus \gamma_1(G)
= \{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}\setminus \gamma_2(F_2).
$$

If $I_\sigma \in I_{k,n} \cap G$ then $\Gamma_1(I_\sigma) = \gamma_1(I_\sigma) = \gamma_2(I_\sigma) = \Gamma_2(I_\sigma)$. The values of $\Gamma_1$ and $\Gamma_2$ on $\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\}\setminus G$ are determined by $\Delta_1$ and $\Delta_2$ respectively. We can establish the equality of $\Gamma_1$ and $\Gamma_2$ on $\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\}\setminus G$ by showing that this set is no larger than $\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}\setminus \gamma_1(G)$. Then, since both $\Delta_1$ and $\Delta_2$ take the $\alpha$th element of $\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\}\setminus G$ to the
αth element of \( \{ J_\sigma \in \mathbf{G} J_{k,n} : \sigma(1) \leq b \} \backslash \gamma_1(\mathbf{G}) \), they must be equal.

\[
\begin{align*}
|\{ J_\sigma \in \mathbf{J}_{k,n} : \sigma(1) \leq b \} \backslash \gamma_1(\mathbf{G})| &= \left| \{ J_\sigma \in \mathbf{J}_{k,n} : \sigma(1) \leq a \} \backslash \gamma_1(\mathbf{G}) \right| + \left| \{ J_\sigma \in \mathbf{J}_{k,n} : a < \sigma(1) \leq b \} \right| \\
&= \left| \{ J_\sigma \in \mathbf{J}_{k,n} : \sigma(1) \leq a \} \backslash \{ \gamma_1(I_\sigma) : I_\sigma \in \mathbf{G} \backslash \mathbf{I}_{k,n} \} \right| \\
&\quad + \left| \{ J_\sigma \in \mathbf{J}_{k,n} : a < \sigma(1) \leq b \} \right| \\
&\geq \left| \{ J_\sigma \in \mathbf{J}_{k,n} : \sigma(1) \leq a \} \backslash \{ \gamma_1(I_\sigma) : I_\sigma \in \mathbf{G} \cap \mathbf{I}_{k,n} \} \right| \\
&\quad + \left| \{ J_\sigma \in \mathbf{J}_{k,n} : a < \sigma(1) \leq b \} \right| \\
&= \left| \{ I_\sigma \in \mathbf{I}_{k,n} : \sigma(1) \leq a \} \backslash \{ I_\sigma \in \mathbf{G} \cap \mathbf{I}_{k,n} : \gamma_1(I_\sigma) \in \mathbf{J}_{k,n} \} \right| \\
&\geq \left| \{ I_\sigma \in \mathbf{I}_{k,n} : \sigma(1) \leq a \} \backslash \mathbf{G} \right|
\end{align*}
\]

(by condition 4)

by conditions 1 and 6)

III. Proof of Theorem 1

Let \( X \) and \( Y \) be \( \omega \)-graphs with dividing sets \( X_0 \) and \( Y_0 \). We will use the structures and definitions developed in §II. Let \( g : Q_1 \to R_1 \) be a bijection such that \( g(Q_1 \cap O_n) = R_1 \cap P_n \) for all \( n < \omega \). Then \( g(Q_0) = R_0 \). For our convenience later in the proof, we will assume that the first \( \mu(n) \) elements of any \( I_{m,n} \) are those elements of \( I_{m,n} \) having an element of \( Q_0 \) as an endpoint.

The homeomorphism we will define is essentially that defined by Norden in [N].

Define \( \Gamma_\phi : I_1 \to J_1 \) by \( \Gamma_\phi(I_\sigma) = J_\sigma \), and \( h_\phi : \bigcup I_1 \to \bigcup J_1 \) by \( h_\phi = \Gamma_\phi \). Then \( h_\phi \) is a homeomorphism. Set \( \theta(\phi) = \phi \).

Let \( E \in M_1 \). Set \( f_E = g \upharpoonright E \) and \( \theta(E) = f_E(E) \). Let \( F_E = \hat{F}_E \) and \( F_{\theta(E)}(E) = \hat{F}_{\theta(E)} \). Each \( I \in F_E \) is adjacent to exactly one element of \( E \) and each element of \( E \) is the endpoint of exactly two elements of \( F_E \). Similarly, each element of \( F_{\theta(E)} \) is adjacent to exactly one element of \( \theta(E) \) and each element of \( \theta(E) \) is the endpoint of exactly two elements of \( F_{\theta(E)} \).

Define \( \gamma_E : F_E \to F_{\theta(E)} \) as follows. Let \( I \in F_E \) and let \( p \in E \) be an endpoint of \( I \). If \( p \) is the right-hand endpoint of \( I \), then set \( \gamma_E(I) \) equal to the element of \( F_{\theta(E)} \) which has \( g(p) \) for its right-hand endpoint. If \( p \) is the left-hand endpoint of \( I \), then set \( \gamma_E(I) \) equal to the element of \( F_{\theta(E)} \) which has \( g(p) \) for its left-hand endpoint. Then \( \gamma_E \) is a bijection. Define \( h_E : (\bigcup I_1) \cup E \to (\bigcup J_2) \cup \theta(E) \) by \( h_E = \Gamma_\phi \cup f_E \). Both \( \Gamma_\phi \) and \( f_E \) are bijections so \( h_E \) is a bijection. It is also a homeomorphism on \( \bigcup I_2 \) because \( \Gamma_\phi \) is. Let \( x \in E \) and let \( V \) be a neighborhood of \( f_E(x) \) in \( Y \). By the definition of \( \gamma_E \) there is a neighborhood \( U \) of \( x \) in \( A^*(x) \cup \{ x \} \) such that \( h_E(U) \subset V \). Thus \( h_E \) is continuous at \( x \). A similar argument shows that \( h_E^{-1} \) is continuous at \( h_E(x) \), so \( h_E \) is a homeomorphism.
Let \( 2 \leq 1 < \omega \) and assume that for all \( k < 1 \) and all \( E \in M_k \),

1. \( f_E : E \to \hat{R}_k \) is a level preserving one-to-one function and \( \theta(E) = f_E(E) \);
2. \( F_E \subset I_{k+1} \) and \( \Theta(E) \subset J_{k+1} \) are finite and \( \gamma_E : F_E \to F_{\theta(E)} \) is a bijection; and
3. the function \( h_E : (\cup I_{k+1}) \cup E \to (\cup J_{k+1}) \cup \theta(E) \) given by \( h_E = \Gamma^* \cup f_E \) is a homeomorphism.

Fix \( E \in M_l \). Each element of \( E \cap Q_l \) is the midpoint of some element of \( I_{l-1} \) and \( h_E'' \), which is defined on \( \cup I_{l-1} \), takes midpoints to midpoints. Thus \( h_E''(p) \in R_l \) for all \( p \in E \cap Q_l \). Define \( f_E : E \to \hat{R}_l \) by

\[
f_E(p) = \begin{cases} h_E'(p) & \text{if } p \in E \cap \hat{Q}_{l-1}, \\ h_E''(p) & \text{if } p \in E \cap Q_l. \end{cases}
\]

Then \( f_E \) is a one-to-one level preserving function. Note that if \( p \in E \cap \hat{Q}_{l-1} \) then \( f_E(p) = h_E'(p) = f_E'(p) \). Extending this backward, we can see that if \( 1 \leq k < l \) and \( p \in E \cap \hat{Q}_k \) then \( f_E(p) = f_E(p) \).

Let \( F_{E1} = A(E \cap Q_l) \) and \( F_{\theta(E)1} = A(\theta(E) \cap R_l) \). Let \( I \in F_{E1} \) and let \( p \in E \cap Q_l \) be an endpoint of \( I \). Then \( f_E(p) = h_E''(p) \in R_l \) and \( h_E''(p) \) is an endpoint of \( h_E''(I) \) because \( h_E'' \) is continuous. Thus \( h_E''(I) \in F_{\theta(E)1} \). A similar argument shows that if \( h_E''(I) \in F_{\theta(E)1} \) then \( I \in F_{E1} \).

Let \( F_{E2} = \{ I \in \hat{F}_E \setminus F_{E1} : h_E'(I) \in \hat{F}_{\theta(E)} \setminus F_{\theta(E)1} \} \) and let \( F_{\theta(E)2} = \{ J \in \hat{F}_{\theta(E)} \setminus F_{\theta(E)1} : h_E''(J) \in \hat{F}_E \setminus F_{E1} \} \). Clearly \( I \in F_{E2} \) if and only if \( h_E'(I) \in F_{\theta(E)2} \). Set \( F_E = F_{E1} \cup F_{E2} \) and \( F_{\theta(E)} = F_{\theta(E)1} \cup F_{\theta(E)2} \). Define \( \gamma_E : F_E \to F_{\theta(E)} \) by

\[
\gamma_E(I) = \begin{cases} h_E''(I) & \text{if } I \in F_{E1}, \\ h_E'(I) & \text{if } I \in F_{E2}. \end{cases}
\]

Then \( \gamma_E \) is a bijection.

Define \( h_E : (\cup I_{l+1}) \cup E \to (\cup J_{l+1}) \cup \theta(E) \) by \( h_E = \Gamma^* \cup f_E \). The function \( h_E \) is a bijection because \( \Gamma^* \) and \( f_E \) are bijections and is a homeomorphism on \( \cup I_{l+1} \) because \( \Gamma^* \) is. If \( p \in E \cap \hat{Q}_l \) then \( A(p) \subset F_{E1} \) and \( h_E(A^*(p) \cup \{ p \}) = h_E''(A^*(p) \cup \{ p \}) \). Now let \( p \in E' \). If \( I \in A_{l+1}(p) \) then \( I \in \hat{F}_E \). Since \( p \) is an endpoint of \( I \) and \( p \in \hat{Q}_{l-1} \), the other endpoint of \( I \) must be an element of \( Q_{l+1} \). Hence \( I \notin F_{E1} \). To show that \( h_E'(I) \in \hat{F}_{\theta(E)1} \), note that \( p \in E' \) and \( h_E'' \) is continuous on \( (\cup I_l) \cup E' \). So \( f_E'(p) = F_{E'}(p) \) is an endpoint of \( h_E'(I) \). But \( f_E' \) is level preserving, so \( f_E'(p) \in \hat{R}_{l+1} \). Again, the other endpoint of \( h_E'(I) \) must be an element of \( R_{l+1} \). Hence \( h_E'(I) \in \hat{F}_{\theta(E)1} \). It follows that \( A_{l+1}(p) \subset F_{E2} \) and \( h_E(A^*_{l+1}(p) \cup \{ p \}) = h_E'(A^*_{l+1}(p) \cup \{ p \}) \). But \( h_E'' \) is a homeomorphism on \( (\cup I_l) \cup E' \) and \( h_E'' \) is a homeomorphism on \( \cup I_{l-1} \), so \( h_E \) is a homeomorphism on \( (\cup I_{l+1}) \cup E \).
Notice that for any \( k < \omega \), \( E \in M_k \), \( x_n \in X_0 \), and \( I_\sigma \in I_{k,n} \), if \( \Gamma_{E}(I_\sigma) \notin J_{k,n} \) then \( \sigma(1) < \mu(n) \) because only the first \( \mu(n) \) elements of \( I_{1,n} \) have endpoints in \( Q_0 \).

For all \( n < \omega \) and all \( E \in M_n \), define \( H_E : S_E \to T_{\theta(E)} \) by \( H_E(A) = \lambda(A \cap X_0) \cup h_E(A \setminus X_0) \). Finally, define \( H : PR[X] \to PR[Y] \) by \( H = \bigcup_{E \in M} H_E \). To show that \( H \) is a bijection it is sufficient to show that \( \theta \) is a bijection. Let \( E, D \in M \) and \( E \neq D \). Then \( \theta(E) = f_E(E) \) and \( \theta(D) = f_D(D) \). Both \( f_E \) and \( f_D \) are level-preserving one-to-one functions, so \( \theta(E) \neq \theta(D) \) if \( E \in M_k \) and \( D \in M_l \) and \( k \neq l \). Assume that \( E, D \in M_k \). Then \( \theta(E) = g(E) \neq g(D) = \theta(D) \) since \( g \) is a bijection. Assume that \( E, D \in M_k \) for some \( k > 1 \). Either \( E \cap Q_k \neq D \cap Q_k \) or \( E' \neq D' \). But the functions \( h_{E'} \), \( h_{E''} \), \( h_{D'} \), and \( h_{D''} \) are all one-to-one, so either \( h_{E''}(E \cap Q_k) \neq h_{D''}(D \cap Q_k) \) or \( h_{E'}(E') \neq h_{D'}(D') \). In either case, \( \theta(E) \neq \theta(D) \).

Let \( A \in S_E \) where \( E \in M_k \) and let \( V \) be a neighborhood of \( H(A) \in Y \). Pick \( a < \omega \) such that if \( t \in A(X) \) then \( \sigma(1) < a \) and if \( t \in A(X) \) then \( \sigma(1) < a \). Let \( m = \max(\{\mu(n) : A(A) \cap I_{1,n} \neq \emptyset \} + 1 \). Pick \( b \in \omega \) such that \( b - m > a \) and

\[
\{\{J_\sigma \in J_{1,n} : \sigma(1) > b\}\} \subset V
\]

for all \( y_n \in H(A) \cap Y_0 \). Set

\[
V_{y_n} = \text{Int}[\bigcup \{J_\sigma \in J_{1,n} : \sigma(1) > b\}]
\]

and set \( V_0 = \bigcup_{p \in H(A) \cap Y_0} V_p \). Pick \( c \in \omega \) such that \( c - m > b \) and if \( x_n \in A \cap X_0 \) and \( p \in Q_1 \cap \text{Int}[\bigcup \{J_\sigma \in J_{1,n} : \sigma(1) > c\}] \), then \( g(p) \in V_{y_n} \). For each \( x_n \in A \cap X_0 \) set \( U_{x_n} = \text{Int}[\bigcup \{J_\sigma \in J_{1,n} : \sigma(1) > c\}] \). Let \( U_0 = \bigcup_{p \in A \cap X_0} U_p \). If \( A \cap X_0 = \emptyset \) then set \( U_0 = \emptyset \). Pick \( r \geq k + 1 \) such that \( h_{E}(A_r(p)) \subset V \) for all \( p \in A \cap X_0 \). Set \( U_p = A_r(p) \cup \{p\} \) for \( p \in A \cap X_0 \) and set \( U_1 = \bigcup_{p \in A \cap X_0} U_p \). Let \( U = U_0 \cup U_1 \). Note that:

1. if \( I_\sigma \cap U_1 \neq \emptyset \) then \( \sigma(1) \leq a \);
2. if \( J_\sigma \cap (H(A) \setminus Y_0) \neq \emptyset \) then \( \sigma(1) \leq a \);
3. if \( I_\sigma \cap U_{x_n} \neq \emptyset \) for some \( x_n \in A \cap X_0 \) then \( I_\sigma \cap I_{1,n} \neq \emptyset \) and \( \sigma(1) > c \);
4. if \( J_\sigma \cap V_{y_n} \neq \emptyset \) for some \( y_n \in H(A) \cap Y_0 \) then \( J_\sigma \cap I_{1,n} \neq \emptyset \) and \( \sigma(1) > b \);
5. if \( p \in A \cap X_0 \) then \( U_p \cap \hat{Q}_k \subset \{p\} \).
6. \( a, b, c \) and \( m \) satisfy condition 1 in Lemmas 4 and 5; and
7. if \( I_\sigma \in I_{1,n} \) and \( m \leq \sigma(1) \) then \( H_D(I_\sigma) \subset \bigcup J_{1,n} \) for any \( 0 < 1 < \omega, n < \omega, \) and \( D \in M \).

The heart of the proof that \( H([A, U]) \subset [H(A), V] \) is contained in Lemmas 6 and 7.

**Lemma 6.** Let \( D \in M_j \) where \( 1 \leq j \leq k \), \( D \subset U \), and \( D \cap U_1 = E \cap \hat{Q}_j \). Let \( C = E \cap \hat{Q}_j \). Then

1. if \( p \in D \cap U_q \) for some \( q \in A \cap X_0 \) then \( f_D(p) \in V_{\lambda(q)} \);
2. if \( p \in D \cap U_q \) for some \( q \in E \) then \( p = q \) and \( f_D(p) = f_E(p) \);
3. if \( I_\sigma \in I_{j+1} \) and \( \sigma(1) \leq a \) then \( \Gamma_D(I_\sigma) = \Gamma_D(I_\sigma) \); and
4. if \( I_\sigma \in I_{j+1,n} \), \( x_n \in A \cap X_0 \), and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{j+1,n} \) and \( \Gamma_D(\sigma)(1) > b \).

Proof. To begin with, let us take note of three useful facts. First, since \( \Gamma_D(I_\sigma) = J_\sigma \) for all \( I_\sigma \in I_1 \), if \( I_\sigma \in I_{1,n} \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) = J_\sigma \in J_{1,n} \) and \( \Gamma_D(\sigma)(1) = \sigma(1) > c > b \). Also, for any \( j \), if \( p \in C \) then \( f_C(p) = f_E(p) \).

Furthermore, if \( I_\sigma \in E_D \) then either \( \sigma(1) \leq a < b \) or \( \sigma(1) > c \).

Let \( j = 1 \). Then \( D \subseteq Q_1 \) and \( D \cap U_1 = E \cap Q_1 \). Let \( p \in D \). If \( p \in U_q \) for some \( q \in A \cap X_0 \), then \( f_D(p) = g(p) \in V_{\delta(q)} \). If \( p \in U_q \) for some \( q \in A \cap X_0 \), then \( q \in E \), \( p = q \), and \( f_D(p) = g(p) = f_C(p) \).

Let \( n < \omega \) and let \( I_\sigma \in I_{2,n} \cap F_D \) with \( \sigma(1) > c \). Let \( p \in D \) be an endpoint of \( I_\sigma \). Since \( \sigma(1) > c \), \( p \) must be in \( U_x \). Then \( f_D(p) \), which is an endpoint of \( \gamma_D(I_\sigma) \), is in \( V_x \). Thus \( \gamma_D(I_\sigma) \in J_{2,n} \) and \( \gamma_D(\sigma)(1) > b > a \).

It follows from \( D \cap U_1 = C \) that \( F_C = \{ I_\sigma \in F_D : \sigma(1) \leq a \} \). Let \( I_\sigma \in F_C \).

Let \( p \in D \) be an endpoint of \( I_\sigma \). Then \( p \) must be an element of \( U_1 \), so \( f_D(p) = f_E(p) = f_C(p) \). Thus \( f_D(p) \) is an endpoint for both \( \gamma_C(I_\sigma) \) and \( \gamma_D(I_\sigma) \). Since both \( \gamma_C \) and \( \gamma_D \) preserve orientation, it must be true that \( \gamma_C(I_\sigma) = \gamma_D(I_\sigma) \).

Also, \( \gamma_D(\sigma)(1) \leq a < b \) because \( f_D(p) \in H(A) \setminus Y_0 \).

By Lemma 4, if \( I_\sigma \in I_{2,n} \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{2,n} \) and \( \Gamma_D(\sigma)(1) > b > a \).

Let \( 2 \leq j \leq k \) and assume that the lemma is valid for all \( 1 \leq i < j \) and all \( D \in M_j \) with \( D \subseteq U \) and \( D \cap U_1 = E \cap Q_i \). Let \( D \in M_j \) with \( D \subseteq U \) and \( D \cap U_1 = E \cap Q_j \). Then \( D' \in M_{j-1} \), \( D' \subseteq U \), and \( D' \cap U_1 = E \cap \hat{Q}_{j-1} = C' \), so the lemma is valid for \( D' \). If \( j = 2 \), then \( D'' = C'' = \emptyset \). If \( j > 2 \), then \( D'' \subseteq M_{j-2}, D'' \subseteq U \), and \( D'' \cap U_1 = E \cap \hat{Q}_{j-2} = C'' \). Thus the lemma is valid for \( D'' \).

Let \( p \in D \cap U_x \) for some \( x_n \in A \cap X_0 \). If \( p \in \hat{Q}_{j-1} \) then \( f_D(p) = f_D'(p) \in V_x \). If \( p \in Q_j \), then \( f_D(p) = h_D''(p) \). Now \( p \) is the midpoint of some element \( I_\sigma \) of \( I_{j-1,n} \), where \( \sigma(1) > c \). But \( \Gamma_D''(I_\sigma) \in J_{j-1,n} \), \( \Gamma_D''(\sigma)(1) > b \), and \( h_D''(p) \) is the midpoint of \( \Gamma_D''(I_\sigma) \). Hence \( f_D(p) \in V_x \).

Let \( p \in D \cap U_q \) for some \( q \in A \cap X_0 \). Then \( q \in E \) and \( q = p \). If \( p \in \hat{Q}_{j-1} \) then \( f_D(p) = f_D'(p) \). If \( p \in Q_j \) then \( f_D(p) = h_D''(p) = \Gamma_D''(p) = \Gamma_C''(p) = h_C''(p) = f_C(p) = f_E(p) \).

Let \( n < \omega \) and let \( I_\sigma \in F_D \cap I_{j+1,n} \) with \( \sigma(1) > c \). Either \( \gamma_D(I_\sigma) = \Gamma_D''(I_\sigma) \) or \( \gamma_D(I_\sigma) = \Gamma_D''(I_\sigma) \). In either case, \( \gamma_D(I_\sigma) \in J_{j+1,n} \) and \( \gamma_D(\sigma)(1) > b > a \).

It follows from the inductive hypotheses that \( F_{C_1} = \{ I_\sigma \in F_{D_1} : \sigma(1) \leq a \} \) and \( F_{C_2} = \{ I_\sigma \in F_{D_2} : \sigma(1) \leq a \} \). Thus \( F_C = \{ I_\sigma \in F_D : \sigma(1) \leq a \} \). Let \( I_\sigma \in F_C \). If \( I_\sigma \in F_{D_1} \) then \( \gamma_D(I_\sigma) = \Gamma_D''(I_\sigma) \). But \( \Gamma_D''(I_\sigma) = \Gamma_C''(I_\sigma) \) so \( \gamma_D(I_\sigma) = \gamma_C(I_\sigma) \). If \( I_\sigma \in F_{D_2} \) then \( \gamma_D(I_\sigma) = \Gamma_D''(I_\sigma) \). But \( \Gamma_D''(I_\sigma) = \Gamma_C''(I_\sigma) \) so \( \gamma_D(I_\sigma) = \gamma_C(I_\sigma) \). In either case, \( \gamma_D(\sigma)(1) \leq a < b \).
By Lemma 4, if \( I_\sigma \in I_{j+1,n} \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{j+1,n} \) and \( \Gamma_D(\sigma)(1) > b \). By Lemma 5, if \( I_\sigma \in I_{j+1} \) and \( \sigma(1) \leq a \), then \( \Gamma_D(I_\sigma) = \Gamma_C(I_\sigma) \).

**Lemma 7.** If \( k < l \), \( D \in M_l \), and \( E \subset D \subset U \), then

1. if \( p \in D \cap U_q \) for some \( q \in A \cap X_0 \) then \( f_D(p) \in V_{\lambda(q)} \);
2. if \( p \in D \cap U_q \) for some \( q \in A \setminus X_0 \) then \( f_D(p) \in V \);
3. if \( I_\sigma \in I_{j+1,\frac{a}{2}} \) for some \( x_n \in A \cap X_0 \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \Gamma_D(\sigma)(1) > b \); and
4. if \( I_\sigma \in I_{j+1} \) and \( \sigma(1) \leq a \) then \( \Gamma_D(I_\sigma) = \Gamma_D^*(I_\sigma) \).

Note that condition 4 implies that \( \gamma_D(\sigma)(1) \leq a \) for all \( I_\sigma \in F_D \) with \( \sigma(1) \leq a \).

**Proof.** The case \( k = 1 \) is given by Lemma 6.

Assume that \( l = k + 1 \). Then \( D' \in M_k \), \( D' \subset U \), and \( D' \cap U_1 = E \). Also, \( D'' \in M_{k-1} \), \( D'' \subset U \), and \( D'' \cap U_1 = E' \). So Lemma 6 holds for \( D' \) and \( D'' \).

Let \( p \in D \cap U_q \) for some \( x_n \in A \cap X_0 \) if \( p \in \hat{Q}_k \) then \( f_D(p) = f_{D'}(p) \in U_n \). Let \( p \in Q_1 \). Then \( p \) is the midpoint of some element \( I_\sigma \) of \( I_{k,n} \) where \( \sigma(1) > c \). Also, \( f_D(p) = h_{D''}(p) \) and \( h_{D''}(p) \) is the midpoint of \( \Gamma_D''(I_\sigma) \). But \( \Gamma_D''(I_\sigma) \in J_{k,n} \) and \( \Gamma_D''(\sigma)(1) > b \). Thus \( f_D(p) \in U_n \).

Let \( p \in D \cap U_q \) for some \( q \in A \setminus X_0 \). Now \( U_q \cap \hat{Q}_1 \subset \{q\} \) so \( p = q \) and \( p \in \hat{Q}_k \). Thus \( f_D(p) = f_{D'}(p) = f_E(p) \in V \).

Let \( I_\sigma \in F_D \cap I_{j+1,n} \) for some \( x_n \in A \cap X_0 \) and let \( \sigma(1) > c \). Either \( \gamma_D(I_\sigma) = \Gamma_D'(I_\sigma) \) or \( \gamma_D(I_\sigma) = \Gamma_D''(I_\sigma) \). In either case, \( \gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \gamma_D(\sigma)(1) > b > a \).

To show that conditions 3 and 4 hold, consider the sets \( F = \{I \in I_{j+1,n} : I \subset \bigcup F_E\} \) and \( G = \{I_\sigma \in F_D : \sigma(1) \leq a\} \). Define \( \gamma \) on \( G \) by \( \gamma(I) = \Gamma_E(I) \). We will show that \( F \subset G \). Let \( I_\sigma \in F \). Then \( \sigma(1) \leq a \) and \( I_\sigma \in F_{\sigma,k+1} \in \hat{F}_E \). Now \( A(E) \subset A(D) \) because \( E \subset D \). Also, \( A(\theta(E)) \subset A(\theta(D)) \). Thus \( I_\sigma \in \hat{F}_D \) and \( h_{D'}(I_\sigma) = h_E(I_\sigma) \in \hat{F}_{\theta(D)} \). If \( I_\sigma \in F_{D,1} \) then there is \( p \in D \cap Q_l \) such that \( p \) is an endpoint of \( I_\sigma \). Then, since \( \sigma(1) \leq a \), \( p \in U_l \). But \( U_l \cap Q_l = \emptyset \), so \( I_\sigma \notin F_{D,1} \). If \( p \in D \cap Q_l \) then \( p \in U_0 \) and \( f_D(p) \in V_0 \). But \( \Gamma_{D'}(\sigma)(1) \leq a \) so \( h_{D'}(I_\sigma) \) cannot have an endpoint in \( \theta(D) \cap R_l \). Therefore \( h_{D'}(I_\sigma) \in \hat{F}_{\theta(D)} \setminus \hat{F}_{\theta(D,1)} \), and \( I_\sigma \in G \). By Lemma 3, \( \Gamma(I) = \Gamma_E(I) \) for all \( I \in I_{j+1,n} \). If \( I \in G \) then \( I \in F_{D,2} \) so \( \gamma(I) = \Gamma_{D'}(I) = \Gamma_E(I) = \gamma(I) \). Thus \( \gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \gamma_D(\sigma)(1) \leq a < b \) for all \( I_\sigma \in F_D \cap I_{l+1,n} \) with \( m \leq \sigma(1) \leq b \). By Lemma 4, if \( I_\sigma \in I_{l+1,n} \) for some \( x_n \in A \cap X_0 \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \Gamma_D(\sigma)(1) > b \). By Lemma 5, \( \Gamma_D(I_\sigma) = \Gamma_D(I_\sigma) = \Gamma_E(I_\sigma) \) for all \( I_\sigma \in I_{l+1,n} \) with \( \sigma(1) \leq a \).

Let \( l \geq k + 2 \) and assume that if \( j = l - 1 \) or \( j = l - 2 \), \( C \in M_j \), and \( E \subset C \subset U \), then the lemma holds for \( C \). Let \( D \in M_l \) with \( E \subset D \subset U \). Then \( D \cap U \cap \hat{Q}_{k+1} = E \). Furthermore \( D' \in M_{l-1} \), \( E \subset D' \subset U \), \( D'' \in M_{l-2} \), and \( E \subset D'' \subset U \). Thus the lemma holds for \( D' \) and \( D'' \).
Let $p \in D \cap U_{x_n}$ for some $x_n \in A \cap X_0$. If $p \in \hat{Q}_{l-1}$ then $f_D(p) \subset f_{D'}(p) \in V_{\gamma_n}$. If $p \in Q_l$ then $p$ is the midpoint of some $I_{a} \in I_{l-1,n}$ with $\sigma(1) > c$. But $f_D(p) = h_{D''}(p)$ is the midpoint of $\Gamma_{D''}(I_{a})$ and $\Gamma_{D''}(I_{a}) \in J_{l-1,n}$ with $\Gamma_{D''}(\sigma)(1) > b$. Hence $f_D(p) \in V_{\gamma_n}$.

Let $p \in D \cap U_q$ for some $q \in A \setminus X_0$. If $p \in \hat{Q}_{l-1}$ then $f_D(p) = f_{D'}(p) \in V$. If $p \in Q_l$ then $f_D(p) = h_{D''}(p) = \Gamma_{D''}(p) = \Gamma_{E}(p) \in V$ because $h_{E}(U_q) \subset V$.

Let $I_{a} \in F_D \cap I_{l+1,n}$ for some $x_n \in A \cap X_0$ and let $\sigma(1) > c$. Either $\gamma_D(I_{a}) = \Gamma_{D'}(I_{a})$ or $\gamma_D(I_{a}) = \Gamma_{D''}(I_{a})$. In either case, $\gamma_D(I_{a}) \in J_{l+1,n}$ and $\Gamma_{D}(\sigma)(1) > b > a$.

To show that conditions 3 and 4 hold, consider the sets $F = \{I \in I_{l+1}: I \subset \bigcup F_E\}$ and $G = \{I \in F_D: \sigma(1) \leq a\}$. Define $\gamma$ on $G$ by $\gamma(I) = \Gamma_{E}(I)$. Let $I_{a} \in F$. Then $I_{a} \in F_{D}$ because $E \subset D$ and $h_{D'}(I_{a}) = h_{E}(I_{a}) \in F_{D}(I_{a})$ because $\theta(E) \subset \theta(D)$. Assume that $I_{a} \notin F_{D'}$. Let $p \in D \cap Q_1$. We will show that $f_{D}(p)$ cannot be an endpoint of $h_{D'}(I_{a})$. If $p \in U_0$, then $f_{D}(p) \in V$. But $\Gamma_{D'}(\sigma)(1) \leq a$ so $f_{D}(p)$ is not an endpoint of $h_{D'}(I_{a})$. If $p \in U_1$ then $p \in I_{\tau}$ for some $I_{\tau} \in I_{k+2}$ with $\tau(1) \leq a$. By the induction hypotheses, $f_{D}(p) = h_{D'}(p) = h_{E}(p) \in h_{E}(I_{\tau})$. If $\sigma \leq k + 2 \neq \tau$ then $I_{a+k+2} \cap I_{\tau} = \emptyset$ so $p$ cannot be an endpoint of any subinterval of $I_{a+k+2}$. If $\sigma \leq k + 2 = \tau$ then $p$ is not an endpoint of $I_{a}$ because $I_{a} \notin F_{D'}$. The assumption that $I_{a} \notin F_{D}$ also implies that $h_{D}(I_{a}) = h_{D'}(I_{a}) = h_{E}(I_{a})$. But $h_{E}^{-1}$ is continuous at $h_{E}(p)$, so $h_{E}(p)$ cannot be an endpoint of $h_{E}(I_{a})$. Therefore $h_{D}(I_{a}) \in F_{D}(I_{a})$. By Lemma 3, $\gamma(I) = \Gamma_{E}(I)$ for all $I \in I_{l+1}$. If $I \in G$ then either $\gamma_{D}(I) = \Gamma_{D'}(I)$ or $\gamma_{D}(I) = \Gamma_{D''}(I)$. In either case, $\gamma_{D}(I) = \Gamma_{E}(I) = \gamma(I)$. Thus $\gamma_{D}(I_{a}) \in J_{l+1,n}$ and $\Gamma_{D}(\sigma)(1) \leq a < b$ for all $I_{a} \in F_{D} \cap I_{l+1,n}$ with $m \leq \sigma(1) \leq b$.

By Lemma 4, if $I_{a} \in I_{l+1,n}$ for some $x_n \in A \cap X_0$ and $\sigma(1) > c$, then $\Gamma_{D}(I_{a}) \subset J_{l+1,n}$ and $\Gamma_{D}(\sigma)(1) > b$. By Lemma 5, if $I_{a} \in I_{l+1}$ and $\sigma(1) \leq a$, then $\Gamma_{D}(I_{a}) = \Gamma_{E}(I_{a})$.

Now let $B \in [A, U]$ and let $B \in S_{D}$. Then $D \subset M_l$ for some $l \geq k$ and $E \subset D \subset U$. Also, $B \setminus X_0 = A \cap X_0$ so $\lambda(B \setminus X_0) = \lambda(A \cap X_0) < V$. Let $p \in B \setminus X_0$. If $p \in D$ then $f_{D}(p) \in V$ by Lemma 7. Assume that $p \notin D$. There is $I_{a} \in I_{l+1}$ such that $p \in I_{a}$. If $p \in U_{x_n}$ for some $x_n \in A \cap X_0$ then $I_{a} \in I_{l+1,n}$ and $\sigma(1) > c$. By Lemma 7, $h_{D}(I_{a}) = \Gamma_{D}(I_{a}) \in J_{l+1,n}$ and $\Gamma_{D}(\sigma)(1) > b$. Thus $h_{D}(p) \in V$. If $p \in U_{q}$ for some $q \in A \setminus X_0$ then $\sigma(1) \leq a$. By Lemma 7, $h_{D}(I_{a}) = \Gamma_{D}(I_{a}) = \Gamma_{E}(I_{a})$. Thus $h_{D}(p) \in V$ because $h_{E}(U_{q}) \subset V$. Therefore $H(B) \in [H(A), V]$ and $H$ is continuous. A similar argument shows that $H^{-1}$ is continuous.

IV. Related results

Corollary 8. If $X$ and $Y$ are $\omega$-graphs and $D$ and $E$ are equipotent discrete subsets of $X$ and $Y$ respectively, then $\bigcup_{p \in D}[p, X]$ is homeomorphic to $\bigcup_{p \in E}[p, Y]$. 
Proof. Extend $D$ and $E$ to dividing sets $X_0$ and $Y_0$ of $X$ and $Y$. Order the sets $X_0$ and $Y_0$ so that $\lambda(D) = E$. Then the homeomorphism defined in the proof of Theorem 1 takes $\bigcup_{p \in D}[p, X]$ to $\bigcup_{p \in E}[p, Y]$, so these two sets are homeomorphic.

The finally results are about spaces other than graphs or $\omega$-graphs. Theorem 2 of [N] shows that points may be removed from certain $T_1$ spaces without affecting its Pixley-Roy hyperspace. The next three lemmas generalize this result. Theorem 12 applies this procedure to $\mathbb{R}^n$.

**Lemma 9.** If $(Z_n : n < \omega)$ is a sequence of disjoint homeomorphic open and closed subsets of $\text{PR}[X]$ such that $\bigcup_{n < \omega} Z_n$ is open and closed in $\text{PR}[X]$, then $\text{PR}[X] \setminus Z_0 \cong \text{PR}[X]$.

**Proof.** For each $n < \omega$ let $H_n : Z_n \to Z_{n+1}$ be a homeomorphism. Define $H : \text{PR}[X] \to \text{PR}[X] \setminus Z_0$ by

$$H(A) = \begin{cases} A & \text{if } A \not\subseteq \bigcup_{n < \omega} Z_n, \\ H_n(A) & \text{if } A \subseteq Z_n. \end{cases}$$

Then $H$ is a homeomorphism.

**Lemma 10.** If $U$ is an open subset of space $X$ and $C$ is closed in $U$ then $\bigcup_{p \in C}[p, U]$ is open and closed in $\text{PR}[X]$.

**Proof.** Clearly $\bigcup_{p \in C}[p, U]$ is an open subset of $\text{PR}[X]$. Let

$$A \in U \setminus \bigcup_{p \in C}[p, U].$$

If $A \not\subseteq U$ then $[A, X]$ is a neighborhood of $A$ that misses $\bigcup_{p \in C}[p, U]$. If $A \subseteq U$ then $A \cap C = \emptyset$, so $[A, U \setminus C]$ is a neighborhood of $A$ in $\text{PR}[X]$ that misses $\bigcup_{p \in C}[p, U]$.

**Lemma 11.** Let $(U_n : n < \omega)$ be a sequence of disjoint open subsets of a space $X$ and let $(C_n : n < \omega)$ be a sequence of subsets of $X$ such that $C_n \subseteq U_n$ and $C_n$ is closed in $U_n$ for all $n < \omega$. Then $\bigcup_{n < \omega} \bigcup_{p \in C_n}[p, U_n]$ is open and closed in $\text{PR}[X]$.

**Proof.** It is clear that $\bigcup_{n < \omega} \bigcup_{p \in C_n}[p, U_n]$ is open in $\text{PR}[X]$. By Lemma 10, each $\bigcup_{p \in C_n}[p, U_n]$ is closed in $\text{PR}[X]$. Let $A \in \text{PR}[X]$. Since $A$ is finite and the $U_n$'s are disjoint, there is a finite subset $B$ of $\omega$ such that $A \cap U_n \neq \emptyset$ if and only if $n \in B$. Then $(\bigcup_{m \in B}[A, U_m]) \cap (\bigcup_{p \in U_n}[p, U_n]) \neq \emptyset$ only if $n \in B$. Thus $\bigcup_{p \in C_n}[p, U_n] : n < \omega$ is locally finite, and $\bigcup_{n < \omega} \bigcup_{p \in C_n}[p, U_n]$ is closed.

**Theorem 12.** Let $0 < n < \omega$ and let $X = \{x \in \mathbb{R}^n : 0 < |x| < 1\}$ where $|x|$ denotes the Euclidean norm. For any $0 < m < \omega$,

$$\text{PR}[\mathbb{R}^n] \cong \text{PR}[m \times \mathbb{R}^n] \cong \text{PR}[\omega \times \mathbb{R}^n] \cong \text{PR}[m \times X] \cong \text{PR}[\omega \times X].$$

**Proof.** We will show that each of these spaces is homeomorphic to $\text{PR}[\mathbb{R}^n]$. Let $D$ be a discrete subset of $\{x \in \mathbb{R} : x \geq 0\}$ which contains 0 and let $\pi : \mathbb{R}^n \to \mathbb{R}$
be the projection onto the first coordinate. Let $L = \{\bar{x} \in \mathbb{R}^n : \pi(\bar{x}) \in D\}$ and let $C = \{\bar{x} \in \mathbb{R}^n : |\bar{x}| \in D\}$. If $D$ is finite then $\mathbb{R}^n \setminus L = (|D| + 1) \times \mathbb{R}^n$ and $\mathbb{R}^n \setminus C = |D| \times X$. If $D$ is infinite then $\mathbb{R}^n \setminus L \approx \omega \times \mathbb{R}^n$ and $\mathbb{R}^n \setminus C \approx \omega \times X$.

Let $U_0 = \mathbb{R}^n$ and let $(U_k : 0 < k < \omega)$ be a sequence of disjoint open balls in $\mathbb{R}^n$, each of which has empty intersection with $L$ and $C$.

Set $C_0 = L$. For every $0 < k < \omega$ let $C_k$ be a subset of $U_k$ which is homeomorphic to $L$. Then $C_k$ is closed in $U_k$ for all $k < \omega$. For each $k < \omega$ set $Z_k = \bigcup_{p \in C_k} [p, U_k]$. By Lemma 10, each $Z_k$ is open and closed in $\text{PR}[\mathbb{R}^n]$. By Lemma 11, $\bigcup_{0 < k < \omega} Z_k$ is open and closed in $\text{PR}[\mathbb{R}^n]$, so $\bigcup_{k < \omega} Z_k$ is open and closed in $\text{PR}[\mathbb{R}^n]$. Clearly each $Z_k$ is homeomorphic to every other $Z_k$, so $\text{PR}[\mathbb{R}^n] \approx \text{PR}[\mathbb{R}^n] \setminus Z_0 \approx \text{PR}[\mathbb{R}^n \setminus L]$. If $D$ is finite then $\text{PR}[\mathbb{R}^n] \approx \text{PR}[(|D| + 1) \times \mathbb{R}^n]$. If $D$ is infinite then $\text{PR}[\mathbb{R}^n] \approx \text{PR}[(\omega \times \mathbb{R}^n)]$.

Now let $C_0 = C$ and for every $k < \omega$ let $C_k$ be a subset of $U_k$ homeomorphic to $C$. Set $Z_k = \bigcup_{p \in C_k} [p, U_k]$ for all $k < \omega$. Again, $(Z_k : k < \omega)$ is a sequence of disjoint homeomorphic open and closed subsets of $\text{PR}[\mathbb{R}^n]$ so $\text{PR}[\mathbb{R}^n] \approx \text{PR}[\mathbb{R}^n] \setminus Z_0 \approx \text{PR}[\mathbb{R}^n \setminus C]$. If $D$ is finite then $\text{PR}[\mathbb{R}^n] \approx \text{PR}[|D| \times X]$. If $D$ is infinite then $\text{PR}[\mathbb{R}^n] \approx \text{PR}[(\omega \times X)]$.

Bibliography


Department of Mathematics, University of Dayton, Dayton, Ohio 45469