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AREA INTEGRAL ESTIMATES FOR CALORIC FUNCTIONS

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ABSTRACT. We study the relationship between the area integral and the parabolic maximal function of solutions to the heat equation in domains whose boundary satisfies a $(\frac{1}{2}, 1)$ mixed Lipschitz condition. Our main result states that the area integral and the parabolic maximal function are equivalent in $L^p(\mu)$, $0 < p < \infty$. The measure μ must satisfy Muckenhoupt's A_∞-condition with respect to caloric measure. We also give a Fatou theorem which shows that the existence of parabolic limits is a.e. (with respect to caloric measure) equivalent to the finiteness of the area integral.

INTRODUCTION

Let $D \subset \mathbf{R}^n$ be a bounded Lipschitz domain and let Ω denote the cylinder $\mathbf{R} \times D$. For $X = (x_0, x)$ and $Y = (y_0, y)$ in $\mathbf{R} \times \mathbf{R}^n$, we let $\delta(X; Y)$ be the parabolic distance defined by $\delta(X; Y) = |x_0 - y_0|^{\frac{1}{2}} + |x - y|$. For $E \subset \mathbf{R}^{n+1}$, set $\delta(X; E) = \inf_{Y \in E} \delta(X; Y)$. For $P \in S \equiv \mathbf{R} \times \partial D$ and $\alpha > 0$, define the parabolic approach region by

$$\Gamma(P, \alpha) = \{Y \in \Omega : \delta(P; Y) < (1 + \alpha)\delta(Y; S)\}.$$

Let u be a caloric function in Ω. By this we mean that u is smooth and satisfies the heat equation

$$\frac{\partial u}{\partial x_0}(X) - \Delta_x u(X) = 0, \qquad X \in \Omega,$$

where we are using $\Delta_x = \sum_{i=1}^n \partial^2 / \partial x_i^2$ to denote the Laplacian in $\mathbf{R}^n$. Define the parabolic maximal function of u by

$$N_\alpha(u)(P) = \sup_{Y \in \Gamma(P, \alpha)} |u(Y)|, \qquad P \in S,$$

and the area integral of u by

$$A_\alpha^2(u)(P) = \int_{\Gamma(P, \alpha)} |\nabla u(Y)|^2 \delta(Y; S)^{-n} dY, \qquad P \in S,$$

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where $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ is the gradient in the n spatial variables. Finally, let x^* be some fixed point in D and let R be the diameter of D. The goal of this paper is to prove the estimate

$$(*) \qquad c\int_S N_\alpha^p(u)(P)\,dP \leq \int_S A_\alpha^p(u)(P)\,dP + R^{n+1}\sum_{k\in\mathbf{Z}} u(kR^2, x^*)^p$$
$$\leq C\int_S N_\alpha^p(u)(P)\,dP$$

where dP denotes surface measure on S and the constants depend on the "Lipschitz character of D", p, α, and the distance from x^* to ∂D.

Estimates of this type are known in the special case when D is the half-space $\mathbf{R}_+^n \equiv \{(x', x_n): x' \in \mathbf{R}^{n-1}, x_n > 0\}$. See the papers of B. F. Jones [J], Calderón and Torchinsky [CT], and Segovia and Wheeden [SW] where the variants of our area integral, the g-function and the g_λ^*-function are also studied. These papers rely on explicit knowledge of the Poisson kernel for $\mathbf{R} \times \mathbf{R}_+^n$ and that it is given by convolution on the boundary. Techniques which are no longer available when D is a Lipschitz domain. Our work is modelled on the method of Burkholder and Gundy [BG] for harmonic functions and extensions of their argument given by Dahlberg [D], and Dahlberg, Jerison and Kenig [DJK]. An examination of the techniques used by these authors shows that if we wish to establish $(*)$, we cannot help but establish estimates similar to $(*)$ in the more general situation when: (i) Ω is a domain whose boundary is given locally as the graph of a function $\phi\colon \mathbf{R} \times \mathbf{R}^{n-1} \to \mathbf{R}$ which satisfies the $\mathrm{Lip}(\frac{1}{2}, 1)$ condition $|\phi(x_0, x') - \phi(y_0, y')| \leq \delta(x_0, x'; y_0, y')$, (ii) surface measure is replaced by a measure μ which satisfies a variant of Muckenhoupt's A_∞-condition with respect to caloric measure. We shall establish this generalization of $(*)$. The conditions (i) and (ii) will be made precise later.

Our interest in the estimate $(*)$ arises from its usefulness in studying solutions of various boundary value problems for the heat equation in Lipschitz cylinders. In [B1], the right-hand inequality of $(*)$ is established for $p = 2$. This estimate is used to study regularity in the initial-Dirichlet problem and the oblique derivative problem for the heat equation. Estimates for the oblique derivative problem with data from L^p may be found in [B2]. One may find applications of area integral estimates for harmonic functions in the papers of Verchota [V] and Kenig and Pipher [KP] where regularity in the Dirichlet problem and the oblique derivative problem for harmonic functions are studied in L^p classes.

The outline of this paper is as follows. In §1, we give definitions and recall a few facts about caloric functions. In §2, we prove a comparison theorem for positive caloric functions. This comparison is used to study the relationship between caloric measure on our domain Ω and the caloric measure on sawtooth domains that arise in the proof of our main theorems. §§3 and 4 are devoted to the proof of (our generalization of) $(*)$. The main step of the proof, the traditional good-λ inequalities, may be found in §3. The statement of our main theorem, Theorem 4.3, and the details needed to complete its proof are in §4. Finally, in §5 we give some easy applications of our results to the question of the existence of parabolic limits for caloric functions.

1. PRELIMINARIES

In so far as possible, we retain the notation used in the introduction. The following changes will be necessary. In the rest of this paper Ω will be a $\mathrm{Lip}(\frac{1}{2}, 1)$ cylinder (defined below) and $S = \partial\Omega$ will be its boundary. We continue to let $X = (x_0, x)$ and Y denote points in $\mathbf{R}^{n+1}$ and will also write $x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$. We use P and Q for points on S and will use the coordinates $P = (p_0, p', p_n) = (p_0, p)$. The definitions given for $\Gamma(P, \alpha)$, $N_\alpha(u)$ and $A_\alpha(u)$ carry over to our more general domains without change. We also remark that we will use $\delta(E; F)$ to denote the distance between two sets.

We begin by defining the class of domains that we are considering. In order to do this, fix $m > 0$ and $r_0 > 0$, let $Z = \{X = (x_0, x): |x_i| < r_0$ for $i = 1, \dots, n-1$, $|x_n| < 2nmr_0$, and $x_0 \in \mathbf{R}\}$ and let $\phi: \mathbf{R}^n \to (-mr_0, mr_0)$ be a function. If $\Omega \subset \mathbf{R}^{n+1}$, we say that (Z, ϕ) is a coordinate cylinder for Ω if

(i) $2Z \cap \partial\Omega = \{(x_0, x', x_n): x_n = \phi(x_0, x')\} \cap 2Z$,

(ii) $2Z \cap \Omega = \{(x_0, x', x_n): x_n > \phi(x_0, x')\} \cap 2Z$,

where $2Z$ denotes the concentric double of Z. We say that Ω is a $\mathrm{Lip}(\frac{1}{2}, 1)$ cylinder with constants m and r_0 if there is a covering of $\partial\Omega$ by coordinate cylinders $\{(Z_i, \phi_i): i = 1, \dots, N\}$ such that the functions ϕ_i satisfy the $\mathrm{Lip}(\frac{1}{2}, 1)$ condition $|\phi_i(x_0, x') - \phi_i(y_0, y')| \le m\delta(x_0, x'; y_0, y')$. The coordinate systems used to define the Z_i's are allowed to differ by a rigid motion in the spatial or x variables.

For the next three constructions, we assume that $r < r_0$ and Q lies in S. We define surface cubes $I_r(Q) \subset S$ by $I_r(Q) = \{(p_0, p', p_n) \in S: |p_0 - q_0| < r^2$, $|p_i - q_i| < r$, for $i = 1, \dots, n-1$ and $|p_n - q_n| < 2nmr\}$. We will use domains $\Psi_r(Q) = \{(x_0, x', x_n) \in \Omega: |x_0 - q_0| < r^2$, $|x_i - q_i| < r$ for $i = 1, \dots, n-1$, and $|x_n - q_n| < 2nmr\}$ and we let $V_r(Q) = (q_0 + 4r^2, q', q_n + 8nmr)$. For the construction of $\Psi_r(Q)$, $I_r(Q)$, and $V_r(Q)$, we are using the coordinate system associated to a coordinate cylinder Z_i which contains Q. Thus when Q lies in several coordinate cylinders, we have several choices for $\Psi_r(Q)$, $I_r(Q)$, and $V_r(Q)$. We ignore this ambiguity as our results will be true for any such choice.

For Harnack's inequality and the maximum principle for caloric functions, we refer the reader to [M] and [Do, p. 268] respectively. For an open set $\Omega \subset \mathbf{R}^{n+1}$, we recall the notion of the parabolic boundary of Ω, $\partial_p\Omega$. This is the collection of $P \in \partial\Omega$ (topological boundary) such that there is a path $\gamma: [0, 1] \to \Omega \cup \{P\}$ with $\gamma(0) = P$ and the first coordinate of γ is a strictly increasing function.

Next we recall that for $\Omega \subset \mathbf{R}^{n+1}$ a nonempty open set and $X \in \Omega$, we may construct the caloric measure ω_Ω^X (see [Do, p. 332]). If f is a bounded continuous function on $\partial_p \Omega$, then the solution in the Perron-Wiener-Brelot sense (PWB-solution) of the Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial x_0} - \Delta_x u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial_p \Omega \end{cases}$$

is given by

$$u(X) = \int_{\partial_p \Omega} f(Q)\, d\omega_\Omega^X(Q).$$

We will frequently drop the domain Ω and the point X at which we are evaluating caloric measure after specifying a particular choice. We recall the Gauss-Weierstrass kernel, which is the fundamental solution for the heat operator,

$$W(X) = \begin{cases} (4\pi x_0)^{-n/2} \exp(\frac{-|x|^2}{4x_0}), & x_0 > 0, \\ 0, & x_0 \leq 0. \end{cases}$$

Using W one constructs, for an open set Ω in $\mathbf{R}^{n+1}$, the Green's function $G = G_\Omega$. For X and Y in Ω, this is defined by

$$G(X;Y) = W(X-Y) - u_Y(X)$$

where $u_Y(X)$ is the PWB-solution of the Dirichlet problem

$$\begin{cases} \frac{\partial u_Y}{\partial x_0}(X) - \Delta_x u_Y(X) = 0, & X \in \Omega, \\ u_Y(Q) = W(Q-Y), & Q \in \partial_p \Omega. \end{cases}$$

We will need the following simple case of the Riesz decomposition for supercaloric functions. For our purposes, a smooth function u is supercaloric if $\frac{\partial u}{\partial x_0}(X) - \Delta u(X) \geq 0$.

Theorem A (see [Do, p. 305]). *Let u be smooth and supercaloric in a neighborhood of Ω, an open subset of $\mathbf{R}^{n+1}$. Then*

$$u(X) = \int_\Omega G(X;Y)\left[\frac{\partial u}{\partial y_0}(Y) - \Delta u(Y)\right] dY + \int_{\partial_p \Omega} u(Q)\, d\omega_\Omega^X(Q).$$

In this paper, we will follow the standard practice of letting c and C denote constants, probably different at each occurrence, which depend on m, r_0 and the collection of coordinate cylinders used to cover S. Other dependencies will be allowed, these will be given in the statement of each theorem, lemma, etc. In each of the following results, ω^X denotes caloric measure on a Lip$(\frac{1}{2}, 1)$-cylinder, Ω.

Lemma B [K, Lemma 1.1]. *For $r < r_0$, we have $\omega^{V_r(P)}(I_r(P)) \geq c$.*

Lemma C [K, Lemma 1.3]. *Let u be nonnegative and caloric in Ω and suppose that u vanishes on $\partial\Omega \setminus I_{r/2}(P)$ and for x_0 near $-\infty$, then for $Y \in \Omega \setminus \Psi_r(P)$,*

$$u(Y) \leq C u(V_r(P))\omega^Y(I_r(P)).$$

For the next three results, and in the rest of this paper, we let

$$R = \sup_s \operatorname{diam}\{x : (s, x) \in \Omega\}$$

and choose a point $x^* \in \mathbf{R}^n$ which satisfies $(x_0, x^*) \in \Omega \setminus \bigcup_i Z_i$ for all $x_0 \in \mathbf{R}$.

Lemma D [W, Lemma 2.2]. *There exists a constant c such that for P satisfying $T \le p_0 \le T + R^2$ and $r < r_0$*

$$\omega^{(T+2R^2, x^*)}(I_r(P)) \le C\omega^{(T+2R^2, x^*)}(I_{r/2}(P)).$$

For $Y \in \Omega$, we let $\hat{Y}$ denote any point in S satisfying $\delta(Y; S) = \delta(\hat{Y}; Y)$.

Lemma E [W, Lemma 2.2]. *There exists a constant c such that if Y satisfies $\delta(Y; S) < r_0$ and $T < y_0 < T + R^2$, then*

$$C^{-1} G(T + 2R^2, x^*; Y) \le \delta(Y; S)^{-n} \omega^{(T+2R^2, x^*)}(I_{\delta(Y;S)}(\hat{Y}))$$
$$\le CG(T + 2R^2, x^*; Y).$$

Theorem F [FGS, Theorem 2]. *Let u and v be positive and caloric in Ω and suppose that u and v vanish on $S \cap \{x_0 > T\}$. Let $a > 0$, then there exists a constant $C = C(a)$ such that for $x_0 > T + aR^2$,*

$$\frac{1}{C} \frac{u(T + aR^2, x^*)}{v(T + aR^2, x^*)} \le \frac{u(X)}{v(X)} \le C \frac{u(T + aR^2, x^*)}{v(T + aR^2, x^*)}.$$

Theorem G [FGS, Theorem 3]. *Let u and v be positive and caloric on $\Psi_{2r}(Q)$ and suppose that u and v vanish on $I_{2r}(Q)$. There exists a constant c such that if $r < cr_0$ and $X \in \Psi_{r/4}(Q)$ then*

$$\frac{u(X)}{v(X)} \le C \frac{u(q_0 + 2r^2, q + re_n)}{v(q_0 - 2r^2, q + re_n)}.$$

2. Properties of caloric measure

By now the basic properties of caloric measure on domains with $\mathrm{Lip}(\frac{1}{2}, 1)$ boundaries are fairly well understood, see the results of [FGS], [K] and [W] listed above. However, we need to be able to work at all sufficiently small scales; while the authors mentioned above were primarily concerned with caloric measure evaluated at a fixed point. Our first goal is to establish a comparison theorem for nonnegative caloric functions that holds for as many values of X as possible. This comparison theorem is closely related to the doubling property of caloric measure and an estimate for caloric measure that is analoguous to an estimate due to Carleson for harmonic measure. These last two results will be our main tools in studying the relation between caloric measure on our original domain Ω and on sawtooth regions (defined below).

We begin with a lemma that gives a simple condition for Harnack's inequality to hold without a time lag.

Lemma 2.1. *Suppose that u is caloric and nonnegative in Ω. Also, assume that u vanishes on $S \setminus I_r(Q)$ and for $x_0 < q_0 - r^2$. Let $J = \{X \in \Omega: \delta(X;S) > ar,\ \delta(X;Q) < (1+\frac{1}{a})r,\ x_0 > q_0 + (1+a)r^2\}$. Then*

$$\sup_J u \leq C_a \inf_J u.$$

Proof. Choose ρ^+ (ρ^-) so that the time coordinate of $V_{\rho^+}(Q)$ $(V_{\rho^-}(Q))$ equals $q_0 + (1+\frac{2}{a})r^2$ $(q_0 + (1+\frac{a}{2})r^2)$. From Harnack's inequality we have

$$\sup_J u \leq C_a u(V_{\rho^+}(Q)) \quad \text{and} \quad u(V_{\rho^-}(Q)) \leq C_a \inf_J u.$$

But Lemma C and Harnack's inequality guarantee that

$$u(V_{\rho^+}(Q)) \leq C_a u(V_{\rho^-}(Q)). \quad \square$$

We state and prove our comparison theorem for nonnegative caloric functions.

Lemma 2.2. *Let u and v be nonnegative and caloric in Ω. Assume that u and v are zero on $S \setminus I_r(Q)$ and vanish for x_0 near $-\infty$. Then if X satisfies $|x-q|^2 < a|x_0 - q_0|$ and $x_0 > q_0 + 4r^2$, we have*

$$\frac{1}{C_a}\frac{u(V_r(Q))}{v(V_r(Q))} \leq \frac{u(X)}{v(X)} \leq C_a \frac{u(V_r(Q))}{v(V_r(Q))}.$$

Proof. Noting Lemma 2.1, Theorem G and Theorem F, it suffices to prove our comparison lemma for the special case when $X = V_\rho(Q)$, $Cr < \rho < cr_0$.

We begin our study of this special case by constructing an auxiliary domain. Let $\gamma = \{(x_0, x', x_n): 2m(|x'| + |x_0|^{\frac{1}{2}}) < -x_n\}$ and define $\Omega_r(Q) = \Omega \setminus (Q + rCe_n + \gamma)$ where C is chosen so that $\Psi_r(Q) \subset \Omega \setminus \Omega_r(Q)$. Observe that for r sufficiently small, $\Omega_r(Q)$ is a $\mathrm{Lip}(\frac{1}{2}, 1)$ cylinder whose constants depend only on those of Ω. Next, we note that by Harnack's inequality, there exists $I_{cr}(P) \subset \partial\Omega_r(Q) \setminus S$ such that

$$\inf_{Y \in I_{cr}(P)} u(Y) \geq c' u(V_r(Q)).$$

While Lemma C implies that

$$\sup_{X \in \partial\Omega_r(Q)\setminus S} u(X) \leq Cu(V_r(Q)).$$

From these inequalities and the maximum principle, we have

$$cu(V_r(Q))\omega^X_{\Omega_r(Q)}(I_{cr}(P)) \leq u(X) \leq Cu(V_r(Q))\omega^X_{\Omega_r(Q)}(I_{Cr}(P)), \qquad X \in \Omega_r(Q).$$

We also have the same estimate for v, thus our lemma will follow from a doubling property for caloric measure applied in $\Omega_r(Q)$. The desired property is: Let $I_r(Q) \subset S$ with $r < cr_0$ and suppose $r < \rho < c'r_0$, then

$$\omega^{V_\rho(Q)}(I_r(Q)) \leq C\omega^{V_\rho(Q)}(I_{r/2}(Q)). \tag{2.3}$$

To establish (2.3), we use Lemma D to see that there is a constant $C = C(b, m)$ such that

$$\omega_{\Psi_{b\rho}(Q)}^{V_\rho(Q)}(I_r(Q)) \le C\omega_{\Psi_{b\rho}(Q)}^{V_\rho(Q)}(I_{r/2}(Q))$$

as long as $b > 5$ and hence $V_\rho(Q) \in \Psi_{b\rho}(Q)$. From the maximum principle, it follows that

$$\omega_{\Psi_{b\rho}(Q)}^{V_\rho(Q)}(I_{r/2}(Q)) \le \omega^{V_\rho(Q)}(I_{r/2}(Q))$$

as long as $\Psi_{b\rho}(Q) \subset \Omega$. Thus the doubling property, (2.3), will follow once we show that for some b,

$$\tfrac{1}{2}\omega^{V_\rho(Q)}(I_r(Q)) \le \omega_{\Psi_{b\rho}(Q)}^{V_\rho(Q)}(I_r(Q)). \tag{2.4}$$

To establish (2.4), we write

$$\omega_{\Psi_{b\rho}(Q)}^{V_\rho(Q)}(I_r(Q)) = \int_{I_r(Q)} 1 - \int_{\partial_p \Psi_{b\rho}(Q)\setminus S} K(P;Y)\, d\omega_{\Psi_{b\rho}(Q)}^{V_\rho(Q)}(Y)\, d\omega^{V_\rho(Q)}(P)$$

where $K(P;Y)$ is the kernel function for Ω normalized by the condition that $K(P; V_\rho(Q)) = 1$. We remind the reader that $K(P;Y) = d\omega^Y(P)/d\omega^{V_\rho(Q)}$ and refer the reader to [K] for additional information about kernel functions. Thus (2.4) will follow once we show that we can choose b large so that

$$\sup_{P\in I_r(Q);\ Y\in\partial_p\Psi_{b\rho}(Q)\setminus S} K(P;Y) \le \frac{1}{2}. \tag{2.5}$$

To establish (2.5), we observe that Lemma C and the normalization of K imply that

$$K(P;Y) \le C, \qquad P \in I_r(Q),\ Y \in \Omega \setminus \Psi_{5\rho}(Q).$$

Using this and the maximum principle, we have

$$K(P;Y) \le C\rho^n W(y_0 - q_0 + 2\rho^2, y - q), \qquad P \in I_r(Q) \text{ and } Y \in \Omega\setminus\Psi_{5\rho}(Q). \tag{2.6}$$

If we choose b large, this completes the proof of (2.4) and hence the proof of (2.3). □

Remark. The comparison between $\omega_{\Psi_{b\rho}(Q)}^{V_\rho(Q)}$ and $\omega^{V_\rho(Q)}$ in (2.4) is a reformulation of Lemma 2.3 in [K]. □

Example. We show that the comparison of Lemma 2.2 cannot hold uniformly outside a paraboloid. We consider the domain $\Omega \equiv \{(x_0, x_1): x_1 > 0\} \subset \mathbf{R}^2$. Let $u(x_0, x_1) = \frac{\partial W}{\partial x_1}(x_0, x_1)$ and $v(x_0, x_1) = \frac{\partial W}{\partial x_1}(x_0 - 1, x_1)$. We see that

$$\frac{u(10, x_1)}{v(10, x_1)} = C\exp(-|x_1|^2/360)$$

which goes to zero as $x_1 \to +\infty$. □

For future reference, we write down two consequences of Lemma 2.2.

Corollary 2.7. *Let $cr_0 > r > 0$ and $Q \in S$. Then for $X \in \Omega$ satisfying $|x-q|^2 < a(x_0 - q_0)$ and $x_0 > q_0 + 4r^2$, we have*

$$\omega_\Omega^X(I_r(Q)) \le C\omega_\Omega^X(I_{r/2}(Q)) \tag{2.8}$$

and if we have $E \subset I_r(Q)$, a Borel set, then

$$C^{-1}\omega_\Omega^X(E) \le \omega_\Omega^{V_r(Q)}(E)\,\omega_\Omega^X(I_r(Q)) \le C\omega_\Omega^X(E). \tag{2.9}$$

The constants depend on a.

Proof. From Lemmas B and 2.1, we have $1 \ge \omega_\Omega^{V_r(Q)}(I_r(Q)) \ge \omega_\Omega^{V_r(Q)}(I_{r/2}(Q)) \ge c$. The estimates (2.8) and (2.9) follow immediately from this observation and Lemma 2.2. □

Before proceeding, we need to introduce auxiliary domains that are the parabolic analogue of the sawtooth regions used in [BG], [D] and [DJK]. For $I = I_r(Q)$ a surface cube with $r < cr_0$, $E \subset I$, a nonempty set, and α sufficiently large, we define

$$\Phi(I, E, \alpha) = \left(\bigcup_{P\in E} \Gamma(P,\alpha)\right) \cap \{(x_0, x) : |x-q| < ar, |x_0 - q_0| < br^2\}.$$

We claim that we may choose a and b so that

(i) $\Phi(I, E, \alpha) = \tilde{\Phi} \cap \{(x_0, x) : |x_0 - q_0| < br^2\}$ where $\tilde{\Phi}$ is a $\mathrm{Lip}(\frac{1}{2}, 1)$ cylinder whose constants $\tilde{m} = \tilde{m}(m, \alpha)$ and $\tilde{r}_0 = c(\alpha, m)r$. We will use S_Φ to denote the lateral boundary of Φ, $S_\Phi = \{(x_0, x) : |x_0 - q_0| < br^2\} \cap \partial\tilde{\Phi}$.

(ii) There exists a point $X_\Phi = (q_0 + br^2, x_\Phi)$ such that $\delta(X_\Phi; \partial_p \Phi) \ge cr$.

(iii) $\partial\Phi \cap \{(x_0, x', x_n) : |x_0 - q_0| < (400r)^2,\ |x_i - q_i| < 400r,\ i = 1, \dots, n-1$ and $|x_n - q_n| < 800\sqrt{n}(m+\tilde{m})r\} = \{P + \tilde{\phi}(P)e_n : P \in I_{400r}(Q)\}$ where $\tilde{\phi} : S \to \mathbf{R}$ is a $\mathrm{Lip}(\frac{1}{2}, 1)$ function.

We do not prove this claim, but after reading Lemma 3.1, the reader should be convinced that its proof is nothing more than an exercise in the triangle inequality. In particular, Lemma 3.1 will illustrate the usefulness of the requirement that α is large.

The next lemma states that on $\partial\Phi \cap S$, caloric measure for Φ and caloric measure for Ω are "the same". This lemma and its proof are generalizations to the caloric case of the Main Lemma of [DJK]. Our reasons for needing this lemma are the same as Dahlberg, Jerison and Kenig's: We cannot hope to obtain a useful comparison between caloric measure and surface measure as S does not, in general, have locally finite n-dimensional Hausdorff measure. Also, an example of Kaufman and Wu [KW] shows that caloric measure and a reasonable substitute for surface measure need not be mutually absolutely continuous.

For the statement of this lemma, recall that x^* and R were defined after Lemma C in §1. Also, measurable will mean Borel measurable.

Lemma 2.10. *Let* $I_r(Q) \subset S \cap \{X : T < x_0 < T + R^2\}$ *and put* $\omega = \omega_\Omega^{(T+2R^2, x^*)}$. *Let* Φ *be one of the domains* $\Phi(I_r(Q), \cdot, \alpha)$. *With* $\nu = \omega_\Phi^{X_\Phi}$, *we have for* F *a measurable subset of* $\partial\Phi \cap S$

$$\frac{\omega(F)}{\omega(I_r(Q))} \leq C(\nu(F))^\kappa .$$

The constants C *and* κ *depend on* α.

Proof. Recall the cube $I_{400r}(Q)$ guaranteed in (iii) of the construction of Φ. We define a map $\pi : I_{400r}(Q) \to \partial\Phi$ given by $\pi(P) = P + \tilde{\phi}(P)e_n$. For a cube $I \subset I_{400r}(Q)$, we let $\pi(I) = \tilde{I}$.

Observe that $E \equiv \partial\Phi \cap S$ is a closed subset of S. The first step is to construct a Whitney decomposition of $I_{10r}(Q) \setminus (S \cap \partial\Phi)$. This gives a sequence of cubes $\{I_j\}_{j=1}^\infty$ where $I_j = I_{r_j}(P_j)$. These cubes satisfy $\delta(I_j; E) \approx r_j$ ($A \approx B$ means that $c \leq A/B \leq C$), $I_j \cap I_k = \varnothing$ and $I_{10r}(Q) \setminus E = \cup_j I_j$. Also observe that $\delta(I_j; \tilde{I}_j) \approx r_j$. We remark that we may choose the Whitney decomposition so that the constant in $\delta(I_j; E) \geq cr_j$ is as large as we wish. The reason for doing this will be made clear in Case 2 below.

We define a measure on $I_{10r}(Q)$ which agrees with ν on E. For $F \subset I_{10r}(Q)$, let

$$\bar{\nu}(F) = \nu(F \cap E) + \sum_{j=1}^\infty \frac{\omega(F \cap I_j)}{\omega(I_j)} \nu(\tilde{I}_j).$$

Our goal is to show that for a surface cube $I \subset I_r(Q)$ and $F \subset I$, we have

$$\frac{\bar{\nu}(F)}{\bar{\nu}(I)} \leq C \frac{\omega(F)}{\omega(I)} \tag{2.11}$$

Then (2.11) and an argument that may be found in [CF], imply a converse to (2.11): There exist $C > 0$ and $\kappa > 0$ depending on the constant in (2.11) such that

$$\frac{\omega(F)}{\omega(I)} \leq C\left(\frac{\bar{\nu}(F)}{\bar{\nu}(I)}\right)^\kappa . \tag{2.12}$$

Thus to establish our lemma, we need to establish (2.11) and show that

$$\bar{\nu}(I_r(Q)) \geq c . \tag{2.13}$$

To establish (2.11), we consider two cases.

Case 1. Let $I = I_\rho(P)$ and assume that if $I_j \cap I \neq \varnothing$, then $r_j \leq 10\rho$.

As a first step towards establishing (2.11) in this case, we claim that $\bar{I} \equiv \pi(I_{11\rho}(P))$, satisfies $\nu(\bar{I}) \approx \bar{\nu}(I)$. To establish this claim, we first show that $\bar{\nu}(I) \leq \nu(\bar{I})$. In fact, from the doubling property of ω, (2.8), we have

$$\bar{\nu}(I) \leq \nu(E \cap I) + \sum_{\{j : I \cap I_j \neq \varnothing\}} \nu(\tilde{I}_j) \leq C\nu(\bar{I}).$$

Since our assumption that $r_j \le 10\rho$ implies that $\cup_{\{j:\, I_j \cap I \neq \varnothing\}} \tilde{I}_j \subset \hat{I}$ where $\hat{I} = \pi(I_{C\rho}(P) \cap I_{10r}(Q))$ for C sufficiently large. To obtain the other inequality, $\nu(\bar{I}) \le C\bar{\nu}(I)$, we note that if $I_j \cap I_{\rho/2}(P) \neq \varnothing$, then there is a surface cube contained in $I_j \cap I$ whose diameter is a fixed fraction of r_j. Thus using the doubling property of ω, we have that $\omega(I \cap I_j) \ge c\omega(I_j)$. Using this, it follows that

$$\bar{\nu}(I) \ge \nu(E \cap I) + c \sum_{\{j:\, I_j \cap I_{\rho/2}(P) \neq \varnothing\}} \nu(\tilde{I}_j) \ge c\nu(\bar{I})$$

where the second inequality follows from the doubling property for ν. This completes the proof of our claim. Note that this claim establishes (2.13).

We let $V' = V_{20\rho}(\pi(P))$ which is defined with respect to the domain Φ. It is clear that $\omega_\Phi^{V'}(I) \ge c$ and that for j such that $I_j \cap I \neq \varnothing$, there is a constant a such that $V' \in \{X : |x - p_j|^2 < a(x_0 - p_{0,j})\}$ where $P_j = (p_{0,j}, p_j)$ (recall that $r_j < 10\rho$). Using our claim and applying (2.9), we have

$$\text{(2.14)} \qquad \begin{aligned} \frac{\bar{\nu}(F)}{\bar{\nu}(I)} &\approx \frac{1}{\nu(\tilde{I})} \left(\nu(F \cap E) + \sum_j \frac{\omega(I_j \cap F)}{\omega(I_j)} \nu(\tilde{I}_j) \right) \\ &\le C \left(\omega_\Phi^{V'}(F \cap E) + \sum_j \frac{\omega_\Omega^{V'}(I_j \cap F)}{\omega_\Omega^{V'}(I_j)} \omega_\Phi^{V'}(\tilde{I}_j) \right). \end{aligned}$$

The maximum principle implies that

$$\text{(2.15)} \qquad \omega_\Phi^{V'}(F \cap E) \le \omega_\Omega^{V'}(F \cap E).$$

To estimate the sum in (2.14), we set $I_j^- = I_{r_j}(P_j - 4r_j^2 e_0)$ and note that

$$\inf_{Y \in \tilde{I}_j} \omega_\Omega^Y(I_j^-) \ge c.$$

Thus from the maximum principle it follows that

$$\omega_\Phi^Y(\tilde{I}_j) \le C\omega_\Omega^Y(I_j^-), \qquad Y \in \Phi.$$

But then applying (2.8) to ω_Ω, we have $\omega_\Omega^{V'}(I_j^-) \le C\omega_\Omega^{V'}(I_j)$, for $I_j \cap I \neq \varnothing$. Combining these last two observations and using (2.15) in (2.14) we obtain

$$\frac{\bar{\nu}(F)}{\bar{\nu}(I)} \le C(\omega_\Omega^{V'}(F \cap E) + \sum_j \omega_\Omega^{V'}(I_j \cap F)) = \omega_\Omega^{V'}(F).$$

Finally using (2.8) we obtain (2.11) in Case 1.

Case 2. Let $I = I_\rho(P)$ and suppose that there exists j_0 such that $I_{j_0} \cap I \neq \varnothing$ and $r_{j_0} > 10\rho$.

If this happens, then we have $E \cap I = \varnothing$ and $r_l \approx r_{j_0}$ for l such that $I_l \cap I \neq \varnothing$. It is at this point that the requirement that the constant in the Whitney decomposition be large is used. Also, $\delta(I_l; I_k) \le C\rho \le C'r_{j_0}$ and hence using the doubling property of ω (respectively ν) we have $\omega(I_l) \approx \omega(I_{j_0})$ (respectively $\nu(\widetilde{I}_l) \approx \nu(\widetilde{I}_{j_0})$) for l such that $I_l \cap I \neq \varnothing$. But then

$$\bar{\nu}(F) \approx \sum_l c_{j_0}\omega(I_\ell \cap F) = c_{j_0}\omega(F)$$

for any $F \subset I$, whence (2.11) follows in Case 2. □

3. Distribution function inequalities

We begin by describing the class of measures we will consider in our estimates. We say that a positive measure μ is in $A_\infty(d\omega)$ if

(i) μ satisfies the doubling condition: $\mu(I_r(P)) \leq C_\mu \mu(I_{r/2}(P))$,

(ii) For $I_r(P) \subset S \cap \{X: T < x_0 < T + R^2\}$ and $E \subset I_r(P)$ a Borel set, we have

$$\frac{\mu(E)}{\mu(I_r(P))} \leq C_\mu \left(\frac{\omega^{(T+2R^2, x^*)}(E)}{\omega^{(T+2R^2, x^*)}(I_r(P))} \right)^{\tau_\mu}.$$

Our next lemma will be used to show that nearby cones have substantial overlap.

Lemma 3.1. *Let α be sufficiently large. There exist constants c and C_α such that if $X \in \Gamma(P, \alpha)$, $\delta(X; S) < cr_0$, $r < cr_0$ and Q satisfies $\delta(P; Q) < r$, then $X + C_\alpha r e_n \in \Gamma(Q, \alpha)$.*

Remark. The lemma is false when α small. The cones $\Gamma(Q, \alpha)$ may be empty for small α.

Proof. Let $X_\rho = X + \rho e_n$. As S is given as the graph of a $\mathrm{Lip}(\frac{1}{2}, 1)$ function near P, it follows that $\delta(X_\rho; S) \geq \delta(X; S) + c\rho$ for $\rho < cr_0$. Hence,

$$\delta(X_\rho; Q) \leq \delta(X; P) + \rho + \delta(P; Q) \leq (1+\alpha)\delta(X_\rho; S) - c(1+\alpha)\rho + \rho + r.$$

Thus if $c(1+\alpha) > 1$, we have $X_\rho \in \Gamma(Q, \alpha)$ for $\rho > [c(1+\alpha) - 1]^{-1} r$. □

We need to define several auxiliary functions before turning to the proof of the main lemmas. For a positive measure μ on S, we let M_μ denote the Hardy-Littlewood maximal operator with respect to μ:

$$M_\mu(f)(P) = \sup_{I \ni P} \frac{1}{\mu(I)} \int_I |f(Q)|\, d\mu(Q).$$

Next we define

$$H_\alpha(\lambda, P) = M_\mu(\chi_{\{A_\alpha(u) > \lambda\}})(P)$$

and

$$D_\alpha(u)(P) = \sup_{Y \in \Gamma(P,\alpha)} \left(\delta(Y; S)|\nabla_y u(Y)| + \delta(Y; S)^2 \left| \frac{\partial u}{\partial y_0}(Y) \right| \right).$$

Lemma 3.2. *Let* $I = I_r(Q) \subset S$ *and assume that* $\alpha'' > 2\alpha'$ *and* α' *is large enough so that the cones* $\Gamma(P, \alpha')$ *satisfy Lemma* 3.1. *Also assume that one of the following hypotheses holds*:

(i) *There exists* P^* *with* $\delta(P^*; Q) \le r/a$ *and* $N_{\alpha'}(u)(P^*) \le \lambda$.

(ii) $r > ar_0$ *and there exists* $X^* \in \bigcap_{Q\in I} \Gamma(Q, \alpha')$ *such that* $|u(X^*)| \le \lambda$.

Then given α *and* $\beta > 1$ *there exists* $\gamma = \gamma(a, \alpha', \alpha'', \alpha, \beta, C_\mu, \tau_\mu) > 0$ *and* $\theta = \theta(C_\mu, \tau_\mu) > 0$ *such that*

$$\alpha\mu(\{P \in I: N_{\alpha'}(u)(P) > \beta\lambda,\ A_{\alpha''}(u)(P) \le \gamma\lambda,\ H_{\alpha''}(\gamma\lambda, P) \le \theta\}) \le \mu(I).$$

Proof. We begin by observing that we may assume u is smooth in a neighborhood of $2Z_i \cap \Omega$ where Z_i is a coordinate cylinder containing I. Simply replace u by $u_\varepsilon(X) = u(X + \varepsilon e_n)$ for ε small. Observe that if u satisfies hypothesis (i), then u_ε satisfies $N_{\alpha'}(u_\varepsilon)(P^*) \le \lambda + o(1)$, as $\varepsilon \to 0^+$. This weaker condition will be enough to carry out the proof given below for ε sufficiently small. Once we have established our lemma for u_ε, several applications of Fatou's lemma yield the conclusion of our lemma for u. For the remainder of this proof, we work with u_ε but we drop the subscript ε.

We choose T so that $I \subset S \cap \{X: T < x_0 < T + R^2\}$. Let $\omega = \omega_\Omega^{(T+2R^2, x^*)}$. We define E_1 by $E_1 = \{P: A_{\alpha''}(u)(P) \le \gamma\lambda,\ \delta(P; I_r(Q)) < (2 + 2\alpha'')r\}$ where γ is to be chosen. Using the doubling property of ω, Lemma D, we have

$$\begin{aligned} \omega(I)\gamma^2\lambda^2 &\ge \int_{E_1} A^2_{\alpha''}(u)(P)\, d\omega(P) \\ &= \int_{O_1} |\nabla u(Y)|^2 \delta(Y; S)^{-n} \int_{E_1} \chi_{\alpha''}(P; Y)\, d\omega(P)\, dY \end{aligned} \tag{3.3}$$

where $O_1 = \bigcup_{P\in E_1} \Gamma(P, \alpha'')$ and $\chi_{\alpha''}(P; Y) = 1$ if $Y \in \Gamma(P, \alpha'')$ and zero otherwise. We let $E_2 = \{P \in I: H_{\alpha''}(\gamma\lambda, P) \le \theta\} \cap E_1$ and set

$$O_2 = \left[\bigcup_{P\in E_2} \Gamma(P, \alpha'')\right] \cap \{Y: \delta(Y; S) \le Cr\}$$

where the constant C is chosen so that $\Phi(I, E_2, \alpha'') \subset O_2$. We claim that we may choose $\theta > 0$, but small, so that

$$\int_{E_1} \chi_{\alpha''}(P; Y)\, d\omega(P) \ge c\omega(I_{\delta(Y;S)}(\hat{Y})), \qquad Y \in O_2, \tag{3.4}$$

where $\hat{Y}$ is any point in S which satisfies $\delta(\hat{Y}; Y) = \delta(Y; S)$. To establish (3.4), suppose that $Y \in \Gamma(\bar{P}, \alpha'')$, with $\bar{P} \in E_2$ and $\delta(Y; S) < r$. Observe that $\{P: \chi_{\alpha''}(P; Y) = 1\} \subset \{P: \delta(P; I_r(Q)) < (2 + 2\alpha'')r\}$, thus

$$\int_{E_1} \chi_{\alpha''}(P; Y)\, d\omega(P) \ge \int_S \chi_{\alpha''}(P; Y)\, d\omega(P) - \int_{\{A_{\alpha''}(u) > \gamma\lambda\}} \chi_{\alpha''}(P; Y)\, d\omega(P)$$

Using the doubling property for ω, it follows that

$$\int_S \chi_{\alpha''}(P; Y)\, d\omega(P) \ge c_1 \omega(I_{\delta(Y;S)}(\hat{Y})).$$

For the other term, we have

$$\int_{\{A_{\alpha''}(u)>\gamma\lambda\}} \chi_{\alpha''}(P;Y)\,d\omega(P) \le \omega(I_{C\delta(Y;S)}(\hat{Y}) \cap \{A_{\alpha''}(u) > \gamma\lambda\})$$
$$\le C_2 H_{\alpha''}(\gamma\lambda;\bar{P})^{\sigma_\mu}\omega(I_{\delta(Y;S)}(\hat{Y}))\,.$$

The first inequality follows since $\{P\colon \chi_{\alpha''}(P;Y) = 1\} \subset \{P\colon \delta(\hat{Y};P) < (2+\alpha'')\delta(Y;S)\}$. To establish the second, we recall that the A_∞-condition and the results of [CF] imply that for $I \subset S \cap \{T < x_0 < T + R^2\}$ and F a measurable subset of I,

$$\frac{\omega(F\cap I)}{\omega(I)} \le C_\mu\left(\frac{\mu(I\cap F)}{\mu(I)}\right)^{\sigma_\mu}$$

for some constants, $C_\mu > 0$, $\sigma_\mu > 0$. Using this inequality, the second estimate follows from the definition of H and the doubling property of ω. Combining these observations, we obtain

$$\int_{E_1} \chi_{\alpha''}(P;Y)\,d\omega(P) \ge (c_1 - C_2\theta^{\sigma_\mu})\omega(I_{\delta(Y;S)}(\hat{Y}))$$

whence (3.4) follows if θ is small. Combining (3.3) and (3.4), we obtain

$$C\omega(I)\gamma^2\lambda^2 \ge \int_{\mathcal{O}_2} |\nabla u(Y)|^2 \delta(Y;S)^{-n}\omega(I_{\delta(Y;S)}(\hat{Y}))\,dY\,. \tag{3.5}$$

Let $\Phi = \Phi(I, E_2, \alpha''/2)$ and recall that $\Phi \subset \{Y\colon |y_0 - q_0| < br^2\}$. We write $\Phi = U_1 \cup U_2$ where $U_1 = \Phi \cap \{Y\colon \delta(Y;I) < r\}$ and $U_2 = \Phi \setminus U_1$. We wish to show that

$$\int_{U_i} G(X_\Phi;Y)|\nabla u(Y)|^2\,dY \le C\gamma^2\lambda^2\,, \qquad i = 1, 2\,. \tag{3.6}$$

To study the case $i = 1$, we will need a slight extension of Lemma E:

$$C\delta(Y;S)^{-n}\omega^{X_\Phi}(I_{\delta(Y;S)}(\hat{Y})) \ge G(X_\Phi;Y)\,, \qquad Y \in U_1\,. \tag{3.7}$$

To establish this, we construct an auxiliary domain, $\Omega_r(Q)$, as at the beginning of the proof of Lemma 2.2. We choose the constant in the definition of this domain so that $U_1 \subset \Omega \setminus \Omega_r(Q)$. Applying Lemma 2.2 to $\omega^{\cdot}(I_{\delta(Y;S)}(\hat{Y}))$ and $G(\cdot\,;Y)$ and using Lemma E, (3.7) follows. To establish (3.6) for $i = 1$, we apply (2.9) to (3.5) and then use (3.7).

To establish (3.6) when $i = 2$, we write $U_2 = U_2' \cup U_2''$ where $U_2' \equiv \{X \in U_2\colon \delta(X;X_\Phi) < \delta(X;\partial_p\Phi)\}$ and $U_2'' \equiv U_2 \setminus U_2'$. For $Y \in U_2''$, we have $|\nabla u(Y)|^2 \le C\gamma^2\lambda^2/r^2$ and since $G(X_\Phi;Y) \le W(X_\Phi - Y)$, it follows that

$$\int_{U_2'} G(X_\Phi;Y)|\nabla u(Y)|^2\,dY \le \gamma^2\lambda^2/r^2 \int_{\{Y\colon\,\delta(X_\Phi;Y)\le cr\}} W(X_\Phi - Y)\,dY \le C\gamma^2\lambda^2\,.$$

To estimate the integral over U_2'', we compare G with W to conclude that $G(X_\Phi;Y) \le Cr^{-n}$ for $Y \in U_2''$. Also, U_2'' is contained in the union of a finite number of cones $\Gamma(Q_i, \alpha'')$ for which $A_{\alpha''}(u)(Q_i) \leq \gamma\lambda$. Finally, $Y \in U_2''$ implies that $\delta(Y; S) \leq Cr$, hence it follows that

$$\int_{U_2''} G(X_\Phi; Y)|\nabla u(Y)|^2 \, dY \leq Cr^{-n} \sum_j \int_{\Gamma(Q_j, \alpha'') \cap U_2''} |\nabla u(Y)|^2 \, dY \leq C'\gamma^2\lambda^2$$

which completes the proof of (3.6) for $i = 2$. Using (3.6) and the fact that $G_\Phi(X_\Phi; Y) \leq G(X_\Phi; Y)$ for Y in Φ, we obtain

$$\int_\Phi G_\Phi(X_\Phi; Y)|\nabla u(Y)|^2 \, dY \leq C\gamma^2\lambda^2.$$

Let $v(Y) = u(Y) - u(X_\Phi)$ and let ω_Φ be the caloric measure for Φ at X_Φ. Applying Theorem A to the supercaloric function $-\frac{1}{2}v(Y)^2$, we obtain

$$\int_{\partial_p \Phi} |v(P)|^2 \, d\omega_\Phi(P) \leq C\gamma^2\lambda^2.$$

From the work of Kemper, [K, Lemma 2.5], it follows that

$$N_\Phi(v)(P) \leq C\left(M_{\omega_\Phi}(v)(P) + \int_{\partial_p \Phi} |v(P)| \, d\omega_\Phi(P)\right)$$

for P satisfying $p_0 < q_0 + 2r^2$ (recall that I was centered at $Q = (q_0, q)$) where, we have defined N_Φ by

$$N_\Phi(v)(P) = \sup_{Y \in \Gamma(P, \alpha''')} |v(Y)|$$

where $\Gamma(P, \alpha''')$ are parabolic approach regions defined for Φ and the opening α''' is chosen so that if $P \in E_2$, then $\Gamma_t(P) \equiv \Gamma(P, \alpha') \cap \{Y : \delta(Y; P) < tr\} \subset \Gamma(P, \alpha''')$ for some t. Thus from the maximal theorem, we have

$$\int_{\partial_p \Phi \cap \{p_0 < q_0 + 2r^2\}} N_\Phi^2(v)(P) \, d\omega_\Phi(P) \leq C\gamma^2\lambda^2. \tag{3.8}$$

In order to proceed, we let $E_3 = E_2 \cap \{P : N_{\alpha'}(u)(P) > \beta\lambda\}$ and observe that $E_3 \subset \partial_p \Phi \cap \partial_p \Omega \cap \{X : x_0 < q_0 + 2r^2\}$. We claim that we may choose γ small so that

$$N_\Phi(v)(P) \geq \left(\frac{\beta - 1}{2}\right)\lambda, \qquad P \in E_3. \tag{3.9}$$

We assume (3.9) for the moment and complete the proof of the lemma. From (3.8) and (3.9), it follows that

$$\omega_\Phi(E_3)\left(\frac{\beta - 1}{2}\right)^2 \lambda^2 \leq C\gamma^2\lambda^2$$

Thus from Lemma 2.10, we obtain

$$\frac{\omega(E_3)}{\omega(I)} \leq C\gamma^{2\kappa}.$$

The lemma follows from the A_∞-condition for μ after possibly choosing γ smaller.

We now turn to the proof of (3.9). We first observe that the estimate

$$D_{\alpha'}(u)(P) \le CA_{\alpha''}(u)(P) \tag{3.10}$$

follows from interior estimates for caloric functions. Using hypothesis (i), (3.10) and Lemma 3.1, it follows that

$$|u(Y)| \le (1 + C_t\gamma)\lambda$$

for $Y \in \Gamma(P, \alpha') \setminus \Gamma_t(P)$ and $P \in E_2$. Hence if we choose γ small, we obtain

$$\sup_{Y\in\Gamma_t(P)} |u(Y)| \ge \beta\lambda, \qquad P \in E_3.$$

Similarly, it follows that $|u(X_\Phi)| \le (1 + C\gamma)\lambda$. Hence after possibly choosing γ smaller, we obtain (3.9). The proof of (3.9) under hypothesis (ii) is similar. We omit the details. □

Remark. For future reference, we note that Lemma 3.2 continues to hold if we define

$$H_\alpha(\lambda, P) = \sup_{\rho<Cr} \frac{1}{\mu(I_\rho(Q))} \int_{I_\rho(Q)} \chi_{\{A_\alpha(u)>\lambda\}}(P)\, d\mu(P)$$

where C is sufficiently large and r is the sidelength of the cube I in Lemma 3.2. □

Lemma 3.11. *Let $I = I_r(Q) \subset S$ and suppose that μ satisfies the A_∞-condition. Let $\lambda > 0$ be given and assume that one of the following two hypotheses holds:*

(i) *There exists P^* with $\delta(P^*; Q) \le ar$ and $A(u)(P^*) \le \lambda$,*
(ii) $r > r_0/a$.

Then for $\alpha, \beta > 1$ there exists $\gamma = \gamma(a, \alpha, \beta, \alpha', \alpha'', C_\mu, \tau_\mu)$ such that

$$\alpha\mu(\{P \in I : A_{\alpha'}(u)(P) > \beta\lambda,\ N_{\alpha''}(u)(P) \le \gamma\lambda\}) \le \mu(I).$$

Proof. As in the previous lemma, we may assume that u is smooth in a neighborhood of I. We let $E = \{P \in I : A_{\alpha''}(u)(P) > \lambda,\ N_{\alpha''}(u)(P) \le \gamma\lambda\}$ where $\gamma > 0$ is to be chosen. We let $\Gamma_t(P) = \Gamma(P, \alpha') \cap \{Y : \delta(Y; P) < tr\}$ and define

$$A_t^2(u)(P) = \int_{\Gamma_t(P)} |\nabla u(Y)|^2 \delta(Y; S)^{-n}\, dY.$$

Our first step is to establish that given $t > 0$, there exists a $\gamma > 0$ such that if $P \in E$, then $A_t^2(u)(P) > ((\beta - 1)/2)^2\lambda^2$. We first establish this claim under hypothesis (i). Towards this end, we write $\Gamma(P, \alpha') \setminus \Gamma_t(P) = U_1 \cup U_2$ where $U_1 \equiv [\Gamma(P, \alpha') \setminus \Gamma_t(P)] \cap \Gamma(P^*, \alpha')$ and $U_2 \equiv [\Gamma(P, \alpha') \setminus \Gamma_t(P)] \setminus \Gamma(P^*, \alpha')$. As $A^2(u)(P^*) \le \lambda^2$, it follows that

$$\int_{U_1} |\nabla u(Y)|^2 \delta(Y; S)^{-n}\, dY \le \lambda^2. \tag{3.12}$$

To estimate the integral over U_2, we use interior estimates to conclude that for $Y \in \Gamma(P, \alpha')$ we have $|\nabla u(Y)| \leq C\delta(Y;S)^{-1}N(u)(P)$. Hence if we can show that

$$(3.13) \qquad \int_{U_2} \delta(Y;S)^{-n-2}\, dY \leq C_t,$$

then it will follow that

$$(3.14) \qquad \int_{U_2} |\nabla u(Y)|^2 \delta(Y;S)^{-n}\, dY \leq C_t \gamma^2 \lambda^2.$$

Thus our claim will follow from (3.12) and (3.14) once we establish (3.13).

To establish (3.13), we choose k which satisfies $2^k tr \leq cr_0 < 2^{k+1} tr$ where cr_0 is the quantity appearing in Lemma 3.1. For $j = 1, \ldots, k$ we let $R_j = U_2 \cap \{Y: 2^{j-1} tr \leq \delta(Y_j; P) < 2^j tr\}$. Using Lemma 3.1, we have

$$\iiint_{R_j} dy_n\, dy'\, dy_0 \leq Cr \iint_{\{(y_0, y'):\ \delta((y_0, y');(p_0, p')) < C2^j tr\}} dy'\, dy_0 \leq Cr(2^j tr)^{n+1}.$$

Hence, it follows that

$$\int_{U_2} \delta(Y;S)^{-n-2}\, dY \leq Ct^{-1} \sum_{j=1}^{k} 2^{-j} + \int_{\Gamma(P, \alpha') \cap \{Y:\ \delta(Y;S) > cr_0\}} dY \leq C_t.$$

This is (3.13). To obtain a lower bound for $A_t^2(u)(P)$ under hypothesis (ii), one only needs to use interior estimates as in estimating the integral over U_2.

We construct the domain $\Phi = \Phi(I, E, \alpha'')$ and choose t small so that for $P \in E$, $\Gamma_t(P) \subset \Phi$. Let $\omega_\Phi = \omega_\Phi^{X_\Phi}$ and define $\chi(P;Y) = 1$ if $Y \in \Gamma_t(P)$ and 0 otherwise. Using our claim and Fubini's theorem, we have

$$\begin{aligned}
\omega_\Phi(E)\left(\frac{\beta - 1}{2}\right)^2 \lambda^2 &\leq \int_E A_t^2(u)(P)\, d\omega_\Phi(P) \\
&= \int_O \int_E \chi(P;Y) |\nabla u(Y)|^2 \delta(Y;S)^{-n}\, d\omega_\Phi(P)\, dY
\end{aligned}$$

where $O = \bigcup_{P \in E} \Gamma_t(P)$. Fix $Y \in O$ and as before, let $\hat{Y}$ denote any point which satisfies $\delta(\hat{Y}; Y) = \delta(Y;S)$. Observe that there exists a constant C such that $\{P \in E: \chi(P;Y) = 1\}$ is contained in $\widetilde{I}_{C\delta(Y;S)}(\hat{Y})$, a surface cube on S_Φ. Hence, using Lemma E and the doubling property (2.8) in Φ, we have

$$\begin{aligned}
\left(\frac{\beta - 1}{2}\right)^2 \omega_\Phi(E) &\leq C \int_O G_\Phi(X_\Phi; Y) |\nabla u(Y)|^2\, dY \\
&\leq C \int_\Phi G_\Phi(X_\Phi; Y) |\nabla u(Y)|^2\, dY.
\end{aligned}$$

Using the Riesz decomposition, Theorem A, we have

$$\left(\frac{\beta - 1}{2}\right)^2 \omega_\Phi(E) \leq C\gamma^2 \lambda^2$$

since Φ was constructed so that $|u| \leq \gamma\lambda$ in Φ. To complete the proof of the lemma, we use Lemma 2.10, the A_∞-condition and then choose γ small. □

4. The main theorem

We begin this section with two lemmas which indicate the role that the sum plays in the estimate $(*)$ and then we state and prove the main theorem. First some notation, let $\Omega_k = \Omega \cap \{X\colon kR^2 < x_0 < (k+1)R^2\}$, let $S_k = \{P \in S\colon kR^2 < p_0 < (k+1)R^2\}$ and define $X_k = ((k+1/2)R^2, x^*)$. Recall that R, the diameter of the spatial slices of Ω, was defined in §1. Finally, we let $\Omega_{k,\varepsilon} = \Omega_k \cap \{X\colon \delta(S;X) > \varepsilon\}$. With this notation available, we can state

Lemma 4.1. *Let $0 < p < \infty$, let μ be a doubling measure and let u be continuous in Ω. Then there is a constant $C = C(\alpha, C_\mu)$ such that*

$$\mu(S_k)u(X_k)^p \leq C \int_{S_k} N_\alpha^p(u)(P)\,d\mu(P).$$

Proof. The set $E \equiv \{P\colon X_k \in \Gamma(P,\alpha)\}$ contains a surface cube $I \subset S_k$ of sidelength comparable to $\delta(X_k;S)$. We have

$$\mu(I)u(X_k)^p \leq \inf_{P\in I} N_\alpha^p(u)(P)\mu(I) \leq \int_{S_k} N_\alpha^p(u)(P)\,d\mu(P).$$

From the doubling property of μ, it follows that $\mu(S_k) \leq C\mu(I)$. The lemma follows from these inequalities. □

Lemma 4.2. *Let u be caloric in Ω. Let p satisfy $0 < p < \infty$. Then there is a constant $C = C(\mu, \varepsilon, \alpha)$ such that*

$$\sup_{Y\in\Omega_{k,\varepsilon}} |u(Y)| \leq |u(X_k)| + C\left(\frac{1}{\mu(S_k)}\int_{S_k} A_\alpha^p(u)(P)\,d\mu(P)\right)^{\frac{1}{p}}.$$

Proof. It suffices to show that

$$\sup_{Y\in\Omega_{k,\varepsilon}} |\nabla u(Y)| + \left|\frac{\partial u}{\partial y_0}(Y)\right| \leq C\left(\frac{1}{\mu(S_k)}\int_{S_k} A_\alpha^p(u)(P)\,d\mu(P)\right)^{\frac{1}{p}}.$$

We first observe that if $Y \in \Gamma(P,\alpha/2)$ and $\delta(Y;S) > \varepsilon$, then $\varepsilon|\nabla u(Y)| + \varepsilon^2|\frac{\partial u}{\partial y_0}(Y)| \leq CA_\alpha(u)(P)$. Next, note that if $Y \in \Omega_{k,\varepsilon}$, then the set $\{P\colon Y \in \Gamma(P,\alpha/2)\}$ contains a surface cube $I \subset S_k$ of sidelength at least $c\varepsilon$. Hence,

$$\mu(I)\left(\varepsilon|\nabla u(Y)| + \varepsilon^2\left|\frac{\partial u}{\partial y_0}(Y)\right|\right)^p \leq C\int_{S_k} A_\alpha^p(u)(P)d\mu(P).$$

Our lemma follows once we observe that $\mu(S_k) \leq C_\varepsilon\mu(I)$ which follows from the doubling of μ. □

We are finally ready to state and prove the main result of this paper.

Theorem 4.3. *Let* u *be caloric in a* $\mathrm{Lip}(\frac{1}{2}, 1)$ *cylinder,* Ω *and let* μ *satisfy the* A_∞*-condition. Then there is a constant* $C = C(\alpha, \mu, p)$ *such that*

$$C^{-1} \int_S N_\alpha^p(u)(P)\, d\mu(P) \le \int_S A_\alpha^p(u)\, d\mu(P) + \sum_{k=-\infty}^{+\infty} \mu(S_k)(u(X_k))^p$$
$$\le C \int_S N_\alpha^p(u)(P)\, d\mu(P).$$

Corollary. *The estimate* $(*)$ *given in the Introduction holds.*

We recall that when $\Omega = D \times \mathbf{R}$, with D a Lipschitz domain, then Fabes and Salsa [FS, Theorem 3.1] have established that surface measure satisfies our A_∞-condition. Hence the Corollary follows.

Proof of Theorem 4.3. We consider the first inequality in Theorem 4.3. Let α'' and α' be as in Lemma 3.2, we will show that

$$\int_{S_k} N_{\alpha'}^p(u)(P)\, d\mu(P) \le C \sum_{j=k-1}^{k+1} \int_{S_j} A_{\alpha''}^p(u)(P)\, d\mu(P) + \mu(S_k) u(X_k)^p. \tag{4.4}$$

Then, we sum on k and use well-known arguments (see [T, pp. 314–317 and 367]) which show that area integrals or parabolic maximal functions defined using different cone openings have comparable L^p-norms. We turn to the proof of (4.4).

We let $E_\lambda = \{P \in S_k : N_{\alpha''}(u)(P) > \lambda\}$ and note that this set is open. Hence, we may find a sequence of surface cubes $I_j = I_{r_j}(P_j)$ which satisfy

(i) $E_\lambda = \bigcup_j I_j$,
(ii) If $r_j \le cr_0$, then $\delta(I_j; S_k \setminus E_\lambda) \le C r_j$,
(iii) $\sum_j \chi_{I_j} \le C$.

Let

$$\lambda_0 = |u(X_k)| + C\left((\mu(S_k))^{-1} \int_{S_k} A_{\alpha''}^p(u)(P) d\mu(P) \right)^{\frac{1}{p}}.$$

Appealing to Lemma 4.2, we see that we may choose the constant in the definition of λ_0 sufficiently large to guarantee that one of the hypotheses of Lemma 3.2 is satisfied for each of the cubes I_j arising in the decomposition of E_λ when $\lambda > \lambda_0$. Hence applying Lemma 3.2, to each of the cubes I_j and summing on j we obtain: For all $\alpha > 1$ and $\beta > 1$, there exists $\gamma > 0$ and $\theta > 0$ such that

$$\alpha\mu(\{P \in S_k : N_{\alpha'}(P) > \beta\lambda;\ H_{\alpha''}(\gamma\lambda, P) \le \theta,\ A_{\alpha''}(u)(P) \le \gamma\lambda\}) \le \mu(E_\lambda) \tag{4.5}$$

when $\lambda > \lambda_0$. Let $T > 0$, and consider

$$\int_0^T \mu(\{P \in S_k : N_{\alpha'}(P) > \lambda\})\lambda^{p-1}\, d\lambda \tag{4.6}$$
$$= \beta^p \left(\int_0^{\lambda_0} + \int_{\lambda_0}^{T/\beta} \mu(\{P \in S_k : N_{\alpha'}(u)(P) > \beta\lambda\})\, d\lambda \right).$$

Recalling the definition of λ_0, we have

$$(4.7)\quad \beta^p \int_0^{\lambda_0} \mu(\{P \in S_k : N_{\alpha'}(u)(P) > \beta\lambda\})\lambda^{p-1}\, d\lambda \le \beta^p C \left(\mu(S_k)|u(X_k)|^p + \int_{S_k} A^p_{\alpha''}(u)(P) d\mu(P)\right).$$

To estimate the second integral on the right of (4.6), we use (4.5) and obtain

$$\begin{aligned}(4.8)\quad \beta^p \int_{\lambda_0}^{T/\beta} & \mu(\{P \in S_k : N_{\alpha'}(u)(P) > \beta\lambda\})\lambda^{p-1}\, d\lambda \\ &\le \frac{\beta^p}{\alpha} \int_{\lambda_0}^{T/\beta} \mu(\{P \in S_k : N_{\alpha'}(u)(P) > \lambda\})\lambda^{p-1}\, d\lambda \\ &+ \beta^p \int_{\lambda_0}^{T/\beta} \mu(\{P \in S_k : A_{\alpha''}(u)(P) > \gamma\lambda\})\lambda^{p-1}\, d\lambda \\ &+ \beta^p \int_{\lambda_0}^{T/\beta} \mu(\{P \in S_k : H_{\alpha''}(\gamma\lambda, P) > \theta\})\lambda^{p-1}\, d\lambda.\end{aligned}$$

Recalling the definition of $H_{\alpha''}$ and using the weak-type (1,1) estimates for the maximal function, we may estimate the last term in equation (4.8) by $\sum_{|j-k|=1} \int_{S_j} A^p_{\alpha''}(u)(P)\, d\mu(P)$ while the second integral on the right-hand side of (4.8) is clearly bounded by this quantity. Choosing β and α so that $\beta^p/\alpha = 1/2$ and combining (4.6) to (4.8) we have

$$\begin{aligned}\int_0^T & \mu(\{P \in S_k : N^p_{\alpha'}(u)(P) > \lambda\})\lambda^{p-1}\, d\lambda \\ &\le C\left(\mu(S_k)u(X_k)^p + \sum_{j=k-1}^{k+1} \int_{S_j} A^p_{\alpha''}(u)(P)\, d\mu(P)\right)\end{aligned}$$

where we have absorbed the first integral on the right of (4.8) into the left-hand side of (4.6). Using the monotone convergence theorem to let $T \to \infty$, we obtain (4.4).

The proof of the right-hand inequality of Theorem 4.3 is similar. We reverse the roles of α'' and α' and then use Lemma 3.11 to establish the analogue of (4.4). While the estimate

$$\sum_k \mu(S_k)u(X_k)^p \le \int_S N^p_{\alpha''}(u)(P)\, d\mu(P)$$

follows from Lemma 4.1. The rest of the details are omitted. □

Remark. For future reference, we state a local version of Theorem 4.3. Let $I_r(Q) \subset S$, the lateral boundary of $\mathrm{Lip}(\frac{1}{2}, 1)$ cylinder Ω, let μ satisfy the A_∞-condition and let u be caloric in a neighborhood (relative to Ω) of $\bigcup_{P\in I_r(Q)} \Gamma(P, \alpha'')$. Then

$$\int_{I_r(Q)} A^p_{\alpha'}(u)(P)\, d\mu(P) \le C \int_{I_r(Q)} N^p_{\alpha''}(u)(P)\, d\mu(P)$$

and

$$\int_{I_{r/2}(Q)} N^p_{\alpha'}(u)(P)\,d\mu(P)$$

$$\leq C_r\left(\int_{I_r(Q)} A^p_{\alpha''}(u)(P)\,d\mu(P) + \mu(I_r(Q))|u(Q+re_n)|^p\right).$$

The cone apertures α' and α'' are as in Lemmas 3.11 and 3.2 respectively. The constant $C = C(\alpha', \alpha'', m, p, C_\mu, \tau_\mu)$ and also depends on r in the second estimate. The proof of these estimates is identical to that of Theorem 4.3. We point out that one will need to use the remark after Lemma 3.2 to establish the second estimate. □

5. Local Fatou theorem

We say that u, a continuous function in Ω, has a parabolic limit at $P \in S$ if

$$\lim_{Y\to P,\ Y\in\Gamma(P,\alpha)} u(Y)$$

exists and is finite for all $\alpha > 0$. If for some $\alpha > 0$ and $\lambda \in \mathbf{R}$, possibly depending on P, u satisfies one of the inequalities

$$\inf_{Y\in\Gamma(P,\alpha)} u(Y) > \lambda \quad \text{or} \quad \sup_{Y\in\Gamma(P,\alpha)} u(Y) < \lambda$$

then we say that u has a one-sided parabolic bound at P. Finally, we say that $N \subset S$ is a nullset for caloric measure if $\omega^X_\Omega(N) = 0$ for all $X \in \Omega$. Observe that Harnack's inequality guarantees that if $N \subset \{X : x_0 < a\} \cap S$, then N is a caloric nullset if and only if $\omega^X_\Omega(N) = 0$ for some $X \in \Omega$ with $x_0 > a$. With the corresponding definition of ω-a.e., we can now state the main result of this section:

Theorem 5.1. *Let $E \subset S$ be measurable and let u be caloric in Ω. The following are equivalent*:

(i) *u has parabolic limits at ω-a.e. point in E.*
(ii) *For ω-a.e. Q in E, there exists $\alpha > 0$ such that $A_\alpha(u)(Q) < \infty$.*
(iii) *u has a one-sided parabolic bound at ω-a.e. point in E.*

Remark. There is no reason to require that u be defined in all of Ω. Our theorem still holds, for example, if u is defined in the finite cylinder $\Omega_T \equiv \Omega \cap \{0 < x_0 < T\}$. The only change needed in the statement is to define the cones by $\Gamma(P, \alpha) = \{Y \in \Omega_T : \delta(Y; P) < (1+\alpha)\delta(Y; \partial_p\Omega_T)\}$. The proof given below also carries over to finite cylinders. □

J. Kemper, [K], has observed that the equivalence of (i) and (iii) when $E = S$ follows from his results and techniques developed by Hunt and Wheeden for their study of harmonic functions in Lipschitz domains [HW]. Our Lemma 2.10 combined with Kemper's result leads quickly to the local result. However, we do not need the full strength of Lemma 2.10; we will only use the qualitative consequence of Lemma 2.10 that on $S \cap S_\Phi$, ω and ω_Φ have the same nullsets. This fact may also be proved using the argument given by Jerison and Kenig [JK, Lemma 6.3] for harmonic measure. We also point out that J. Hattemer [H] has studied the relationship between the existence of parabolic limits and the finiteness of the area integral at the initial surface $\partial\Omega \cap \{X: x_0 = 0\}$ for domains $\Omega \subset (0,\infty) \times \mathbf{R}^n$. Jones and Tu [JT] have established the equivalence of (i) and (iii) for domains in $\mathbf{R}^2$ which are slightly smoother than ours. They require the Lipschitz exponent for the time variable to be strictly larger than $1/2$ while we allow the exponent to equal $1/2$. Finally, we remark that J. Lewis and J. Silver [LS] have studied caloric measure on domains $\{x > \phi(t)\} \subset \mathbf{R}^2$ when ϕ belongs to the Besov space $\Lambda_{1/2}^{\infty,2}$. They show that the projection onto $\mathbf{R}$ of a caloric measure $\omega^{(t,x)}$ and surface measure are mutually absolutely continuous on $(-\infty, t)$.

We begin the proof of Theorem 5.1 with two lemmas which show that modulo caloric nullsets, parabolic boundedness or finiteness of the area integral for one cone opening imply the same property for any cone opening.

Lemma 5.2. *Let ∇u be bounded on compact subsets of Ω and let $E \subset S$ be a measurable set such that for each P in E, there exists $\alpha > 0$ for which $A_\alpha(u)(P) < \infty$. Then for ω-a.e. P in E and all $\beta > 0$, $A_\beta(u)(P) < \infty$.*

Proof. Let $I = I_r(\bar{Q})$ be a surface cube and choose β, α and λ satisfying $\beta > \alpha > 0$ and $\lambda > 0$. Let $E' = \{P \in I \cap E: A_\alpha(u)(P) < \lambda\}$. We recall the function $\chi_\alpha(P;Y) = 1$ if $Y \in \Gamma(P,\alpha)$ and zero otherwise. Let $\omega = \omega_\Omega^{V_r(\bar{Q})}$ and observe that the doubling property of ω guarantees that if $Y \in \Gamma(P,\alpha)$ with $P \in I_r(\bar{Q})$ and $\delta(Y;S) < r$, then there is a constant C_0 such that

$$\omega(B(P,(2+\beta)\delta(Y;S))) \leq C_0\omega(B(P,\alpha\delta(Y;S)))$$

where $B(P,r) \equiv \{Q \in S: \delta(P;Q) < r\}$. The constants $2+\beta$ and α are chosen since the triangle inequality guarantees that

$$B(\hat{Y},\alpha\delta(Y;S)) \subset \{P: \chi_\alpha(P;Y) = 1\} \subset B(\hat{Y},(2+\alpha)\delta(Y;S))$$

where $\hat{Y}$ is any point which satisfies $\delta(Y;\hat{Y}) = \delta(Y;S)$. Let $E'_t = \{P \in E': \omega(B(Q,\rho) \cap E') \geq (1-(1/2C_0))\omega(B(Q,\rho))$ for all $\rho < t$ and $B(Q,\rho) \ni P\}$. We let $O_\alpha = \bigcup_{P\in E'} \Gamma(P,\alpha)$ and set $O_{\beta,t} = [\bigcup_{P\in E'_t} \Gamma(P,\beta)] \cap \{Y: \delta(Y;S) < t/(2+\beta)\}$. We claim that

$$O_{\beta,t} \subset O_\alpha, \tag{5.3}$$

$$\int_S \chi_\beta(P;Y)\,d\omega(P) \leq \frac{1}{2}\int_{S\cap E'} \chi_\alpha(P;Y)\,d\omega(P), \quad Y \in O_{\beta,t}. \tag{5.4}$$

To establish (5.3), suppose that $Y \in O_{\beta,t}$ and hence $Y \in \Gamma(P,\beta)$ for some $P \in E'_t$ and $\delta(Y;S) < t/(2+\beta)$. We have

$$\begin{aligned}(5.5)\quad \omega(B(\hat{Y},\alpha\delta(Y;S)) \setminus E') &\le \omega(B(\hat{Y},(2+\beta)\delta(Y;S)) \setminus E') \\ &\le \frac{1}{2C_0}\omega(B(\hat{Y},(2+\beta)\delta(Y;S))) \le \frac{1}{2}\omega(B(\hat{Y},\alpha\delta(Y;S)))\end{aligned}$$

where the second inequality follows since $P \in B(\hat{Y},(2+\beta)\delta(Y;S)) \cap E'_t$ and $(2+\beta)\delta(Y;S) < t$. The third inequality is just our choice of C_0. Assertion (5.3) follows since (5.5) shows that $E' \cap \{P\colon \chi_\alpha(P;Y) = 1\} \supset E' \cap B(\hat{Y},\alpha\delta(Y;S))$ has positive caloric measure and hence is nonempty. To establish (5.4), we again use (5.5) to see that if $Y \in O_{\beta,t}$ then

$$\begin{aligned}\int_{E'} \chi_\alpha(P;Y)\,d\omega(P) &\ge \omega(B(\hat{Y},\alpha\delta(Y;S)) \cap E') \\ &\ge \frac{1}{2}\omega(B(\hat{Y},\alpha\delta(Y;S))) \ge \frac{1}{2C_0}\int_S \chi_\beta(P;Y)\,d\omega(P)\end{aligned}$$

as desired.

Now we can easily complete the proof of this lemma. Set

$$A^2_{\beta,t}(u)(P) = \int_{\Gamma(P,\beta)\cap O_{\beta,t}} |\nabla u(Y)|^2 \delta(Y;S)^{-n}\,dY.$$

Using Fubini's theorem, (5.3) and (5.4), we have

$$\begin{aligned}\int_{E'_t} A^2_{\beta,t}(u)(P)\,d\omega(P) &\le \int_{O_{\beta,t}} |\nabla u(Y)|^2 \delta(Y;S)^{-n} \int_S \chi_\beta(P;Y)\,d\omega(P)dY \\ &\le 2\int_{O_\alpha} |\nabla u(Y)|^2 \delta(Y;S)^{-n} \int_{E'} \chi_\alpha(P;Y)\,d\omega(P)dY \\ &\le 2\int_{E'} A^2_\alpha(u)(P)\,d\omega(P)\end{aligned}$$

which shows that $A_{\beta,t}(u)$ and hence $A_\beta(u)$ is finite ω-a.e. on E'_t. This suffices to prove the lemma since we may choose a countable sequence of values for β,α,t,λ and the cube I such that ω-a.e. point in E belongs to a sequence of sets of the form E'_t for which β is arbitrarily large. □

Lemma 5.6. *Let u be bounded on compact subsets of Ω and let E be a measurable subset of S for which u is parabolically bounded below at each $P \in E$. Then for ω-a.e. $P \in E$ and all $\beta > 0$,*

$$\inf_{Y \in \Gamma(P,\beta)} u(Y) > -\infty.$$

Proof. We begin by chosing $I = I_r(\bar{Q})$ a surface cube, λ,α, and β satisfying $\lambda \in \mathbf{R}$ and $\beta > \alpha > 0$. We set

$$E' = \{P \in I \cap E\colon u(Y) > \lambda \text{ for all } Y \in \Gamma(P,\alpha)\}$$

We let $t < r$ and construct E'_t and then O_α and $O_{\beta,t}$ just as in Lemma 5.2. Since $u \geq \lambda$ on O_α, from (5.3), it follows that $u \geq \lambda$ for $Y \in \Gamma(P,\beta) \cap \{Y: \delta(Y;S) < t\}$. As in Lemma 5.2, this suffices to prove the lemma. □

Proof of Theorem 5.1. We will show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). If u has a parabolic limit at P, then $N_\beta(u)(P) < \infty$ for all $\beta > 0$. Choose β large so that we may construct the domains $\Phi(\cdot,\cdot,\beta)$. Choose a cube $I = I_r(Q) \subset S$ and let $E' = \{P \in I: N_\beta(u)(P) < \lambda\}$ for some $\lambda > 0$. It suffices to show that for some $\alpha > 0$, that $A_\alpha(u)(P) < \infty$ for ω-a.e. $P \in E'$. To establish this, we construct the domain $\Phi = \Phi(I, E', \beta)$ and observe that $|u| \leq \lambda$ in Φ. Hence, applying the remark after the proof of Theorem 4.3, we obtain that for γ sufficiently large,

$$\int_{\widetilde{I}} \widetilde{A}_\gamma^2(u)(P)\,d\omega_\Phi^{X_\Phi}(P) < \infty$$

where $\widetilde{A}_\gamma$ is the area integral defined for the domain Φ. Using this observation and Lemma 2.10, it follows that $\widetilde{A}_\gamma(u)(P) < \infty$ for ω-a.e. P in E'. Finally choosing α small, it is obvious that we have $A_\alpha(u)(P) < \infty$ whenever $\widetilde{A}_\gamma(u)(P) < \infty$.

(ii) $\Rightarrow$ (iii). From Lemma 5.2, we see that our hypothesis implies that if $\beta > 0$, then $A_\beta(u)(P) < \infty$ for ω-a.e. P in E. As before, choose β large, $\lambda > 0$ and $I = I_r(\bar{Q}) \subset S$ a surface cube. Let $E' = \{P \in E \cap I: A_\beta(u)(P) \leq \lambda\}$ and let $E'_t = \{P \in I \cap E: \omega(E' \cap B(Q,\rho)) > (1-(1/2C_0))\omega(B(Q,\rho))$ for $\rho < t$ and $B(Q,\rho) \ni P\}$ where ω is caloric measure for Ω at $V_r(\bar{Q})$, $t/r > 0$ is small and C_0 is chosen so that $\omega(B(P,(2+\beta)\rho)) < C_0\omega(B(P,\beta\rho))$ for $\rho < r$. Note that with these choices, we may define $O_{\beta,t}$ as in the proof of Lemma 5.2 and we have (5.4) with $\alpha = \beta$. Also, note that it suffices to show that for some $\alpha > 0$, $N_\alpha(u)(P) < \infty$ for ω-a.e. P in $E'_t \cap I_{r/2}(Q)$ since we may write E as the countable union of such sets.

Let $\Phi = \Phi(I_r, E'_t, \beta)$. Applying the remark after the proof of Theorem 4.3, it follows that for γ large,

$$\int_{\widetilde{I}_{r/2}} \widetilde{N}_{\gamma/2}^2(u)(P)\,d\omega_\Phi^{X_\Phi}(P) \leq C\left(\int_{\widetilde{I}} \widetilde{A}_\gamma^2(u)(P)\,d\omega_\Phi^{X_\Phi}(P) + u(X_\Phi)^2\right) \tag{5.7}$$

where $\widetilde{A}$ and $\widetilde{N}$ are defined with reference to Φ. This will complete the argument if we can show that the right-hand side of (5.7) is finite. Arguing as in the proof of Lemma 3.11, we have

$$\int_{\widetilde{I}} \widetilde{A}_\gamma^2(u)(P)\,d\omega_\Phi^{X_\Phi}(P) \leq C\int_\Phi G_\Phi(X_\Phi;Y)|\nabla u(Y)|^2\,dY. \tag{5.8}$$

We show that the integral on the right-hand side of this inequality is finite. Using Lemma 2.1 and (3.7) from the proof of Lemma 3.2 to estimate G, and then (5.4), we have

$$\begin{aligned}\int_{O_{\beta,t}} &|\nabla u(Y)|^2 G(V_r(\bar{Q});Y)\,dY\\ &\leq C\int_{O_{\beta,t}} |\nabla u(Y)|^2 \delta(Y;S)^{-n}\int_S \chi_\beta(P;Y)\,d\omega(P)\,dY\\ &\leq C'\int_{O_{\beta,t}} |\nabla u(Y)|^2 \delta(Y;S)^{-n}\int_{S\cap E'} \chi_\beta(P;Y)\,d\omega(P)\,dY\\ &= \int_{E'} A_\beta^2(u)(P)\,d\omega(P) < \infty\end{aligned}$$

The maximum principle and then Lemma 2.1 imply that, at least for t/r small

$$G_\Phi(X_\Phi;Y) \leq G(X_\Phi;Y) \leq CG(V_r(\bar{Q});Y), \qquad Y \in O_{\beta,t}.$$

Combining these last two observations, we see that the right-hand side of (5.8) is finite as desired.

(iii) $\Rightarrow$ (i). As before, we choose $\lambda > 0$, β large and $I \subset S$ a surface cube. Let $E' = \{P \in I \cap E\colon \inf_{Y\in\Gamma(P,\beta)} u(Y) > -\lambda\}$ and construct the domain $\Phi = \Phi(I, E', \beta)$. It suffices to show that u has parabolic limits ω-a.e. in E'. Using Theorem 2.6 in [K] we see that u has parabolic limits through Φ ω_Φ-a.e. on S_Φ and hence ω-a.e. in E'. As β may be arbitrarily large, this establishes the existence of parabolic limits through Ω as desired. Of course, the points where u is parabolically bounded above may be handled similarly. □

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