ON A PROBLEM OF S. MAZUR

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Abstract. In this work a generalization of Mazur's problem concerning the continuity of linear functionals is given.

S. Mazur asked (about 1935) the following question [7, Problem 24]: In a Banach space $E$ an additive functional $f$ is given with the property that, for any continuous function $\varphi : [0,1] \rightarrow E$ the function $f \circ \varphi$ is Lebesgue-measurable. Is $f$ continuous? This question was answered affirmatively by I. Labuda and R. D. Mauldin in [3] by the following theorem:

Theorem 1. Let $E$ be a Banach space, $F$ a Hausdorff topological vector space, $f : E \rightarrow F$ an additive operator. If $f \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi : [0,1] \rightarrow E$, then $f$ is continuous.

The following more general theorem is due to Z. Lipecki [4].

Theorem 2. Let $G, H$ be Hausdorff topological abelian groups, $G$ is metrizable, complete, connected and locally arcwise connected, and let $f : G \rightarrow H$ be a homomorphism. If $f \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi : [0,1] \rightarrow G$, then $f$ is continuous.

Recently, R. Ger presented similar results concerning convex functionals [2].

The aim of this paper is to give another generalization of Mazur's problem. Namely, we show, that if in the original problem $f$ is an exponential polynomial, then the statement remains valid.

First we collect some necessary facts about polynomials and exponential polynomials on groups. Most of these results can be found in [5, 6]. Let $G$ be an abelian group, $H$ a complex linear space. The function $p : G \rightarrow H$ is called a polynomial if for some nonnegative integer $N$ we have

$$\Delta_{y_1,\ldots,y_{N+1}}^{N+1} p(x) = 0$$

for all $x, y_1, \ldots, y_{N+1}$ in $G$. The smallest integer $N$ with this property is called the degree of $p$ and is denoted by $\deg p$. It is well known [1] that any

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function $p$ satisfying (1) can uniquely be represented in the form
\[ p(x) = A_N(x, \ldots, x) + A_{N-1}(x, \ldots, x) + \cdots + A_1(x) + A_0 \]
for all $x$ in $G$, where $A_k: G^k \to H$ is a $k$-additive and symmetric function ($k = 1, 2, \ldots, N$) and $A_0$ is in $H$. For the sake of simplicity we shall use the notation
\[ A^{(k)}(x) = A_k(x, \ldots, x) \]
for all $x$ in $G$, that is $A^{(k)}$ is the diagonalization of the $k$-additive and symmetric function $A_k$, $(k = 1, 2, \ldots, N)$.

Let $C$ denote the set of complex numbers. The function $m: G \to C$ is called an exponential if for all $x, y$ in $G$ we have
\[ m(x + y) = m(x)m(y) \]
and $m$ is not identically zero. That is, exponentials are just the homomorphisms of $G$ into the multiplicative group of nonzero complex numbers.

The function $f: G \to H$ is called an exponential polynomial if it has a representation
\[ f(x) = \sum_{k=1}^{n} p_k(x)m_k(x) \]
for all $x$ in $G$, where $p_k: G \to H$ is a polynomial and $m_k: G \to C$ is an exponential $(k = 1, \ldots, n)$. It is well known [6] that if in (2) we have $m_i \neq m_j$ for $i \neq j$ then the representation (2) for $f$ is unique.

In order to prove our main theorem for exponential polynomials we first consider the case of polynomials.

**Theorem 3.** Let $G$ be a metrizable topological abelian group which is complete, connected and locally arcwise connected, further let $H$ be a metrizable locally convex complex topological vector space and $p: G \to H$ a polynomial. If $p \circ \phi$ is Lebesgue-measurable for any continuous function $\phi: [0, 1] \to G$, then $p$ is continuous.

**Proof.** Let $p = A^{(N)} + q$, where $N \geq 1$ is an integer, $A_N: G^N \to H$ is $N$-additive and symmetric and $q: G \to H$ is a polynomial of degree at most $N-1$. It is enough to show that $A_N$ is continuous, then by induction we have the statement. It is well known [1] that
\[ A_N(x_1, x_2, \ldots, x_N) = \frac{1}{N!} \Delta^{N}_{x_1, \ldots, x_N} p(0) \]
\[ = \frac{1}{N!} \sum_{i_1 < \cdots < i_k} (-1)^{N-k} p(x_{i_1} + \cdots + x_{i_k}) \]
which implies that the function $t \to A_N(\phi(t), x_2, \ldots, x_N)$ is Lebesgue-measurable for any continuous function $\phi: [0, 1] \to G$, and for any fixed $x_2, \ldots, x_N$ in $G$. Using the symmetry of $A_N$ and Theorem 2 we have that $A_N$ is continuous in each variable. From the theorem of Baire it follows that $A_N$ is...
continuous at least at one point. Then, using the connectedness of \( G \), it follows from Theorem 4.2 in [5] that \( A_N \) is continuous.

**Theorem 4.** Let \( G \) be a metrizable topological abelian group which is complete, connected and locally arcwise connected, further let \( H \) be a metrizable locally convex complex topological vector space and \( f : G \to H \) an exponential polynomial. If \( f \circ \varphi \) is Lebesgue-measurable for any continuous function \( \varphi : [0,1] \to G \), then \( f \) is continuous.

**Proof.** Let \( f = \sum_{k=1}^n p_k m_k \), where \( n \geq 1 \) is an integer, \( p_k : G \to H \) is a polynomial and \( m_k : G \to C \) is an exponential \((k = 1, 2, \ldots, n)\), \( m_i \neq m_j \) for \( i \neq j \) and \( p_k = A_k^{(N_k)} + q_k \), where \( A_k^{(N_k)} : G^{N_k} \to H \) is \( N_k \)-additive and symmetric, \( q_k : G \to H \) is a polynomial of degree at most \( N_k - 1 \), \( A_k^{(N_k)} \neq 0 \) \((k = 1, 2, \ldots, n)\). We show that \( m_k, A_k^{(N_k)} \) is continuous \((k = 1, 2, \ldots, n)\). By induction on \( n \), first let \( n = 1 \), \( f = p_1 m_1 \). Here we use induction on the degree of \( p_1 \). If \( \deg p_1 = 0 \), then \( p_1 \) is constant and \( p_1 \neq 0 \). It is very easy to see, that in this case the property of \( f \) implies that \( m_1 \circ \varphi \) is Lebesgue-measurable for any continuous function \( \varphi : [0,1] \to G \), hence by Theorem 2, \( m_1 \) is continuous. Then \( p_1 \circ \varphi \) is Lebesgue-measure for any continuous function \( \varphi : [0,1] \to G \), and by Theorem 3, \( p_1 \) is continuous and hence \( f \) is continuous. If \( \deg p_1 \geq 1 \) then \( p_1 \) is nonconstant, hence there exists \( y \) for which \( \Delta_j p_1 \neq 0 \) is identically zero and \( \deg \Delta_j p_1 < \deg p_1 \). On the other hand

\[
\Delta_j p_1(x)m_1(x) = m_1(-y)p_1(x + y)m_1(x + y) - p_1(x)m_1(x)
\]

\[
= m_1(-y)f(x + y) - f(x)
\]

that is the function \( (\Delta_j p_1 m_1) \circ \varphi \) is Lebesgue-measurable for any continuous function \( \varphi : [0,1] \to G \) which implies the statement of the theorem for \( n = 1 \).

Let now \( n \geq 2 \). We show, that, for example, \( m_2 \) and \( A_2^{(N_2)} \) are continuous. Let \( y \) be an element for which \( m_1(y) \neq m_2(y) \). Then we have

\[
\Delta_j^{N_1+1}(f m_1^{-1})(x) = m_1(x)^{-1} \sum_{j=0}^{N_1+1} \binom{N_1+1}{j} (-1)^{N_1+1-j} m_1(y)^{-j} f(x + jy)
\]

by the definition of difference operators. From this equation we infer that the function \([m_1 \Delta_j^{N_1+1}(f m_1^{-1})] \circ \varphi \) is Lebesgue-measurable for any continuous function \( \varphi : [0,1] \to G \), by the same property of \( f \). On the other hand

\[
(f m_1^{-1})(x) = p_1(x) + \sum_{k=2}^n p_k(x)(m_k m_1^{-1})(x)
\]
holds for all \( x \) in \( G \). By taking differences we have

\[
\Delta_{j_1}^{N_{j_1}+1} (f m_1^{-1})(x) = \sum_{k=2}^{n} \Delta_{j_1}^{N_{j_1}+1} \left[ (m_k m_1^{-1})(A_k^{(N_k)}) + q_k \right](x)
\]

\[
= \sum_{k=2}^{n} \sum_{j=0}^{N_{j_1}+1} \binom{N_{j_1}+1}{j} (-1)^{N_{j_1}+1-j} m_k(x) m_1(x)^{-1} m_k(y)^j m_1(y)^{-j}
\]

\[
\times (A_k^{(N_k)}(x + jy) + q_k(x + jy))
\]

\[
= m_1(x)^{-1} \sum_{k=2}^{n} m_k(x) \left[ \sum_{j=0}^{N_{j_1}+1} \binom{N_{j_1}+1}{j} (-1)^{N_{j_1}+1-j} m_k(y)^j
\]

\[
\times m_1(y)^{-j} (A_k^{(N_k)}(x) + q_k^{*, j,y}(x)) \right]
\]

\[
= m_1(x)^{-1} \sum_{k=2}^{n} m_k(x) [(m_k(y)m_1(y)^{-1} - 1)^{N_{j_1}+1} A_k^{(N_k)}(x) + q_k^{*, j,y}(x)],
\]

where \( q_k^{*, j,y} : G \to H \) is a polynomial of degree at most \( N_k - 1 \) and \( q_k^{*, j,y} : G \to H \) is a polynomial of degree at most \( N_k - 1 \) \( k = 2, \ldots, n; j = 0, 1, \ldots, N_{j_1} + 1 \). We have a representation for the exponential polynomial \( m_1 \Delta_{j_1}^{N_{j_1}+1} (f m_1^{-1}) \) from which we infer—by the above consideration—that \( m_k \) and its polynomial coefficient is continuous. It follows that \( (m_k(y)m_1(y)^{-1} - 1)^{N_{j_1}+1} A_k^{(N_k)} \) must be continuous, and especially—as \( m_2(y) \neq m_1(y) \)—the function \( A_2^{(N_2)} \) is continuous. Hence the theorem is proved.

References


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