TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 316, Number 1, November 1989

ON THE VARIETY OF PLANE CURVES OF DEGREE d WITH δ NODES AND κ CUSPS

PYUNG-LYUN KANG

ABSTRACT. Let $\mathbf{P}^N$ be the projective space which parametrizes all plane curves of degree d and $V(d,\delta,\kappa)$ the subvariety of $\mathbf{P}^N$ consisting of all reduced and irreducible plane curves of degree d with δ nodes and κ cusps as their only singularities. In this paper we prove that $V(d,\delta,\kappa)$ is irreducible if $\kappa \leq 3$, except possibly when $\kappa = 3$ and $d = 5$ or 6.

1. INTRODUCTION

In this paper we study the locus $V(d,\delta,\kappa)$ of reduced and irreducible plane curves of degree d with δ nodes and κ cusps as their only singularities.

Let $\mathbf{P}^N$ be the projective space parametrizing plane curves of degree d, and $V_{d,g}$ the locally closed subset of $\mathbf{P}^N$ of reduced and irreducible plane curves of degree d and geometric genus g. Harris [H1] recently proved that the variety $V_{d,g}$ is irreducible. Knowing this fact, Diaz and Harris [DH1] showed that

(1.1) If $W \subset V_{d,g}$ is any subvariety of codimension 1, and $D \in W$ a general point, then the singularities of D are either (1) m nodes, (2) $m-1$ nodes and one cusp (CU), (3) $m-2$ nodes and one tacnode (TN), or (4) $m-3$ nodes and one ordinary triple point (TR), where $m = \frac{1}{2}(d-1)(d-2) - g$.

They asked in [DH2] whether those loci are irreducible. In this paper we prove that the variety $V(d,\delta,1)$, CU in (1.1), is irreducible. We essentially follow the same idea as that in [H1]; first we show $V(d, \frac{1}{2}(d-1)(d-2)-1, 1)$, the locus of reduced and irreducible rational plane curves of degree d with $\frac{1}{2}(d-1)(d-2)-1$ nodes and one cusp, is irreducible; second we show that there exists only one component of $V(d,\delta,1)$ which contains

$$V(d, \tfrac{1}{2}(d-1)(d-2)-1, 1)$$

in its closure; finally we prove that every component of $V(d,\delta,1)$ admits a degeneration to $V(d, \frac{1}{2}(d-1)(d-2)-1, 1)$.

Using the same argument as above we prove the more general theorem.

Received by the editors April 4, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14H10.
Key words and phrases. Plane curves, cusps, families of plane curves with nodes and cusps.

(1.2) **Theorem.** *The variety* $V(d, \delta, \kappa)$ *is irreducible if* $\kappa \leq 2$. *If* $\kappa = 3$, *then, except possibly when* $d = 5$ *or* 6, $V(d, \delta, \kappa)$ *is irreducible.*

Remark. Ziv Ran has recently and independently shown that $V(d, \delta, 1)$ is irreducible. He has also shown that the two other divisors TN and TR of $V_{d,g}$ in (1.1) are irreducible.

(1.3) *Definitions and notations.* We mean by the (geometric) genus $g(C)$ of a reduced curve C the genus of its normalization; in particular, if C has irreducible components $C_1, C_2, \ldots, C_k$, then

$$g(C) = \sum_i g(C_i) - k + 1.$$

If C is a reduced plane curve, then the class $c(C)$ of C is the degree of the dual curve C^* in $(\mathbf{P}^2)^*$, the projective plane of lines of $\mathbf{P}^2$. If a reduced plane curve C has δ nodes and κ cusps as its only singularities, then $g(C) = \frac{1}{2}(d-1)(d-2) - \delta - \kappa$ and $c(C) = d(d-1) - 2\delta - 3\kappa$.

By a nice curve with given singular points we mean a curve all other singularities of which besides given ones are nodes.

Acknowledgment. I would like to thank my thesis advisor Dr. Steven Diaz for his helpful advice and careful proofreading throughout the preparation of this paper.

2

In this section we prove the first two claims before (1.2) in the Introduction.

(2.1) **Theorem.** *Assume* $d \geq 3$. *Then* $V(d, \frac{1}{2}(d-1)(d-2) - \kappa, \kappa) \subset V_{d,0}$ *is irreducible if* $\kappa \leq \min(d-2, \frac{1}{2}(d+1))$.

Proof. Note that any rational plane curve of degree d can be realized as a projection π_Λ of a rational normal curve C in the projective space $\mathbf{P}^d$ from a $(d-3)$-plane Λ.

Fix a rational normal curve C and a projective plane $\mathbf{P}^2$ to project onto once and for all. Let $\mathrm{Gr}(d-3, \mathbf{P}^d)$ be the Grassmannian of $(d-3)$-planes of $\mathbf{P}^d$ and $\mathscr{V}$ the dense subset in $\mathrm{Gr}(d-3, \mathbf{P}^d)$ of $(d-3)$-planes which occur in the above fact and avoid the fixed projective plane. Then we have a map π from $\mathscr{V}$ to $V_{d,0}$ modulo $\mathrm{Aut}(\mathbf{P}^2)$ defined by $\pi(\Lambda) = \pi_\Lambda(C)$ for $\Lambda \in \mathscr{V}$. It is therefore enough to show that $\pi^{-1}(V(d, \frac{1}{2}(d-1)(d-2) - \kappa, \kappa))$ is irreducible in $\mathrm{Gr}(d-3, \mathbf{P}^d)$. In fact, since any automorphism of C is given by an automorphism of $\mathbf{P}^d$, we have $\pi_\Lambda(C) = \pi_{\Lambda'}(C)$ if and only if $\Lambda = A(\Lambda')$ where A lies in $\mathrm{Aut}(C) \subset \mathrm{Aut}(\mathbf{P}^d)$.

Let $\Lambda \in \pi^{-1}(V(d, \frac{1}{2}(d-1)(d-2) - \kappa, \kappa))$. So Λ meets κ distinct tangent lines, say, t_i of C away from C, where t_i are tangent lines of C at $p_i \in C$, for $i = 1, \ldots, \kappa$. Let $\sigma_2(t_i) = \{(d-3)\text{-planes meeting } t_i\}$ which is a codimension two Schubert variety of $\mathrm{Gr}(d-3, \mathbf{P}^d)$ (see [GH] for notations).

Then $\Lambda \in \bigcap_{i=1}^{\kappa} \sigma_2(t_i)$ and a general $(d-3)$-plane in $\bigcap_{i=1}^{\kappa} \sigma_2(t_i)$ meets only κ distinct tangent lines, otherwise (i.e. if it meets more than κ distinct tangent lines of C) it will be a general member of a subvariety with higher codimension (cf. [EH, Theorem (2.3)]).

Let C_κ be the κth symmetric product of C and Δ be the diagonal of C_κ, i.e., $\Delta = \{(p_1, \dots, p_\kappa) \in C_\kappa | p_i = p_j$ for some distinct i and $j\}$. Note that $C_\kappa \backslash \Delta$ is irreducible. Since, from what we observed before,

$$\pi^{-1}(V(d, \tfrac{1}{2}(d-1)(d-2) - \kappa, \kappa))$$

can be considered as a fiber space over $C_\kappa \backslash \Delta$ with a fiber over a point $(p_1, \dots, p_\kappa)$ in $C_\kappa \backslash \Delta$ an open and dense set of $\bigcap_{i=1}^{\kappa} \sigma_2(t_i)$ consisting of $(d-3)$-planes which meet t_i transversely away from C, it is enough to show that $\bigcap_{i=1}^{\kappa} \sigma_2(t_i)$ is irreducible and of constant dimension. For this purpose we now fix for a while κ distinct points $p_1, \dots, p_\kappa$ on C and the tangent lines t_i of C at p_i. Let $x_i \in t_i$, $i = 1, \dots, \kappa$, and $\sigma_3(x_i)$ be the Schubert variety of $(d-3)$-planes containing x_i which is codimension three in $\mathrm{Gr}(d-3, \mathbf{P}^d)$. Consider $\bigcap_{i=1}^{\kappa} \sigma_3(x_i)$ consisting of $(d-3)$-planes which contain the plane S_κ spanned by $x_1, \dots, x_\kappa$, then $\bigcap_{i=1}^{\kappa} \sigma_3(x_i) \subset \bigcap_{i=1}^{\kappa} \sigma_2(t_i)$. On the other hand, any $\Lambda \in \bigcap_{i=1}^{\kappa} \sigma_2(t_i)$ will lie in $\bigcap_{i=1}^{\kappa} \sigma_3(x_i)$ where $x_i = \Lambda \cap t_i$. We therefore have

$$\bigcup_{(x_1, \dots, x_\kappa) \in t_1 \times \cdots \times t_\kappa} \bigcap_{i=1}^{\kappa} \sigma_3(x_i) = \bigcap_{i=1}^{\kappa} \sigma_2(t_i).$$

We can now consider in $\mathscr{V}$ $\bigcap_{i=1}^{\kappa} \sigma_2(t_i)$ as the fiber space over $t_1 \times \cdots \times t_\kappa$ the product of κ lines with a fiber $\bigcap_{i=1}^{\kappa} \sigma_3(x_i)$ over a point $(x_1, \dots, x_\kappa) \in t_1 \times t_2 \times \cdots \times t_\kappa$, i.e., each fiber consisting of general member of $\bigcap_{i=1}^{\kappa} \sigma_3(x_i)$. We claim that $\bigcap_{i=1}^{\kappa} \sigma_3(x_i)$ is a nonempty irreducible subvariety of $\mathrm{Gr}(d-3, \mathbf{P}^d)$ of dimension $3(d-\kappa-2)$ whenever $d-\kappa-2 \geq 0$ and $2\kappa - 2 \leq d-1$. To see this, choose $(x_1, \dots, x_\kappa) \in t_1 \times \cdots \times t_\kappa$. Then we claim that $\{x_1, \dots, x_\kappa\}$ spans a $(\kappa-1)$-plane S_κ. Suppose not, then $\{x_1, \dots, x_\kappa\}$ would span at most a $(\kappa-2)$-plane and together with $p_1, \dots, p_\kappa$ at most a $(2\kappa-2)$-plane since we may assume that no x_i is on C, which meets C at 2κ points counting multiplicities. But any n-plane, $n \leq d-1$, meets a rational normal curve of degree d in at most $n+1$ points. This gives a contradiction once we have $2\kappa - 2 \leq d-1$. Call S_κ the $(\kappa-1)$-plane spanned by $x_1, \dots, x_\kappa$. Considering the projection $\pi_{S_\kappa} : \mathbf{P}^d \to \mathbf{P}^{d-\kappa}$ from S_κ we can easily check that $\bigcap_{i=1}^{\kappa} \sigma_3(x_i) \simeq \mathrm{Gr}(d-\kappa-3, \mathbf{P}^{d-\kappa})$, which confirms that $\bigcap_{i=1}^{\kappa} \sigma_3(x_i)$ is irreducible of dimension $3(d-\kappa-2)$. (Note that $\bigcap_{i=1}^{\kappa} \sigma_3(x_i) = \{S_\kappa\}$ when $d-\kappa-3 = -1$.) Since the base space $t_1 \times \cdots \times t_\kappa$ and the fibers are irreducible, so is the total space $\bigcap_{i=1}^{\kappa} \sigma_2(t_i)$. Q.E.D.

(2.2) **Theorem.** *Assume that $\kappa \leq \min(d-4, \frac{1}{2}(d-1))$. Then there exists only one component of $V(d, \delta, \kappa)$ which contains $V(d, \frac{1}{2}(d-1)(d-2) - \kappa, \kappa)$ in its closure.*

Proof. Through a point E in $V(d, \frac{1}{2}(d-1)(d-2)-\kappa, \kappa)$, there exist

$$\binom{\frac{1}{2}(d-1)(d-2)-\kappa}{\frac{1}{2}(d-1)(d-2)-\kappa-\delta}$$

sheets of the closure of $V(d, \delta, \kappa)$ each of which corresponds to smoothing some $\frac{1}{2}(d-1)(d-2)-\kappa-\delta$ nodes. Since $V(d, \frac{1}{2}(d-1)(d-2)-\kappa, \kappa)$ is irreducible, it is enough to show that these sheets come from the same component of $V(d, \delta, \kappa)$, which is equivalent to showing that we can interchange the $\frac{1}{2}(d-1)(d-2)-\kappa-\delta$ nodes as E varies along a closed arc in $V(d, \frac{1}{2}(d-1)(d-2)-\kappa, \kappa)$.

For the rest of the proof we are continuing with the previous set up of (2.1). Say that $E = \pi_\Lambda(C)$. Note that the nodes and the cusps of E correspond to the intersection points of Δ with secant lines and tangent lines of C, respectively.

Let $\{x_1, \dots, x_\kappa\} = \Lambda \cap TC$, where TC is the tangent variety of C. Note that $x_i \notin C$. Define $\pi_{S_\kappa} : \mathbf{P}^d \to \mathbf{P}^{d-\kappa}$ as before. We then claim that the image C_κ of C in $\mathbf{P}^{d-\kappa}$ is a nondegenerate rational curve in $\mathbf{P}^{d-\kappa}$ of degree d with κ cusps as its only singularities. It is easy to see that it has no more cusps since Λ meets only κ distinct tangent lines of C transversely. Now suppose that C_κ has (at best) a node. Then the $(\kappa-1)$-plane S_κ would meet a secant line of C besides κ tangent lines of C, which is impossible since S_κ with the κ tangent lines and the secant line above will span at most (2κ)-plane. This means that there exists a (2κ)-plane, if $2\kappa \le d-1$, meeting a rational normal curve C at $2\kappa+2$ points, which is a contradiction.

Since no secant lines and tangent lines other than t_i meet S_κ, they map via π_{S_κ} to the corresponding lines of C_κ. And $\pi_{S_\kappa}(\Lambda)$ meets $\frac{1}{2}(d-1)(d-2)-\kappa$ distinct secant lines of C_κ away from C_κ since $E = \pi_{\Lambda_\kappa} \cdot \pi_{S_\kappa}(C)$, $\Lambda_\kappa = \pi_{S_\kappa}(\Lambda)$. Let $\mathscr{C}(C)$ and $\mathscr{C}(C_\kappa)$ be the chodal varieties of C in $\mathbf{P}^d$ and C_κ in $\mathbf{P}^{d-\kappa}$, respectively. Suppose now that $\pi = \pi_{S_\kappa|\mathscr{C}(C)} : \mathscr{C}(C) \to \mathscr{C}(C_\kappa)$ is generically one to one under our assumption $\kappa \le \min(d-4, \frac{1}{2}(d-1))$. Now {hyperplanes of $\mathbf{P}^d$ containing S_κ} = {hyperplanes of $\mathbf{P}^{d-\kappa}$}. Thus the degree of $\mathscr{C}(C_\kappa)$ is equal to $\frac{1}{2}(d-1)(d-2)-\kappa$, the degree of $\mathscr{C}(C)-\kappa$, since $\mathscr{C}(C_\kappa)$ misses κ points on $\mathscr{C}(C)$ under the projection π_{S_κ}. As the nodes of E correspond to the intersection of Λ and the secant lines of C, they correspond to the intersection of Λ_κ and the secant lines of C_κ. Then applying the uniform position theorem [ACGH, H3] to $\mathscr{C}(C_\kappa)$ we get that the monodromy action on the points of intersection of (smooth locus) of $\mathscr{C}(C_\kappa)$ with a general $(d-\kappa-3)$-plane P in $\mathbf{P}^{d-\kappa}$ is the full symmetric group. But a $(d-\kappa-3)$-plane P in $\mathbf{P}^{d-\kappa}$ with S_κ spans a $(d-3)$-plane Λ the projection π_Λ of C from which gives a curve in $V(d, \frac{1}{2}(d-1)(d-2)-\kappa, \kappa)$. So we may assume, in the beginning of this proof, that $E = \pi_\Lambda(C)$ where Λ occurs in the above way. So we are done once we show that $\pi : \mathscr{C}(C) \to \mathscr{C}(C_\kappa)$ is generically one to one.

Suppose not. Then, for a general point q of a general secant line L of $\mathscr{C}(C_\kappa)$, there is another secant line through q. Now take L_1 and L_2 two general secant lines meeting at $q \notin C_\kappa$. Let $X_1 = \{p \in C_\kappa:$ some secant line $(\neq L_1, L_2)$ through p meets $L_1 \backslash q\}$; and $X_2 = \{p \in C_\kappa:$ some secant line $(\neq L_1, L_2)$ through p meets $L_2 \backslash q\}$. Then both X_1 and X_2 are dense in C_κ, so we can choose $p \in C_\kappa$ whose two second lines meet L_1 and L_2 away from C_κ, so we can choose $p \in C_\kappa$ whose two secant lines meet L_1 and L_2 away from C_κ and q. Consider the 3-plane P_3 spanned by L_1, L_2 and p. Then P_3 meets C_κ at seven points. If P_3 is a hyperplane of $\mathbf{P}^{d-\kappa}$ we stop here. If not, we do the same thing and find a point $p' \in C_\kappa$ at which two secant lines of C_κ meet L_1 and L_2, respectively, away from C_κ, q and those intersection points of L_i with the lines we get in the previous steps to get a $(2+m)$-plane of $\mathbf{P}^{d-\kappa}$ meeting C_κ at $4+3m$ points until the $(2+m)$-plane becomes a hyperplane of $\mathbf{P}^{d-\kappa}$, i.e., $m = d-\kappa-3$ and $4+3m = 3d-3\kappa-5$. Since $\deg(C_\kappa) = d$, π cannot be multiple to one if $3d-3\kappa-5 > d$, equivalently, if $\kappa < (2d-5)/3$. Note that $\kappa < (2d-5)/3$ if $\kappa \leq \min(d-4, \frac{1}{2}(d-1))$ except when $d=7$ and $\kappa = 3$. If $d = 7$, $\kappa = 3$ and π is multiple to one, then we cannot have in $\mathbf{P}^3$ a rational curve of degree 7 with 3 cusps and one node because any such curve in $\mathbf{P}^3$ would be a projection of $C_\kappa = C_3 \subset \mathbf{P}^4$ from a point q on some general secant line of C_3 and there are at least two secant lines through q. But now consider a rational plane curve D of degree 5 with three cusps and three nodes, blow up $\mathbf{P}^2$ at two nodes and four smooth points of D (no three collinear nor all on a conic), and then embed it in $\mathbf{P}^3$ via the system of cubics through the 6 blown up points to get a rational curve of degree 7 in $\mathbf{P}^3$ with three cusps and one node. Therefore $\pi: \mathscr{C}(C) \to \mathscr{C}(C_\kappa)$ is generically one to one.

(2.3) *Remark.* Even though (2.1) or (2.2) does not include some cases of Theorem (1.2) when $d = 4$ and $d = 5$, we can see that they are still valid.

(1) If $d = 4$ and $\kappa = 3$, then $V(4,0,3)$ is the only possibility and it is irreducible because it is the family of duals of elements of $V(3,1)$, the locus of reduced and irreducible plane curves of degree 3 having one node as their only singular point.

(2) If $d - \kappa = 3$ in the proof of (2.2) then $\mathscr{C}(C_\kappa)$ is $\mathbf{P}^3$ or dense in $\mathbf{P}^3$ and any $(d-3)$-plane in $\bigcap_{i=1}^{\kappa} \sigma_3(x_i)$ maps to a point. So we cannot use the uniform position theorem as we did in (2.2). But if $d = 4$ and $\kappa = 1$, a $(d-3)$-plane Λ which is a line in $\mathbf{P}^4$ maps to a point in $\mathbf{P}^3$ through which two secant lines pass. Now we have a two-fold covering from $\mathscr{C}(C)$ to (a dense set) of $\mathbf{P}^3$. Considering the monodromy action to this map we can show that the monodromy group is S_2, the full symmetric group on two elements.

If $d = 5$ and $\kappa = 2$, then any $\Lambda \in \bigcap_{i=1}^{\kappa} \sigma_3(x_i)$ maps via π_{S_κ} to a point in $\mathbf{P}^3$ at which four secant lines meet. So we have a four-fold covering from $\mathscr{C}(C)$ to a (dense set of) $\mathbf{P}^3$. If $d = 5$ and $\kappa = 2$ then $V(5,4,2)$ which consists of rational curves, $V(5,3,2)$, $V(5,2,2)$, $V(5,1,2)$ and $V(5,0,2)$ are the only possibilities. Considering the monodromy action to the four-fold covering we can show that (2.2) remains true for $V(5,3,2)$ and $V(5,1,2)$. To get $V(5,0,2)$ from $V(5,4,2)$ we have to smooth all four nodes, so there exists only one branch of $V(5,0,2)$ through each member of $V(5,4,2)$. For $V(5,2,2)$, we prove it by direct computation: first note on three singular points of degree 5 curve are collinear; pick four points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$, first two for cusps and the others for nodes, and show by explicitly writing down equations that the family of curves in $V(5,2,2)$ which have their nodes and cusps fixed as above is irreducible. Since we can always move by automorphisms of $\mathbf{P}^2$ four general points to the above four points, it is done.

(3) If $d = 4$ and $\kappa = 2$, there is again only one sheet of $V(4,0,2)$ in its closure through each general member of $V(4,1,2)$.

3

Let $U(d,\delta,\kappa)$ be the locus of points of $\mathbf{P}^N$ corresponding to reduced plane curves of degree d with δ nodes and κ cusps as their only singularities. In this section we want to estimate the dimensions of some subloci of $V(d,\delta,\kappa)$ or $U(d,\delta,\kappa)$ which we will use in the next section. We begin this by introducing a slightly larger locus than $U(d,\delta,\kappa)$. Let $U^{d,g,c}$ ($V^{d,g,c}$ resp.) in $\mathbf{P}^N$ be the locus consisting of reduced (and irreducible resp.) plane curves of degree d, genus g and class c. Then $V^{d,g,c}$ is a union of some components of $U^{d,g,c}$ whose general member is irreducible. Diaz and Harris then showed

(3.1) **Theorem** [DH1, (1.2)]. *If $c \geq 2g-1$, then any component W of $V^{d,g,c}$ has dimension $d+c-g+1$ and any singular point of a general member C of $V^{d,g,c}$ is either a node or a cusp. In fact, C has δ nodes and κ cusps where $g = \frac{1}{2}(d-1)(d-2) = \delta - \kappa$ and $c = d(d-1) - 2\delta - 3\kappa$.*

Unless confusion arises we mean $g = \frac{1}{2}(d-1)(d-2) - \delta - \kappa$ and $c = d(d-1) = 2\delta - 3\kappa$ in the context of $U^{d,g,c}$ or $U(d,\delta,\kappa)$. Note that the condition $c \geq 2g-1$ in (3.1) can be read as $\kappa \leq 2d-1$ and $d+c-g+1$ as $g+3d-\kappa-1$. By examining Theorem (3.1) for each d one by one we get

(3.2) **Corollary.** *Let $\kappa \leq 11$. If W is a component of $U^{d,g,c}$, then* $\dim W = d+c-g+1$ *and the general member of W has δ nodes and κ cusps as its only singularities.*

Throughout this section the main tools are the following propositions.

(3.3) **Proposition** [H1]. *Let S be a smooth rational surface, D a divisor class on S, and $W \subset |D|$ a family of reduced curves of geometric genus g in the linear system $|D|$. Assume that for a general member $E \in W$ the restriction to each component of E of the anti-canonical divisor class $-K_S$ contains an effective divisor of positive degree. Then*

(1) $\dim W \leq g - D \cdot K_S - 1$.

(2) *If* $\dim W = g - D \cdot K_S - 1$, *then the general member of W has only nodes as its singularities*; *and*

(3) *if* $\dim W = g - D\cdot K_S - 1$ *and* $C \subset S$ *is any curve, the general member of* W *meets* C *transversely.*

Note that if $\eta\colon X \to \mathbf{P}^2$ is a map, by $\eta(X)$ we will not mean the reduced image, but rather the image with multiplicities, so that $\deg \eta(X) = \deg(\eta^*\mathcal{O}_{\mathbf{P}^2}(1))$.

(3.4) **Proposition** [H1]. *Let* S *be a smooth rational surface. Let* $\pi\colon \Xi \to B$ *be a family of curves of geometric genus* g *with* B *irreducible, and* $\eta\colon \Xi \to S$ *a map such that if* $X_b = \pi^{-1}(b)$ *is a general fiber,* η *is not constant on any component* Z *of* X_b, *and the restriction of* $-K_S$ *to the reduced image* Z' *of* Z *is linearly equivalent on* Z' *to a positive effective divisor, and let* W *be the locus of images* $\{\eta(X_b)\}_{b\in B}$ *of fibers of* π *under the map. Then*

(1) $\dim W \le g - \deg(\eta^* K_S\,|_{X_b}) - 1$; *and*

(2) *if equality holds in* (1) *then the map* $\eta \mid X_b$ *is birational onto its image, and the conlusions of parts* (2) *and* (3) *of Proposition* (3.3) *above apply to* $\eta(X_b)$ *accordingly.*

(3.5) **Definition.** Let C be a reduced curve on a nonsingular surface S, q a singular point of C. Take a sufficiently small neighborhood N of q so that, if $h\colon C' \to C$ is the normalization of C over N and $C_1', C_2', \dots, C_n'$ are the components of C', each C_i' contains only one preimage of q. Define then

$$\overline{m}_q(C) = \sum_i [\operatorname{mult}_q h(C_i') - 1] \quad \text{and} \quad \overline{m}(C) = \sum_{q\in C} \overline{m}_q(C).$$

If C has only one singular branch at q then $\overline{m}_q(C) \ge 1$, and with equality if and only if the singular branch of C is locally given by $y^2 = x^{2n+1}$ for some positive integer n.

If W is an equigeneric family of reduced curves on a nonsingular surface S and has $\overline{m}(C)$ constant for all $C \in W$ then we call W an equiclassical family. (Cf. [DH1] for details, in particular for the case $S = \mathbf{P}^2$.)

(3.6) **Proposition.** *Let* S *be a smooth surface,* D *a divisor class, and* $W \subset |D|$ *an equiclassical family of reduced and irreducible curves* C *in* S *of genus* g *and* $\overline{m}(C) = \kappa$. *Then*

$$\dim W \le \max(g - K_S\cdot D - \kappa - 1, g).$$

Proof. Zariski [Z2] established the following fact: let $W \subset |D|$ be a family of curves in a smooth surface, then the tangent space $T_C W$ to W at a general point $C \in W$ be identified as a subspace of $H^0(\mathcal{O}_C(C)\otimes I)$, where I is the equisingular ideal of C. Let H be the equiclassical ideal of C (see [DH1] for details) then $I \subset H$. Thus

$$\dim W \le h^0(\mathcal{O}_C(C)\otimes I) \le h^0(\mathcal{O}_C(C)\otimes H).$$

Let $\varphi\colon C' \to C$ be the normalization of C. Then there exist two divisors A' and R' on C' of degrees $2(p_a(C) - p_g(C))$ twice the difference between the arithmetic and the geometric genera of C and $\overline{m}(C) = \kappa$ respectively, so that there is an inclusion map

$$H^0(\mathscr{E}_C(C) \otimes H) \hookrightarrow H^0(C', \varphi^*\mathscr{O}_C(C) - A' - R') \quad \text{(cf. [DH1])}.$$

Note that $\deg(\varphi^*\mathscr{O}_C(C) - A' - R') = C \cdot C - 2\{\frac{1}{2}(C \cdot C + K_S \cdot C) + 1 - g\} - \kappa = -K_S \cdot C - 2 + 2g - \kappa$. If $-K_S \cdot C \geq \kappa + 1$ then $\varphi^*\mathscr{O}_C(C) - A' - R'$ is nonspecial. Applying the Riemann-Roch theorem to $\varphi^*\mathscr{O}_C(C) - A' - R'$ on C' we have $\dim W \leq \deg(\varphi^*\mathscr{O}_C(C) - A' - R') - g + 1 = -K_S \cdot C - 2 + 2g - \kappa - g + 1 = g - K_S \cdot C - \kappa - 1$. If $-K_S \cdot C \leq \kappa = \overline{m}(C)$, then $\deg(\varphi^*\mathscr{O}_C(C) - A' - R') \leq 2g - 2$ and, by Riemann-Roch,

$$\dim W \leq \deg(\varphi^*\mathscr{O}_C(C) - A' - R') - g + 1 + h^1(\varphi^*\mathscr{O}_C(C) - A' - R') \leq g,$$

since

$$\begin{aligned} h^1(\varphi^*\mathscr{O}_C(C) - A' - R') &= h^0(K_{C'} - \varphi^*\mathscr{O}_C(C) + A' + R') \\ &\leq \deg(K_{C'} - \varphi^*\mathscr{O}_C(C) + A' + R') + 1. \quad \square \end{aligned}$$

Let $U^{d,g}$ ($V^{d,g}$ resp.) in $\mathbf{P}^N$ be the closure of the locus of reduced (and irreducible resp.) plane curves of degree d and genus g. Unlike the degeneration of $V^{d,g}$ to $V^{d,0}$ in [H1] we have to control cusps too. Therefore we consider here various subloci of $U(d, \delta, \kappa)$ or $U^{d,g,c}$ defined by requiring that for a general member the cusps lie at certain fixed points. To begin with we choose and then fix κ distinct points $Q_1, \ldots, Q_\kappa$ no three or which are collinear and a line L missing them. Now define

(3.7) $U^{d,g}_{m,\kappa}$ ($V^{d,g}_{m,\kappa}$ resp.) as the closure of the locus of all reduced (and irreducible resp.) plane curves C of degree d and genus g which do not contain L, have a contact of order m with L at a point p (a smooth point p resp.) of C, and satisfy $\overline{m}_{Q_i}(C) \geq 1$ at the fixed κ points Q_i.

For the rest of this paper $\kappa \leq 3$, and κ points Q_i and L are fixed unless otherwise specified.

(3.8) **Lemma.** *Let X be a component of $U^{d,g}_{m,\kappa}$. For $\kappa \leq 3$ we have*

(1) *if X is nonempty, then*

(a) $\dim X = g + 3d - 3\kappa - m$; *and*

(b) *if C is a general point of X, ($m \geq 2$) the curve C has a unique point p of intersection multiplicity m with L; in a neighborhood of p, C is a union of smooth arcs $C^1, \ldots, C^k$ having contact of orders $a_1, \ldots, a_k$ with L respectively, $\sum a_i = m$, having minimal order of contact among themselves:*

$$(C^i \cdot C^j)_p = \min(a_i, a_j),$$

and ($m \geq 1$) C has cusps at Q_i and the remaining singularities of C are all nodes.

(2) *Let* $\pi: \Xi \to B$ *be a family of smooth curves of genus* g *with* B *irreducible,* $\eta: \Xi \to \mathbf{P}^2$ *a map whose restriction to a general fiber* X_b *of* Ξ *has degree* d *, is not constant on any component of* X_b *, does not map any component of* X_b *to* L *,* $\eta(X_b)$ *has a singular branch at each* Q_i *and has contact of order* m *with* L *at some point* $p \in L$ *. Let* $X = \{\eta(X_b)\}_{b\in B} \subset \mathbf{P}^N$ *be the locus of images. Then* $\dim(X) \le g + 3d - 3\kappa - m$ *, and if* $\dim(X) = g + 3d - 3\kappa - m$ *,* $\eta \mid X_b$ *is birational and* X *is dense in a component of* $U^{d,g}_{m,\kappa}$ *.*

Proof. We begin the proof by introducing the following: for a component W of $U^{d,g,c}$, let

W_m = (the closure in $U^{d,g}$ of) $\{C \in W \mid C$ does not contain L and $(C \cdot L)_p = m$ for some point $p \in L\}$; and

$W_{m,\kappa}$ = (the closure in $U^{d,g}$ of) $\{C \in W \mid C$ satisfies $\overline{m}_{Q_i}(C) \ge 1$, does not contain L and $(C \cdot L)_p = m$ for some point $p \in L\}$.

We first consider a component X of $U^{d,g}_{m,\kappa}$ whose general member C is irreducible. Suppose $\overline{m}(C) = \kappa' \ge \kappa$. Then we may assume $X \subset V^{d,g,c'}$ where $c' = 2(d-1) + 2g - \kappa'$. Let W' be a component of $V^{d,g,c'}$. Fix now a point $p \in L$ and let W'^0_m be the locus of curves C in W'_m having intersection number m with L at p specifically but no cusps in general on L. Since $\dim V^{d,g,c'} = g + 3d - \kappa' - 1$ and $(L \cdot C)_p = m$ is expressed by m equations on the coefficients of C, we have $\dim W'^0_m \ge g + 3d - \kappa' - m - 1$ if nonempty.

To prove that $\dim W'^0_m \le g + 3d - \kappa' - m - 1$, we blow up $\mathbf{P}^2$ at p m times in the direction of L: let S_1 be the blow-up of $\mathbf{P}^2$ at p, E_1 the exceptional divisor, and $p_1 \in E_1$ the point corresponding to L; define blow-ups S_i by letting S_{i+1} be the blow-up of S_i at p_i, E_{i+1} the exceptional divisor, and $p_{i+1} \in E_{i+1}$ the point of intersection of E_{i+1} with the proper transform of L. We blow up exactly m times if p is a smooth point of C; otherwise we may stop before. Let C be a general member of W'^0_m, C_i the proper transform of C in S_i, and b_{i+1} the multiplicity of C_i at p_i. Letting $\pi_i: S_m \to S_i$ and $\pi: S_m \to \mathbf{P}^2$ we have on S_m that $C_m \sim \pi^* C - \sum_i^m b_i \pi_i^* E_i$ with $\sum_i^m b_i = m$ and

$$K_{S_m} = \pi^*(-3H) + \sum_i^m \pi_i^* E_i$$

where H is a hyperplane divisor of $\mathbf{P}^2$. Therefore

$$-K_{S_m} \cdot C_m = 3d - m$$

which is at least $2d$ (when $m = d$). Applying (3.6) to the family $\{C_m \mid C \in W'^0_m\}$ of curves in the above, we get

$$\dim W'^0_m \le g + 3d - m - \kappa' - 1$$

and

$$\dim W'_m \le g + 3d - m - \kappa'$$

(in fact an equality). Since there is an automorphism of $\mathbf{P}^2$ which maps the line L to itself and three general points to another three general points,

$$\dim X \le g + 3d - m - \kappa' - 2\kappa \le g + 3d - m - 3\kappa$$

with equality only if $\kappa' = \kappa$, consequently only if $\overline{m}_{Q_i}(C) = 1$.

For (1b), therefore, we may assume that X is a component of $W_{m,\kappa}$ where W is a component of $V^{d,g,c}$ with $c = 2(d-1) + 2g - \kappa$. Repeating the previous dimension count by replacing W' by W and κ' by κ, we have

$$\dim W_{m,\kappa} = g + 3d - 3\kappa - m.$$

We can also see, from the dimension counts in the previous paragraph, that if C is a general member of $W_{m,\kappa}$, then C is again a general one of W_m^0. Thus we can easily remove the possibility that $\overline{m}(C_m) > \kappa$. Since we require that $\overline{m}_{Q_i}(C) \ge 1$ for all i, C satisfies $\overline{m}_{Q_i}(C) = 1$ and has only one singular branch C_i at each Q_i possibly with smooth branches through each Q_i. Now suppose that a general curve C in $W_{m,\kappa}$ has a more complicated singular point than a cusp, say, at Q_1. To count the dimension of a family of such curves C we fix a line T_1 through Q_1 as a tangent line of a singular branch of C and assume that C has, locally at Q_1, r smooth branches transverse to T_1 and s smooth branches tangent to T_1 as well as one singular branch C_s. Let x and y be a local coordinate of $\mathbf{P}^2$ at q_1 and T_1 be $\{y = 0\}$. Then we blow up S_m as follows:

(3.8.1) Let S_{m+1} be the blow-up of S_m at Q_1, E_{m+1} the exceptional divisor and $Q_1^1 \in E_{m+1}$ the point corresponding to T_1; if a curve C in $\mathbf{P}^2$ has an ordinary cusp at Q_1 $(y^2 = x^3)$, then its proper transform C_{m+1} in S_{m+1} meets E_{m+1} with multiplicity $(E_{m+1} \cdot C_{m+1})_{Q_1^1} = 2$ at a smooth point Q_1^1 of C_{m+1}; blowing up S_{m+1} at Q_1^1 (call it S_{m+2}) and again S_{m+2} at Q_1^2 the point corresponding to E_{m+1} on S_{m+2} to get S_{m+3}. Then on S_{m+3}, if a singular branch of a general curve C has a cusp at Q_1, the proper transforms C_{m+3} of C in W_m^0 with only Q_1 and T_1 fixed form an equiclassical family of curves of genus g with $\overline{m}(C_{m+3}) = \kappa - 1$. Let $\pi_i \colon S_{m+3} \to S_i$ and E_i the exceptional curve of $S_i \to S_{i-1}$ for $1 \le i \le m+3$, here $S_0 = \mathbf{P}^2$. Then

$$C_{m+3} \simeq \pi^* C - \sum_i^m b_i \pi_i^* E_i - (r + s + 2)\pi_{m+1}^* E_{m+1} - (s+1)\pi_{m+2}^* E_{m+2} - E_{m+3}$$

where $\sum_i^m b_i = m$. So $-K_{m+3} \cdot C_{m+3} = 3d - m - r - 2s - 4$. (From now on K_i is the canonical divisor of S_i.) If the local equation of a singular branch is given by $y^2 = x^{2n+1}$, $n \geq 2$, then the proper transform C_{m+3} of C is equivalent to

$$\pi^* C - \sum_i^m \pi_i^* E_i - (r+s+2)\pi_{m+1}^* E_{m+1} - (s+2)\pi_{m+2}^* E_{m+2}$$

with $\overline{m}(C_{m+3}) = \kappa - 1$ provided that $n = 2$ or $\overline{m}(C_{m+3}) = \kappa$ otherwise. In either case, $-\kappa_{m+3} \cdot C_{m+3} = 3d - m - r - 2s - 4$ which is big enough to apply (3.6) since $r + 2s + 2n + 1 \leq d$ by considering the intersection multiplicity of T_1 and C at Q_1. Thus $\dim\{C \in W_m^0|$ with Q_1 and T_1 fixed$\} \leq g + 3d - m - r - 2s - 4 - \kappa$ if $n = 1$ or 2; or $\dim\{C \in W_m^0|$ with Q_1 and T_1 fixed$\} \leq g + 3d - m - r - 2s - 5 - \kappa$ if $n \geq 3$. Removing the restriction on T_1 and moving Q_1 to any point we want, $\dim W_m^0 \leq g + 3d - m - r - 2s - \kappa - 1$ if $n = 1$ or 2; or $\dim W_m^0 \leq g + 3d - m - r - 2s - \kappa - 2$ if $n \geq 3$, which is always less than $g + 3d - m - \kappa - 1$ the actual dimension of W_m^0 provided either that r or s is not zero, or that $n \geq 3$.

Now suppose that the local equation of C at Q_1 is $y^2 = x^5$. Then we blow up S_m as follows: let S_{m+1} be the blow-up of S_m at Q_1, E_{m+1} the exceptional curve, Q_1^1 the point corresponding to T_1, C_{m+1} the proper transform of C in S_{m+1}; then C_{m+1} has an ordinary cusp at Q_1^1; now blow up S_{m+1} beginning at Q_1^1 three times as we did in (3.8.1) to resolve a cusp, call S_{m+4} the final blown up surface. On S_{m+4}, $\{C_{m+4}|C \in W_m^0$ with Q_1 and T_1 fixed$\}$ forms an equiclassical family of genus g and $\overline{m}(C_{m+4}) = \kappa - 1$. Again by (3.6), since $-K_{m+4} \cdot C_{m+4} = 3d - 6 - m$ where K_{m+4} is the canonical divisor of S_{m+4}, this family has dimension at most $g + (3d - 6 - m) - (\kappa - 1) - 1$, therefore $\dim W_m^0 \leq g + 3d - \kappa - m - 3$, which again gives a contradiction.

Suppose this time that a general member C of $W_{m,\kappa}$ has a singular point other than a node away from Q_i and L. Then we consider the sublocus of W_m^0 of curves C through a fixed point q (not p nor in L) which is a singular point with $\overline{m}_q(C) = 0$ but not a node of C, so either all branches at q are transverse or some are not. By blowing up S_m at q (once if all branches are transverse, twice if some are tangential; see (3.10)) and counting dimension of each component in a similar way we get a contradiction again since the dimension of W_m^0 goes down at least by 1.

We now claim that if C is a general member of W_m^0 then C_m in S_m must meet the union $E = \bigcup \widetilde{E}_i$ of the $\widetilde{E}_i$ transversely: otherwise we blow up those points (including the intersections $\widetilde{E}_i \cap \widetilde{E}_j$) at which C_m and E do not meet transversely and apply another dimension count to get a contradiction. In particular, C_m will meet each $\widetilde{E}_i$ transversely at $b_i - b_{i+1}$ distinct points other than $\widetilde{E}_i \cap \widetilde{E}_{i-1}$ or $\widetilde{E}_i \cap \widetilde{E}_{i+1}$. Then a branch C^i of C whose proper transform in C_m crosses $\widetilde{E}_{a_i}$ will then be smooth and have contact of order a_i with L at p; two branches C^i, C^j, since their proper transforms cross E at different points, will have contact of order $\min(a_i, a_j)$.

If a general curve C has a point $q \neq p$ in L of intersection multiplicity bigger than 1 with L then, by blowing up S_m at q as we did for p, we get a contradiction as well.

Let $W^0_{m,\kappa} = \{C \in W_{m,\kappa} | (C \cdot L)_p = m$ for a fixed point p in $L\}$. Then if $\kappa \leq 2$, $\dim W^0_{m,\kappa} = g + 3d - m - 3\kappa - 1$ since there is an automorphism fixing two points Q_1 and Q_2, and mapping p to any point in L except $L \cap L'$, where L' is the line through Q_1 and Q_2.

To prove the lemma in case that C is reducible, suppose that C has k components C_i of genus g_i and degree d_i with a contact of multiplicity m_i with L at a fixed point p in L. If $\kappa = 1$ there is an easy way to see the proof of lemma; we may blow up S_m at Q_1 three times as in (3.8.1); then on S_{m+3} $-K_{m+3}$ the anticanonical divisor is equivalent to an effective divisor and any irreducible curve in S_{m+3} other than the proper transform $\widetilde{L}$ of L or $\widetilde{E}_i$ of E_i meets $-K_{m+3}$ positively; applying now Propositions (3.3) and (3.4) to a family of curves in S_{m+3} in the divisor class $|\pi^* dH - \sum_i^m b_i \pi_i^* E_i - 2\pi^*_{m+1} E_{m+1} - \pi^*_{m+2} E_{m+2} - E_{m+3}|$ we have part (1) and (2).

If $\kappa = 2$ we may assume either that C_1 has two cusps and the other components have no cusps, or that each C_i has a cusp at Q_i for $i = 1, 2$ and the others have no cusps at all. Then if the former case happens C_1 moves in a family of dimension at most $g_1 + 3d_1 - m_1 - 7$ and C_i by [H1, Lemma (2.4)] at most $g_i + 3d_i - m_i - 1$ which is the dimension of $(V^{d,g})^0_m$ for $i \geq 2$, therefore C in a family of dimension at most $3d + g - m - 7$ since $g = \sum g_i - k + 1$. If the latter case happens then C_i moves in a family of dimension at most $g_i + 3d_i - m_i - 4$ for $i = 1$ and 2; $g_i + 3d_i - m_i - 1$ for $i \geq 3$, therefore C at most $3d + g - m - 7$. If we have an equality in either case, then all C_i must meet transversely except at p the point of intersection number m between C and L and no C_i meet at a point in L except p: if not, we blow up S_m a proper number of times at each of those possible unwanted points and apply another dimension count to get a contradiction. We also have equalities for all C_i. From the case that C is irreducible and [H1, Lemma (2.4)] we have the desired results.

If $\kappa = 3$, it is enough to consider the following three possibilities: (i) each C_i has a cusp at Q_i for $i = 1, 2, 3$ and the other components have no cusp; (ii) C_1 has a cusp at Q_1, C_2 has cusps at Q_2 and Q_3 and the others have no cusps; (iii) C_1 has all three cusps and the others have no cusps at all. In (i) and (ii), the proofs are exactly the same as the case $\kappa = 2$ since $\dim W^0_{m,\kappa} = g + 3d - m - 3\kappa - 1$ if $\kappa \leq 2$. For (iii) we know, from the case of $V^{d,g}_{m,\kappa}$, that C_1 moves in a family of dimension at most $g_1 + 3d_1 - m_1 - 9$ without fixing a point p in L if m_1 is positive. (If m_1 is zero, it is easy to see that C_1 moves in a family of dimension at most $g_1 + 3d_1 - 10$ and then choose $p \in L$ for the remaining C_i.) Then for each C_1, $p \in L$ is automatically fixed and C_i moves in a family of dimension at most $g_i + 3d_i - m_i - 1$ for $i \geq 2$. Therefore C moves in a family of dimension at most $(g+k-1)+3d-m-9-(k-1) = g+3d-m-9$, and with equality holding only if we have equality at each stage. From the case of irreducible curves, [H1, Lemma (2.4)] and the above proof of $\kappa = 2$ for transverse intersections between components of C, a general curve satisfies the description in (3.8) with cusps at Q_i.

Part (2) follows easily from part (1). Suppose that X_b has components $X^1, \dots, X^k, X^j$ of genus g_j, and $\eta|_{X^j}$ has degree d_j, has contact of order m_j with L at p, and has degree n over its image. Then divide the cases as before according to how many Q_i the reduced image of each $\eta(X^j)$ has. Q.E.D.

This time we consider the locus $U^{d,g}_{m,l,\kappa}$ of curves C in $U^{d,g}$ which do not contain L, satisfy $\overline{m}_{Q_i}(C) \geq 1$ (so $c(C) \leq c$) and $\sum_i (C \cdot L)_{p_i} = m$ for some collection of l points $p_1, \dots, p_l$ of L. Then we have

(3.9) **Lemma.** *Let* $\kappa \leq 3$. *Then*

(1) $\dim U^{d,g}_{m,l,\kappa} \leq g + 3d - 3\kappa - m + l - 1$,

(2) *let* $\pi \colon \Xi \to B$ *be a family of smooth curves of genus* g, $\eta \colon \Xi \to \mathbf{P}^2$ *a map whose restriction to a general fiber* X_b *of* π *has degree* d, *is not constant on any component of* X_b, *does not map any component of* X_b *to* L, *for some set of distinct points* $p_1, \dots, p_l$ *in* L *has total order of contact*

$$\sum_i \sum_{q \in \eta^{-1}(p_i)} \operatorname{mult}_q(\eta^* L) = m$$

and on its reduced image C_b *of* $\eta(X_b)$ $\overline{m}_{Q_i}(C_b) \geq 1$. *Let* $X = \{\eta(X_b)\}_{b \in B} \subset \mathbf{P}^N$ *be the locus of images. Then*

$$\dim X \leq g + 3d - 3\kappa - m + l - 1.$$

Proof. The proof of part (1) is very similar to that of part (1) of (3.8): we fix l points $p_1, p_2, \dots, p_l$ and blow up $\mathbf{P}^2$ at $p_1, \dots, p_l$ at most m times in the direction of L, i.e., blow up $\mathbf{P}^2$ at p_i until the proper transform of a general curve C does not meet the proper transform of L. Call the blown up space S_m. The family $V^{d,g,0}_{m,l} = \{C \in V^{d,g} \mid \sum_i (C \cdot L)_{p_i} = m$ for fixed l points $p_1, \dots, p_l$ and C has κ points (not fixed) away from L at each of which $\overline{m}(C) \geq 1\}$ gives a family of curves $\overline{C}$ the proper transform of C in S_m of genus g and $\overline{m}(\overline{C}) \geq \kappa$ with $\overline{C} \cdot (-K_m) = 3d - m$. By (3.6) $\dim V^{d,g,0}_{m,l} \leq g + 3d - m - \kappa - 1$, so $\dim V^{d,g}_{m,l,\kappa} \leq g + 3d - m - 3\kappa + l - 1$.

The case of reducible curves and (2) follow from (1) just as in (3.8). Q.E.D.

(3.10) **Lemma.** *Let* $TN^{d,g}(\kappa)$ *be the closure in* $V^{d,g}$ *of the locus of reduced and irreducible curves of degree* d *and genus* g *having* κ ($\kappa \leq 3$) *tacnodes away from* L *and satisfying*

$$\sum_i (C \cdot L)_{p_i} = m,$$

for some collection of l *points* $p_1, \dots, p_l$ *in* L. *Then*

(1) $\dim TN^{d,g}(\kappa) \le g + 3d - m - \kappa + l - 1$.

Let $TN^{d,g}_{\kappa}(\kappa)$ *be the sublocus of* $TN^{d,g}(\kappa)$ *whose members have their tacnodes at the fixed* κ *distinct points* Q_i, $1 \le i \le \kappa$. *Since there is an (the) automorphism of* $\mathbf{P}^2$ *which maps* L *to itself and moves three general points to another three general points of* $\mathbf{P}^2$ *we have*

(2) $\dim TN^{d,g}_{\kappa}(\kappa) \le g + 3d - m - 3\kappa + l - 1$.

We mean by a tacnode a point where a curve C *has two smooth branches meeting tangentially.*

Proof. Let $\kappa = 1$. Fixing then l specific points $p_1, \dots, p_l$ in L, one point Q_1 in $\mathbf{P}^2$ and a line T_1 through Q_1, we consider $TN^0_{\bar{1}}(1)$ (the closure of) the sublocus of $TN^{d,g}_1(1)$ of curves C with $\sum_i (L\cdot C)_{p_i} = m$ and T_1 the common tangent line of two branches of C at Q_1 which meet tangentially. Let S be the blow-up of $\mathbf{P}^2$ at $p_1, p_2, \dots, p_l$ as in the proof of (3.9). (Note that it was called S_m, but from now on, for the sake of notation we fix S the blow-up of $\mathbf{P}^2$ at the fixed l points p_i in L.) Let S_1 be the blow-up of S at Q_1, E_1 the exceptional curve of the blow-up $S_1 \to S$, Q_1' the point on E_1 corresponding to T_1, and S_2 the blow-up of S_1 at Q_1'. On S_2 the proper transform $\overline{C}_2$ of a general curve $C \in TN^0_{\bar{1}}(1)$ is equivalent to $\pi^*\overline{C} - 2\pi^*E_1 - 2\pi^*E_2$ where $\overline{C}$ is the proper transform of C of the blow-up $S \to \mathbf{P}^2$. Here π represents blow-up maps from S_2 to S_1, S, or $\mathbf{P}^2$ according to context. Since $K_{S_2} = \pi^*K_S + \pi^*E_1 + E_2$ (so, $-K_{S_2}$ is equivalent to an effective divisor $\pi^*H + \widetilde{T}_1 + \widetilde{L}$ where $\widetilde{T}_1$ and $\widetilde{L}$ are the proper transforms of T_1 and L respectively, and H is a hyperplane divisor of $\mathbf{P}^2$), $-K_{S_2}\cdot C_2 = 3d - m - 4$ which is always positive since d is at least 4 for a plane curve of degree d to have a tacnode. By Proposition (3.3) we have $\dim TN^0_{\bar{1}}(1)$ is at most $g + 3d - m - 5$.

If $\kappa = 2$, then d is at least 5. Adopting the previous setting we fix the second point Q_2 away from both Q_1 and L, and a line T_2 in general distinct from T_1 through Q_2. Now consider $TN^0_{\bar{2}}(2)$ (the closure of) the sublocus of $TN^{d,g}_2(2)$ consisting of curves satisfying $\sum_i (C\cdot L)_{p_i} = m$ and each T_i, $i = 1$ or 2, is the common tangent line of two smooth branches of C at Q_i. Blow up S_2 in the above at Q_2 twice as we did S at Q_1, getting S_4. On S_4 we still have effective anticanonical divisor $-K_4$ having positive intersection number with C_4 the proper transform of a general member C. In fact by choosing T the line through Q_1 and Q_2, $-K_4 \sim \widetilde{L} + 2\widetilde{T} + \widetilde{E}_1 + \widetilde{E}_3$ and $-K_4\cdot C_4 = 3d - m - 8$, where $\sim$ representing the proper transform in S_4 of a corresponding divisor. Again Proposition (3.3) tells that

$$\dim TN^0_{\bar{2}}(2) \le g + 3d - m - 9 \quad \text{and} \quad \dim TN^0_2(2) \le g + 3d - m - 7.$$

If $\kappa = 3$ then $d \ge 5$ too. If we blow up all three tacnodes we will have trouble in making first the anticanonical divisor effective and second $3d - m - 12$

positive. Instead of fixing three tacnodes, therefore, we only fix two of them, say Q_1 and Q_2, together with lines T_i through Q_i as in the case $\kappa = 2$. Then on S_4 we have a family of curves C_4 of genus g and $C_4 \cdot -K_4 = 3d - m - 8$, so of dimension at most $g + 3d - m - 9$ with equality holding if and only if C_4 is a nodal curve. Since C_4 is not a nodal curve, the above family has dimension at most $g + 3d - m - 10$, and then $g + 3d - m - 4$ by removing the restrictions on Q_i. Thus $\dim TN_3^{d,g}(3) \leq g + 3d - m - 10 + l$. Q.E.D.

(3.11) *Remark.* If $\kappa \geq 2$ and one (or two) of Q_i is (are) replaced by cusp(s) in (3.10) then on S_2 or on S_4, depending on the number of tacnodes, we do the same dimension counts as we did in the proof of (3.8) due to Proposition (3.6) to get the same upper bound of possible dimension of a corresponding locus.

This time we introduce the locus $U_{m,l}^{d,g}(\kappa)$ consisting of curves in $U^{d,g}$, a general member C of which does not contain L, has one singular branch or two smooth branches meeting tangentially at each of the fixed κ points Q_i, and for some collection of l points $p_1, \ldots, p_l$ in L satisfies $\sum_i (C \cdot L)_{p_i} = m$.

(3.12) **Lemma.** *Let W be a component of $U_{m,l}^{d,g}(\kappa)$ for $\kappa \leq 3$.*

(1) (a) *If $\kappa \geq 2$ and a general member C of W has two irreducible components which are tangent to each other at least at two points of Q_i, or ($\kappa = 3$) C has three components each of which is tangent to the others (see* (xiii) *below), then*

$$\dim W \leq g + 3d - m - 3\kappa + l.$$

(b) *Otherwise*; $\dim W \leq g + 3d - m - 3\kappa + l - 1$.

(2) *There is a corresponding parametric version of* (1).

Proof. If a general member C of a component W of $U_{m,l}^{d,g}(\kappa)$ has a singular branch at each Q_i, see (3.9). If a general member C is irreducible, see (3.10) and (3.11).

Now assume that a general curve C is reducible. In this case we only consider the possible cases for $\kappa = 3$. If $\kappa = 1$ or 2 we will have much fewer cases than $\kappa = 3$, which will eventually be included in what we will consider from now on. Note that it is enough to consider the components of C through Q_i because the remaining ones are in general of nodal curves with some contact with L (cf. see the proof of (3.8) when C is reducible and [H1, Lemma (2.5)]). Without further mention we assume that each component C_j of C has degree d_j and genus g_j with $\sum_i (C_j \cdot L)_{p_i} = m_j$ for some collection of l points in L, where $\sum_j d_j = d$, $\sum_j g_j - n + 1 = g$, n the number of components of C, and $\sum_j m_j = m$. Then C is one of the following:

(a) C has two components C_1 and C_2

(i) with C_1 having a cusp or a tacnode at each of Q_1 and Q_2, and C_2 a cusp or a tacnode at Q_3;

(ii) with C_1 having a cusp or a tacnode at Q_1, C_2 a cusp or a tacnode at Q_2, and C_1 and C_2 meeting tangentially at Q_3;

(iii) with C_1 having a cusp or a tacnode at each of Q_1 and Q_2, and C_1 and C_2 meeting tangentially at Q_3;
(iv) with C_1 and C_2 meeting tangentially at Q_1 and Q_2, and C_1 having a cusp or a tacnode at Q_3;
(v) with C_1 and C_2 meeting tangentially at Q_1, Q_2 and Q_3.

(b) C has three components C_1, C_2 and C_3
(vi) with each C_i having a cusp or a tacnode at each Q_i;
(vii) with C_1 having a cusp or a tacnode at each of Q_1 and Q_2, and C_2 and C_3 meeting tangentially at Q_3;
(viii) with C_1 having a cusp or a tacnode at Q_1 and being tangent to C_3 at Q_3, and C_2 having a cusp or a tacnode at Q_2;
(ix) with C_1 having a cusp or a tacnode at Q_1 and meeting C_2 and C_3 tangentially at Q_2 and Q_3 respectively;
(x) with C_1 having a cusp or a tacnode at Q_1 and meeting C_2 tangentially at Q_2, and C_2 being tangent to C_3 at Q_3;
(xi) with C_1 and C_2 meeting tangentially at Q_1 and Q_2, and C_3 having a cusp or a tacnode at Q_3;
(xii) with C_1 meeting C_2 at Q_1 and Q_2 and C_3 at Q_3.
(xiii) with C_1 tangent to C_2 at Q_3, C_2 tangent to C_3 at Q_1 and C_3 tangent to C_1 at Q_2.

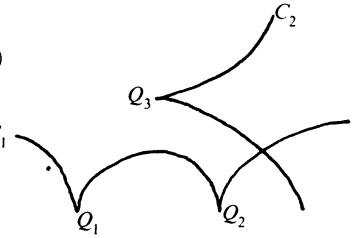
(c) C has four components
(xiv) with C_i, $i = 1$ or 2, having a cusp or a tacnode at Q_i, and C_3 and C_4 meeting tangentially at Q_3;
(xv) with C_4 being tangent to C_1 and C_2 at Q_1 and Q_2 respectively, and C_3 having a cusp or a tacnode at Q_3;
(xvi) with C_1 having a cusp or a tacnode at Q_1 and meeting C_2 at Q_2 tangentially, and C_3 and C_4 meeting tangentially at Q_3;
(xvii) with C_1 and C_2, C_2 and C_3, C_3 and C_4 meeting tangentially at Q_1, Q_2 and Q_3 respectively;
(xviii) with C_4 being tangent to C_1, C_2 and C_3 at Q_1, Q_2 and Q_3 respectively;
(xvix) with C_1 and C_2 being tangent at Q_1 and Q_2, C_3 and C_4 at Q_3.

(d) C has five components
(xx) with C_1 being tangent to C_2 at Q_1, C_3 to C_4 at Q_2, and C_5 having a cusp or a tacnode at Q_3;
(xxi) with C_4 being tangent to C_1 and C_2 at Q_1 and Q_2 respectively, and C_5 to C_3 at Q_3.

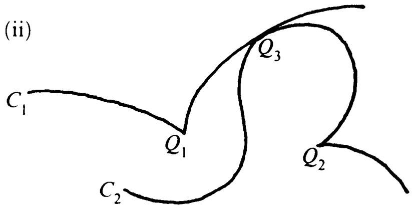
(e) C has 6 components
(xxii) with C_1 being tangent to C_2 at Q_1, C_3 to C_4 at Q_2, and C_5 to C_6 at Q_3.

In cases (i) and (vi), all components meet transversely, so apply (3.9) or (3.10) to each C_i.

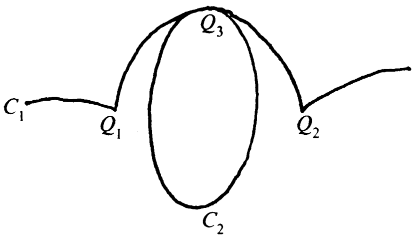
(i)



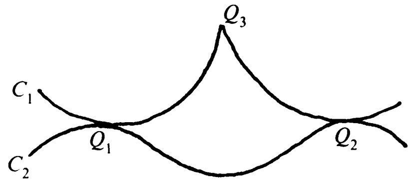
(ii)



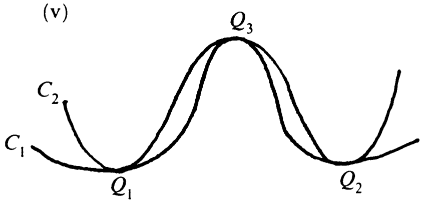
(iii)



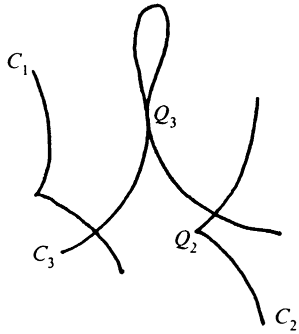
(iv)



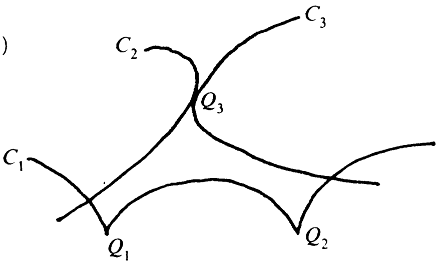
(v)



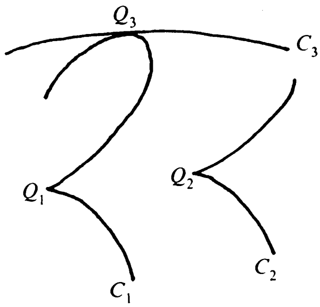
(vi)



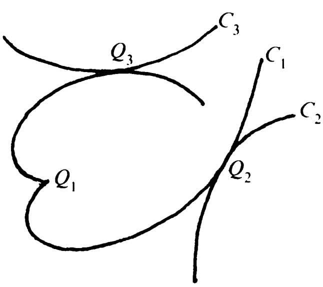
(vii)



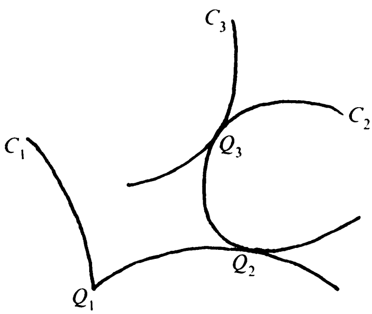
(viii)



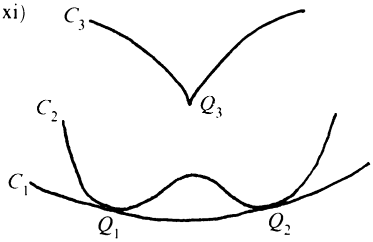
(ix)



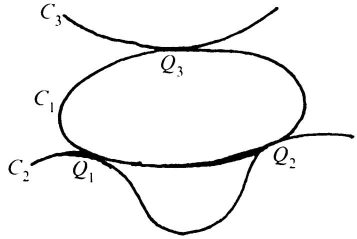
(x)



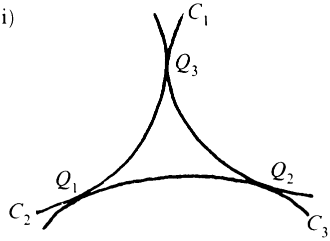
(xi)



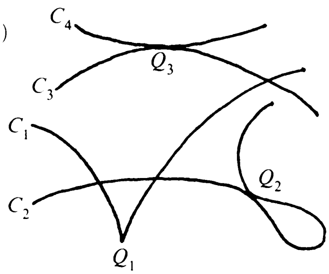
(xii)



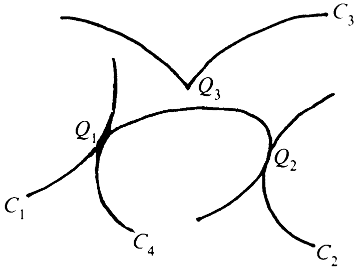
(xiii)



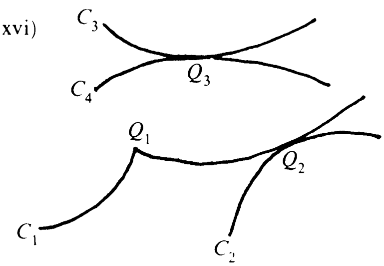
(xiv)



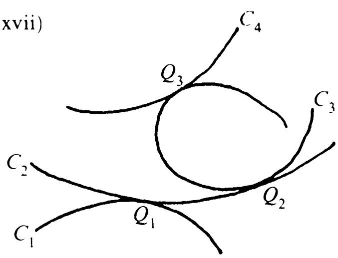
(xv)



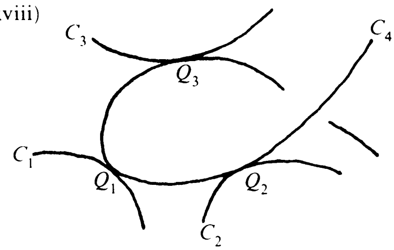
(xvi)



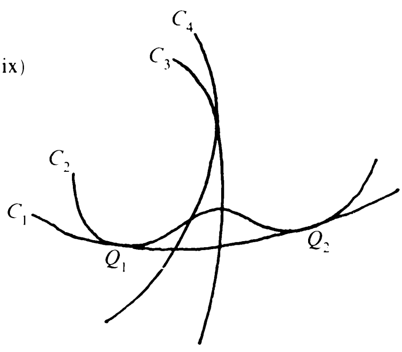
(xvii)



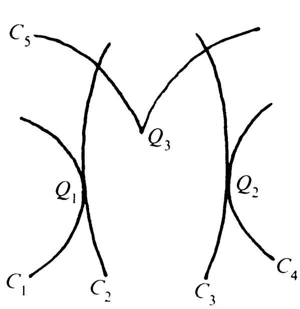
(xviii)



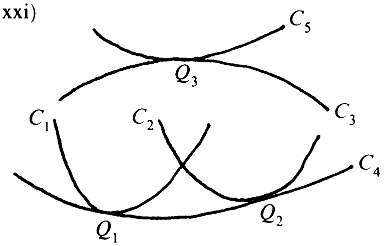
(xix)



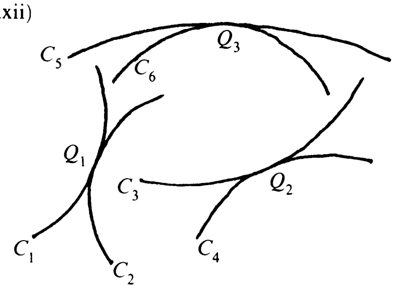
(xx)



(xxi)



(xxii)



Proofs of (ii), (iii), (iv), (viii) and (ix) are similar; we blow up some of Q_i without fixing others and use the existence of an automorphism. We only prove (ii) here. Choose a line T_3 through Q_3 and l points p_i in L. Blow up $\mathbf{P}^2$ at p_i as in (3.9) and twice at Q_3 in the direction of T_3. We now apply (3.3) on this blown up space the anticanonical divisor of which is equivalent to an effective divisor and has an intersection number $3d_j - m_j - 2$ with the proper transform $\widetilde{C}_j$ of C_j, $j = 1$ or 2, to get that C_j (without fixing Q_j) moves in a family of dimension at most $g_j + 3d_j - m_j - 4$ since $\widetilde{C}_j$ is not a nodal curve. Therefore $C = C_1 \cup C_2$ moves in a family of dimension at most $g + 3d - m + l - 6$ by moving a line T_3 and l points p_i in L. Since there is an automorphism which maps L to itself, fixes Q_3 and moves two points ($\neq Q_3$ or $\notin L$) to another two points ($\neq Q_3$ or $\notin L$), the curve C in (ii) moves in a family of dimension at most $g + 3d - m + l - 10$.

Proof of (vii). Consider C_1 as a curve in $V^{d,g}_{m_1,l_1,2}$, $TN^{d,g}_{m_1,l_1,2}(2)$ or in (3.11). Then C_1 moves in a family of dimension at most $g_1 + 3d_1 - m_1 - 7 + l_1$. With each C_1, l_1 ($\leq l$) points $p_i \in L$ are chosen. If $l_1 < l$, choose extra $l - l_1$ points in L. We may assume that $d_2 \leq d_3$ (then d_2 is possibly 1 and $d_2 \geq 2$). Then C_2 as a curve simply through Q_3 with some contact with L moves in a family of dimension at most $g_2 + 3d_2 - m_2 - 2$ and C_3 with the tangent line of C_2 at Q_3 in a family of dimension at most $g_3 + 3d_3 - m_3 - 3$. Therefore C moves in a family of dimension at most $(g+2) + 3d - m - 12 + l_1 + (l - l_1)$.

Proofs of (x), (xv), (xvi), (xvii), and (xviii) are similar too. We only give the proof of (xvi). Consider C_1 as a curve that has a cusp at Q_1 and passes through Q_2. Then applying (3.3) on the blow-up of S at Q_1 as in (3.8.1) and once at Q_2 we get that C_1 moves in a family of dimension at most $g_1 + 3d_1 - m_1 - 5$; with the tangent line of each C_1 at Q_2 we see that C_2 moves in a family of dimension at most $g_2 + 3d_2 - m_2 - 3$ (again if C_2 is a line then it is the tangent line of each C_1 in general missing preassigned l points in L); similarly C_3 moves in a family of dimension at most $g_3 + 3d_3 - m_3 - 2$ and C_4 in a family of dimension at most $g_4 + 3d_4 - m_4 - 3$.

Proofs of (xiv), (xx), (xxi) and (xxii): with preassigned l points in L we take care of each C_j in appropriate order. We here prove (xiv). By (3.9) or (3.10), C_j, ($j = 1$ or 2, moves in a family of dimension at most $g_j + 3d_j - m_j - 4$.

For C_3 and C_4 we do exactly the same thing as we have done for C_2 and C_3 in case (vii).

Proof of (xiii). Assume $d_1 \le d_2 \le d_3$. If C_1 is a line through Q_2 and Q_3, then d_2 and d_3 are at least 2. Blowing up S twice at Q_2 (at Q_3 resp.) and once at Q_1, C_3 (C_2 resp.) moves in a family of dimension at most $g_3 + 3d_3 - m_3 - 4$ ($g_2 + 3d_2 - m_2 - 4$ resp.). Since we may assume that C_1 misses preassigned p_i C moves in a family of dimension at most $(g+2) + 3(d-1) - m - 8$. (In fact we can get dimension bound as $g + 3d - m - 10$ unless both C_2 and C_3 are conics.)

In general we may assume that $2 \le d_1 \le d_2 \le d_3$. Then C_1 as a curve through Q_2 and Q_3 moves in a family of dimension at most $g_1 + 3d_1 - m_1 - 3$. Now choose the tangent lines T_j of C_1 at Q_j, $j = 2, 3$. Then C_2 (C_3 resp.) moves in a family of dimension at most $g_2 + 3d_2 - m_2 - 4$ ($g_3 + 3d_2 - m_3 - 4$ resp.): blow up S twice at Q_3 (Q_2 resp.) in the direction of T_3 (T_2 resp.) and once at Q_1. Again we can have $g + 3d - m - 10$ unless all three are conics.

Proofs of the remaining cases (v), (xi), (xii) and (xix). These are the cases in which two components of C meet tangentially at least at two points. The trouble only happens when both curves are conics ((xi), (xii), (xiii)) or when one is a conic and the other is a cubic (v); otherwise we can prove similarly as case (xvi). We here give the proof of (xi), but the proofs of the others are very similar.

Proof of (xi). Choose l points p_i in L and let S be the blow-up of $\mathbf{P}^2$ at p_i as in (3.9) or (3.10). If C_1 is a line then C_2 has degree at least 4. So C_2 moves in a family of curves of dimension at most $g_2 + 3d_2 - m_2 - 5$: apply (3.3) to the blow-up of S twice both at Q_1 and at Q_2 in the direction of the line through Q_1 and Q_2. Since C_3 has one cusp or one tacnode, by (3.9) or (3.10) C_3 moves in a family of curves of dimension at most $g_3 + 3d_3 - m_3 - 4$, and thus C in a family of dimension at most $g + 3d - m - 10$ with l points in L fixed.

In general assume that $2 \le d_1 \le d_2$. As a curve through Q_1 and Q_2 with $\sum_i (C_1 \cdot L)_{p_i} = m_i$, C_1 moves in a family of dimension at most $g_1 + 3d_1 - m_1 - 3$: apply (3.3) to the blow-up of S at Q_1 and Q_2. Now fix only one tangent line T_1 of C_1 at Q_1. Then C_2, as a curve tangent to T_1 and through Q_2, moves in a family of dimension at most $g_2 + 3d_2 - m_2 - 4$ (in fact, $g_2 + 3d_2 - m_2 - 5$ unless $d_2 = m_2 = 2$), therefore C in a family of dimension at most $g + 3d - m - 9$ with l points in L fixed. Q.E.D.

The dimension of the locus we will need finally in the next section is the following: let $T^{d,g}_{m,l,\kappa}$ be the locus of reduced plane curves C in $U^{d,g}$ which have an n-tuple point at Q_1 and singular points at the remaining Q_i, do not contain the line L, and satisfy $\sum_i (C \cdot L)_{p_i} = m$ for some collection of l points $p_1, \dots, p_l$ in L. Then

(3.13) **Lemma.** (1) (a) $\dim T^{d,g}_{m,l,\kappa} \le g + 3d - m - 2\kappa - n + l + 1$.

(b) *Let W be a component of $T^{d,g}_{m,l,\kappa}$ whose members C have a singular branch at each Q_i for $i \geq 2$, then*

$$\dim W \leq g + 3d - m - n - 3(\kappa - 1) + l - 1 .$$

In particular, if $n = 3$, $\dim W \leq g + 3d - m - 3\kappa + l - 1$.

(2) *There is a corresponding parametric version.*

Proof. We may assume that C does not contain lines through two of Q_i; in this case the lines are fixed, so we may assume that these lines would miss the preassigned p_i or that some p_i are fixed by these lines among l points in L. Choose l points in L. Let S be the blow-up of $\mathbf{P}^2$ at p_i as in (3.9). Let S_3 be the blow-up of S at Q_1, Q_2 and Q_3. Then the proper transform of any component of C has positive intersection number with the anticanonical divisor of S_3. Applying (3.3), we get $\dim T^{d,g}_{m,l,\kappa} \leq g + 3d - m - 2\kappa - n + 1$ since $-K_3 \cdot \widetilde{C} = 3d - m - n - 2(\kappa - 1)$, $\widetilde{C}$ the proper transform of C in S_3.

To show (b), we blow up S only at Q_1 without fixing the remaining Q_i. Let S_1 be the blow-up of S at Q_1. Then it can be easily checked that any component of the proper transform of C satisfies the condition of (3.6). Applying (3.6) to each component and adding each possible dimension, we conclude that C, without fixing Q_i for $i \geq 2$, moves in a family of dimension at most $g + 3d - m - n - (\kappa - 1) - 1$ since $-K_{S_1} \cdot \widetilde{C} = 3d - m - n$ and $\overline{m}(\widetilde{C}) = \kappa - 1$, $\widetilde{C}$ the proper transform of C in S_1. Moving first l points in L and using an automorphism of $\mathbf{P}^2$, we have $\dim W \leq g + 3d - m - n - 3(\kappa - 1) + l - 1$. □

4. Proof of Theorem (1.2)

Due to Theorems (2.1), (2.2) and Remark (2.3), Theorem (1.2) will follow once we show that any component W of $V(d, \delta, \kappa)$ contains in its closure a component of $V(d, \frac{1}{2}(d-1)(d-2) - \kappa, \kappa)$.

For this purpose we first show the following proposition asserting the existence of certain degenerations in the families $V_{m,\kappa}(d, \delta, \kappa)$ when $\kappa \leq 3$, where $V_{m,\kappa}(d, \delta, \kappa) = V(d, \delta, \kappa) \cap V^{d,g}_{m,\kappa}$, i.e., an open dense set of $V^{d,g}_{m,\kappa}$.

(4.1) **Proposition.** *Let W be a component of $V_{m,\kappa}(d, \delta, \kappa)$. Then W contains in its closure either*

(1) *a component of $V_{m+1,\kappa}(d, \delta, \kappa)$; or*

(2) *a component of $U^{d,g-1}_{m,\kappa}$ whose general member is a curve C that either*

  (a) *is smooth at its point of intersection multiplicity m with L, and has at most two irreducible components; or*

  (b) *(if $m \geq 2$) has exactly two branches at its unique point p of intersection multiplicity m with L, with each of its (at most two) irreducible components containing p.*

Proof. Let X be the intersection of the closure of W with $g+3d-3\kappa-m-1$ general hypersurfaces in the space $\mathbf{P}^N$ of curves of degree d. Applying the semistable reduction theorem we may make a series of finite base changes $B \xrightarrow{\alpha} X$ and blow-ups and blow-downs to arrive at a 1-parameter family $\pi: \Xi \to B$ of nodal curves of arithmetic genus g, with smooth total space Ξ, a map $\eta: \Xi \to \mathbf{P}^2$ mapping the fibers $X_b = \pi^{-1}(b)$ of π to the corresponding plane curves $\alpha(b) \in X$, and Ξ minimal with respect to these properties; i.e., there are no rational components of fibers of π meeting the rest of the fibers in only one point and on which η is constant. If $m > 1$, for general $b \in B$, then $\eta^*(L)|X_b$ will have single point p_b of multiplicity m; by the assumption that Ξ is smooth, these points will extend to a section Γ of π. If $m = 1$, we can (after possibly further base changes, blowing-ups and blowing-downs) similarly arrive at a section Γ of π meeting the general fiber X_b in a point of $\eta^{-1}(L)$. There are also κ sections Γ_i of π for $1 \le i \le \kappa$: for general $b \in B$, X_b will have single point q_{b_i} which maps to Q_i and again these points will extend to a section γ_i of π since Ξ is smooth.

Case (i). All the fibers of π are smooth.

In this case we claim that W must contain in its closure a component of $V_{m+1,\kappa}(d, \delta, \kappa)$ of genus g. To see this, write the divisor η^*L as $\eta^*L = m\Gamma + \Gamma'$. Note that we cannot have $\Gamma' = 0$, since η^*L has positive self intersection and Γ^2 nonpositive (cf. [H4]). On the other hand η^*L must be connected [Ha, Chapter 3, Exercise 11.3], so $\Gamma \cap \Gamma' \neq \emptyset$. Now, for any $b_0 \in \pi(\Gamma \cap \Gamma')$, we have $\mathrm{mult}_{p_{b_0}}(\eta^*L|_{X_{b_0}}) \ge m+1$. Thus the map $\eta|X_{b_0}$ will have contact of order at least $m+1$ with L at p_{b_0}. (See [H1] for the above.)

Also each Γ_i meets X_{b_0} at one point q_i because all the fibers are smooth. Since η is regular, $C_0 = \eta(X_{b_0})$ has at least one branch with multiplicity at least 2 at $Q_i = \eta(q_i)$. So, $\overline{m}_{Q_i}(C_0) \ge 1$. Therefore C_0 moves in a family $V^{d,g}_{m+1,\kappa}$ of dimension $g + 3d - 3\kappa - (m+1)$ and we conclude from Lemma (3.8) that C_0 is a general member of a component of $V^{d,g}_{m+1,\kappa}$ which is a nice curve with cusps at Q_i and has a contact of order $m+1$ with L at a smooth point of C_0.

Case (ii). Some fiber X_0 of π is singular.

Let $\mathscr{L}$ be the pullback line bundle $\eta^*\mathscr{O}_{\mathbf{P}^2}(1)$. Let $p_0 \in X_0$ be the point of intersection $X_0 \cap \Gamma$ (Note that p_0 is a smooth point since Ξ is smooth.) Let $Y_0' \subset X_0$ be the connected component of $\eta^{-1}(L) \cap X_0$ containing p_0; this will either be p_0 itself or a union of irreducible components of X_0. Let Y_0 be the closure of the complement of Y_0' in X_0 and write $Y_0' \cap Y_0 = \{p_1, \dots, p_k\}$ (so that, in particular, if $Y_0' = \{p_0\}$, then $k = 1$ and $p_1 = p_0$). Let a_i be the multiplicity $\mathrm{mult}_{p_i}(\eta^{-1}(L)|_{Y_0})$ with which p_i appears in the divisor $\eta^{-1}(L)$

restricted to Y_0 and let $n = \sum_{i=1}^{k} a_i$. Write

$$\deg(\mathscr{L}|_{Y_0'}) = \alpha, \qquad \deg(\mathscr{L}|_{Y_0}) = d - \alpha,$$

and let β be the sum of the degrees of $\mathscr{L}$ on the components of Y_0 in $\eta^{-1}(L)$. Then from [H1], we know $g(y_0) \leq p_a(y_0) \leq g-k+1$ with equality holding only if Y_0 is smooth and Y_0' is of arithmetic genus zero. Secondly $n \geq \eta^{-1}(L)\cdot Y_0 \geq m - \alpha$.

In addition to the above we consider $\eta^{-1}(Q_i)$ or Γ_i. Since $(\Gamma_i \cdot X_0) = 1$, each Γ_i meets Y_0 at a smooth point q_i away from components of $\eta^{-1}(L)$. Let Y_i' be the connected component of $\eta^{-1}(Q_i) \cap X_0$ containing q_i.

Case (iia). Assume $Y_i' = \{q_i\} = \Gamma_i \cap X_0$ for all i. Then as in the case (i), $\overline{m}_{Q_i}(C_0) \geq 1$.

Let Y be the disjoint union of the normalizations of the components of Y_0 not contained in $\eta^{-1}(L)$ on which η is nonconstant. Write $\eta\colon Y \to \mathbf{P}^2$ for the composite $Y \to Y_0 \to \mathbf{P}^2$. The image of η is of degree $d - \alpha - \beta$ and has k points with a total order of contact with L at least $m - \alpha$ and $\overline{m}_{Q_i}(\eta(Y)) \geq 1$ since $Q_i \notin L$. Thus, by Lemma (3.9), map $\eta|_Y$ moves in a family of dimension at most $3(d-\alpha-\beta)+(g-k+1)-3\kappa-(m-\alpha)+k-1$ which we expect is not less than $3d+g-3\kappa-m-1$. Thus $-2\alpha-3\beta \geq -1$, which implies that $\alpha = \beta = 0$.

Having $\alpha = 0$, all of Y_0' must map to the point $\eta(p_0) = p$ and, from the minimality of Ξ, Y_0' is not of arithmetic genus zero when $k = 1$. Thus $\eta(Y)$ moves in a family $U_{m,\kappa}^{d,g_1}$ of dimension at most $3d + g_1 - 3k - m$ where $g_1 = \text{genus}(Y_0) \leq \min(g-1, g-k+1)$. By dimension count of a family of $\eta(Y)$ again by Lemma (3.8) we have

$$(*) \qquad 3d + g_1 - 3\kappa - m \geq 3d + g - 3\kappa - m - 1.$$

Therefore $g_1 = g-1$ and $k \leq 2$. Having an equality in $(*)$, $\eta(Y)$ is a general member of a component of $U_{m,\kappa}^{d,g-1}$ having intersection number m with L at a point p. If $k=1$, p is a smooth point of $\eta(Y)$ and $\eta(Y)$ is simply a curve of degree d with $\delta+1$ nodes and κ cusps at Q_i as its only singularities; if $k=2$, $\eta(Y)$ has a double point at p consisting of two smooth branches with contact of order a, κ cusps at Q_i, $\delta+1-a$ nodes elsewhere, and no other singularities. In any case $\eta(Y)$ has at most two components otherwise X_0 would drop its genus at least by 2. In particular Y_0 is smooth, so if Y_0 has two components then they are disjoint but meet Y_0' since X_0 is connected. Both components of $\eta(Y)$ thus go through p.

Case (iib). X_0 is singular and $Y_i' \supsetneqq \{q_i\}$ for some i.

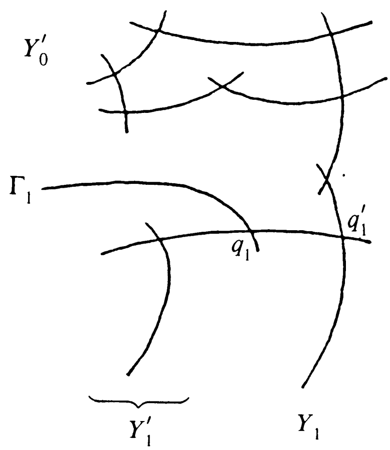
Say $i = 1$. As we did in the analysis of $\eta^{-1}(L)$, let Y_1' be the connected component of $\eta^{-1}(Q_1) \cap Y_0$ containing q_1; since we are assuming $Y_1' \supsetneqq \{q_1\}$, this is a union of irreducible components of X_0. Let Y_1 be the closure of the complement of Y_1' in Y_0. Say $Y_1' \cap Y_1 = \{q_1^1, \dots, q_1^{n_1}\}$. Then

$$g(Y_1) \le p_a(Y_1) = p_a(Y_0) - p_a(Y_1') - n_1 + 1 \le p_a(Y_0) - n_1 + 1$$

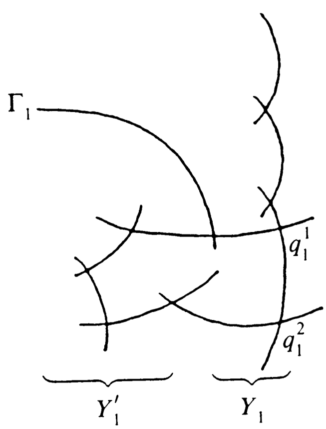
equality only if Y_1 is smooth and Y_1' is rational. Note that Y_1' is not rational when $n_1 = 1$ due to the minimality of Ξ. So,

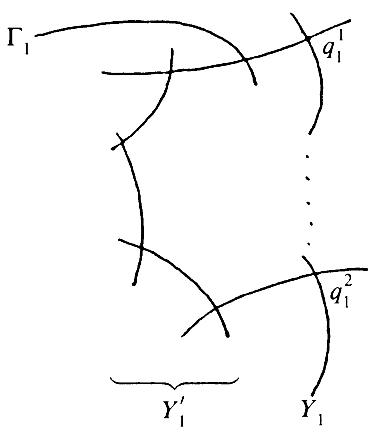
$$(**) \qquad g(Y_1) \le \min\{p_a(Y_0) - 1, p_a(Y_0) - n_1 + 1\} \le g - k - n_1 + 2.$$

(1) Assume $n_1 = 1$. Then Y_1' is not rational because of the minimality of Ξ. The picture of X_0 in this case looks like the following: We therefore see that there is a branch of $C_0 = \eta(X_0)$ through Q_1 on which $\overline{m}_{Q_1} \ge 1$: look at a small neighborhood Z of q_1^1, then $\mathrm{mult}_{\eta(q_1^1)}(\eta(Z)) \ge 2$, and therefore $\overline{m}'_{Q_1}(\eta(Z)) \ge 1$.



(2) Assume $n_1 = 2$. Then X_0 (near Γ_1) looks like either one of the following two pictures.





Now take a small neighborhood N_i of q_1^i in Y_1 on which q_1^i is the only point of $\eta^{-1}(Q_1)$ respectively. Assume $N_1 \cap N_2 = \varnothing$; if the image of any of N_i is not smooth at Q_1, then $\overline{m}_{Q_1}(\eta(N_i)) \geq 1$; if the images of both N_i are smooth, then they meet at Q_1 with contact at least two since a cusp cannot be degenerated to a node.

(3) Let $n_1 \geq 3$. We consider N_i as before at q_1^i and assume $N_i \cap N_j = \varnothing$ for $i \neq j$. If one of them has a nonsmooth image at Q_1, then $\overline{m}_{Q_i}(C_0) \geq 1$. If all the images of N_i are smooth at Q_1, then either two of them meet with contact at least two at Q_1 or all of them meet transversely. So the only new case we see here is that the map $\eta|_{X_0}$ has a triple point (in general an n_1-tuple point) at Q_1.

If Y_1 is not smooth and $Y_i' \supsetneqq \{q_i\}$ for some $i \geq 2$ in any possibility of case (iib), do the same analysis we have done here; starting with Y_1, getting Y_2', Y_2 and $n_2 = \#\{Y_2' \cap Y_2\}$, and then in Y_2 (if necessary) Y_3', Y_3 and n_3. Then

$$(***)\quad g(Y_2) \leq p_a(Y_2) \leq \min\{p_a(Y_1)-1, p_a(Y_1)-n_2+1) \leq g-k-n_1-n_2+3.$$

Except for the case that at least two of Q_i lie in (2) or (3) of case (iib) and $\eta(X_0)$ admits no singular branches at Q_2 or Q_3, let Y be the disjoint union of the normalizations of the components of Y_1 not contained in $\eta^{-1}(L)$ or $\eta^{-1}(Q_1)$ on which η is not constant. Write $\eta\colon Y \to \mathbf{P}^2$ for the composite $Y \to Y_1 \to \mathbf{P}^2$. Then the map $\eta|_Y$ lies in one component of the closure of the locus described in (3.12)(b) if $n_1 \leq 2$ or in (3.13)(b) if $n_1 \geq 3$ with degree $d-\alpha-\beta$ and genus at most $g-k-n_1+2$, therefore moves in a family of dimension at most

$$(g-k-n_1+2)+3(d-\alpha-\beta)-3\kappa-(m-\alpha)+k-1,$$

which we expect is at least $g+3d-3\kappa-m-1$. Thus $-2\alpha-3\beta \geq n_1-2 \geq -1$ since $n_1 \geq 1$, which implies that $\alpha = \beta = 0$. Having $\alpha = 0$, k points of $Y_0' \cap Y_0$ are mapped to one point $\eta(p_0)$ and, due to the minimality of Ξ, Y_0' is not arithmetic genus zero when $k=1$, so $g(Y_0) \leq \min(g-1, g-k+1)$ and $g(Y_1) \leq \min(g-2, g-k-n_1+2)$ (see $(**)$). Letting g_1 the genus of Y_1 and counting dimension again we get $g_1 \geq g-1$, which is impossible.

In the case we have to consider Y_2, let Y be the disjoint union of the normalization of the components of Y_2 not contained in $\eta^{-1}(L)$ or $\eta^{-1}(Q_i)$ for $i=1,2$, on which η is not constant. Write $\eta\colon Y \to \mathbf{P}^2$ for the composite $Y \to Y_2 \to \mathbf{P}^2$. We may assume $n_1 \geq n_2 \geq 2$. If $n_1 = n_2 = 2$, we can say that $\eta(Y)$ has tacnodes at Q_1 and Q_2, and has a singular point at Q_3 other than a node from our previous analysis. By (3.12)(a), it moves in a family of curves of dimension at most

$$\begin{aligned}(g-k-n_1-n_2+3)+3(d-\alpha-\beta)-3\kappa-(m-\alpha)+k\\ \geq g+3d-3\kappa-m-1.\end{aligned}$$

So

$$-2\alpha - 3\beta \geq n_1 + n_2 - 4 = 0.$$

Thus $\alpha = \beta = 0$ and k points of $Y_0' \cap Y_0$ are mapped to one point $\eta(p_0)$. As before, $g(Y_0) \leq \min(g-1, g-k+1)$ and $g(Y_2) \leq \min(g-3, g-k-n_1-n_2+3)$ (see $(***)$). Letting $g_2 = g(Y_2)$ and counting dimension again, we get $g_2 \geq g-2$, which is a contradiction. If $n_1 \geq 3$ and $n_2 \geq 2$, we may now assume that the map $\eta|_Y$ admits a triple point at Q_1 and a tacnode at Q_2 due to the dimension counts in the last section. Then the map $\eta|_Y$ moves in a family of dimension at most

$$(g-k-n_1-n_2+3) + 3(d-\alpha-\beta) - (m-\alpha) - 2\kappa - 2 + k$$
$$\geq g + 3d - m - 3\kappa - 1.$$

So $-2\alpha - 3\beta \geq n_1 + n_2 - \kappa - 2 \geq 0$ (in fact, it is only worth considering when $\kappa = 3$). Thus $\alpha = \beta = 0$ as in the previous cases. But if $n_1 \geq 3$ and $n_2 \geq 2$, $g_2 = g(Y_2) \leq \min(g-1-n_1-n_2+2,\ g-k-n_1-n_2+3) \leq g-4$. New dimension count shows that $g_2 \geq g-3$.

The only possibilities in case (ii) therefore are the ones that occur in (iia). □

(4.2) To complete the proof, we follow the same argument as Harris used in [H1, §4] for $V^{d,g}$. Let $X \subset \mathbf{P}^N$ be any closed family of plane curves whose general member is reduced and irreducible of geometric genus g having cusps at the fixed points. We have then a rational map $\varphi = \varphi_X : X \to \overline{\mathfrak{M}}_g$ where $\overline{\mathfrak{M}}_g$ is the moduli space of stable curves of genus g. We denote $\Phi = \Phi_X \subset X \times \overline{\mathfrak{M}}_g$ the graph of this map and by φ_1, φ_2 the projections of Φ to X and $\overline{\mathfrak{M}}_g$. Let $\Delta_0 \subset \overline{\mathfrak{M}}_g$ be the closure of the locus of irreducible singular curves. We say that the family X degenerates to Δ_0 if $\varphi(X)$ meets Δ_0, and that X degenerates to Δ_0 in codimension one in cusp preserving manner if the inverse image $\varphi^{-1}(\Delta_0) = \varphi_1(\varphi_2^{-1}(\Delta_0))$ in X of Δ_0 has codimension one in X and a general element of some component of $\varphi^{-1}(\Delta_0)$ has cusps at the limits of cusps of the general member of X.

We observe that since a multiple of the divisor Δ_0 in $\overline{\mathfrak{M}}_g$ is Cartier, $\varphi^{-1}(\Delta_0)$ can fail to be of codimension one in X only if it is empty or if the general fiber of $\varphi^{-1}(\Delta_0)$ over its image in X is positive dimensional. It follows that X degenerates to Δ_0 in codimension one in cusp preservng manner whenever there is an isolated point (C, D) of $\varphi_2^{-1}(\Delta_0) \cap \varphi_1^{-1}(C)$ and C has cusps at the limits of the points where the general member of X has cusps.

One circumstance in which this will be the case is the following: After making a finite base change $\alpha: B \to \Phi$ there will exist a family $\pi: \Xi \to B$ of stable curves and a rational map $\eta: \Xi \to \mathbf{P}^2$ carrying a general fiber $X_b = \pi^{-1}(b)$ of π to the corresponding plane curve. Suppose now that for some point $(C_0, D_0) \in \Phi$ the map $\eta_{|D_0}: D_0 \to C_0$ is defined and birational on every component of D_0 except possibly rational curves containing three or fewer branches of nodes of D_0, having cusps at the limits of cusps of the general member of X—for example, $D_0 \to C_0$ is just a partial normalization with cusps of C_0 desingularized in D_0. Then clearly we cannot vary the isomorphism class of D_0 without varying the curve C_0 as well, so that (C_0, D_0) is an isolated point of the fiber $\varphi_1^{-1}(C_0)$ in Φ. If in addition $D_0 \in \Delta_0$ we will call (C_0, D_0) a cusp preserving good degeneration of the family X; by what we have just said a family that admits a cusp preserving good degeneration will degenerate to Δ_0 in codimension one in cusp preserving manner. Note that if $Y \subset X$ is a subfamily of curves, generically of the same genus g and of having cusps at the same points where the general member of X has them, and Y admits a cusp preserving good degeneration, then X does.

A basic observation is the following:

(4.3) **Proposition.** *Let W be a component of $V_{m,\kappa}^{d,g}$. Then the following are equivalent:*

(a) *W admits a cusp preserving good degeneration;*

(b) *W degenerates to Δ_0 in codimension one in cusp preserving manner; and*

(c) *W contains a component of $U_{m,\kappa}^{d,g-1}$ of the type described in* (4.1)(2), *whose general member is irreducible.*

Proof. Same as in [H1, (4.1)].

Using this equivalence, we may deduce from (4.1) as in [H1, (4.2)] the following:

(4.4) **Proposition.** *For $g \geq 1$, every component of $V_{m,\kappa}^{d,g}$ admits a cusp preserving good degeneration.*

Proof. Same as in [H1, (4.2)] except that all curves (or some components of them) in $\mathbf{P}^2$ occurring in the proof have cusps at the fixed points Q_i and their partial normalizations desingularize cusps. Q.E.D.

Now note that any general member of a component of $U_{1,\kappa}^{d,g-1}$ whose general member is irreducible is in fact a curve in a component W' of $V(d, \delta+1, \kappa)$. Since any component W of $V(d, \delta, \kappa)$ contains in its closure a curve in W', we see that it contains in its closure W' from the fact that any component of $V(d, \delta+1, \kappa)$ is smooth and the local picture of $V(d, \delta, \kappa)$ at the point of $V(d, \delta+1, \kappa)$. By induction on genus any component W of $V(d, \delta, \kappa)$ contains in its closure a component of $V(d, \frac{1}{2}(d-1)(d-2) - \kappa, \kappa)$.

REFERENCES

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of curves*, vol. 1, Springer-Verlag, New York, Heidelberg and Berlin, 1985.

[DH1] S. Diaz and J. Harris, *Ideals associated to deformations of singular plane curves*, Trans. Amer. Math. Soc. **309** (1988), 433–468.

[DH2] ——, *Geometry of the Severi variety*, Trans. Amer. Math. Soc. **309** (1988), 1–34.

[EH] D. Eisenbud and J. Harris, *Divisors on general curves and cuspidal rational curves*, Invent. Math. **74** (1983), 371–418.

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience, New York, 1978.

[H1] J. Harris, *On the Severi problem*, Invent. Math. **84** (1986), 445–461.

[H2] ——, *Curves and their moduli*, Notes of lectures given at the Amer. Math. Soc. Summer Institute, (1986), Preprint.

[H3] ——, *The genus of space curves*, Math. Ann. **249** (1980), 191–204.

[H4] ——, *Families of smooth curves*, Duke Math. J. **51** (1984), 409–419.

[Ha] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, Heidelberg and Berlin, 1977.

[Z1] O. Zariski, *Algebraic surfaces*, Springer-Verlag, Berlin, 1971.

[Z2] ——, *Algebraic system of plane curves*, Amer. J. Math. **104** (1982), 209–226.

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

Current address: Department of Mathematics, Pohang Institute of Science and Technology, Pohang, 790–330, Korea