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INTEGRAL DUBROVIN VALUATION RINGS

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ABSTRACT. In the preceding paper, Dubrovin valuation rings integral over their centers in central simple algebras were characterized by value functions. Here, these value functions are used to give a method for extending integral Dubrovin valuation rings in generalized crossed product algebras. Several applications of this extension theorem are given, including new and more natural proofs of some theorems on valued division algebras over Henselian fields.

1. INTRODUCTION

This paper is a sequel to [M_2]. In previous papers [D_1, D_2, BG, W_1] a strong case has been made that Dubrovin valuation rings are a very good generalization to central simple algebras of the classical valuation rings on fields and division algebras. While Dubrovin valuation rings do not come equipped with a valuation, it was shown in [M_2] that there is a value function associated to every Dubrovin valuation ring integral over its center. We give here several applications of this result which use value functions in order to construct integral Dubrovin valuation rings. These lead to generalizations (with much better proofs) of some theorems from [JW and W_2].

The arguments given here show the advantages of working with Dubrovin valuation rings on central simple algebras, even if one were primarily interested in valuation rings in division rings. For, with crossed product constructions and with tensor products, the end result is usually a central simple algebra, and it can be very difficult to get back to the underlying division algebra. It was precisely this difficulty that made some of the proofs in [JW] awkward and indirect.

Our main result here is Theorem 2.1 on extending integral Dubrovin valuation rings to generalized crossed product algebras. This theorem is applied in §3 to give generalizations to integral Dubrovin valuation rings of the following theorems on valued division algebras over Henselian fields: the inertial extension theorem [JW, Theorem 3.1], the homological description of inertially split division algebras [JW, Theorem 5.6(b)] and the description of the value group

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and residue ring of the tensor product of an inertially split and a totally ramified division algebra [JW, Theorem 6.3]. In addition, we give another tensor product theorem generalizing [W_2, Proposition 2]. This generalization was invoked already in [W_2] in determining the possible residue rings of division algebras over a Henselian field. Finally, in §4 we construct a few examples showing the need for some of the hypotheses for the theorems in §3.

The notation and terminology of this paper will follow [M_2]. In particular, if B is a Dubrovin valuation ring of a central simple F-algebra S, and if $V = B \cap F$ (a valuation ring of F), $J(B)$ is the Jacobson radical of B, $\overline{B} = B/J(B)$, S^* is the group of units of S, $st(B) = \{s \in S^* \mid sBs^{-1} = B\}$, $\Gamma_B = st(B)/B^*$, the (linearly ordered) value group of B, and $\theta_B : \Gamma_B/\Gamma_V \to \mathcal{G}(Z(\overline{B})/\overline{V})$ is the map induced by conjugation by elements of $st(B)$. If (F_h, V_h) is the Henselization of (F, V) then n_B is the matrix size of $S \otimes_F F_h$, t_B is the matrix size of $\overline{B}$ and $s_B = n_B/t_B$ (an integer by [W_1, Theorem D]).

2. The main theorem

In this section we prove the main theorem of this paper. The generality of Theorem 2.1 makes it somewhat cumbersome to state. However, no less general formulation would cover all the applications given in §3. Since the theorem deals with generalized crossed products, we give a very brief introduction to generalized crossed products.

Recall that a central simple F-algebra S is called a crossed product when S contains a subfield K such that K/F is Galois and $[K : F] = \sqrt{[S : F]} = \deg(S)$. In this case S has a direct sum decomposition as a K-vector space in which multiplication is described completely by K, the Galois group $G = \mathcal{G}(K/F)$, and a certain 2-cocycle in $Z^2(G, K^*)$.

A more general situation is for S to contain a Galois extension K of F which may have $[K : F] < \sqrt{[S : F]}$. Such an algebra is then called a *generalized crossed product* for K over F. Generalized crossed products are discussed in [T, J_1, J_2 and KY]. Just as for crossed products, generalized crossed products can be described "homologically": let S be a central simple F-algebra and $K \subseteq S$ a subfield Galois over F with $G = \mathcal{G}(K/F)$. Let $C = C_S(K)$, the centralizer of K in S. So C is a central simple K-algebra and $\deg(S) = \deg(C)[K : F]$. By the Noether-Skolem theorem there are $x_\sigma \in S^*$ for $\sigma \in G$ such that $x_\sigma a x_\sigma^{-1} = \sigma(a)$ for all $a \in K$. For $\sigma, \tau \in G$, $x_\sigma x_\tau x_{\sigma\tau}^{-1}$ centralizes K, so lies in C^*. Let $\omega : G \to \mathrm{Aut}(C)$ and $f : G \times G \to C^*$ be defined by

$$\omega_\sigma = \omega(\sigma) = \mathrm{inn}(x_\sigma)|_C, \qquad f(\sigma, \tau) = x_\sigma x_\tau x_{\sigma\tau}^{-1},$$

where $\mathrm{inn}(s)$ is the map on S given by $a \mapsto sas^{-1}$ for $s \in S^*$. We then have

(1) $\omega_\sigma(a) = \sigma(a)$ for $a \in K$,
(2) $\omega_\sigma \omega_\tau = \mathrm{inn}(f(\sigma, \tau))\omega_{\sigma\tau}$ for all $\sigma, \tau \in G$,
(3) $\omega_\sigma(f(\tau, \rho)) f(\sigma, \tau\rho) = f(\sigma, \tau) f(\sigma\tau, \rho)$ for all $\sigma, \tau, \rho \in G$.

Furthermore, $S = \bigoplus_{\sigma\in G} Cx_\sigma$, and multiplication is given by

$$(*) \qquad (cx_\sigma)(dx_\tau) = [c\omega_\sigma(d)f(\sigma,\tau)]x_{\sigma\tau}.$$

The couple (ω, f) satsifying (1)–(3) is called a *generalized cocycle.* Conversely, starting with a central simple K-algebra C and a generalized cocycle (ω, f) one can construct an algebra T as $T = \bigoplus_{\sigma\in G} Cx_\sigma$ with multiplication given by $(*)$ (extended bilinearly to T). It is known that T is a central simple F-algebra containing copies of C and K with $C = C_T(K)$. This generalized crossed product for K over F is denoted $(C, G, (\omega, f))$. Just as in the case of ordinary cocycles there is a notion of equivalence of generalized cocycles; changing a generalized cocycle to an equivalent one corresponds to changing the choice of the x_σ (to $c_\sigma x_\sigma$ where $c_\sigma \in C$). Furthermore, one calls a generalized cocycle *normalized* when $x_{\mathrm{id}} = 1$, that is when $\omega_{\mathrm{id}} = \mathrm{id}$ and $f(\mathrm{id}, \tau) = f(\tau, \mathrm{id}) = 1$ for all $\tau \in G$. The proofs of these statements are straightforward and can be found in the references given above.

Given a generalized crossed product $S = (C, G, (\omega, f))$ one would like to find out information about Dubrovin valuation rings of S from information known about Dubrovin valuation rings of C and vice versa. The following result does this for integral Dubrovin valuation rings.

For this result we set up some notation. Let C be a central simple K-algebra. Suppose C has an integral Dubrovin valuation ring A, i.e. A is a Dubrovin valuation ring integral over its center. Let $W = A \cap K$. Suppose K is a Galois extension of a field F with $G = \mathcal{G}(K/F)$, $|G| = n$, $V = W \cap F$ and that W/V is indecomposed. Let (ω, f) be a normalized generalized cocycle and S the generalized crossed product $(C, G, (\omega, f))$. Say $S = \bigoplus_{\sigma\in G} Cy_\sigma$, with multiplication given by $(*)$. We first notice that since W/V is indecomposed, $\sigma(W) = W$ for all $\sigma \in G$, so $y_\sigma A y_\sigma^{-1}$ is a Dubrovin valuation ring lying over W in C, hence by [W$_1$, Theorem A] there are $c_\sigma \in C^*$ with $y_\sigma A y_\sigma^{-1} = c_\sigma A c_\sigma^{-1}$. So if $x_\sigma = c_\sigma^{-1} y_\sigma$ $(x_{\mathrm{id}} = 1)$ then $x_\sigma A x_\sigma^{-1} = A$ and $S = \bigoplus_{\sigma\in G} Cx_\sigma$. Replacing the y_σ by the x_σ corresponds to replacing (ω, f) by an equivalent cocycle, as mentioned above. So this change does not affect the algebra S. We assume (ω, f) has been modified this way. It was shown in [M$_2$, Corollary 2.5] there is a value function w on C corresponding to A. Using w, we define a function w' on S by $w'(\sum_{\sigma\in G} c_\sigma x_\sigma) = \min_\sigma\{w(c_\sigma) + w'(x_\sigma)\}$, where $w'(x_\sigma) = \frac{1}{n}\sum_{i=1}^n w(f(\sigma^i, \sigma))$ in the divisible hull Δ_A of Γ_A. Set $B = \{s \in S \mid w'(s) \geq 0\}$.

Theorem 2.1. *With the notation above, w' is a value function on S and the map $\alpha\colon G \to \Delta_A/\Gamma_A$ given by $\alpha(\sigma) = w'(x_\sigma) + \Gamma_A$ is a homomorphism. Let $I = \ker(\alpha)$. Suppose that the ω_σ induce distinct automorphisms of $Z(\overline{A})$ for $\sigma \in I$. Then B is an integral Dubrovin valuation ring with*

$$\Gamma_B = \Gamma_A + \langle\{w'(x_\sigma) \mid \sigma \in G\}\rangle,$$

hence $\Gamma_B/\Gamma_A = im(\alpha) \cong G/I$. *Furthermore,* $\overline{B}$ *is a generalized crossed product* $(\overline{A}, I', (\overline{\omega}, g))$, *where* I' *is a group isomorphic to* I *and consists of the automorphisms of* $Z(\overline{A})$ *induced by the* ω_σ. *So* $Z(\overline{B}) = \mathscr{F}(I') \subseteq Z(\overline{A})$. *In addition,* B *is a compatible extension of* A, $\delta(B) = \delta(A)\delta(W/V)$, $\theta_B(w'(x_\sigma) + \Gamma_V) = \overline{\omega}_\sigma|_{Z(\overline{B})}$ *for* $\sigma \in G$, *and the following diagram is commutative*:

$$\begin{array}{ccc} \Gamma_A/\Gamma_V & \xrightarrow{\theta_{A'}} & \mathscr{G}(Z(\overline{A})/\overline{W}) \\ \downarrow & & \downarrow \text{res} \\ \Gamma_B/\Gamma_V & \xrightarrow{\theta_B} & \mathscr{G}(Z(\overline{B})/\overline{V}), \end{array}$$

where $\theta_{A'}$ *is the map induced by* θ_A.

Proof. Since we are assuming that the x_σ have been chosen to satisfy $x_\sigma A x_\sigma^{-1} = A$, it is clear that $w \circ \omega_\sigma = w$, hence $f(\sigma, \tau) \in st(A)$ for all $\sigma, \tau \in G$. The equation

$$w'(x_\sigma) + w'(x_\tau) = w(f(\sigma, \tau)) + w'(x_{\sigma\tau}) \tag{2.2}$$

can be seen as follows: let $c = \deg(C)$, $s = \deg(S)$ and $n = [K : F]$. So $cn = s$. As $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$, if $v = w|_K$, a valuation on K, we have

$$v(Nrd(x_\sigma)) + v(Nrd(x_\tau)) = v(Nrd(f(\sigma, \tau))) + v(Nrd(x_{\sigma\tau})). \tag{2.3}$$

If N is the reduced norm on C, then $Nrd|_C = N_{K/F} \circ N$ by [R, 9.14], and since $x_\sigma^n = \prod_{i=1}^n f(\sigma^i, \sigma) \in C$,

$$\begin{aligned} v(Nrd(x_\sigma)) &= \frac{1}{n} v(Nrd(x_\sigma^n)) = \frac{1}{n} v(N_{K/F}(N(x_\sigma^n))) \\ &= v(N(x_\sigma)) = \sum_{i=1}^n v(N(f(\sigma^i, \sigma))) = c \sum_{i=1}^n \frac{1}{c} v(N(f(\sigma^i, \sigma))) \\ &= c \sum_{i=1}^n w(f(\sigma^i, \sigma)) = sw'(x_\sigma) \end{aligned}$$

as W/V is indecomposed and $w(t) = \frac{1}{c} v(N(t))$ for $t \in st(A)$. Also

$$\begin{aligned} v(Nrd(f(\sigma, \tau)) &= v(N_{K/F}(N(f(\sigma, \tau))) \\ &= nv(N(f(\sigma, \tau))) = \frac{s}{c} v(N(f(\sigma, \tau)) = sw(f(\sigma, \tau)). \end{aligned}$$

Using this in (2.3) and dividing by s gives (2.2). This shows α is a homomorphism. We verify that w' satisfies the axioms for a value function given in [M $_2$, Definition 2.1]:

(1) $w'(s + t) \geq \min\{w'(s), w'(t)\}$: Say $s = \sum_\sigma c_\sigma x_\sigma$ and $t = \sum_\sigma d_\sigma x_\sigma$. So $w'(s+t) = \min_\sigma\{w(c_\sigma + d_\sigma) + w'(x_\sigma)\}$. Suppose the minimum occurs at τ and that $w(c_\tau) \leq w(d_\tau)$. Then

$$\begin{aligned} w'(s + t) &= w(c_\tau + d_\tau) + w'(x_\tau) \geq w(c_\tau) + w'(x_\tau) \\ &\geq \min_\sigma\{w(c_\sigma) + w'(x_\sigma)\} = w'(s) \\ &\geq \min\{w'(s), w'(t)\}. \end{aligned}$$

(2) $w'(st) \geq w'(s) + w'(t)$: First, if $s = c_\sigma x_\sigma$ and $t = d_\tau x_\tau$ then

$$\begin{aligned} w'(st) &= w'(c_\sigma \omega_\sigma(d_\tau) f(\sigma, \tau) x_{\sigma\tau}) = w(c_\sigma \omega_\sigma(d_\tau) f(\sigma, \tau)) + w'(x_{\sigma\tau}) \\ &\geq w(c_\sigma) + w(\omega_\sigma(d_\tau)) + w(f(\sigma, \tau)) + w'(x_{\sigma\tau}) \\ &= w(c_\sigma) + w(d_\tau) + w'(x_\sigma) + w'(x_\tau) = w'(s) + w'(t) \end{aligned}$$

by (2.2). In general, write $s = \sum_{\sigma \in G} s_\sigma$ and $t = \sum_{\tau \in G} t_\tau$ where $s_\sigma = c_\sigma x_\sigma$ and $t_\tau = d_\tau x_\tau$. Then

$$\begin{aligned} w'(st) &= w'\left(\sum_{\sigma,\tau} s_\sigma t_\tau\right) \geq \min_{\sigma,\tau}\{w'(s_\sigma t_\tau)\} \\ &\geq \min_{\sigma,\tau}\{w'(s_\sigma) + w'(t_\tau)\} = \min_\sigma\{w'(s_\sigma)\} + \min_\tau\{w'(t_\tau)\} \\ &= w'(s) + w'(t). \end{aligned}$$

(3) $im(w') = w'(st(w'))$: Recall that $st(w') = \{s \in S^* \mid w'(s^{-1}) = -w'(s)\}$. By the definition of w', $im(w') = im(w) + \langle\{w'(x_\sigma) \mid \sigma \in G\}\rangle$. Now $im(w) = w(st(w)) \subseteq w'(st(w'))$ since $w'|_C = w$. Because $x_\sigma^{-1} = f(\sigma^{-1}, \sigma)^{-1} x_{\sigma^{-1}}$, $w'(x_\sigma^{-1}) = -w(f(\sigma^{-1}, \sigma)) + w'(x_{\sigma^{-1}})$. Also, by (2.2), since $x_{\sigma^{-1}} x_\sigma = f(\sigma^{-1}, \sigma)$, $w'(x_{\sigma^{-1}}) + w'(x_\sigma) = w(f(\sigma^{-1}, \sigma))$. Thus $w'(x_\sigma^{-1}) = -w(x_\sigma)$, so $x_\sigma \in st(w')$. Therefore $im(w') \subseteq w'(st(w'))$, hence $im(w') = w'(st(w'))$. Thus w' is a value function on S.

Let $B = \{s \in S \mid w'(s) \geq 0\}$ and $J = \{s \in S \mid w'(s) > 0\}$. To prove B is a Dubrovin valuation ring, by [M_2, Theorem 2.4] we need to see that B/J is simple Artinian. We will show B/J is a generalized crossed product over $\overline{A}$. Now $I = \ker(\alpha) = \{\sigma \in G \mid w'(x_\sigma) \in \Gamma_A\}$. So, for $\sigma \in I$ there is an $a_\sigma \in st(A) \subseteq C^*$ with $w'(x_\sigma) = w'(a_\sigma)$. We now modify the x_σ (and modify (ω, f) to an equivalent generalized cocycle) by replacing x_σ by $a_\sigma^{-1} x_\sigma$ for $\sigma \in I$. With the new x_σ we have $w'(x_\sigma) = 0$ for $\sigma \in I$. Thus $w(f(\sigma, \tau)) = 0$ for $\sigma, \tau \in I$, by (2.2). As $A^* = st(A) \cap (A - J(A))$ by [W_1, Lemma 3.3], $f(\sigma, \tau) \in A^*$ for $\sigma, \tau \in I$.

Let $\overline{\omega}_\sigma$ be the automorphism of $\overline{A}$ induced by conjugation by x_σ, and $\psi_\sigma = \overline{\omega}_\sigma|_{Z(\overline{A})}$. The ψ_σ are distinct (for $\sigma \in I$) by assumption. Let $I' = \{\psi_\sigma \mid \sigma \in I\}$ and $g(\psi_\sigma, \psi_\tau) = \overline{f(\sigma, \tau)} \in \overline{A}^*$ for σ, $\tau \in I$. We show I' is a group and $(\overline{\omega}, g)$ is a generalized cocycle for $Z(\overline{A})/\mathcal{F}(I')$. First, to show $\overline{\omega}_\sigma \overline{\omega}_\tau = \operatorname{inn}(g(\psi_\sigma, \psi_\tau))\overline{\omega}_{\sigma\tau}$, for $a \in A$,

$$\begin{aligned} \overline{\omega}_\sigma \overline{\omega}_\tau(\overline{a}) &= \overline{\omega}_\sigma(\overline{x_\tau a x_\tau^{-1}}) = \overline{x_\sigma x_\tau a x_\tau^{-1} x_\sigma^{-1}} = \overline{\omega_\sigma \omega_\tau(a)} \\ &= \overline{f(\sigma, \tau)\omega_{\sigma\tau}(a) f(\sigma, \tau)^{-1}} = \overline{f(\sigma, \tau)}\,\overline{\omega_{\sigma\tau}(a)}\,\overline{f(\sigma, \tau)}^{-1} \\ &= g(\psi_\sigma, \psi_\tau)\overline{\omega}_{\sigma\tau}(\overline{a}) g(\psi_\sigma, \psi_\tau)^{-1} = (\operatorname{inn}(g(\psi_\sigma, \psi_\tau))\overline{\omega}_{\sigma\tau})(\overline{a}) \end{aligned}$$

as (ω, f) is a generalized cocycle. Furthermore, if $\overline{a} \in Z(\overline{A})$, $\psi_{\sigma\tau}(\overline{a}) = \overline{\omega}_{\sigma\tau}(\overline{a}) \in Z(\overline{A})$, so $g(\psi_\sigma, \psi_\tau)\overline{\omega}_{\sigma\tau}(\overline{a}) g(\psi_\sigma, \psi_\tau)^{-1} = \overline{\omega}_{\sigma\tau}(\overline{a})$. Therefore $\psi_\sigma \psi_\tau =$

$\psi_{\sigma\tau}$. This shows I' is a group and the map $I \to I'$ given by $\sigma \mapsto \psi_\sigma$ is a homomorphism. Since the ψ_σ are distinct, $I \cong I'$. Second, we need to show $\overline{\omega}_\sigma(g(\psi_\tau, \psi_\rho))g(\psi_\sigma, \psi_{\tau\rho}) = g(\psi_\sigma, \psi_\tau)g(\psi_{\sigma\tau}, \psi_\rho)$. To see this,

$$\begin{aligned}\overline{\omega}_\sigma(g(\psi_\tau, \psi_\rho))g(\psi_\sigma, \psi_{\tau\rho}) &= \overline{\omega}_\sigma(\overline{f(\tau, \rho)})\overline{f(\sigma, \tau\rho)} = \overline{\omega_\sigma(f(\tau, \rho))f(\sigma, \tau\rho)} \\ &= \overline{f(\sigma, \tau)}\,\overline{f(\sigma\tau, \rho)} = g(\psi_\sigma, \psi_\tau)g(\psi_{\sigma\tau}, \psi_\rho).\end{aligned}$$

Thus $(\overline{\omega}, g)$ is a generalized cocycle; let T be the central simple algebra $(\overline{A}, I', (\overline{\omega}, g))$. We identify I' with I when used as index sets. Thus, we write $T = \bigoplus_{\sigma\in I}\overline{A}y_\sigma$ with $y_\sigma \overline{a} y_\sigma^{-1} = \overline{\omega}_\sigma(\overline{a})$ for $\overline{a} \in \overline{A}$ and $y_\sigma y_\tau y_{\sigma\tau}^{-1} = g(\psi_\sigma, \psi_\tau)$. If $\sum_{\sigma\in G} c_\sigma x_\sigma \in B$ then $w(c_\sigma) + w'(x_\sigma) \geq 0$ for all $\sigma \in G$. Hence $c_\sigma \in A$ for $\sigma \in I$ since $w'(x_\sigma) = 0$. For $\sigma \notin I$, $w'(x_\sigma) \notin \Gamma_A$, so $w(c_\sigma) + w'(x_\sigma) > 0$. Define $h\colon B \to T$ by $h(\sum_{\sigma\in G} c_\sigma x_\sigma) = \sum_{\sigma\in I} \overline{c_\sigma} y_\sigma$. Clearly h is an additive group homomorphism, and $\ker(h) = \{\sum_{\sigma\in G} c_\sigma x_\sigma \in B \mid c_\sigma \in J(A)$ for all $\sigma \in I\} = \{\sum_{\sigma\in G} c_\sigma x_\sigma \in S \mid w'(\sum_{\sigma\in G} c_\sigma x_\sigma) > 0\} = J$. Surjectivity of h is also clear. To see h preserves multiplication, if $\sum_{\sigma\in G} c_\sigma x_\sigma$ and $\sum_{\tau\in G} d_\tau x_\tau \in B$,

$$\sum_{\sigma\in G} c_\sigma x_\sigma \cdot \sum_{\tau\in G} d_\tau x_\tau = \sum_{\rho\in G}\left(\sum_{\sigma\tau=\rho} c_\sigma \omega_\sigma(d_\tau) f(\sigma, \tau)\right) x_\rho.$$

If $\rho \in I$ and $\sigma\tau = \rho$ with $\sigma \notin I$ then $w'(c_\sigma x_\sigma)$ and $w'(d_\tau x_\tau) > 0$; so $w(c_\sigma \omega_\sigma(d_\tau) f(\sigma, \tau)) = w'((c_\sigma x_\sigma)(d_\tau x_\tau)) \geq w'(c_\sigma x_\sigma) + w'(d_\tau x_\tau) > 0$. So

$$\begin{aligned} h\left(\sum_{\sigma\in G} c_\sigma x_\sigma \cdot \sum_{\tau\in G} d_\tau x_\tau\right) &= \sum_{\rho\in I}\sum_{\sigma\tau=\rho} \overline{c_\sigma \omega_\sigma(d_\tau) f(\sigma, \tau)}\, y_\rho \\ &= \sum_{\sigma, \tau\in I} \overline{c_\sigma}\, \overline{\omega}_\sigma(\overline{d_\tau}) g(\psi_\sigma, \psi_\tau) y_{\sigma\tau} = \sum_{\sigma\in I} \overline{c_\sigma} y_\sigma \cdot \sum_{\tau\in I} \overline{d_\tau} y_\tau. \end{aligned}$$

Hence h is a ring homomorphism, and so $B/J \cong T$. Thus by [M_2, Theorem 2.4], B is a Dubrovin valuation ring of S. From the construction of w' it is clear that $\Gamma_B = im(w') = \Gamma_A + \langle\{w'(x_\sigma)\}\rangle$, so $\Gamma_B/\Gamma_A = \mathrm{im}(\alpha) \cong G/I$ and $\overline{B} = B/J = (\overline{A}, I', (\overline{\omega}, g))$. Since $w'|_C = w$, $J(A) = J \cap A$ and $st(A) = st(w) \subseteq st(w') = st(B)$. Hence B/A is compatible. The commutativity of the diagram is clear. By definition of θ_B and the isomorphism $\Gamma_B/\Gamma_V \cong im(w')$, since $x_\sigma \in st(B)$, $\theta_B(w'(x_\sigma) + \Gamma_V)$ is the map on $Z(\overline{B})$ induced by conjugation by x_σ, so $\theta_B(w'(x_\sigma) + \Gamma_V) = \overline{\omega}_\sigma|_{Z(\overline{B})} = \psi_\sigma$. Finally, for the information about the defect, we have

$$\begin{aligned}[S : F] &= |G|[C : K][K : F] \\ &= |G/I||I||\Gamma_A : \Gamma_W|[\overline{A} : \overline{W}]\delta(A)|\Gamma_W : \Gamma_V|[\overline{W} : \overline{V}]\delta(W/V) \\ &= (|\Gamma_B : \Gamma_A||\Gamma_A : \Gamma_W||\Gamma_W : \Gamma_V|)\,(|I|[\overline{A} : \overline{W}][\overline{W} : \overline{V}])\delta(A)\delta(W/V) \\ &= |\Gamma_B : \Gamma_V|[\overline{B} : \overline{V}]\delta(A)\delta(W/V).\end{aligned}$$

So, $\delta(B) = \delta(A)\delta(W/V)$. □

An alternative way to define w' above is to set $w'(x_\sigma) = \frac{1}{s}w(Nrd(x_\sigma))$. It is shown in the proof of Theorem 2.1 that $x_\sigma \in st(B)$, so by [M$_2$, Proposition 2.6] this formula for $w'(x_\sigma)$ is valid. While this definition would simplify the proof somewhat, it does not lend itself as well to actual computations.

Corollary 2.4. *With notation of Theorem* 2.1, *let* H *be the kernel of the natural homomorphism* $\mathcal{G}(K/F) \to \mathcal{G}(\overline{W}/\overline{V})$. *Suppose* $I \cap H = 0$. *Then for* $\sigma \in I$ *the* ω_σ *induce distinct automorphisms of* $Z(\overline{A})$, *hence the conclusions of Theorem* 2.1 *apply.*

Proof. The assumption that $I \cap H = 0$ means that the $\sigma \in I$ induce distinct automorphisms of $\overline{W}$. Since $\overline{W} \subseteq Z(\overline{A})$, this assures that the ω_σ induce distinct automorphisms of $Z(\overline{A})$, which is what is needed for Theorem 2.1. □

3. Applications of the main theorem

In this section we will make use of Theorem 2.1 to generalize results of [JW and W$_2$] to integral Dubrovin valuation rings. The first consequence is a generalization of [JW, Theorem 3.1] for Galois inertial extensions. We will follow the notation of §2. Note that if K/F is inertial (= unramified) with respect to W/V then the H of Corollary 2.4 is zero, so the condition $I \cap H = 0$ is automatically satisfied. This is the case here.

Theorem 3.1. *Suppose* S *is a central simple* F*-algebra,* K/F *an inertial Galois extension relative to the valuation rings* W/V. *Let* A *be a Dubrovin valuation ring in* $S \otimes_F K$ *with* $A \cap K = W$, *and* B *a Dubrovin valuation ring in* S *with* $B \cap F = V$. *Then,* $Z(\overline{A}) = Z(\overline{B})\overline{W}$, $\overline{A} \sim \overline{B} \otimes_{Z(\overline{B})} Z(\overline{B})\overline{W}$, $\Gamma_A/\Gamma_W = \theta_B^{-1}(\mathcal{G}(Z(\overline{B})/(Z(\overline{B}) \cap \overline{W})))$, $\delta(A) = \delta(B)$ *and* $M_n(\overline{B})$ *is a generalized crossed product over* $\overline{A}$, *where* $n = [K : F]$. *Furthermore,* $s_B \leq s_A$. *In particular, if* A *is integral then* B *is integral.*

Proof. First suppose A is integral. We have $K \cong F \otimes_F K \subseteq S \otimes_F M_n(F) = M_n(S)$ and $S \otimes_F K = C_{M_n(S)}(F \otimes_F K)$. So $M_n(S)$ is a generalized crossed product over $S \otimes_F K$ with group G. Thus by Corollary 2.4, $M_n(S)$ contains an integral Dubrovin extension B_0 of V. Hence by [D$_1$, Theorem 7, p. 280], S contains a Dubrovin valuation ring B_1/V and $B_0 \cong M_n(B_1)$. B_1 is integral as B_0 is integral. By [W$_1$, Theorem A], $B_1 \cong B$. Since $\overline{B_0} \cong M_n(\overline{B})$ is a generalized crossed product over $\overline{A}$, $\overline{A} = C_{\overline{B_0}}(Z(\overline{A}))$; hence $\overline{A} \sim \overline{B_0} \otimes_{Z(\overline{B_0})} Z(\overline{A}) \sim \overline{B} \otimes_{Z(\overline{B})} Z(\overline{A})$. $Z(\overline{A})/Z(\overline{B})$ is a Galois extension with group I', and no element of I' other than the identity fixes $\overline{W}$ since $\psi_\sigma|_{\overline{W}} = \overline{\sigma} \neq \text{id}$ for $\sigma \neq \text{id}$. Hence $\mathcal{G}(Z(\overline{A})/Z(\overline{B})\overline{W}) = \{id\}$, so $Z(\overline{B})\overline{W} = Z(\overline{A})$. From the diagram in 2.1, since θ_A and θ_B are surjective we see that the image of Γ_A/Γ_W in $\mathcal{G}(Z(\overline{B})/\overline{V})$ is $\mathcal{G}(Z(\overline{B})/(Z(\overline{B}) \cap \overline{W}))$, so $\Gamma_A/\Gamma_W = \theta_B^{-1}(\mathcal{G}(Z(\overline{B})/(Z(\overline{B}) \cap \overline{W})))$. Finally, since W/V is inertial, $\delta(W/V) = 1$, so $\delta(A) = \delta(B_0) = \delta(B)$.

Now drop the assumption of integrality. Let (F_h, V_h) and (K_h, W_h) be the Henselizations of (F, V) and (K, W) respectively. Let D and E be the underlying division algebras of $S \otimes_F F_h$ and $(S \otimes_F K) \otimes_K K_h = S \otimes_F K_h$. Since F_h and K_h are Henselian there are invariant valuation rings $R \subseteq D$ and $R' \subseteq E$ lying over V_h, W_h. By [W$_1$, Theorem B], $\overline{B} = \overline{R}$, $\overline{A} = \overline{R'}$, $\Gamma_B = \Gamma_R$, $\Gamma_A = \Gamma_{R'}$, $\delta(B) = \delta(R)$, $\delta(A) = \delta(R')$, $\theta_B = \theta_R$ and $\theta_A = \theta_{R'}$. Therefore applying what has been proven above to R and $M_r(R')$, where r is the matrix size of $D \otimes_{F_h} K_h$, show the conclusions proven for A integral above hold in general.

To see $s_B \le s_A$, let $V \subseteq V_1 \subseteq V_2$ be overrings of V in F and set $B_i = BV_i$, $W_i = WV_i$ and $A_i = AW_i = AV_i$. Since W/V is inertial, W_i/V_i is inertial. The above arguments applied to B_2 and A_2 show $Z(\overline{B_2}) \subseteq Z(\overline{A_2})$. Let $\widetilde{V_1} = V_1/J(V_2)$ and $\widetilde{W_1} = W_1/J(W_2)$. So $\widetilde{W_1}/\widetilde{V_1}$ is also inertial. Let l_i (resp. l_i') be the number of extensions of $\widetilde{V_1}$ (resp. $\widetilde{W_1}$) to $Z(\overline{B_2})$ (resp. $Z(\overline{A_2})$). Since $\widetilde{W_1}/\widetilde{V_1}$ is indecomposed, l_i is the number of extensions of $\widetilde{W_1}$ to $Z(\overline{A_2})$. As $Z(\overline{B_2}) \subseteq Z(\overline{A_2})$, $\widetilde{W_1}$ extends in at most l_i ways to $Z(\overline{B_2})$. Therefore $l_i' \le l_i$. Applying this to [W$_1$, Theorem D] the l_i for B are less than or equal to the corresponding l_i' for A, so $s_B \le s_A$. □

The following corollary was pointed out to us by Tignol.

Corollary 3.2. *Suppose S is a central simple F-algebra and V is a valuation ring of F. If S is split by a Galois extension K of F containing an inertial extension of V, then every Dubrovin valuation ring B of S lying over V is integral over V.*

Proof. Say $S \otimes_F K \cong M_k(K)$. If W is the inertial extension of V to K, then $M_k(W)$ is a Dubrovin valuation ring of $M_k(K)$ integral over W. Theorem 3.1 shows B is integral over V. □

In Theorem 3.1 we related the Dubrovin valuation rings B of S and A of $S \otimes_F K$ by showing $M_n(B)$ is a compatible extension of A (after modifying the choice of B). A more direct approach would be to choose A and B so that A is a compatible extension of B. However, it turns out that this is often not possible. We will show in Proposition 3.4 exactly when this can be done.

Lemma 3.3. *Let $S \subseteq T$ be central simple algebras. Then there is a value function w on T (with B_w a Dubrovin valuation ring) such that $w|_S$ is a value function (with $B_{w|_S}$ a Dubrovin valuation ring) iff there is a compatible extension A/B of integral Dubrovin valuation rings with B a Dubrovin valuation ring of S and A of T.*

Proof. Suppose w is a value function on T with $w|_S$ a value function on S. If A is the integral Dubrovin valuation ring of T corresponding to w and B corresponding to $w|_S$, then it is clear that A/B is a compatible extension. Conversely, suppose A/B is compatible. Let w be the value function on T

corresponding to A. Then $B = A \cap S = \{s \in S \mid w(s) \geq 0\}$ and $J(B) = \{s \in S \mid w(s) > 0\}$. So if $v = w|_S$ then $B_v = B$ and $J_v = J(B)$ where B_v and J_v are as in [M_2, Lemma 2.2]. Therefore $B_v/J_v = \overline{B}$ is simple Artinian. To show v is a value function on S and B is a Dubrovin valuation ring, by [M_2, Theorem 2.4] it remains to show $\mathrm{im}(v) = v(st(v))$. Take $s \in S$. Since B is integral, $BsB = \beta B$ for some $\beta \in st(B)$. So $AsA = \beta A$, therefore $v(s) = w(s) = w(\beta)$. $w(\beta^{-1}) = -w(\beta)$ as $\beta \in st(B) \subseteq st(A) = st(w)$. Thus $\beta \in st(v)$, and $v(s) = w(\beta) = v(\beta) \in (st(v))$. Therefore v is a value function on S. □

Proposition 3.4. *Let S, K, F, W, V be as in Theorem 3.1, and set $T = S \otimes_F K$. Suppose A is integral Dubrovin in T with $A \cap K = W$ and B integral Dubrovin in S with $B \cap F = V$. Then we can choose A so that A/B is compatible iff the following equivalent conditions hold:*

(1) *$Z(\overline{B})$ and $\overline{W}$ are linearly disjoint over $\overline{V}$,*
(2) *$Z(\overline{B}) \cap \overline{W} = \overline{V}$,*
(3) *$\Gamma_A = \Gamma_B$,*
(4) *$A \cong B \otimes_V W$.*

Proof. By Theorem 3.1, $\Gamma_A = \theta_B^{-1}(\mathscr{G}(Z(\overline{B})/(Z(\overline{B}) \cap \overline{W}))) \subseteq \Gamma_B$. Thus $\Gamma_A = \Gamma_B$ iff $Z(\overline{B}) \cap \overline{W} = \overline{V}$ iff $Z(\overline{B})$ and $\overline{W}$ are linearly disjoint over $\overline{V}$, as $Z(\overline{B})/\overline{V}$ is a normal extension and $\overline{W}/\overline{V}$ is separable. When these happen, $\overline{B} \otimes_{\overline{V}} \overline{W}$ is simple, so $B \otimes_V W$ is Dubrovin by [M_2, Theorems 3.3 and 3.4], and is a compatible extension of B. Conversely, if $A' = B \otimes_V W$ is Dubrovin, let $I = J(B) \otimes_V W$. Then, $I = J(B) \otimes_V W + B \otimes_V J(W)$ since $J(W) = J(V)W$. Also, $I \subseteq A'$ as W is a flat V-module. Then

$$A'/I \cong \overline{B} \otimes_{\overline{V}} \overline{W} = \overline{B} \otimes_{Z(\overline{B})} (Z(\overline{B}) \otimes_{\overline{V}} \overline{W}).$$

Now $Z(\overline{B}) \otimes_{\overline{V}} \overline{W}$ is a direct sum of fields, so A'/I is semisimple. As $I \subseteq J(A')$ (since $I \neq A'$) this implies that $I = J(A')$, so A'/I is actually simple, hence $Z(\overline{B}) \otimes_{\overline{V}} \overline{W}$ is a field. Thus $Z(\overline{B})$ and $\overline{W}$ are linearly disjoint over $\overline{V}$. On the other hand, if A/B is compatible then [W_1, §3] (or Lemma 3.3) shows $\Gamma_B \subseteq \Gamma_A$. Since $\Gamma_A \subseteq \Gamma_B$ we get $\Gamma_A = \Gamma_B$. □

In [JW, Theorem 3.1] Jacob and the second author prove a theorem about inertial scalar extensions of Henselian valued fields. By using Dubrovin valuation rings we now give a more natural proof of [JW, Theorem 3.1] in the case of a Galois inertial extension.

Corollary 3.5. *Let (F, V) be a Henselian valued field, (K, W) an inertial Galois extension of (F, V), D an F-central division algebra and E the underlying division algebra of $D \otimes_F K$. If V_D, V_E are the invariant valuation rings of D, E lying over V, W respectively then*

(1) $Z(\overline{V_E}) = Z(\overline{V_D})\overline{W}$.
(2) $\overline{V_E} \sim \overline{V_D} \otimes_{Z(\overline{V_D})} Z(\overline{V_D})\overline{W}$.

(3) $\Gamma_{V_E} = \theta_{V_D}^{-1}(\mathcal{G}(Z(\overline{V_D})/Z(\overline{V_D}) \cap \overline{W}))$.
(4) $\delta(V_E) = \delta(V_D)$.
(5) *The following diagram is commutative*:

$$\begin{array}{ccc} \Gamma_{V_E}/\Gamma_W & \xrightarrow{\theta_{V_E}} & \mathcal{G}(Z(\overline{V_E})/\overline{W}) \\ \downarrow & & \downarrow{\scriptstyle \text{res}} \\ \Gamma_{V_D}/\Gamma_V & \xrightarrow{\theta_{V_D}} & \mathcal{G}(Z(\overline{V_D})/\overline{V}). \end{array}$$

Proof. Since F is Henselian, every central simple algebra T with $F \subseteq Z(T)$ and $[T : F] < \infty$ contains an integral Dubrovin valuation ring lying over the (Henselian) valuation ring V of F. As $\Gamma_{M_n(B)} = \Gamma_B$, $\overline{M_n(B)} = \overline{B}$, $\delta(M_n(B)) = \delta(B)$ and $\theta_{M_n(B)} = \theta_B$ for any Dubrovin valuation ring B, by [W$_1$, Corollary 3.6, Theorems B and C], this corollary follows from Theorem 3.1 and the diagram of Theorem 2.1. □

The next two results deal with inertially split algebras, that is central simple F-algebras S such that S has a splitting field K which is an inertial extension of F. In [JW, §5] such algebras are described homologically (for F Henselian). Theorem 2.1 allows these results to be seen via Dubrovin valuation rings and generalized.

Theorem 3.6. *Let K/F be an inertial Galois extension relative to the valuation rings W/V and let $G = \mathcal{G}(K/F)$. Let $S = (K, G, f)$ be a crossed product algebra for some $f \in Z^2(G, K^*)$. Define α and w' as in Theorem 2.1, (with $A = W$). If B is the integral Dubrovin valuation ring of S associated to w' then $\Gamma_B/\Gamma_V = im(\alpha)$, $\overline{B} = (\overline{W}, \overline{I}, g)$ for some $g \in Z^2(\overline{I}, \overline{W}^*)$, where $I =$ ker(α) and $\overline{I}$ is the image of I in $\mathcal{G}(\overline{W}/\overline{V})$, and B is defectless. Furthermore, if α_0 is the map $\mathcal{G}(Z(\overline{B})/\overline{V}) \to \mathcal{G}(\overline{W}/\overline{V}) \to \Gamma_B/\Gamma_V$ induced by α, then $\theta_B = \alpha_0^{-1}$.*

Proof. Here $C = K$ and so B is defectless since W is. Everything follows immediately from Theorem 2.1. □

An easy calculation shows that the formula for α is the same as given in [JW, Remark 5.5]. Hence we get a new proof of Theorem 5.6(b) in [JW]:

Corollary 3.7. *Let (F, V) be a Henselian valued field and D an inertially split division algebra with $F = Z(D)$. Write $D \sim (K, G, f)$ for any splitting field K of D with K inertial Galois over F, and let $\alpha\colon G \to \Delta_V/\Gamma_V$ be the map of Theorem 2.1. If V_D is the invariant valuation ring of D lying over V, then V_D/V is defectless, $\Gamma_{V_D}/\Gamma_V = im(\alpha)$ and $Z(\overline{V_D}) = \overline{\mathcal{F}(\text{ker}(\alpha))}$.*

Proof. Since D is inertially split and F is Henselian, $D \sim (K, G, f)$ for some K/F inertial Galois. The corollary then follows from Theorem 3.6. □

In [JW, §6] it is shown that a tame division algebra D over a Henselian field can be decomposed up to Brauer equivalence into an inertially split division algebra tensored with a totally ramified division algebra. Then information on the value group and the center of the residue division algebra for the valuation ring of D is obtained from the tensor factors. We now generalize this to integral Dubrovin valuation rings.

First, to set up notation, let (F, V) be a valued field, K, L Galois extensions of F with $G = \mathcal{G}(L/F)$, $H = \mathcal{G}(K/F)$, $n = |G|$ and $m = |H|$ such that V extends to $V_K \subseteq K$ and $V_L \subseteq L$ with V_K/V tame totally ramified and V_L/V inertial. Take $f \in Z^2(G, L^*)$, $g \in Z^2(H, K^*)$, $S = (L, G, f)$, $T = (K, H, g)$ and suppose T is a totally ramified division algebra with respect to the extension of v to T with invariant valuation ring B_T. S has an integral Dubrovin valuation ring B_S by Theorem 3.6.

Theorem 3.8. *With S and T as above, $S \otimes_F T$ contains an integral Dubrovin valuation ring B such that B is defectless, $\Gamma_B = \Gamma_{B_S} + \Gamma_{B_T}$, $\ker(\theta_B) = \Gamma_{B_T}$, $\overline{B} = (\overline{V_L}, G_0, h)$ for some $G_0 \subseteq G$, and $Z(\overline{B}) = \mathcal{F}(\theta_{B_S}(\Gamma_{B_S} \cap \Gamma_{B_T})/\Gamma_V) \subseteq Z(\overline{B_S})$. Furthermore, the following diagram commutes:*

$$\begin{array}{ccc} \Gamma_{B_S}/\Gamma_V & \xrightarrow{\theta_{B_S}} & \mathcal{G}(Z(\overline{B_S})/\overline{V}) \\ \downarrow & & \downarrow \text{res} \\ \Gamma_B/\Gamma_V & \xrightarrow{\theta_B} & \mathcal{G}(Z(\overline{B})/\overline{V}). \end{array}$$

Proof. Write $S = \bigoplus_{\sigma \in G} Lx_\sigma$, $T = \bigoplus_{\tau \in H} Ky_\tau$ where the x_σ, y_τ have the usual crossed product properties. We assume f and g have been normalized and that f has also been modified to suppose $f(\sigma, \tau) \in st(B_S)$ as in Theorem 2.1. Let w be the value function on S as in Theorem 2.1 and for convenience also write w for the valuation on T. Since $y_\tau^m = \prod_{i=1}^m g(\tau^i, \tau)$, $w(y_\tau) = \frac{1}{m}\sum_{i=1}^m w(g(\tau^i, \tau))$. Furthermore, the $w(y_\tau)$ are distinct mod Γ_{V_K}, and so $w\left(\sum_{\tau \in H} k_\tau y_\tau\right) = \min_\tau\{w(k_\tau) + w(y_\tau)\}$. To see this, if $w(y_\tau) \in \Gamma_{V_K}$, we can modify y_τ to assume $w(y_\tau) = 0$. Thus for all $a \in V_K$, $\overline{\tau(a)} = \overline{y_\tau a y_\tau^{-1}} = \overline{y_\tau}\,\overline{a}\,\overline{y_\tau}^{-1} = \overline{a}$ as $\overline{B_T} = \overline{V}$, a field. So $\overline{\tau(a)/a} = 1$ for all $a \in K^*$. But the proof of [JW, Proposition 2.2] yields $\tau = 1$. Let $M = LK$. Since $L \cap K$ is both totally ramified and inertial over F, $L \cap K = F$. Thus M/F is Galois with Galois group $G \times H$. Furthermore, M/F is indecomposed with respect to V, and the extension V_M of V to M has $\overline{V_M} = \overline{V_L}$ and $\Gamma_{V_M} = \Gamma_{V_K}$. By [R, 29.9, 29.16], $S \otimes_F T = (M, G \times H, h)$ where h is the 2-cocycle defined by $h((\sigma_1, \tau_1), (\sigma_2, \tau_2)) = f(\sigma_1, \sigma_2)g(\tau_1, \tau_2)$. Let $(M, G \times H, h) = \bigoplus_{(\sigma, \tau) \in G \times H} Mz_{\sigma\tau}$ and define w' on $S \otimes_F T = \bigoplus_{(\sigma, \tau) \in G \times H} Mz_{\sigma\tau}$ as in Theorem 2.1. Then $w'(z_{\sigma\tau}) = w(x_\sigma) + w(y_\tau)$. To see this,

$$\begin{aligned} w'(z_{\sigma\tau}) &= \frac{1}{nm}\sum_{i=1}^{nm} w(h((\sigma, \tau)^i), h(\sigma, \tau)) = \frac{1}{nm}\sum_{i=1}^{nm} w(f(\sigma^i, \sigma)g(\tau^i, \tau)) \\ &= \frac{1}{nm}\sum_{i=1}^{nm} w(f(\sigma^i, \sigma)) + \frac{1}{nm}\sum_{i=1}^{nm} w(g(\tau^i, \tau)) \\ &= \frac{1}{n}\sum_{i=1}^{n} w(f(\sigma^i, \sigma)) + \frac{1}{m}\sum_{i=1}^{m} w(g(\tau^i, \tau)) = w(x_\sigma) + w(y_\tau) \end{aligned}$$

since $|G| = n$ and $|H| = m$. The kernel of the canonical map $\mathscr{G}(M/F) \to \mathscr{G}(\overline{V_M}/\overline{V})$ is H. Let I be the kernel of $\alpha\colon G \times H \to \Delta/\Gamma_{V_M} = \Delta/\Gamma_{V_K}$ given by $\alpha(\sigma, \tau) = w'(z_{\sigma\tau}) + \Gamma_{V_M}$. If $\tau \in H \cap I$, $w'(z_{1\tau}) + \Gamma_{V_M} = w(y_\tau) + \Gamma_{V_M} = 0$, so $w(y_\tau) \in \Gamma_{V_M} = \Gamma_{V_K}$, which implies $\tau = 1$. Thus $I \cap H = 0$, so by Corollary 2.4, Theorem 2.1 applies.

Let $B \subseteq S \otimes_F T$ be the integral Dubrovin valuation ring obtained from w'. Then $\Gamma_B/\Gamma_{V_K} = \mathrm{im}(\alpha) = (\Gamma_{B_S} + \Gamma_{B_T})/\Gamma_{V_K}$, so $\Gamma_B = \Gamma_{B_S} + \Gamma_{B_T}$. B/V is defectless since V_M/V is defectless. Let $\overline{I}$ be the image of I in $\mathscr{G}(\overline{V_M}/\overline{V}) = G$. Then $\overline{B} = (\overline{V_M}, \overline{I}, k)$ for some cocycle k. Since $\overline{I} = \{\sigma \in G \mid w(x_\sigma) \in \Gamma_{B_T}\} \supseteq \{\sigma \in G \mid w(x_\sigma) \in \Gamma_{V_L}\} = \mathscr{G}(\overline{V_L}/Z(\overline{B_S}))$, $Z(\overline{B}) = \mathscr{F}(\overline{I}) \subseteq Z(\overline{B_S})$. $\theta_{B_S}(w(x_\sigma)) = \overline{\sigma}|_{Z(\overline{B_S})}$ and $\theta_B(w'(z_{\sigma\tau})) = \overline{(\sigma, \tau)}|_{Z(\overline{B})} = \overline{\sigma}|_{Z(\overline{B})}$. Since $w'(z_{\sigma\tau}) = w(x_\sigma) + w(y_\tau)$, $\theta_B(w(x_\sigma)) = \theta_B(w'(z_{\sigma 1})) = \overline{\sigma}|_{Z(\overline{B})}$, which is the restriction of $\theta_{B_S}(w(x_\sigma))$ to $Z(\overline{B})$. So the diagram commutes. To determine $Z(\overline{B})$ note that $\Gamma_{B_S}/\Gamma_V = \{w(x_\sigma) + \Gamma_V \mid \sigma \in G\}$ and $\theta_{B_S}(w(x_\sigma) + \Gamma_V) = \overline{\sigma}|_{Z(\overline{B_S})}$. So $\theta_{B_S}((\Gamma_{B_S} \cap \Gamma_{B_T})/\Gamma_V) = \{\overline{\sigma}|_{Z(\overline{B_S})} \mid w(x_\sigma) \in \Gamma_{B_T}\} = \{\overline{\sigma}|_{Z(\overline{B_S})} \mid \sigma \in \overline{I}\}$. Therefore $\mathscr{F}(\theta_{B_S}((\Gamma_{B_S} \cap \Gamma_{B_T})/\Gamma_V)) = \mathscr{F}(\overline{I}) \cap Z(\overline{B_S}) = \mathscr{F}(\overline{I}) = Z(\overline{B})$ since $\mathscr{F}(\overline{I}) \subseteq Z(\overline{B_S})$. Finally to see $\Gamma_{B_T} = \ker(\theta_B)$, since $\Gamma_{B_T} = \Gamma_{V_K} + \langle w(y_\tau) \mid \tau \in H\rangle$ and $\theta_B(w(y_\tau) + \Gamma_V) = \mathrm{id}$, $\Gamma_{B_T} \subseteq \ker(\theta_B)$. For the reverse inclusion, take $\alpha \in \ker(\theta_B)$. Since $\Gamma_B = \Gamma_{B_S} + \Gamma_{B_T}$, $\alpha = w(x_\sigma) + \gamma$ with $\gamma \in \Gamma_{B_T}$. Then $w(x_\sigma) \in \ker(\theta_B)$. So we see from above that $\sigma \in \overline{I}$, hence $w(x_\sigma) \in \Gamma_{B_T}$. □

Corollary 3.9 [JW, Theorem 6.3]. *Let (F, V) be a Henselian valued field, S an inertially split division algebra over F and T a tame, totally ramified division algebra over F. If $D \sim S \otimes_F T$ and V_D, V_S, V_T are the invariant valuation rings of D, S, T respectively lying over V, then $\Gamma_{V_D} = \Gamma_{V_S} + \Gamma_{V_T}$, $\Gamma_{V_T}/\Gamma_V = \ker(\theta_{V_D})$, $Z(\overline{V_D}) = \mathscr{F}(\theta_{V_S}((\Gamma_{V_S} \cap \Gamma_{V_T})/\Gamma_V))$, and the following diagram is commutative:*

$$\begin{array}{ccc} \Gamma_{V_S}/\Gamma_V & \xrightarrow{\theta_{V_S}} & \mathscr{G}(Z(\overline{V_S})/\overline{V}) \\ \downarrow & & \downarrow \mathrm{res} \\ \Gamma_{V_D}/\Gamma_V & \xrightarrow{\theta_{V_D}} & \mathscr{G}(Z(\overline{V_D})/\overline{V}). \end{array}$$

Proof. Say L_0 is an inertial splitting field of S. If L is the normal closure of L_0/F, then L/F is inertial Galois and $S \sim (L, G, f)$ for some $f \in Z^2(G, L^*)$, where $G = \mathscr{G}(L/F)$. Draxl's decomposition theorem [Dr, Theorem 1] shows T is a tensor product of symbol algebras, hence a crossed product. Thus, Theorem 3.8 applies to $(L, G, f) \otimes_F T$. Since this algebra is isomorphic to some $M_k(D)$, any Dubrovin valuation ring of it lying over V is isomorphic to $M_k(V_D)$ by [W$_1$, Theorem A]. The corollary then follows because $\overline{M_k(V_D)} \cong M_k(\overline{V_D})$ and $\Gamma_{M_k(V_D)} = \Gamma_{V_D}$ by [W$_1$, Corollary 3.5]. □

In [W$_2$] the second author determines those division algebras which arise as the residue division ring of an invariant valuation ring in a finite dimensional F-central division algebra. One step in the proof is to decompose a given valued division algebra (in the Brauer group $\mathrm{Br}(F)$) into a tensor product of other division algebras, such that the residue division ring of each piece can be easily calculated. Given two such pieces, say D_1 and D_2, it is seen that $\Gamma_{V_{D_1}} \cap \Gamma_{V_{D_2}} = \Gamma_V$, but $\overline{V_{D_1}} \otimes_{\overline{V}} \overline{V_{D_2}}$ is not neccesarily a division algebra. However, there is a common subfield L of D_1 and D_2 inertial over F with $\overline{V_L} \subseteq Z(\overline{V_{D_i}})$ such that $\overline{V_{D_1}} \otimes_{\overline{V_L}} \overline{V_{D_2}}$ is a division algebra. The second author then proves [W$_2$, Proposition 2], a generalization of [M$_1$, Theorem 1], to describe the underlying division algebra of $D_1 \otimes_F D_2$.

Theorem 3.11 below generalizes [M$_2$, Theorem 3.3] and [W$_2$, Proposition 2] in the context of Dubrovin valuation rings. The generalization is complicated by the fact that contractions of Dubrovin valuation rings to subalgebras need not be Dubrovin valuation rings. The hypothesis (4) of Theorem 3.11 is needed to go from the original central simple algebra to a centralizer. The examples in §4 show the pathology that can arise in this situation.

Lemma 3.10. *Let S be a central simple F-algebra, B an integral Dubrovin valuation ring over V in S, $L \subset S$ an inertial Galois extension of F with respect to W/V. Suppose $\overline{W} \subseteq Z(\overline{B})$ and either B is invariant or for all overrings V_i of V in F, $\overline{WV_i} \subseteq Z(\overline{BV_i})$. If $C = C_S(L)$ and $A \subseteq C$ is a Dubrovin valuation ring over W then A is integral, $\overline{A} = \overline{B}$, $\Gamma_A = \ker(\rho) \subseteq \Gamma_B$, where $\rho\colon \Gamma_B \to \mathcal{G}(\overline{W}/\overline{V})$ is the map induced by θ_B. Furthermore, A/W is defectless iff B/V is defectless.*

Proof. If B is invariant then $A = B \cap C$ and the lemma is clear by using [M$_1$, Theorem 2] and [JW, Lemma 1.8]. So suppose B is not invariant. First, to show A is integral we use [W$_1$, Theorem F]. Let $B \subseteq B_i \subseteq B_j$ be two overrings of B with $V_i = B_i \cap F$ and $V_j = B_j \cap F$. Since W/V is inertial, $W_i = WV_i$ and $W_j = WV_j$ are inertial over V_i and V_j respectively. Let $A_i = AW_i$ and $A_j = AW_j$, overrings of A. By Theorem 3.1 we get $Z(\overline{A_j}) = Z(\overline{B_j})\overline{W_j} = Z(\overline{B_j})$. Let $\widetilde{V_i} = V_i/J(V_j) \subseteq \overline{V_j}$ and $\widetilde{W_i} = W_i/J(W_j) \subseteq \overline{W_j}$. Since B/V is integral, $\widetilde{V_i}$ extends uniquely to $Z(\overline{B_j})$. Thus the diagram below shows $\widetilde{W_i}$ extends uniquely to $Z(\overline{A_j})$. Thus by [W$_1$, Theorem F], A/W is integral.

$$\begin{array}{ccc} & & Z(\overline{A_j}) = Z(\overline{B_j}) \\ & & \cup \\ \widetilde{W_i} & \subset & \overline{W_j} \\ \cup & & \cup \\ \widetilde{V_i} & \subset & \overline{V_j} \end{array}$$

To determine Γ_A and $\overline{A}$ we "Henselize" and use [W$_1$, Theorem B]. If (F_h, V_h) is the Henselization of (F, V) and (L_h, W_h) is the Henselization of (L, W), then $L_h = LF_h$. Let D, E be the underlying division algebras of $S \otimes_F F_h$ and $C \otimes_L L_h$ respectively. Since F_h and L_h are Henselian, there are invariant valuation rings $R \subseteq D$ and $R' \subseteq E$ extending V_h and W_h respectively. So [W$_1$, Theorem B] shows that $\Gamma_B = \Gamma_R$, $\Gamma_A = \Gamma_{R'}$, $\overline{B} \sim \overline{R}$

and $\overline{A} \sim \overline{R'}$. By Corollary 3.5, $\overline{R'} \sim \overline{D}$, so $\overline{A} \sim \overline{B}$ as $\overline{W} \subseteq Z(\overline{B})$. Also, $\Gamma_{R'} = \theta_R^{-1}(\mathcal{G}(Z(\overline{R})/\overline{W_h}))$, hence $\Gamma_A = \ker(\rho)$. $\delta(A) = \delta(R') = \delta(R) = \delta(B)$, so A is defectless iff B is defectless. The only thing remaining is to show $\overline{A} = \overline{B}$. Since $\overline{A} \sim \overline{B}$ it suffices to show $t_A = t_B$. Since A and B are integral it is then enough to show $n_A = n_B$. By definition of n_A and n_B, $S \otimes_F F_h = M_{n_B}(D)$ and $C \otimes_L L_h = M_{n_A}(E)$. Since $C \sim S \otimes_F L$, dimension count shows $S \otimes_F L = M_l(C)$, where $l = [L : F]$. Therefore $(S \otimes_F L) \otimes_L L_h = M_{ln_A}(E)$. But $(S \otimes_F L) \otimes_L L_h = S \otimes_F L_h = S \otimes_F (F_h \otimes_{F_h} L_h) = (S \otimes_F F_h) \otimes_{F_h} L_h = M_{n_B}(D \otimes_{F_h} L_h)$. Since $\overline{W_h} = \overline{W} \subseteq Z(\overline{B}) = Z(\overline{R})$ and F_h is Henselian, L_h embeds in D, so $D \otimes_{F_h} L_h = M_l(E)$. Thus $M_{ln_A}(E) = M_{n_B l}(E)$, so $n_A = n_B$. □

Theorem 3.11. *Let S_1 and S_2 be central simple F-algebras, $B_i \subseteq S_i$ integral Dubrovin valuation rings with $B_1 \cap F = B_2 \cap F = V$. Suppose S_1 and S_2 contain a common subfield L which is Galois over F, $l = [L : F]$, and L has an extension W of V for which W/V is inertial. Suppose further that*

(1) $\overline{W} \subseteq Z(\overline{B_1})$, $Z(\overline{B_2})$,

(2) $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$,

(3) $\overline{B_1} \otimes_{\overline{W}} \overline{B_2}$ *is simple Artinian,*

(4) *Either the B_i are both invariant, or for all overrings $V_i \supseteq V$ in F, $\overline{WV_i} \subseteq Z(\overline{B_1 V_i})$, $Z(\overline{B_2 V_i})$,*

(5) B_1/V *is defectless.*

If $S = S_1 \otimes_F S_2$ and B is a Dubrovin valuation ring in S lying over V, then B is integral, $\overline{B} = M_l(\overline{B_1} \otimes_{\overline{W}} \overline{B_2})$, $\Gamma_B = \{\gamma_1 + \gamma_2 \in \Gamma_{B_1} + \Gamma_{B_2} \mid \rho_1(\gamma_1) = \rho_2(\gamma_2)\}$, where $\rho_i : \Gamma_{B_i} \to \mathcal{G}(\overline{W}/\overline{V})$ is given by $\rho_i(bB_i^*)(\overline{w}) = \overline{bwb^{-1}}$.

Proof. Let $C_i = C_{S_i}(L)$ and A_i Dubrovin over W in C_i. By Lemma 3.10, A_i is integral over W, $\overline{A_i} = \overline{B_i}$ and $\Gamma_{A_i} = \Gamma_{B_i}$. Thus $\Gamma_{A_1} \cap \Gamma_{A_2} = \Gamma_V = \Gamma_W$ and $\overline{A_1} \otimes_{\overline{W}} \overline{A_2}$ is simple. Furthermore A_1/W is defectless since B_1/V is defectless. By [M$_2$, Theorem 3.3] if $C = C_1 \otimes_L C_2$ and A is the integral Dubrovin valuation ring of C described by that theorem, $\Gamma_A = \Gamma_{A_1} + \Gamma_{A_2}$ and $\overline{A} = \overline{A_1} \otimes_{\overline{W}} \overline{A_2}$. We write $S_i = (C_i, G, (\omega_i, f_i))$, with (ω_i, f_i) chosen as in Theorem 2.1 so that $f_i(\sigma, \tau) \in st(A_i)$ for all σ, $\tau \in G$. By the product theorem [T, Theorem 1.6] or [J$_2$, 1.15] or [KY, Theorem 3], $S = S_1 \otimes_F S_2 \sim (C, G, (\omega_1 \otimes \omega_2, f_1 \otimes f_2))$. Let $S_0 = (C, G, (\omega_1 \otimes \omega_2, f_1 \otimes f_2))$. By dimension count, $S = M_l(S_0)$ and $S_0 \otimes_F L = M_l(C)$, so $S \otimes_F L = M_{l^2}(C)$. Since $S \otimes_F L$ contains the integral Dubrovin valuation ring $M_{l^2}(A)$, S contains an integral Dubrovin valuation ring B by Theorem 3.1. As $S = M_l(S_0)$, S_0 contains an integral Dubrovin valuation ring B_0 with $B \cong M_l(B_0)$. That corollary gives $\overline{A} \sim M_{l^2}(\overline{A}) \sim \overline{B} \otimes_{Z(\overline{B})} Z(\overline{B})\overline{W} = \overline{B}$ as $\overline{W} \subseteq Z(\overline{B})$. Because

$$\Gamma_A = \theta_B^{-1}(\mathcal{G}(Z(\overline{B})/(Z(\overline{B}) \cap \overline{W}))) = \theta_B^{-1}(\mathcal{G}(Z(\overline{B})/\overline{W})),$$

$|\Gamma_B : \Gamma_A| = l$. As $\delta(A) = \delta(B)$, using [W$_1$, Theorem C] applied to $[S : F]$ and $[C : L]$ we see that $[\overline{B} : \overline{V}] = l^2[\overline{A} : \overline{V}]$, so $\overline{B} = M_l(\overline{A}) = M_l(\overline{B_1} \otimes_{\overline{W}} \overline{B_2})$.

To determine Γ_B, we have to look more closely at S_0. Say $S_1 = \bigoplus_{\sigma\in G} Cx_\sigma$, $S_2 = \bigoplus_{\sigma\in G} Cy_\sigma$ and $S_0 = \bigoplus_{\sigma\in G} Cz_\sigma$ where the x_σ, y_σ, z_σ correspond to (ω_1, f_1), (ω_2, f_2) and $(\omega_1\otimes\omega_2, f_1\otimes f_2)$ respectively. Since $f_i(\sigma,\tau)\in st(A_i)$, $f_1(\sigma,\tau)\otimes f_2(\sigma,\tau)\in st(A)$. If w_1, w_2, w are the value functions on C_1, C_2, C corresponding to A_1, A_2, A and w_1', w_2', w' are the value functions on S_1, S_2, S_0 corresponding to B_1, B_2, B_0 respectively, then

$$\begin{aligned} w'(z_\sigma) &= \frac{1}{l}\sum_{i=1}^{l} w(h(\sigma^i,\sigma)) = \frac{1}{l}\sum_{i=1}^{l} w(f_1(\sigma^i,\sigma)\otimes f_2(\sigma^i,\sigma)) \\ &= \frac{1}{l}\sum_{i=1}^{l} w_1(f_1(\sigma^i,\sigma)) + \frac{1}{l}\sum_{i=1}^{l} w_2(f_2(\sigma^i,\sigma)) \\ &= w_1'(x_\sigma) + w_2'(y_\sigma). \end{aligned}$$

Therefore $\Gamma_B = \Gamma_A + \langle w'(z_\sigma)\rangle \subseteq (\Gamma_{A_1} + \langle w_1(x_\sigma)\rangle) + (\Gamma_{A_2} + \langle w_2(y_\sigma)\rangle) = \Gamma_{B_1} + \Gamma_{B_2}$. Since $\rho_1(w_1(x_\sigma)) = \rho_2(w_2(y_\sigma)) = \overline{\sigma}$ it follows that $\Gamma_B = \{\gamma_1+\gamma_2 \in \Gamma_{B_1}+\Gamma_{B_2} \mid \rho_1(\gamma_1) = \rho_2(\gamma_2)\}$. □

This result has a couple of corollaries which deal with invariant valuation rings and arbitrary Dubrovin valuation rings. Note that $\overline{B_1}\otimes_{\overline{W}}\overline{B_2}$ being simple is equivalent to $Z(\overline{B_1})$ and $Z(\overline{B_2})$ linearly disjoint over $\overline{W}$.

Corollary 3.12. *Suppose (F, V) is Henselian, D_1, D_2 F-central division algebras, $V_{D_i}\subseteq D_i$ invariant valuation rings lying over V with V_{D_1}/V defectless, $\Gamma_{V_{D_1}}\cap\Gamma_{V_{D_2}} = \Gamma_V$, $\mathscr{L} = Z(\overline{V_{D_1}})\cap Z(\overline{V_{D_2}})$ separable over $\overline{V}$ and $Z(\overline{V_{D_1}})$ and $Z(\overline{V_{D_2}})$ linearly disjoint over $\mathscr{L}$. If D is the underlying division algebra of $D_1\otimes_F D_2$, then $\Gamma_{V_D} = \{\gamma_1+\gamma_2 \in \Gamma_{V_{D_1}}+\Gamma_{V_{D_2}} \mid \rho_1(\gamma_1) = \rho_2(\gamma_2)\}$ where $\rho_i\colon \Gamma_{V_{D_i}} \to \mathscr{G}(\mathscr{L}/\overline{V})$ is as above and $\overline{V_D}\sim\overline{V_{D_1}}\otimes_{\mathscr{L}}\overline{V_{D_2}}$.*

Proof. Because $\mathscr{L}$ is separable over $\overline{V}$ and F is Henselian, there is a unique (up to isomorphism) inertial extension L of F with $\overline{V_L} = \mathscr{L}$, which lies in both D_1 and D_2. As $Z(\overline{V_{D_1}})_{\text{sep}}$ is abelian Galois over $\overline{V}$ by [JW, Proposition 1.7] and $\mathscr{L}\subseteq Z(\overline{V_{D_1}})_{\text{sep}}$, $\mathscr{L}/\overline{V}$ is Galois, hence L/F is Galois. The linear disjointness of $Z(\overline{V_{D_1}})$ and $Z(\overline{V_{D_2}})$ over $\mathscr{L}$ implies that $\overline{V_{D_1}}\otimes_{\overline{\mathscr{L}}}\overline{V_{D_2}}$ is simple. Since $D_1\otimes_F D_2 \cong M_n(D)$ for some D, the corollary then follows directly from Theorem 3.11 and [W$_1$, Corollary 3.6]. □

Corollary 3.13. *Let (F, V) be any valued field, S_1 and S_2 central simple F-algebras and $B_i\subseteq S_i$ Dubrovin valuation rings over V. Suppose B_1/V is defectless, $\Gamma_{B_1}\cap\Gamma_{B_2} = \Gamma_V$, $\mathscr{L} = Z(\overline{B_1})\cap Z(\overline{B_2})$ separable over $\overline{V}$, and $Z(\overline{B_1})$ and $Z(\overline{B_2})$ linearly disjoint over $\mathscr{L}$. Then if B is a Dubrovin valuation ring in $S_1\otimes_F S_2$ lying over V, we have $\Gamma_B = \{\gamma_1+\gamma_2 \mid \rho_1(\gamma_1) = \rho_2(\gamma_2)\}$ and $\overline{B}\sim\overline{B_1}\otimes_{\mathscr{L}}\overline{B_2}$.*

Proof. This follows easily from Corollary 3.12 and the Henselization theorem [W$_1$, Theorem B]. □

The conditions in Corollaries 3.12 and 3.13 that $Z(\overline{B_1}) \cap Z(\overline{B_2})$ is separable over $\overline{V}$ and $Z(\overline{B_1})$ and $Z(\overline{B_2})$ be linearly disjoint over the $Z(\overline{B_1}) \cap Z(\overline{B_2})$ occur quite often. For instance if either $Z(\overline{B_1})$ or $Z(\overline{B_2})$ is separable over $\overline{V}$ then both conditions hold. Hence if B_1/V is tame then these conditions hold.

4. Examples

In Theorem 3.11 it is somewhat unsatisfying to have to include hypothesis (4). However, the second example below will show that this is necessary. The difficulty shows up in Lemma 3.10. Given C a subalgebra of a central simple algebra S and an integral Dubrovin valuation ring B in S one would like to obtain an integral Dubrovin valuation ring in C. This is not always possible, as shown in the next example. This shows one particular difficulty with Dubrovin valuation rings, that contractions of Dubrovin valuation rings to subalgebras need not be Dubrovin.

Example 4.1. *D an F-central division algebra, $L \subset D$ a Galois inertial field extension of F with respect to the valuation rings W/V, B integral Dubrovin in D lying over V with $\overline{W} \subseteq Z(\overline{B})$, $C = C_D(L)$, but C contains no integral Dubrovin valuation ring lying over W. Thus $B \cap C$ is not a Dubrovin valuation ring.*

Proof. Let

$$F = \mathbf{Q}(x, y), \qquad L = F(\sqrt{2}), \qquad D = \left(\frac{2, 1+x}{F}\right) \otimes_F \left(\frac{3, y}{F}\right).$$

Let V be composite of the usual rank two valuation ring on F with the 5-adic valuation ring on $\mathbf{Q}$. (That is, V is the valuation ring inherited from $\mathbf{Q}((x))((y))$.) Since 2 is not a square mod 5, $\mathbf{Z}_{(5)}$ extends uniquely to $\mathbf{Q}(\sqrt{2})$, hence V extends uniquely to an inertial valuation ring W of L. D can be seen to be a division ring in much the same way as in the examples in [M_2] (by considering the y-adic valuation ring of F), but since knowing whether D is a division ring is not needed, the proof will be omitted.

Now $\left(\frac{2, 1+x}{V}\right)$ is an Azumaya algebra over V with residue ring $\left(\frac{2, 1}{\overline{V}}\right) \cong M_2(\mathbf{Z}/5\mathbf{Z})$. By [JW, Ex. 4.3] $\left(\frac{3, y}{F}\right)$ is a nicely semiramified division algebra over F with residue division algebra $\overline{F}(\sqrt{3}) = (\mathbf{Z}/5\mathbf{Z})(\sqrt{3})$. If B is the tensor product of the above Azumaya algebra with this invariant valuation ring, then B is an integral Dubrovin valuation ring over V with residue ring $M_2(\mathbf{Z}/5\mathbf{Z}) \otimes_{\mathbf{Z}/5\mathbf{Z}} (\mathbf{Z}/5\mathbf{Z})(\sqrt{3}) = M_2((\mathbf{Z}/5\mathbf{Z})(\sqrt{3})) = M_2((\mathbf{Z}/5\mathbf{Z})(\sqrt{2}))$ by [M_2, Theorems 3.3 and 3.4]. Thus $\overline{W} = (\mathbf{Z}/5\mathbf{Z})(\sqrt{2}) \subseteq Z(\overline{B})$.

Let $C = C_D(L) = L \otimes_F \left(\frac{3, y}{F}\right) = \left(\frac{3, y}{L}\right)$. If W' is the usual rank two valuation ring on L with residue field $\mathbf{Q}(\sqrt{2})$, then C is a nicely semiramified division algebra over L with residue division algebra $\overline{W'}(\sqrt{3}) = \mathbf{Q}(\sqrt{2}, \sqrt{3})$. Now the 5-adic valuation ring of $\mathbf{Q}$ extends in two ways to $\mathbf{Q}(\sqrt{2}, \sqrt{3})$. Let A be the composite of the nicely semiramified valuation ring in C with one of the extensions of $\mathbf{Z}_{(5)}$ to $\mathbf{Q}(\sqrt{2},\sqrt{3})$. By [W$_1$, Th. F], since $\mathbf{Z}_{(5)}$ extends in more than one way to $\mathbf{Q}(\sqrt{2},\sqrt{3})$, A is not integral. This shows that the inequality $s_B \le s_A$ in Theorem 3.1 can be strict. □

As a preliminary to the next example, for F, L, V, W as in the above example, let $D = \left(\frac{2,x}{F}\right) \otimes_F \left(\frac{3,y}{F}\right)$ and V' the usual rank two valuation ring of F with residue field $\mathbf{Q}$. By [JW, Ex. 4.3], $\left(\frac{2,x}{F}\right)$ and $\left(\frac{3,y}{F}\right)$ are nicely semiramified division algebras over V' with value groups $\frac{1}{2}\mathbf{Z}\times\mathbf{Z}$, $\mathbf{Z}\times\frac{1}{2}\mathbf{Z}$ and residue division algebras $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{3})$ respectively. Thus by [JW, Ex. 4.3], D is a nicely semiramified division algebra whose invariant valuation ring B' satisfies $\overline{B'} = \mathbf{Q}(\sqrt{2},\sqrt{3})$. $\mathbf{Z}_{(5)}$ extends in two ways to $\overline{B'}$. If B is the composite of B' and one of these extensions, then B is a noninvariant Dubrovin valuation ring of D over V with $t_B = 1$. Thus $n_B \ne t_B$, so B is not integral. We will use this to give the following example.

Example 4.2. *D_1, D_2 F-central division algebras, $B_i \subseteq D_i$ integral Dubrovin over V, $L \subset D_1$, D_2 inertial Galois over F (with respect to $W \subseteq L$) with $\Gamma_{B_1}\cap\Gamma_{B_2} = \Gamma_V$, $\overline{W} \subseteq Z(\overline{B_i})$, $\overline{B_1}\otimes_{\overline{W}}\overline{B_2}$ simple and B_1/V defectless, but there is no integral Dubrovin valuation ring in $D_1\otimes_F D_2$ lying over V.*

Proof. Let F, L, V, W be as above, and set

$$D_1 = \left(\frac{2,11}{F}\right)\otimes_F\left(\frac{3,7y}{F}\right), \qquad D_2 = \left(\frac{2,11x}{F}\right)\otimes_F\left(\frac{3,7}{F}\right).$$

Thus $L \subseteq D_i$. $\left(\frac{2,11}{F}\right)$ and $\left(\frac{3,7}{F}\right)$ contain Azumaya algebras A_1 and A_2 respectively with $\overline{A_1} = \overline{A_2} = M_2(\mathbf{Z}/5\mathbf{Z})$. The algebras $\left(\frac{3,7y}{F}\right)$ and $\left(\frac{2,11x}{F}\right)$ contain nicely semiramified invariant valuation rings C_1 and C_2 respectively with $\Gamma_{C_1} = \mathbf{Z}\times\frac{1}{2}\mathbf{Z}\times\mathbf{Z}$, $\Gamma_{C_2} = \mathbf{Z}\times\mathbf{Z}\times\frac{1}{2}\mathbf{Z}$ and $\overline{C_1} = \overline{C_2} = \overline{W}$ respectively, as $(\mathbf{Z}/5\mathbf{Z}(\sqrt{2})) = (\mathbf{Z}/5\mathbf{Z}(\sqrt{3})) = \overline{W}$. If $B_i = A_i \otimes_V C_i$, then by [M$_2$, Theorems 3.3 and 3.4], B_i is integral Dubrovin, $\overline{B_i} = M_2(\overline{W})$ and $\Gamma_{B_i} = \Gamma_{C_i}$. So all the above conditions hold. However

$$D_1\otimes_F D_2 \cong \left(\frac{2,11}{F}\right)\otimes_F\left(\frac{2,11x}{F}\right)\otimes_F\left(\frac{3,7y}{F}\right)\otimes_F\left(\frac{3,7}{F}\right)$$
$$\cong M_2\left(\left(\frac{2,x}{F}\right)\right)\otimes_F M_2\left(\left(\frac{3,y}{F}\right)\right) \cong M_4(D)$$

where D is the division algebra described before this example. Since D has no integral Dubrovin valuation ring over V, neither does $D_1\otimes_F D_2$. □

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