TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 317, Number 1, January 1990

REPRESENTING SETS OF ORDINALS AS COUNTABLE UNIONS OF SETS IN THE CORE MODEL

MENACHEM MAGIDOR

ABSTRACT. We prove the following theorems.

Theorem 1 ($\neg 0^{\#}$). *Every set of ordinals which is closed under primitive recursive set functions is a countable union of sets in* L.

Theorem 2. (No inner model with an Erdös cardinal, i.e. $\kappa \to (\omega_1)^{<\omega}$.) *For every ordinal* β, *there is in* K *an algebra on* β *with countably many operations such that every subset of* β *closed under the operations of the algebra is a countable union of sets in* K.

0. INTRODUCTION

(See [Je2] for basic notation and terminology.) Jensen's famous covering theroem for L [De-Jen] claims that if $0^{\#}$ does not exist then every set of ordinals X is included in a set of ordinals $Y \in L$ such that $|Y| \leq |X| + \aleph_1$.

In some sense the meaning of this theorem is that if $0^{\#}$ does not exist, then the universe of sets V is not too far from the constructible universe L. We can consider the set Y in the theorem to be an approximation for X from above.

This paper deals with the problem of "approximating the set X by a set in L from below". For instance consider the following problem: *Assume* $0^{\#}$ *does not exist* ($\neg 0^{\#}$) *and* X *is uncountable. Does* X *necessarily contain an uncountable subset which is in* L?

The answer to this problem as stated is obviously "No". Consider a subset of $\aleph_1$, introduced by forcing over L, using finite conditions. The resulting set does not contain any infinite set in L. This example indicates how to modify the statement of the problem. The generic set we obtained by forcing has many "holes" in it. For instance there are many ordinals α such that α belongs to our set but $\alpha + 1$ does not, etc., and if we close the set under the two simple functions $\alpha \to \alpha + 1$ and $\alpha \to \alpha - 1$ if α is successor and α otherwise, then the resulting set is itself in L because it is the ordinal ω_1.

It seems that the problem should be modified by assuming that the given set is closed under a collection of simple enough functions. In this case it turns out

Received by the editors February 9, 1987 and, in revised form, May 16, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03E45; Secondary 03E35.

that the answer to the problem may be "yes" and we get:

Theorem 1.1 ($\neg 0^{\#}$). *Let X be a set of ordinals closed under primitive recursive set functions. (Namely whenever $\alpha_1, \ldots, \alpha_n \in X$ and f is a primitive recursive set function, then, if $f(\alpha_1, \ldots, \alpha_n)$ is an ordinal, $f(\alpha_1, \ldots, \alpha_n) \in X$.) Then X is a countable union of sets in L.*

It follows that if X is uncountable, it contains an uncoutable set in L. Actually it will be clear from the proof of Theorem 1.1 that one could relax the closure condition for X by assuming "X is closed under any set function obtained by composition from rudimentary functions and the function $\alpha \to L_\alpha$".

Jensen's covering theorem was expanded by the joint work of Dodd and Jensen. In [Do-Jen] they defined a very natural inner model, which they called "the core model" or alias K. The structure theory as well as the combinatorial properties of L are valid also in K, which, in the absence of $0^{\#}$, is L. Most important the following covering theorem holds for K (see [Do]): Assume there is no inner model with a measurable cardinal, then every set of ordinals X is included in a set Y, such that $Y \in K$ and $|Y| \leq |X| + \aleph_1$. Thus the assumption $\neg 0^{\#}$ is replaced by the weaker assumption "No inner model with a measurable cardinal". One would expect Theorem 1.1 to generalize when we replace L by K and $\neg 0^{\#}$ by "No inner model with a measurable cardinal".

The first observation concerning this possible generalization of Theorem 1.1 is that it does not hold if we stick with the assumption "X is closed under primitive recursive set functions". Assume that $0^{\#}$ exists, but still there is no inner model with a measurable cardinal. We always have $0^{\#} \in K$ (if $0^{\#}$ exists). By forcing over K we introduce a subset of ω_1 with no infinite subset which is in K. Let a be this set. Since $0^{\#} \in K$ we can enumerate the canonical indiscernibles for L in K. Let $\langle \alpha_i | i < \omega_1 \rangle \in K$ be an enumeration of the first ω_1 of them. Let B be $\{\alpha_i | i \in a\}$ and $\overline{B}$ the closure of B under primitive recursive set funtions. Since the primitive recursive set functions are absolute between V and L, and since $C = \{\alpha_i | i < \omega_1\}$ is a set of indiscernibles for L it is clear that $\overline{B} \cap C = B$. $\overline{B}$ cannot be the union of countably many sets in K, because otherwise the set $\overline{B} \cap C$ will contain an uncountable set in K. Hence $\{i | \alpha_i \in B\}$ contain an infinite (actually uncountable) set in K. This last set is a. Thus we derived a contradiction from the assumption that Theorem 1.1 generalizes to K, verbatim.

It seems that if any generalization of Theorem 1.1 is possible it must use a wider class of functions. But no matter what family of functions one uses the following argument shows that the assumption "No inner model with a measurable cardinal" must be strengthened. Assume there is an Erdös cardinal (A cardinal κ such that $\kappa \to (\omega_1)^{<\omega}$). By a theorem of Jensen (see [Don-Jen-Ko]) if κ is Erdös in V it is Erdös in K. Now one can repeat the argument above to show that if one forces over K and introduces a subset of ω_1 by using

finite conditions, then for every family of less than κ functions, (which may belong to the generic extension) one can find a set closed under these functions, which is not a countable union of sets in K. (Consider in K the structure $\mathscr{A} = \langle \kappa, \mathscr{P}, \models, \tau_1, \ldots \tau_p \ldots \rangle$, where $\mathscr{P}$ is the forcing notion and τ_i are terms whose realizations form the given sequence of functions. Let $\langle \alpha_i | i < \omega_1 \rangle$ be a sequence of indiscernibles for the structure $\mathscr{A}$. Let $a \subseteq \omega_1$ be the subset of ω_1 introduced by forcing with $\mathscr{P}$. Then the closure of $\langle \alpha_i | i \in a \rangle$ is the required set.)

It turns out that the existence of an Erdös cardinal is the right assumption for pushing the argument above, because we have:

Theorem 2.2. *Assume there is no Erdös cardinal in K. Then for every ordinal β one can define in K a countable collection of functions on β such that every subset of β closed under the given family of functions is a countable union of sets in K.*

The study of these problems was intiated by a work of Baumgartner. Baumgartner considered the problem of the size of a closed unbounded subset of $P_\kappa(\lambda)$ where $\lambda > \kappa$ (see [Je1] for definition and basic facts). Baumgartner [Ba] proved that every closed unbounded subset of $P_{\omega_1}(\lambda)$, where $\lambda > \omega_1$, is always of the maximal possible size, namely λ^ω. The next problem is the size of clubs (= closed unbounded subsets) in $P_{\omega_2}(\lambda)$. Starting from a model with an Erdös cardinal he was able to get a model of Set Theory in which the size of every club in $P_{\omega_2}(\lambda)$ was the maximal possible, namely λ^{ω_1}, without this bound being trivial; namely he can have cases for which $\lambda^\omega < \lambda^{\omega_1}$ (for instance $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1}$ is large). He asked whether the large cardinal assumption he was using was necessary.

Theorem 2.2 implies that Baumgartner was using exactly the right assumption. Because if there is no inner model with Erdös cardinal, then by Theorem 2.2 we have a club in $P_{\omega_2}(\lambda)$, such that every member of it is a countable union of sets in K. Hence, we have a club of size at most λ^ω (note that Baumgartner's proof for $P_{\omega_1}(\lambda)$ actually shows that for $P_{\omega_2}(\lambda)$ the size must be at least λ^ω). Baumgartner's model gives an example of a model in which Theorem 2.2 fails for $\beta = \aleph_3$.

An obvious generalization of Theorem 2.2 is obtained by replacing the assumption

$$K \models \text{ There is no } \kappa \text{ such that } \kappa \to (\omega_1)^{<\omega}$$

by (for instance)

$$K \models \text{ There is no } \kappa \text{ such that } \kappa \to (\omega_2)^{<\omega}.$$

The reader should have no difficulty in verifying that the conclusion is now weakened to "Every subset of β closed under the given family of functions is the union of $\aleph_1$ sets in K". For some β's we can prove a better theorem:

Theorem 3.1. *Assume there is no inner model with a measurable cardinal. Let $\beta < \aleph_{\omega_2}$. Then (in V) one can define a countable set of functions such that every subset of β closed under these functions is a union of $\aleph_1$ sets in K.*

We have to make a few remarks about the proofs. The Jensen and the Dodd-Jensen covering theorems were proved using the fine structure theory developed by Jensen ([Jen], see also [De]) for the constructible universe and generalized to K by Dodd and Jensen [Do-Jen]. Silver developed an alternative approach using what came to be known as the Silver machines. In 1978 Silver and the author independently found a proof of the covering theorem for L, which uses neither the fine structure theory nor the Silver machines. Silver extended this approach to the core model and the Dodd-Jensen covering theorem. The proofs of Theroems 1.1, 2.1, 3.1 and 4.1 are presented in terms of these ideas. Since no presentation of the core model theory along these lines was published yet, we inserted a special section at the end of the paper, called §A, which summarizes the basic definitions and facts about K. The reader may note that the proofs of Theroems 1.1 and 2.1 can be modified to give a fine structure free proof of the Jensen and the Dodd-Jensen covering theorems respectively. (We hope to publish those proofs sometime.) Also, see the remarks at the end of §1. The basic definitions of notions like "iterable structure", "mouse", etc. are little different than the usual definition, so we recommend that the reader look at §A.

Our notation and terminology are standard. We are going to denote cardinals like $\aleph_1, \aleph_2, \ldots$ etc. interchangeably by $\aleph_1, \aleph_2, \ldots$ and $\omega_1, \omega_2, \ldots$ etc., when usually we use the alephs when we want to stress the "cardinal character" of the ordinal, and the omegas when we want to stress its "ordinal character".

Unless otherwise stated, when we write $\aleph_1, \omega_1, P(x)$ (the power set of x), $|Y|$ (the cardinality of Y), etc, we refer to these notions in the sense of the whole universe V. We shall use a superscript when we want to relativize a notion to a particular model. Thus $H^K(\lambda)$ is $H(\lambda)$ (the set of all sets hereditarily of cardinality less than λ) in the sense of K.

We say that a subset of an ordinal β, X, is closed under the function g if X is closed under $g\lceil\beta$, where $g\lceil\beta$ is defined as $g(x_1, \ldots, x_n)$ if $g(x_1, \ldots, x_n) \in \beta$ and 0 otherwise. If A is a set then $[A]^n$ is the set of all subsets of A of cardinality n. If A is a set of ordinals, we can alternatively consider $[A]^n$ to be the set of all increasing sequences of members of A of length n. $[A]^{<\omega}$ is defined to be $\bigcup_{n<\omega}[A]^n \cdot \kappa \to (\lambda)^{<\omega}$ means that for every function $f : [\kappa]^{<\omega} \to 2$ $\exists A \subseteq \kappa$, the order type of A is λ and for all $n < \omega$ f is constant on $[A]^n$. If $\kappa \to (\omega_1)^{<\omega}$ we say that κ is an Erdös cardinal. $L_\alpha[A]$ is the usual hierarchy of constructible sets from a predicate A. (The usual L_α hierarchy is $L_\alpha[\phi]$.) The main fact we shall need about the $L_\alpha[A]$'s is Lemma 1.2, namely that the usual well-ordering is (uniformly) Σ_1 in the structure $\langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$. (Note that we assume that our formulas can mention A.) Σ_ω is the union of the Σ_n's.

For each Σ_n formula $\Phi(x, x_1, \ldots, x_n)$ we introduce the canonical Skolem term $\tau_\Phi(x_1, \ldots, x_n)$, which for a structure with a definable well-ordering denotes the canonical Skolem function h_Φ, where

$$h_\Phi(x_1, \ldots, x_n) = \text{The minimal } x \text{ (in a distinguised well-ordering) such that } \Phi(x, x_1, \ldots, x_n) \text{ holds if such } x \text{ exist and } 0 \text{ otherwise.}$$

(Note that h_Φ is usually not a Σ_n-function.) A Σ_n Skolem term is obtained from canonical Skolem terms by composition. (We usually confuse a Skolem term and the function it denotes in a given structure. Whenever it will be important to distinguish the structure in which the given term is interpreted we shall use superscripts. Thus $\tau^{\mathscr{A}}$ is the function denoted by τ in the structure $\mathscr{A}$.)

Let $\mathscr{A} = \langle A, \in, \ldots \rangle$ and $\mathscr{B} = \langle B, \in, \ldots \rangle$ be two structures. $j : \mathscr{A} \to \mathscr{B}$ is a Σ_n embedding $(0 \leq n \leq \omega)$ if for every Σ_n formula Φ and $x_1, \ldots, x_e \in \mathscr{A}$

$$\mathscr{A} \models \Phi(x_1, \ldots, x_e) \quad \text{iff} \quad \mathscr{B} \models \Phi(j(x_1), \ldots, j(x_e)).$$

A trivial but key observation is:

Lemma 0.1. *Let $\mathscr{A} = \langle A, \in, \ldots \rangle$ and $\mathscr{B} = \langle B, \in, \ldots \rangle$ be two structures, having a Σ_1 well-ordering defined by the same formula. Let $j : \mathscr{A} \to \mathscr{B}$ be a Σ_n embedding $(1 \leq n \leq \omega)$. Then for every Σ_n Skolem term τ and $x_1, \ldots, x_e \in A$*

$$\tau^{\mathscr{B}}(j(x_1), \ldots, j(x_e)) = j(\tau^{\mathscr{A}}(x_1, \ldots, x_e))$$

("*Σ_n Skolem functions commutes with Σ_n embeddings*").

Proof. It is enough to prove the lemma for canonical Skolem terms because it will then follow, for a general Skolem term by induction on a sequence of Skolem terms, constructing the given term by composition from canonical terms.

For a Σ_n formula $\Phi(x, x_1, \ldots, x_n)$, $h_\Phi^{\mathscr{A}}$ is definable in $\mathscr{A}$ by the following Boolean combination of Σ_n formulas:

$$\begin{aligned} h_\Phi^{\mathscr{A}}(x_1, \ldots, x_e) = y \quad \text{iff} \quad & \exists x \Phi(x, x_1, \ldots, x_e) \to \Phi(y, x_1, \ldots, x_e) \\ & \wedge \forall z (z < y \to \neg \Phi(z, x_1, \ldots, x_e)) \\ & \wedge \forall x \neg \Phi(x, x_1, \ldots, x_e) \to y = 0. \end{aligned}$$

Since j preserves Σ_n formulas it preserves any Boolean combination of them. □

It follows from Lemma 0.1 that if $\mathscr{A}$ is a Σ_n elementary substructure of $\mathscr{B}$ (namely the identity map is a Σ_n embedding) and $1 \leq n \leq \omega$, then $\mathscr{A}$ is closed under the Σ_n Skolem functions of $\mathscr{B}$. Hence given $A \subseteq B$ there exists the minimal Σ_n substructure of $\mathscr{B}$ containing A, which is the closure of A under the Σ_n Skolem functions of $\mathscr{B}$. We shall denote this closure by $H_n^{\mathscr{B}}(A)$. We shall drop the superscript $\mathscr{B}$ whenever it is understood from the context.

Lemma 0.1, while being a very simple observation, is the key idea in eliminating the fine structure from our arguments. (The uniformization theorem of [Jen] and the closure under Σ_n functions is replaced by closure under Σ_n Skolem functions, which as we observed before are usually not Σ_n definable.)

1. Every prim-recursive closed set is a countable union of constructible sets

In this section we prove

Theorem 1.1 $(\neg 0^{\#})$. *Every set of ordinals which is closed under primitive recursive set functions is a countable union of constructible sets.*

Before we start the proof of Theorem 1.1 we need a few basic facts about the L_α hierarchy. These facts are well known, though at least for Lemmas 1.3 and 1.4 we could not find a direct reference. For future application we have to state these facts for $L_\alpha[A]$ hierarchy, where A is any predicate. The cases of Lemmas 1.2, 1.3 and 1.4 used in this section are obtained by taking $A = \varnothing$.

Lemma 1.2. *The canonical well-ordering of $L[A]$ restricted to $L_\alpha[A]$ is (uniformly in α and A) Σ_1 over the structure $\langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$.*

This lemma for the L_α hierarchy is Lemma 24 in Chapter 2 of [De]. The proof for $L_\alpha[A]$ is the same.

Lemma 1.3. *If $\langle M, \in, A \cap M\rangle$ is a Σ_1 substructure of $\langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$, then it is isomorphic to a structure of the form $\langle L_\gamma[B], \in, B\rangle$.*

Note. This lemma is well known for limit α. See for instance [De, Chapter 2].

Proof. By trivial modification of [Bo] we can prove the following

Basic Claim. There exists a sentence $\Phi(A)$ such that a transitive structure of the form $\langle B, \in, A\rangle$ is of the form $L_\alpha[A]$ iff $\langle B, \in, A\rangle \models \Phi(A)$.

Let $\langle M, \in, M \cap A\rangle$ be a Σ_1 substructure of $\mathscr{A} = \langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$. Let j be the Σ_1 embedding from the transitive collapse of $\langle M, \in, M \cap A\rangle$, $\mathscr{B} = \langle H, \in, B\rangle$ onto M.

We first note that if $x \in M$ and $x \in L_\gamma[A]$, $\gamma < \alpha$, then $x \in L_{\gamma'}[A]$ where $L_{\gamma'}[A] \in M$ for some $\gamma' < \alpha$. (This follows from the fact that the following Σ_1 formula holds for x in $L_\alpha[A]$: $\exists T(T$ is transitive $\wedge\, x \in T \wedge \langle T, \in, A \cap T\rangle \models \Phi(A)$.) It is also clear from the Basic Claim that if $L_\gamma[A] \in M$, then $j^{-1}(L_\gamma[A])$ has the form $L_\delta[B]$. It follows that if α is limit then H has the form $L_\gamma[B]$ for some γ. If α is nonlimit, say $\alpha = \beta + 1$, then it follows that $j^{-1}(M \cap L_\beta[A])$ is $L_\gamma[B]$ for some γ. Let $x \in H$. $j(x)$ is first order definable over some $\langle L_\rho[A], \in, A \cap L_\rho[A]\rangle$ $(\rho < \alpha)$ using the formula $\psi(z, x_1, \ldots, x_n)$.

We get

$$(*)\qquad \begin{aligned} &\mathscr{A} \models \exists T \exists x_1 \exists x_2, \dots, \exists x_n \\ &\quad (T \text{ is transitive } \wedge x_1 \in T \wedge \cdots \wedge x_n \in T \\ &\quad \wedge \langle T, \in, A \cap T\rangle \models \Phi(A) \wedge j(x) \subseteq T \\ &\quad \wedge \forall y \in T(y \in j(x) \leftrightarrow \langle T, \in, A \cap T\rangle \models \psi(z, x_1, \dots, x_n)). \end{aligned}$$

Since this is a Σ_1 formula, the same formula holds for x in $\mathscr{B}$, hence x is first order definable over some $L_\delta[B]$, such that $L_\delta[B] \in H$. It is easily verified that $\delta \leq \gamma$. We actually proved that $H \subseteq L_{\gamma+1}[B]$.

If $H \subseteq L_\gamma[B]$ then $H = L_\gamma[B]$ and we are finished. If not $\exists x \in H - L_\gamma[B]$. It follows immediately that $j(x) \in L_{\beta+1}[A] - L_\beta[A]$. Hence T, witnessing $(*)$ for x, must be $L_\beta[A]$. It follows that $L_\beta[A] \in M$. Therefore $j^{-1}(L_\beta[A]) = L_\gamma[B]$ but then for every formula $\psi(z, x_1, \dots, x_n)$ and $y_1, \dots, y_n \in L_\gamma[B]$

$$\begin{aligned} \mathscr{A} \models \exists x(x \subseteq L_\beta[A] \wedge \forall y \in L_\beta[A](y \in x \leftrightarrow \langle L_\beta[A], \in, A \cap L_\beta[A]\rangle \\ \models \psi(y, j(y_1), \dots, j(y_n)))). \end{aligned}$$

Since this is a Σ_1 formula

$$\begin{aligned} \mathscr{B} \models \exists x(x \subseteq L_\gamma[B] \wedge \forall y \in L_\gamma[B](y \in x \leftrightarrow \langle L_\gamma[B], \in, B \cap L_\gamma[B]\rangle \\ \models \psi(y, y_1, \dots, y_n))). \end{aligned}$$

The last claim means that $L_{\gamma+1}[B] \subseteq H$; therefore $H = L_{\gamma+1}[B]$. □

Lemma 1.4. *Let $\{\mathscr{A}_i, f_{ij} | i \in I, i \leq j\}$ be a directed system indexed by the partially ordered directed set I, where each $\mathscr{A}_i$ has the form $\langle L_{\beta_i}[A_i], \in, A_i \cap L_{\beta_i}[A_i]\rangle$ and the embeddings f_{ij} $(i \leq j)$ are Σ_0 embeddings. Then the directed limit of the system, if it is well founded, is a structure of the form $\mathscr{B} = \langle L_\gamma[A], \in, A \cap L_\gamma[A]\rangle$.*

Proof. The directed limit of our directed system is clearly isomorphic to some structure of the form $\mathscr{A} = \langle H, \in, A\rangle$, where H is transitive. We have to show that $H = L_\gamma[A]$ for some γ. Let δ be $\sup\{\delta' | L_{\delta'}[A] \in H\}$. Clearly $L_\delta[A] \subseteq H$. Note also that if δ is nonlimit then $L_\delta[A] \in H$. We claim that either $H = L_\delta[A]$ or $H = L_{\delta+1}[A]$. Recall that if $i < j$, then f_{ij} is a Σ_0 embedding of $\mathscr{A}_i$ into $\mathscr{A}_j$. Let $f_{i\infty}$ be the canonical embedding of $\mathscr{A}_i$ into the directed limit of our system $\mathscr{A}$.

We first show $H \subseteq L_{\delta+1}[A]$. Let $x \in H$, hence $x = f_{i\infty}(y)$ for some $i \in I$ and $y \in \mathscr{A}_i$. By the assumption about $\mathscr{A}_i$, $\exists \gamma_i \in \mathscr{A}_i$ such that y is first order definable over $\langle L_{\gamma_i}[A_i], \in, A_i \cap L_{\gamma_i}[A_i]\rangle$ using the formula $\psi(z, z_1, \dots, z_k)$ and the parameters $z_1, \dots, z_k$.

By the Basic Claim of Lemma 1.3 and since $f_{i\infty}$ is a Σ_0 embedding, $f_{i\infty}(L_{\gamma_i}[A_i])$ has the form $L_{\delta'}[A] \in H$ and $x = f_{i\infty}(y)$ is first order definable over $\langle L_{\delta'}[A], \in, A \cap L_{\delta'}[A]\rangle$ using the formula ψ and the parameters $f_{i\infty}(z_1), \dots, f_{i\infty}(z_k)$. Hence $x \in L_{\delta'+1}[A] \subseteq L_{\delta+1}[A]$. We have proved $H \subseteq L_{\delta+1}[A]$.

Now we distinguish two cases.

Case I. For every $i, g \in \mathscr{A}_i$, $\exists k \geq i$, $\exists \gamma \in \mathscr{A}_k$ such that $\gamma < \beta_k$ and $f_{ik}(y) \in L_\gamma[A_k]$.

In this case for every $x \in H$ if $x = f_{i\infty}(y)$ we can assume without loss of generality that $\exists \gamma \in \mathscr{A}_i$ such that $\gamma < \beta_i$ and $y \in L_\gamma[A_i]$ (by replacing i by k). Using the Basic Claim, $f_{i\infty}(L_\gamma[A_i]) = L_{\delta'}[A]$ for some $\delta' \leq \delta$. Hence $x = f_{i\infty}(y) \in L_{\delta'}[A] \subseteq L_\delta[A]$, and we have proved $H \subseteq L_\delta[A]$, which implies $H = L_\delta[A]$.

Case II. For some $x \in H$, $x = f_{i\infty}(y)$, there is *no* k such that $f_{ik}(y) \in L_\gamma[A_k]$ where $\gamma < \beta_k$. It follows that for all $i \leq k$, β_k is a successor ordinal, say $\beta_k = \gamma_k + 1$ and $f_{ik}(y) \in L_{\gamma_k+1}[A_k] - L_{\gamma_k}[A_k]$.

For $i \leq k$ we clearly must have $f_{k\infty}(L_{\gamma_k}[A_k]) = L_{\delta'}[A]$ for some fixed $\delta' \leq \delta$. One could easily verify that we must have $\delta' = \delta$. (Otherwise $\delta' + 1 \leq \delta$, whether δ is successor or limit; we get that $L_{\delta'+1}[A] \in H$. But then for some k' and some $\eta < \beta_{k'}$ which can be assumed to be $\geq k$ we have $f_{k'\infty}(L_\eta[A_{k'}]) = L_{\delta'+1}[A]$. But then clearly $\beta_k > \eta > \gamma_k$, a contradiction.)

Let $x_1, \ldots, x_n \in H$ and ψ be a first order formula. We claim

$$s = \{z \mid \langle L_\delta[A], \in, A \cap L_\delta[A]\rangle \vDash \psi(z, x_1, \ldots, x_n)\} \in H.$$

Let $i \leq k$ be large enough so that for some $y_1, \ldots, y_n \in A_k$ $x_l = f_{k\infty}(y_l)$ for all $1 \leq l \leq n$. Let $t = \{z \mid \langle L_{\gamma_k}[A_k], \in, A_k \cap L_{\gamma_k}[A_k]\rangle \vDash \psi(z, y_1, \ldots, y_n)\}$, $t \in L_{\gamma_k+1}[A_k] = L_{\beta_k}[A_k]$; clearly $f_{k\infty}(t) = s$. Hence $s \in H$. We have shown $L_{\delta+1}[A] \subseteq H$. Hence $H = L_{\delta+1}[A]$. □

Proof of Theorem 1.1. Let X be a subset of $\beta = \sup X$ such that X is closed under primitive recursive set functions. Note that β is a limit ordinal. Using the closure property of X one could easily verify that there exists $X \subseteq Y \subseteq L_\beta$ such that Y is a Σ_1 substructure of $\langle L_\beta, \in\rangle$ and $Y \cap \beta = X$. (Pick $Y = \bigcup_{\gamma \in X} H_\omega^{L_\gamma}(X \cap \gamma)$. Since the ω-Skolem functions for L_γ are really primitive recursive functions if we add γ as an extra variable, we get $Y \cap \beta = X$. Y is easily verified to be a Σ_1 substructure of L_β, since X is cofinal in L_β.) By Lemma 1.2 let j be an isomorphism between $\langle L_\alpha, \in\rangle$ and Y for some α. Note that $j''\alpha = X$. For technical covenience we also set $j(\alpha) = \beta$.

Now we prove by induction on $\gamma \in X \cup \{\beta\}$ that $X \cap \gamma$ is a countable union of constructible sets. If γ is a successor ordinal, say $\gamma = \delta + 1$, then by the closure property of X, $\delta \in X$. By the induction assumption, $X \cap \delta$ is a countable union of constructible sets, therefore $X \cap \gamma = X \cap \delta \cup \{\delta\}$ is such a union. If γ is a limit ordinal such that the cofinality of the order type of $X \cap \gamma$ (which is $j^{-1}(\gamma)$) is ω then again the claim follows easily from the induction assumption. So the interesting case is when $\eta = j^{-1}(\gamma)$, which is the order type of $X \cap \gamma$, has cofinality $> \omega$. Note that $j \restriction L_\eta$ is a Σ_0 embedding of $\langle L_\eta, \in\rangle$ into $\langle L_\gamma, \in\rangle$ which we can assume to be different than the identity, otherwise $X \cap \gamma = \eta$ and the claim for γ is trivial since $X \cap \gamma \in L$.

We introduce the crucial definition:

Definition. (a) η decomposes at ρ $(\rho \geq \eta)$ if $H_\omega^{L_\rho}(A \cup \mu) \supseteq \eta$ for some $\mu < \eta$ and some countable subset $A \subseteq L_\rho$. ("The L_ρ Skolem hull of $A \cup \mu$ covers η".) Note that we are not assuming that $A \in L$.

(b) If η decomposes at some ρ, let $\rho(\eta)$ be the minimal ρ such that η decomposes at ρ and let $n(\eta)$ be the minimal n, $1 \leq n \leq \omega$, such that $H_n^{L_\rho}(A \cup \mu) \supseteq \eta$ for some countable subset of $A \subseteq L_\rho$ and some $\mu < \eta$.

We distinguish two cases:

Case I. η never decomposes. We shall get a contradiction by producing a nontrivial elementary embedding of L into L, which implies the existence of $0^\#$. Note that the fact that η does not decompose implies that η is a cardinal in L, hence $P^L(\delta) \subseteq L_\eta$ for all $\delta < \eta$.

We get the elementary embedding by the usual ultrapower construction (see [De]). Let κ be the first ordinal moved by j. Let U be the ultrafilter on $P^L(\kappa)$ defined by j, namely

$$A \in U \leftrightarrow \kappa \in j(A).$$

(Recall that $P^L(\kappa) \subseteq L_\eta$.) Once we have U, we can form the ultrapower L^κ/U (the ultrapower is formed by functions from κ into L, which are in L). L is canonically embedded into L^κ/U. We shall get the required nontrivial elementary embedding of L into L if we show that L^κ/U is well founded.

Assume otherwise. Let $\langle [f_n]_U | n < \omega \rangle$ be a decreasing '$\in$' sequence in L^κ/U where $[f]_U$ is the equivalence class of f modulo U. The f_n's are functions in L whose domain is κ, hence $\{f_n | n < \omega\} \subseteq L_\rho$ for some ρ. Let N be the Skolem hull in L_ρ of $\kappa \cup \{f_n | n < \omega\}$. $\langle N, \in \rangle$ is isomorphic to a structure of the form $\langle L_\xi, \in \rangle$ by an isomorphism which is the identity on κ. Let g_n be the image of f_n under this collapsing isomorphism. Note that

$$L_\xi = H_\omega^{L_\xi}(\{g_n | n < \omega\} \cup \kappa).$$

We claim that we must have $\xi < \eta$. Otherwise

$$\eta \subseteq L_\xi = H_\omega^{L_\xi}(\{g_n | n < \omega\} \cup \kappa),$$

which shows, since $\kappa < \eta$, that η decomposes at ξ. Hence we get $g_n \in L_\eta$. Note that

$$\{\alpha | f_{n+1}(a) \in f_n(\alpha)\} = \{\alpha | g_{n+1}(\alpha) \in g_n(\alpha)\} \in U.$$

Therefore by definition of U, $\kappa \in j(\{\alpha | g_{n+1}(\alpha) \in g_n(\alpha)\})$. Since the g_n's are in the domain of j, we get $j(g_{n+1})(\kappa) \in j(g_n)(\kappa)$ which yields an $\in$ decreasing sequence in L_β, a contradiction. Case I is complete.

Case II. η decomposes at some ρ. Let $\rho = \rho(\eta)$ $n = n(\eta)$. We distinguish three subcases.

Case IIa. $1 < n < \omega$. Let $m = n - 1$. We shall represent $\langle L_\rho, \in\rangle$ as a directed limit of structures of the form $\langle L_\xi, \in\rangle$ for $\xi < \eta$, where the embeddings associated with the directed system are members of L_η, and they are Σ_m embeddings.

The indices of the directed system are pairs of the form (p, μ), where p is a finite subset of L_ρ and $\mu < \eta$. The structure associated with the index (p, μ) is $\langle L_{\xi(p,\mu)}, \in\rangle$ and is defined to be the transitive collapse of $H_m^{L_\rho}(p \cup \mu)$. Denote the collapsing map by $h_{p,\mu}$. The fact that $m < n$ implies immediately that $\xi = \xi(p, \mu) < \eta$, because otherwise $\eta \subseteq H_m^{L_\xi}(p' \cup \mu)$, where p' is the set of collapses of members of p, ξ is obviously less or equal to ρ, and we get a contradiction either to the minimality of ρ or of n.

Let I be the set of indices of our directed system. I is partially ordered by $(p, \mu) \leq (q, \delta)$ if $p \subseteq q$ and $\mu \leq \delta$. I is clearly a directed partially ordered set. If $i < k$ we have to define the embedding f_{ij} from $L_{\xi(i)}$ into $L_{\xi(k)}$. Let $i = (p, \mu)$ and $k = (q, \delta)$, $i \leq k$, imply that $p \cup \mu \subseteq q \cup \delta$. Hence $H_m^{L_\rho}(p \cup \mu) \subseteq H_m^{L_\rho}(q \cup \delta)$. We can define f_{ik} by $h_k \circ h_i^{-1}$. One can easily verify that if $i_1 \leq i_2 \leq i_3$ then $f_{i_1 i_3} = f_{i_2 i_3} \circ f_{i_1 i_2}$. Also $\langle L_\rho, \in\rangle$ is isomorphic to the directed limit of the directed system. We shall denote the directed system by $\mathrm{DS}(\rho, \eta, m)$.

Note that one can define f_{ik} equivalently as follows: let $x \in L_{\xi(i)}$, $x = h_i(y)$, where $i = (p, \mu)$, $k = (q, \delta)$, and $y \in H_m^{L_\rho}(p \cup \mu)$. Hence $y = \tau^{L_\rho}(\vec{p}, \mu_1, \ldots, \mu_l)$ for some Σ_m Skolem term τ and $\mu_1, \ldots, \mu_l < \mu$. Hence $x = \tau^{L_{\xi(i)}}(h_i'' \vec{p}, \mu_1, \ldots, \mu_l)$. Define $f_{ik}(x) = \tau^{L_{\xi(k)}}(h_k'' \vec{p}, \mu_1, \ldots, \mu_l)$. It is easily verified that this definition of $f_{ik}(x)$ is independent of the particular choice of τ and $\mu_1, \ldots, \mu_l$. It also follows easily that this definition is equivalent to the previous definition.

The merit of this definition is that it makes clear that f_{ik} is very simply definable from $\xi(i), \xi(k)$ and the two finite sets $h_i'' p$ and $h_k'' p$. We actually proved.

Lemma 1.5. *If $i, k \in I$, $i < k$, then $f_{ik} \in L_\eta$.*

L_ρ is represented as the directed limit of $\mathrm{DS} = \mathrm{DS}(\rho, \eta, m)$, where all the structures and embeddings are in L_η. Let $f_{i\infty}$ be the canonical embedding of $L_{\xi(i)}$ into L_ρ. Note that $f_{i\infty}$ is exactly h_i^{-1}. Using the embedding j we transform our directed system to a similar directed system whose structures and embeddings are in L_γ.

The transformed directed system, $\mathrm{TDS} = \mathrm{TDS}(\rho, \eta, m)$, will have the same set of indices like DS, the structure associated with $i \in I$ will be $j(\langle L_{\xi(i)}, \in\rangle) = \langle L_{j(\xi(i))}, \in\rangle$. If $i < k$, $i, k \in I$, the embedding $g_{ik} : L_{j(\xi(i))} \to L_{j(\xi(k))}$ will be $j(f_{ik})$. TDS is easily verified to be a directed system with Σ_m embeddings.

Lemma 1.6. *The directed limit of TDS is well founded.*

Proof. Assume otherwise, then there exists a sequence in I, $i_1 < i_2 < \cdots$, such that if we denote $\xi(i_k)$ by ξ_k and $f_{i_l i_k}$ by f_{lk}, the directed system

$$(*) \qquad L_{j(\xi_1)} \xrightarrow{j(f_{12})} L_{j(\xi_2)} \xrightarrow{j(f_{23})} L_{j(\xi_3)} \to \cdots$$

has an ill founded directed limit.

Let $i_k = (p_k, \mu_k)$. Let $A = \bigcup_{k<\omega} p_k$ and $\mu = \bigcup_{k<\omega} \mu_k$. Since $\mathrm{cf}(\eta) > \omega$, $\mu < \eta$. Let M be $H_m^{L_\rho}(A \cup \mu)$. M is collapsed to a transitive structure by a map h. Let $\langle L_{\xi(\infty)}, \in\rangle$ be the transitive isomorph of M. By minimality of ρ and n we can show as above that $\xi(\infty) < \eta$. (Note that A is countable). For each $k < \omega$ define $f_{k\infty}\colon L_{\xi_k} \to L_{\xi(\infty)}$ by $h \circ h_k^{-1}$. (Note that $H_m^{L_\rho}(p_k \cup \mu_k) \subseteq H_m^{L_\rho}(A \cup \mu)$.) An argument similar to the proof of Lemma 1.5 shows that $f_{k\infty} \in L_\eta$ for all $k < \omega$. Clearly for $k < l < \omega$ the following diagram commutes:

$$\begin{tikzcd}
 & L_{\xi\infty} & \\
L_{\xi k} \arrow[ur, "f_{k\infty}"] \arrow[rr, "f_{kl}"] & & L_{\xi l} \arrow[ul, "f_{l\infty}"']
\end{tikzcd}$$

Hence, since all relevant structures and embeddings are in L_η, the following diagram also commutes:

$$\begin{tikzcd}
L_{j\xi(\infty)} & & & \\
L_{j(\xi_1)} \arrow[u, "j(f_{k\infty})"] \arrow[r, "j(f_{12})=g_{12}"] \arrow[urr, "j(f_{2\infty})"] & L_{j(\xi_2)} \arrow[r, "j(f_{23})=g_{23}"] \arrow[ull, "j(f_{3\infty})"'] & L_{j(\xi_3)} \arrow[ull, bend right]
\end{tikzcd}$$

Therefore the directed limit of the system $(*)$ can be embedded into $L_{j(\xi(\infty))}$. This contradicts the assumption that the directed limit of $(*)$ is ill founded. □

By Lemmas 1.4 and 1.6 the directed limit of TDS is of the form $\langle L_\delta, \in\rangle$ for some δ. Let $g_{i\infty}$ be the canonical embedding of $L_{j(\xi_i)}$ into L_δ. We shall show that L_ρ can be embedded into L_δ by an embedding extending $j \restriction L_\eta$. For $x \in L_\rho$ define $\tilde{j}(x)$ as follows: Pick any $i = (p, \mu) \in I$ such that $x \in H_m^{L_\rho}(p \cap \mu)$ ($x \in p$ is sufficient) and define $\tilde{j}(x) = g_{i\infty}(j(h_i(x)))$. It can easily be verified that $\tilde{j}$ is well defined (namely $\tilde{j}(x)$ does not depend on the particular choice of i) and that $\tilde{j}$ is a Σ_m embedding of L_ρ into L_δ. The next lemma claims that $\tilde{j}$ is even a better map, i.e. a Σ_{m+1} embedding.

Lemma 1.7. *$\tilde{j}$ is a Σ_n embedding.*

Proof. Let $\Phi(x, \vec{y})$ be a Π_m formula. If $L_\rho \models \exists x \Phi(x, y_1, \ldots, y_k)$, then fix x witnessing this fact. Hence $L_\delta \models \Phi(\tilde{j}(x), \tilde{j}(y_1), \ldots, \tilde{j}(y_k))$ because $\tilde{j}$ is a Σ_m embedding. Therefore $L_\delta \models \exists x \Phi(x, \tilde{j}(y_1), \ldots, \tilde{j}(y_k))$.

For the other direction assume

$$L_\delta \models \exists x \Phi(x, \tilde{j}(y_1), \ldots, \tilde{j}(y_k)).$$

Fix $x \in L_\delta$ witnessing this fact. Since L_δ is the directed limit of TDS, $x \in$ Image $g_{i\infty}$ for some $i \in I$. Without loss of generality we can assume that i is large enough so that also $\tilde{j}(y_1), \ldots, \tilde{j}(y_k)$ are in the image of $g_{i\infty}$. (This is equivalent to $y_1, \ldots, y_k$ being in the image of $f_{i\infty}$.) By definition of $\tilde{j}$, for $1 \leq l \leq k$ we have $\tilde{j}(y_l) = g_{i\infty}(j(h_i(y_l)))$. Hence (since $g_{i\infty}$ is a Σ_m embedding)

$$L_{j(\xi(i))} \models \Phi(g_{i\infty}^{-1}(x), j(h_i(y_1)), j(h_i(y_2)), \ldots, j(h_i(y_k))),$$

j is a Σ_0 embedding of L_η into L_β and $\xi(i) < \eta$, hence

$$L_{\xi(i)} \models \exists x \Phi(x, h_i(y_1), \ldots, h_i(y_k)).$$

But then applying $f_{i\infty}$, which is a Σ_m embedding and fixing an x witnessing the last fact

$$L_\rho \models \Phi(f_{i\infty}(x), y_1, \ldots, y_k).$$

(Recall that $f_{i\infty} = h_i^{-1}$ which implies $L_\rho \models \exists x \Phi(x, y_1, \ldots, y_k)$.) □

Lemma 1.8. *$\tilde{j}$ extends $j \restriction L_\eta$.*

Proof. Let $x \in L_\eta$. Since η is a limit ordinal we can pick $\mu < \eta$, $x \in L_\mu$. Let $i = (\{x\}, \mu)$. Note that $L_\mu \subseteq H_m^{L_\rho}(\{x\} \cup \mu)$. Hence $h_i \restriction L_\mu$ is the identity. Actually for every $i < l$ $h_l \restriction L_\mu$ is the identity, hence $f_{ll'} \restriction L_\mu$ is the identity for every $i < l < l'$.

By definition, $\tilde{j}(x) = g_{i\infty}(j(h_i(x))) = g_{i\infty}(j(x))$. Note that $j(x) \in j(L_\mu)$ and that $g_{ll'} = j(f_{ll'}) \restriction j(L_\mu) =$ identity for $i < l < l'$. It follows that since $j(L_\mu)$ is transitive, $g_{i\infty} \restriction j(L_\mu)$ is the identity. Hence $\tilde{j}(x) = g_{i\infty}(j(x)) = j(x)$. □

Let $\xi = \sup j''\eta = \sup \tilde{j}''\eta$. It follows from Lemma 1.8 that $\tilde{j}''\rho \cap \xi = j''\eta = X \cap \gamma$. We are ready now to conclude the proof of Case IIa.

By definition of ρ, we have $H_n^{L_\rho}(A \cup \mu) \supseteq \eta$ for some countable $A \subseteq L_\eta$ and some $\mu < \eta$. Let $B = \tilde{j}''A$. By Lemmas 0.1 and 1.7 it follows that

$$\tilde{j}''\eta = H_n^{L_\delta}(B \cup \tilde{j}''\mu) \cap \xi.$$

But $\tilde{j}''\eta = X \cap \gamma$ and $\tilde{j}''\mu = X \cap j(\mu)$. Hence

$$(*) \qquad X \cap \gamma = H_n^{L_\delta}(B \cup (X \cap j(\mu))) \cap \xi.$$

Since $\mu < \eta$, $j(\mu) < j(\eta) = \gamma$. By the induction assumption $X \cap j(\mu)$ is a countable union of constructible sets, say $X \cap j(\mu) = \bigcup_{k<\omega} C_k$, where $C_k \in L$ and we can assume without loss of generality that a union of finitely many C_k's is also of the form C_j for some j. It follows from $(*)$ that

$$X \cap \gamma = \bigcup_{\substack{p \in [B]^{<\omega} \\ k<\omega}} H_n^{L_\delta}(p \cup C_k) \cap \xi.$$

For each fixed p and k, $H_n^{L_\delta}(p \cup C_k)$ is in L. Hence $X \cap \gamma$ is a countable union of sets in L, because B is countable (hence $[B]^{<\omega}$ is countable). Case IIa is complete.

Case IIb. $n = 1$, we first claim that ρ must be a limit ordinal. This follows from the following lemma.

Lemma 1.9. *Let $\alpha \geq \omega$ and let $A \subseteq L_{\alpha+1}$. Then there is a set $\overline{A} \subseteq L_\alpha$ such that $|\overline{A} - A| \leq |A - L_\alpha| + \aleph_0$, $L_\alpha \cap A \subseteq \overline{A}$ and such that $H_1^{L_{\alpha+1}}(A) \cap L_\alpha \subseteq H_\omega^{L_\alpha}(\overline{A})$.*

(If ρ is a successor then $\rho = \delta + 1$, but by Lemma 1.9, $\eta \subseteq H_1^{L_{\delta+1}}(A \cup \mu) \subseteq H_\omega^{L_\delta}(\overline{A})$ where $\overline{A}$ is as in the lemma, but $\overline{A} = B \cup \mu$ where B is countable, and we get a contradiction to the definition of ρ.)

Proof of Lemma 1.9. For each formula $\psi(x, \vec{y})$ and a finite sequence $\vec{p} \subseteq L_\alpha$, $a_\alpha^{\psi, p}$ is the set $\{z | z \in L_\alpha, \langle L_\alpha, \in\rangle \vDash \psi(z, \vec{p})\}$. Each $x \in L_{\alpha+1} - L_\alpha$ is a definable subset of L_α. So for each $x \in A$ pick ψ_x and finite $p(x) \subseteq L_\alpha$ such that x is $a_\alpha^{\psi_x, p(x)}$ and let $\overline{A}$ be $(A \cap L_\alpha) \cup \{p(x) | x \in L_{\alpha+1} - L_\alpha, x \in A\}$. All the claims about $\overline{A}$ are clear except the main one: $H_1^{L_{\alpha+1}}(A) \cap L_\alpha \subseteq H_\omega^{L_\alpha}(\overline{A})$. Let $C = H_\omega^{L_\alpha}(\overline{A})$, and let

$$B = \{a_\alpha^{\psi, \vec{p}} | \vec{p} \subseteq C,\ \psi \text{ any first order formula}\}.$$

Note that $B \cap L_\alpha = C$ (because if a subset of L_α, definable over L_α using $\vec{p} \subseteq C$, is actually in L_α, it belongs to $H_\omega^{L_\alpha}(C) = C$). By definition of $\overline{A}$, $A \subseteq B$ hence it is enough to show that B is a Σ_1 substructure of $L_{\alpha+1}$ because then it follows that $H_1^{L_{\alpha+1}}(\overline{A}) \subseteq B$, hence $H_1^{L_{\alpha+1}}(A) \cap L_\alpha \subseteq B \cap L_\alpha = H_\omega^{L_\alpha}(\overline{A})$. C is an elementary substructure of $\langle L_\alpha, \in\rangle$, hence $\langle C, \in\rangle$ is isomorphic to a structure of the form $\langle L_\beta, \in\rangle$. Let $j\colon L_\beta \to L_\alpha$ be the inverse of this isomorphism. j can be easily extended to an embedding of $L_{\beta+1}$ into $L_{\alpha+1}$ by mapping $a_\beta^{\psi, \vec{p}}$ to $a_\alpha^{\psi, j(\vec{p})}$. The image of j extended to $L_{\beta+1}$ is clearly B. We shall get that B is a Σ_1 substructure of $L_{\alpha+1}$ if we prove

Lemma 1.10. *Let $\psi(x_1, \ldots, x_k)$ be a Σ_0 formula and let $\psi_1, \psi_2, \psi_3, \ldots, \psi_k$ be any formulas of set theory. Then there exists a formula Φ such that for all α and all $\vec{p}_1, \ldots, \vec{p}_k$ finite sequences of members of L_α $\psi(a_\alpha^{\psi_1, \vec{p}_1}, \ldots, a_\alpha^{\psi_k, \vec{p}_k})$ iff $L_\alpha \vDash \Phi(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k)$.*

Proof. By induction on the structure of ψ, the only interesting cases being the atomic case and the (bounded) quantifier case,

$$x_i = x_j \quad \text{by } \forall y[\psi_i(y, \vec{p}_i) \leftrightarrow \psi_j(y, \vec{p}_j)],$$
$$x_i \in x_j \quad \text{by } \exists z[\forall y[y \in z \leftrightarrow \psi_i(y, \vec{p}_i)] \wedge \psi_j(z, \vec{p}_j)].$$

If ψ is $\exists y \in x_i\ \psi'(y, x_1, \ldots, x_k)$, let Φ' be the formula satisfying the lemma for ψ' and the formulas $z \in y$, $\psi_1, \ldots, \psi_k$ (we consider y to be $\vec{p}$) and let Φ be the formula $\exists y[\psi_i(y, \vec{p}_i) \wedge \Phi'(y, \vec{p}_1, \ldots, \vec{p}_n)]$. $\square$

It follows from Lemma 1.10 that j is a Σ_0 embedding. It is a Σ_1 embedding because if $L_{\alpha+1} \models \exists x \psi(x, j(y_1), \dots, j(y_n))$, where ψ is Σ_0, then $y_i = a_\beta^{\psi_i, \vec{p}_i}$ for some ψ_i and $\vec{p}_i \subseteq L_\beta$ and $j(y_i) = a_\alpha^{\psi_i, j(\vec{p}_i)}$. x is $a_\alpha^{\psi_0, \vec{p}_0}$ for some $\vec{p}_0 \subseteq L_\alpha$. Let Φ be the formula corresponding to ψ and $\psi_0, \dots, \psi_n$ by Lemma 1.10. Then

$$L_{\alpha+1} \models \psi(a_\alpha^{\psi_0, \vec{p}_0}, a_\alpha^{\psi_1, j(\vec{p}_1)}, \dots, a_\alpha^{\psi_n, j(p_n)}).$$

Hence

$$L_\alpha \models \Phi(\vec{p}_0, j(\vec{p}_1), \dots, j(\vec{p}_n))$$

which implies

$$L_\alpha \models \exists \vec{y} \Phi(\vec{y}, j(\vec{p}_1), \dots, j(\vec{p}_n)).$$

$j \restriction L_\beta$ is an elementary embedding, therefore

$$(*) \qquad L_\beta \models \exists \vec{y} \Phi(\vec{y}, \vec{p}_1, \dots, \vec{p}_n).$$

Hence

$$L_{\beta+1} \models \exists x \psi(x, y_1, y_2, \dots, y_n).$$

(Take $x = a_\beta^{\psi_0, \vec{y}}$ where $\vec{y}$ witnesses $(*)$.) The other direction ($L_{\beta+1} \models \exists x \psi \to L_{\alpha+1} \models \exists x \psi$) follows easily from the fact that j is Σ_0. This proves Lemma 1.9. □

We consider two subcases of Case IIb.

Case IIb1. $\mathrm{cf}(\rho) > \omega$. The structure of the proof in this case is like the proof in Case IIa. We shall represent $\langle L_\rho, \in \rangle$ as a directed limit of structures of the form $\langle L_\xi, \in \rangle$ $\xi < \eta$, where all embeddings are Σ_0 embeddings. The directed system is $\mathrm{DS} = \mathrm{DS}(\rho, \eta, 0)$. The set of indices I are all triples of the form (p, μ, ν) where $\nu < \rho$, $p \subseteq L_\nu$ (p finite), and $\mu < \eta$. The partial order I is $(p, \mu, \nu) \leq \langle \bar{p}, \bar{\mu}, \bar{\nu} \rangle$ if $\nu \leq \bar{\nu}$, $p \subseteq \bar{p}$, $\mu \leq \bar{\mu}$ and if $\nu < \bar{\nu}$ then $\nu \in \bar{p}$. The structure associated with (p, μ, ν) will be the transitive collapse of $H_\omega^{L_\nu}(p \cup \mu)$. As in Case IIa this transitive collapse has the from $\langle L_{\xi(p, \mu, \nu)}, \in \rangle$ where $\xi(p, \mu, \nu) < \eta$. The embeddings between $L_{\xi(i)}$ and $L_{\xi(j)}$ if $i, j \in I$, $i < j$, are defined as in Case IIa noting that if $(p, \mu, \nu) < (\bar{p}, \bar{\mu}, \bar{\nu})$ then $H_\omega^{L_\nu}(p \cup \mu) \subseteq H_\omega^{L_\nu}(\bar{p} \cup \bar{\mu})$ and $H_\omega^{L_\nu}(p \cup \mu)$ is a Σ_0 elementary substructure of $H_\omega^{L_\nu}(\bar{p} \cup \bar{\mu})$. The fact that L_ρ is the directed limit of $\mathrm{DS}(\rho, \eta, 0)$ uses the fact that ρ is limit. (For every $x \in L_\rho$ we have to find $\nu < \rho$ such that $x \in L_\nu$.)

Now we are ready to define the transformed directed system $\mathrm{TDS} = \mathrm{TDS}(\rho, \eta, 0)$, where TDS has the same set of indices I, and the structure associated with $i \in I$ is $\langle j(L_{\xi(i)}), \in \rangle$. The embeddings are as before: if $i < k$, then $g_{ik} = j(f_{ik})$. As in Lemma 1.6 we can prove that the directed system TDS has a well-founded limit.

This proof uses $\mathrm{cf}(\rho) > \omega$. The argument is as follows: If $i_n = (p_n, \mu_n, \nu_n)$ $(n < \omega)$ form an increasing sequence in I such that the limit of

$$(*) \qquad L_{j(\xi(i_0))} \xrightarrow{j(f_{i_0 i_1})} L_{j(\xi(i_1))} \xrightarrow{j(f_{i_1 i_2})} L_{j(\xi(i_2))}$$

is ill founded, then if $\nu = \sup_{n<\omega} \nu_n$ we have $\nu < \rho$ because $\mathrm{cf}(\rho) > \omega$. Let $A = \bigcup_{n<\omega} p_n$ and $\mu = \sup_{n<\omega} \mu_\eta$. By minimality of ρ, $H_\omega^{L_\nu}(A \cup \mu)$ collapses to some $\langle L_\xi, \in\rangle$ where $\xi < \eta$. Like in Lemma 1.6 one can show that the directed limit of $(*)$ can be Σ_0 embedded into $j(L_\xi)$.

Let $\langle L_\delta, \in\rangle$ be the transitive isomorph of the directed limit of TDS. As in Case IIa $j \restriction L_\eta$ can be extended to a Σ_1 embedding $\tilde{j}$ of $\langle L_\rho, \in\rangle$ into $\langle L_\delta, \in\rangle$. Now the proof of Case IIb1 concludes as in Case IIa. □

Case IIb2. $\mathrm{cf}(\rho) = \omega$. Let $\rho = \sup\langle \rho_n | n < \omega\rangle$ where each $\rho_n < \rho$. For each ρ_n separately consider the directed system $\mathrm{DS}(\rho_n, \eta, \omega)$ defined as having the indices (p, μ) where $p \subseteq L_{\rho_n}$, p finite, and $\mu < \eta$. The structure associated with (p, μ) is the transitive collapse of $H_\omega^{L_{\rho_n}}(p \cup \mu)$. By minimality of ρ this structure has the form $\langle L_\xi, \in\rangle$ for $\xi < \eta$. The embeddings are defined as in Case IIa, and are Σ_ω (namely, elementary embeddings.) We define the transformed directed system $\mathrm{TDS}(\rho_n, \eta, \omega)$ as before and we can get (using $\rho_n < \rho$) that it has a well-founded directed limit isomorphic to $\langle L_{\delta_n}, \in\rangle$ for some δ_n. $j \restriction L_\eta$ can be extended to an elementary embedding $\tilde{j}_n$ from L_{ρ_n} into L_{δ_n}.

Let A be a countable set and $\mu < \eta$ be such that $H_1^{L_\rho}(A \cup \mu) \supseteq \eta$. Note that since the well-ordering of L is such that if $\alpha < \beta$ every element of L_α proceeds all the elements of $L_\beta - L_\alpha$, and if a Σ_1 sentence holds in L_ρ it holds in L_{ρ_n} for some $n < \omega$ we get for every $B \subseteq L_\rho$

$$H_1^{L_\rho}(B) = \bigcup_{n<\omega} H_1^{L_{\rho_n}}(B \cap L_{\rho_n}).$$

In particular

$$(*) \qquad \eta = \bigcup_{n<\omega} \eta \cap H_1^{L_{\rho_n}}((A \cap L_{\rho_n}) \cup \mu).$$

Let X_n be $j''(\eta \cap H_1^{L_{\rho_n}}(A \cap L_{\rho_n} \cup \mu))$. Since $X \cap \gamma = j''\eta$, $(*)$ implies that $X = \bigcup_{n<\omega} X_n$. $\tilde{j}_n$ extends $j \restriction \eta$ and $\tilde{j}_n$ is elementary, hence by Lemma 0.1 $X_n = \xi \cap H_1^{L_{\delta_1}}(j_n''(A \cap L_{\rho_n}) \cup j''\mu)$ where ξ is $\sup j''\eta$. (We used $\tilde{j}_n''\mu = j''\mu$.) $j''\mu = X \cap j(\mu)$ and by the induction assumption $X \cap j(\mu) = \bigcup_{n<\omega} Y_n$ where $Y_n \in L$.

We can assume without loss of generality that any finite union of Y_n's is some Y_k. It follows that

$$X_n = \bigcup_{\substack{k<\omega \\ p \in [A \cap L_{\rho_n}]^{<\omega}}} \xi \cap H_1^{L_{\delta_n}}(\tilde{j}_n''p \cup Y_k).$$

For fixed k, $\xi \cap H_1^{L_{\delta_n}}(\tilde{j}_n''p \cup Y_k)$ is in L. Hence, since $A \cap L_{\rho_n}$ is countable, X_n is a countable union of constructible sets. Therefore $X \cap \gamma = \bigcup_{n<\omega} X_n$ is a countable union of constructible sets. Case IIb2 is complete.

Case IIc. $n = \omega$. For each $k < \omega$ we separately consider the directed system $\mathrm{DS}_k = \mathrm{DS}(\rho, \eta, k)$ defined as in Case IIa. For each $k < \omega$, $j \restriction L_\eta$ can be extended to an embedding $j_k : L_\rho \to L_{\delta_k}$ for some δ_k (L_{δ_k} is the directed limit of the transformed directed system TDS_k) such that j_k is Σ_k.

For some $\mu < \eta$ and countable $A \subseteq L_\rho$,

$$\eta \subseteq H_\omega^{L_\rho}(\mu \cup A) = \bigcup_{k<\omega} H_k^{L_\rho}(\mu \cup A).$$

Let $\xi = \sup j''$, $\eta = \sup X \cap \gamma$ and $B_k = j''(\eta \cap H_k^{L_\rho}(\mu \cup A))$. Clearly $X \cap \gamma = \bigcup_{k<\omega} B_k$. j_k is a Σ_k embedding, hence

$$\begin{aligned} B_k &= H_k^{L_{\delta_k}}(j_k''\mu \cup j_k''A) \cap \xi = H_k^{L_{\delta_k}}(j''\mu \cup j_k''A) \\ &= H_k^{L_{\delta_k}}(X \cap j(\mu) \cup j_k''A); \end{aligned}$$

here $X \cap j(\mu)$ is $\bigcup_{i<\omega} Y_i$, where $Y_i \in L$ (by the induction assumption). We can again assume that the set $\{Y_i | i < \omega\}$ is closed under finite unions. Therefore

$$B_k = \bigcup_{\substack{p \in [A]^{<\omega} \\ i<\omega}} H_k^{L_{\delta_k}}(Y_i \cup p),$$

as in previous cases B_k is a countable union of constructible sets, hence $X \cap \gamma$ is a countable union of constructible sets. Case IIc is complete. □

As promised in the introduction we shall briefly describe how the arguments of the proof of Theorem 1.1 can be modified to a proof of the Jensen covering theorem. So let X be a subset of $\beta = \sup(X)$ (we assume $\neg 0^\#$). We would like to find $Y \in L$, $X \subseteq Y$ and $|Y| \leq |X| + \aleph_1$. We prove the existence of such Y by induction on β. Without loss of generality we can assume that X is the set of ordinals, X^*, of a Σ_1 substructure of L_β. (We can always cover the original X by such elementary substructure without increasing the cardinality, but later we shall have to assume that this substructure is closed enough, so we may be forced to get to a substructure of cardinality $\aleph_1 + |X|$.) X^* is isomorphic to L_η, and as before let j be the isomorphism $j : L_\eta \to L_\beta$.

The definition of "η decomposes at ρ" is now replaced by "η is not a cardinal at ρ if for some $\mu < \eta$, a *finite* subset $A \subseteq L_\rho$."

$H_\omega^{L_\rho}(A \cup \mu) \supseteq \eta$, $\rho(\eta)$, and $n(\eta)$ are defined as above.

Case I is now: η is always a cardinal. We can extend j to be a Σ_1 embedding of L into L. The well-foundedness of the ultrapower is handled by assuming that X is closed enough, so that a counterexample is already in L_η. (The fact that we can close X enough by performing $\aleph_1$ many steps is the main technical lemma whose proof we omit.)

In Case IIa: We define $\mathrm{DS}(\rho, \eta, m)$ and $\mathrm{TDS}(\rho, \eta, m)$ as above. TDS has a well-founded directed limit again by making X closed enough, so we extend to $\tilde{j} : L_\rho \to L_\delta$ where $\tilde{j}$ is Σ_n.

Now $X = H_n^{L_\delta}(j''A \cup j''\mu) \cap \beta$ for some $\mu < \eta$, a finite $A \subseteq L_\rho$. By the induction assumption, $j''\mu$ (which is included in X) is covered by some $Y^* \in L$, $|Y^*| \leq |X|$, so $X \subseteq H_n^{L_\delta}(j''A \cup Y^*) \cap \beta$. The last set is in L.

In Case IIb we argue as in Case IIb1, noting that ρ is limit.

In Case IIc we define $\mathrm{DS}(\rho, \eta, < \omega)$ to be indexed by $\langle k, \rho, \mu \rangle$ where $k < \omega$, $\rho \subseteq L_\rho$, ρ finite and $\mu < \eta$ where the corresponding structure is the collapse of $H_k^{L_\rho}(\rho \cup \mu)$. The argument is as above.

2. Generalizing from L to K

The main theorem of this section is

Theorem 2.1. *Assume there is no inner model with a cardinal κ satisfying $\kappa \to (\omega_1)^{<\omega}$. Then for each ordinal β, one can define in K a family of countably many functions such that every subset of β closed under these functions is a countable union of sets in K.*

We shall need the following theorem by Jensen [Don-Jen-Ko]. (The version as stated in [Don-Jen-Ko] is a little bit different but the same proof yields this modified version.) Let f be a function $f : [\rho]^{<\omega} \to 2$. Let $A \subseteq \rho$. We say that A is a nice homogeneous set for f if for all $\rho_1 < \cdots < \rho_n$, $\bar{\rho}_1 < \bar{\rho}_2 < \cdots < \bar{\rho}_n$, $\rho_i, \bar{\rho}_i \in A$ and $\xi_1, \ldots, \xi_k < \min(\rho_1, \bar{\rho}_1)$

$$f(\xi_1, \ldots, \xi_k, \rho_1, \ldots, \rho_n) = f(\xi_1, \ldots, \xi_k, \bar{\rho}_1, \ldots, \bar{\rho}_n).$$

Theorem 2.2 [Don-Jen-Ko]. *Let ρ be an ordinal. There exists $f : [\rho]^{<\omega} \to 2$ and a closed unbounded subset $C \subseteq \rho$, with f, $C \in K$ (in fact f, C are uniformly definable in K from ρ), such that if there exists in V a nice homogeneous set A for f such that $A \subseteq C$ and the order type of A is ω_1^V then $K \models \rho \to (\omega_1^V)^{<\omega}$.*

The rest of this section is devoted to the

Proof of Theorem 2.1. Assume that every γ satisfies $K \models \gamma \nrightarrow (\omega_1)^{<\omega}$. By Theorem 2.2 let f_γ, C_γ witness the truth of Theorem 2.2. (Hence by assumption there is no nice homogeneous set for f_γ contained in C_γ of order type ω_1^V.) Let β be our given ordinal and consider the structure $\mathscr{L} = \langle H^K(\beta^{++}), \in, \beta, F_\beta, D_\beta \rangle$, where F_β is the function satisfying

$$F_\beta(\gamma, \{\rho_1, \ldots, \rho_k\}) = f_\gamma(\{\rho_1, \ldots, \rho_k\})$$

for $\gamma \leq \beta$ and D_β is a binary predicate $D_\beta(\gamma, \delta)$ iff $\delta \in C_\gamma$. Let $\langle g_i | i < \omega \rangle$ be a sequence of canonical Skolem functions for the structure $\mathscr{L}$. (We assume that the g_i are closed under composition.)

Main Claim. Every $X \subseteq \beta$ which is closed under $g_i \restriction \beta$ (for all $i < \omega$) is a countable union of sets in $\mathbf{K}$. (Recall that $g_i \restriction \beta(\rho_1, \ldots, \rho_n) = g(\rho_1, \ldots, \rho_n)$ if $g(\rho_1, \ldots, \rho_n) \in \beta$ and 0 otherwise.)

Proof of the Main Claim. Let j be an isomorphism between some ordinal and X. By assumption about X, there is a $Y \subseteq H^K(\beta^{++})$ such that $Y \cap \beta = X$ and Y is the domain of an elementary substructure of $\mathscr{L}$. Hence j can be extended to an elementary embedding of some transitive structure $\langle A, \in, \delta, \overline{F}_\delta, \overline{D}_\delta \rangle$ into $\mathscr{L}$, where $j''\delta = X$.

As in the proof of Theorem 1.1 we shall prove by induction on $\gamma \in X \cup \{\beta\}$ that $X \cap \gamma$ is a countable union of sets in $\mathbf{K}$. So let $\gamma \in X \cup \{\beta\}$ where the induction assumption is already verified for all ordinals in $X \cap \gamma$. Let $\alpha = j^{-1}(\gamma)$. Note that we can assume without loss of generality that $\mathrm{cf}(\alpha) > \omega$, otherwise the induction assumption will immediately get $X \cap \gamma$ as a countable union of sets in $\mathbf{K}$. Also we can assume that $j \restriction \alpha$ is not the identity, otherwise $X \cap \gamma$ is an ordinal. The structure of the proof is similar to the proof of Theorem 1.1.

Definition 2.3. α decomposes at the mouse m if m is an n mouse based at some ordinal $> \alpha$, and $H^m_{n+1}(B \cup \rho) \supseteq \alpha$ for some countable set of parameters $B \subseteq m$ and some $\rho < \alpha$. (See §A for basic definitions about mice.)

The crucial place where we use the fact that there is no Erdös cardinal in K is the following lemma:

Lemma 2.4. *Let m be a mouse based at some $\rho < \alpha$ such that $m \notin A$; then there is a countable iterate of m, m_η, which is based at $\rho_\eta < \alpha$, but $m_{\eta+1}$ is based at $\rho_{\eta+1}$, where $\alpha \leq \rho_{\eta+1}$.*

Proof. Let $\langle m_\eta | \eta < \omega_1 \rangle$ be the first ω_1 iterates of m, where m_η is based at ρ_η. We claim that we cannot have $\rho_\eta < \alpha$ for all $\eta < \omega_1$. Assume otherwise and let $\kappa = \sup\{\rho_\eta | \eta < \omega_1\}$. Note that, by our assumption, $\kappa \leq \alpha$. Let $\bar{f}_\kappa, \overline{C}_\kappa$ be the collapses of $f_{j(\kappa)}$, $C_{j(\kappa)}$ respectively. Let m_λ be an iterate of m to some regular $\lambda > \alpha$. We claim that $\bar{f}_\kappa$, $\overline{C}_\kappa \in m_\lambda$.

Since $\bar{f}_\kappa, \overline{C}_\kappa \in A$ and A is elementarily embedded in $H^K(\beta^{++})$ there exists some mouse $t \in A$ based at some ordinal $> \kappa$ such that $\bar{f}_\kappa \in t$ and $\overline{C}_\kappa \in t$. If either $\bar{f}_\kappa$ or $\overline{C}_\kappa$ are not in m_λ we must have $m < t$ because when we iterate t to λ, t_λ contains $\bar{f}_\kappa$ and $\overline{C}_\kappa$ but m_λ does not. But using the fact that A is a model of ZF^- and m is based at $\rho < \alpha$, one can easily show that $m \in A$, which is a contradiction. Hence $\bar{f}_\kappa, \overline{C}_\kappa \in m_\lambda$.

Since $\bar{f}_\kappa, \overline{C}_\kappa \in m_\lambda$, all but finitely many of $\langle \rho_\eta | \eta < \omega_1 \rangle$ are indiscernibles with respect to $\bar{f}_k$ and to $\overline{C}_\kappa$. (They actually form a nice set of indiscernibles.) Since $\overline{C}_\kappa$ and $\langle \rho_\eta | \eta < \omega_1 \rangle$ are closed unbounded subsets of κ and $\mathrm{cf}(\kappa) = \omega_1$, we have ω_1 many ρ_η's satisfying $\rho_\eta \in \overline{C}_\kappa$, but by the indiscernibility we get that all ρ_η except finitely many are in $\overline{C}_\kappa$. Therefore we get that, for some $\eta_0 < \omega_1$, $\{\rho_\eta | \eta_0 < \eta < \omega_1\}$ forms a nice set of indiscernibles for $\bar{f}_\kappa$ which is a subset of $\overline{C}_\kappa$. j is an elementary embedding $j(\bar{f}_\kappa) = f_{j(\kappa)}$, $j(\overline{C}_\kappa) = C_{j(\kappa)}$. Hence the set $\{j(\rho_\eta) | \eta_0 < \eta < \omega_1\}$ forms a nice set of indiscernibles for $f_{j(\kappa)}$

which is a subset of $C_{j(\kappa)}$ contradicting the definition of $f_{j(\kappa)}$ and $C_{j(\kappa)}$. Recall tht we assumed that $f_{j(\kappa)}$ has no nice homogeneous subset of $C_{j(\kappa)}$ in V.

Hence we proved that for some countable η, $\alpha \leq \rho_\eta$. The first such η cannot be a limit ordinal because otherwise $\rho_\eta = \sup_{\eta'<\eta} \rho_{\eta'}$, and since $\rho_{\eta'} < \alpha$ for $\eta' < \eta$ we get $\rho_\eta = \alpha$, but the cofinality of α is larger than ω. So we cannot have $\alpha = \sup_{\eta'<\eta} \rho_{\eta'}$. Hence the minimal η such that $\alpha \leq \rho_\eta$ is a successor ordinal, say $\eta = \xi + 1$. Therefore $\rho_\xi < \alpha$ and $\alpha \leq \rho_{\xi+1}$, which proves the lemma. □

Corollary 2.5. *Let m be a mouse based at some $\rho < \alpha$ such that $m \notin A$. Then α decomposes at some countable iterate of m (which can be assumed of course to be a mouse).*

Proof. Let $\eta < \omega_1$ be as in Lemma 2.4. Let $m_{\eta'}$, $\eta' \leq \eta + \omega$, be the iterates of m where $m_{\eta'}$ is based at $\rho_{\eta'}$. $m_{\eta+\omega}$ is of course a mouse, together with the natural indiscernibles $\langle \rho_{\eta'} | \eta < \eta' < \eta + \omega \rangle$. Note that $\alpha < \rho_{\eta+\omega}$ since $\alpha \leq \rho_{\eta+1}$. m is the Σ_{n+1} Skolem hull of ρ_0 (for some $0 \leq n \leq \omega$) and some finite $p \subseteq m$. (Recall $\rho_0 < \alpha$.) Let h be the canonical embedding of m into $m_{\eta+\omega}$. Then h is a Σ_{n+1} embedding, hence $h''m \subseteq H_{n+1}^{m_{\eta+\omega}}(\rho_0 \cup h''(p))$. By the definition of the iterates,

$$m_{\eta+\omega} = H_n^{m_{\eta+\omega}}(\{\rho_{\eta'} | \eta' < \eta + \omega\} \cup h''m)$$

(where $n + 1 = \omega$ for $n = \omega$). Hence

$$\begin{aligned} \alpha \subseteq m_{\eta+\omega} &= H_{n+1}^{m_{\eta+\omega}}(\{\rho_{\eta'} | \eta' < \eta + \omega\} \cup \rho_0 \cup h''p) \\ &\subseteq H_{n+1}^{m_{\eta+\omega}}(\{\rho_{\eta+k} | 0 < k < \omega\} \cup \rho_\eta \cup h''p). \end{aligned}$$

This clearly shows that α decomposes at $m_{\eta+1}$ because $\rho_\eta < \alpha$. □

We resume the proof of the Main Claim and as in the proof of Theorem 1.1 we distinguish two cases.

Case I. α does not decompose at any mouse. By Corollary 2.5 we get that for every mouse m, if m is based at $\rho < \alpha$ then $m \in A$. (Otherwise α decomposes at some mouse which is an iterate of m.) In this cas we shall get a nontrivial elementary embedding of K into K. Hence by a theorem of Dodd and Jensen (see [Do]) there exists an inner model with a measurable cardinal. Since every measurable is Erdös, we got a contradiction.

Let κ be the first point moved by j. Define an ultrafilter on $P^K(\kappa)$, U, by $B \in U \leftrightarrow \kappa \in j(B)$. Note that $P^K(\kappa) \subseteq A$ since by our assumption α does not decompose at any mouse m. In particular α is a cardinal in K. Since $\kappa < \alpha$, every subset of κ in K belongs to some mouse based at $\rho < \alpha$, but then that mouse is in A, hence the given subset of κ is in A. Therefore U is defined on all subsets of κ in K.

Once we have U we can form the ultrapower K^κ/U. We claim that this ultrapower is well founded. Assume that $\langle [f_i]_U | i < \omega \rangle$ is an $\in$ decreasing

sequence in K^κ/U with each $f_i \in K$ (where $[f_i]_U$ is the equivalence class of the function f_i modulo U). We can find an ω mouse m based at μ, $\mu > \kappa$, such that $f_i \in m$ for $i < \omega$. Let C_m be the set of indiscernibles for m. We can assume without loss of generality that there is p such that $m = H^m_\omega(p \cup \mu)$, where p is a finite subset of m. Let n be the transitive collapse of $t = H^m_\omega(C_m \cup p \cup \{\mu\} \cup \{f_i | i < \omega\} \cup \kappa)$. Note that n is an ω mouse because if $x \in t$ then X is definable in m from $C_m \cup p$ and finitely many members of μ. Since t is an elementary substructure of m, these finitely many members of μ are in t. Hence $t = H^t_\omega(C_m \cup p \cup (t \cap \mu))$, but then $n = H^n_\omega(f''C_m \cup f''p \cup f(\mu))$ where f is the collapse function on t.

Let $\bar{f}_i \in n$ be $f(f_i)$, the collapse of f_i, and $\overline{C}_m, \bar{p}, \bar{\xi}_\mu$ be $f''C_m, f''p$, $f(\mu)$ respectively. Since the collapse map is the identity on κ we have

$$(*) \qquad \{\eta | \eta < \kappa, \bar{f}_{i+1}(\eta) \in \bar{f}_i(\eta)\} = \{\eta | \eta < \kappa; f_{i+1}(\eta) < f_i(\eta)\} \in U.$$

n is based at ξ. $\xi \neq \alpha$ because clearly $\mathrm{cf}(\xi) = w \neq \mathrm{cf}(\alpha)$. If $\xi > \alpha$ we have a mouse at which α decomposes because

$$n = H^n_\omega(\overline{C}_m \cup \bar{p} \cup \{\bar{\mu}\} \cup \kappa \cup \{\bar{f}_i | i < \omega\}) \supseteq \alpha.$$

Therefore, by our assumption, $\xi < \alpha$, hence it follows that $n \in A$. Therefore $j(\bar{f}_i)$ is defined and by $(*)$ and the definition of U

$$j(\bar{f}_{i+1})(\kappa) \in j(\bar{f}_i)(\kappa)$$

for $i < \omega$ which is clearly a contradiction. Case I is complete.

Case II. α decomposes at some mouse m. Let m be a mouse which is minimal in the pre-well-ordering of mice, among mice at which α decomposes. m is an n mice for some $1 \leq n \leq \omega$ and, for some countable set of parameters $B \subseteq m$ and some $\rho < \alpha$, $H^m_{n+1}(B \cup \rho) \supseteq \alpha$. Let k be the minimal ordinal $1 \leq k \leq \omega$ such that $H^m_k(B \cup \rho) \supseteq \alpha$ for some countable $B \subseteq m$ and some $\rho < \alpha$. (Note that $k \leq n + 1$.)

Lemma 2.6. *For some finite $p \subseteq m$, $m = H^m_k(p \cup \mu)$, where m is based at μ.*

Proof. Without loss of generality we can assume $k < n + 1$ (because the case $k = n + 1$ follows from m being an n mouse). Note that $\alpha < \mu$. We have to show that $H^m_k(p \cup \mu) = m$ for some finite $p \subseteq m$. Assume it fails. Let τ be a Σ_k Skolem term and let p be a finite sequence from m. Define a function $f_{\tau,p}$ on α^l as follows (l is determined of course by the arity of τ and the length of p):

$$f_{\tau,p}(\alpha_1, \ldots, \alpha_l) = \begin{cases} \tau(\alpha_1, \ldots, \alpha_l, p) & \text{if } \tau(\alpha_1, \ldots, \alpha_l, p) \in \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Claim. $f_{\tau,p} \in m$. Otherwise consider $H^m_k(\mu \cup p)$. Let t be its transitive collapse. Note that t is clearly a k mouse. Since the collapse map is the identity on μ, the measure in t is the same as in m, namely if $m = \langle L_\eta[U], \in, U\rangle$ then

$t = \langle L_{\bar{\eta}}[U], \in, U\rangle$ where clearly $\bar{\eta} \leq \eta$. If $\bar{\eta} = \eta$ we get a contradiction to our assumption because then $m = H_k^m(\mu \cup p')$, where p' is the transitive collapse of p. Hence $\bar{\eta} < \eta$. But then one can easily show that $f_{\tau,\vec{p}} \in L_{\bar{\eta}+1}[U] \subseteq m$ (using the fact that the collapse map is the identity on α and that t is embedded in m by a Σ_k embedding). Hence we get a contradiction and the claim is proved.

Let $B \subset m$ be countable and let $\rho < \alpha$ such that $H_k^m(B \cup \rho) \supseteq \alpha$. Then clearly for every member of α, δ, there is a Skolem term τ and a finite sequence $\rho_1, \ldots, \rho_n, p \subseteq B$ such that

$$\tau(\rho_1, \ldots, \rho_l, p) = \delta = f_{\tau,p}(\rho_1, \ldots, \rho_l).$$

Hence

$$\alpha \subseteq H_1^m(\{f_{\tau,p} | \tau \text{ a Skolem term}, p \subseteq B\} \cup \rho).$$

We showed that $k = 1$. We want to show that if $m = \langle L_\eta[U], \in, U\rangle$ then η is a limit ordinal.

Claim. η is a limit ordinal.

Proof. Assume otherwise. Say $\eta = \delta + 1$. Let E be the set $\{f_{\tau,p} | \tau \text{ a Skolem term}, p \subseteq B\}$. By minimality of m, there must be $f \in E$ such that $f \in L_\eta - L_\delta$. Let t be the transitive collapse of $H_1^m(\{f\} \cup \mu)$. Clearly the collapse map is the identity on μ, hence on α, hence on f. Therefore $f \in t$, t has the form $L_{\eta'}[U]$ for some $\eta' \leq \eta$. Since $f \in t$ we have $\eta' = \eta$, hence $t = m$. We have obtained that m is a Σ_1 closure of a finite set and μ, a contradiction. □

We actually claim more than that:

Claim. $\langle L_\eta[U], \in, U\rangle$ is an admissible structure.

Proof. Let E be the set $\{f_{\tau,p} | \tau \text{ a Skolem term}, p \subseteq B\}$. by minimality of m there is no $\eta' < \eta$ such that $E \subseteq L_{\eta'}[U]$. Let $z \in L_\eta[U]$ and let $\Phi(x, y)$ be a Σ_1 formula such that $\forall x \in z \exists y \Phi(x, y)$. $z \in L_{\eta'}$ for some $\eta' < \eta$. Hence there exists some $f \in E$ such that $f \notin L_{\eta'}$. Consider the transitive collapse of $H_1^m(\{f\} \cup \mu)$. Since by our assumption m is not the Σ_1 closure of a finite set and μ, and since the collapse map is the identity on μ, this transitive collapse has the form $\langle L_\delta[U], \in, U\rangle$ for some $\delta < \eta$. Let ξ be the minimal such that $f \in L_\xi[U]$. Clearly $\xi \in H_1^m(\{f\} \cup \mu)$, and $z \in L_\xi[U]$. Also $L_\xi[U] = H_1^{L_\xi}(\{f\} \cup \mu)$. Hence $H_1^m(\{f\} \cup \mu)$ contains $L_\xi[U]$, therefore the collapse of $H_1^m(\{f\} \cup \mu)$ is the identity on $L_\xi[U]$ (hence on z) $H_1^m(\{f\} \cup \mu)$ is a Σ_1 elementary substructure of m, so $L_\delta[U] \models \forall x \in z \exists y \Phi(x, y)$ and the proof of the claim is concluded. □

Let $\langle g_\xi | \xi < x\rangle$ be an enumeration of all functions from α^l into α in m (for $l < \omega$) enumerated by the canonical well-ordering of m. Since this well-ordering is defined such that $L_{\eta'}[U]$ is an initial segment, where $\eta' < \eta$, and since η is a limit ordinal, by the admissibility of m we get that $\langle g_\xi | \xi < x'\rangle \in m$

for $x' < x$. So one can easily verify that the function $F(\xi) = g_\xi$ is Σ_1 in m (using α as parameter). But $\alpha < \mu$. This implies that $x < \mu$ (because otherwise two indiscernibles in C_m above α will be mapped to the same function on α^l).

We are ready for our final contradiction. Consider the transitive collapse of $H_1^m(\mu)$. It contains g_ξ for all $\xi < x$. In particular it contains E. This transitive collapse must be of the form $L_{\eta'}[U]$ for $\eta' \leq \eta$. But, since it contains E, $\eta' = \eta$. Hence m is the Σ_1 closure of μ, which is a contradiction. □

Lemma 2.7. *Let m and k be as above. Assume $1 < k \leq \omega$, and let t be the transitive collapse of $H_{k-1}^m(D \cup \rho)$ for some $\rho < \alpha$ and D a countable set containing the indiscernibles for m, C_m. Then if t is a mouse we have $t \in A$. (If $k = \omega$ replace $k - 1$ by any natural number.)*

Proof. Let t be based at κ. We claim that $\kappa < \alpha$. We cannot have $\kappa = \alpha$ since the cofinality of κ is ω, hence, if our claim fails, $\alpha < \kappa$. Then $\alpha \subseteq H_{k-1}^t(\overline{D} \cup \rho)$, where $\overline{D}$ is the collapse of D, hence α decomposes at t. Since $t \leq m$ this is a contradiction to the minimality of m or the minimality of k (in case t is equivalent to m). Hence $\kappa < \alpha$. By Lemma 2.5, if $t \notin A$, α decomposes at some countable iterate of t, using $k - 1$ closure. (Note that the countable iterate of t is the Σ_{k-1} closure of ρ, $\overline{D}$ and the indecernibles introduced by the iteration.) Since this countable iterate of t, $\bar{t}$, is equivalent to t and $\bar{t} \leq m$, we get a contradiction either to the minimality of m or of k. So we prove $t \in A$. □

The corresponding lemma for the case $k = 1$ is

Lemma 2.8. *Let $k = 1$ and m be as before, say $m = \langle L_\eta[U], \in, U\rangle$. Then for all $\bar{\eta} < \eta$ and countable $D \subseteq L_{\bar{\eta}}[U]$, $C_m \subseteq D$ and $\rho < \alpha$, if t is the transitive collapse of $H_\omega^{L_\eta[U]}(D \cup \rho)$, and t is a ω mouse then $t \in A$.*

Proof. Let t be based at κ. Like the proof of Lemma 2.7 we can show that $\kappa < \alpha$ and t is clearly less than m because t is embedded into $L_{\bar{\eta}}[U]$. Hence α does not decompose at any countable iterate of t. By Corollary 2.5, $t \in A$. □

We are now ready to handle Case II which like the proof of Theorem 1.1, splits into three subcases.

Case IIa. $1 < k < \omega$. As in the corresponding case in the proof of Theorem 1.1 we shall represent m as a directed limit of mice lying in A, where the embeddings are in A and they are Σ_{k-1} embeddings. This directed system is denoted by $\mathrm{DS}(m, k-1, \alpha)$. The indices of the directed system are pairs of the form (p, ρ), where p is a finite subset of m and $\rho < \alpha$. The structure associated with the pair (p, ρ) is the transitive collapse of $H_{k-1}^m(C_m \cup p \cup \rho)$. Denote it by $m_{(p,\rho)}$. $m_{(p,\rho)}$ is clearly a $k - 1$ mouse and by Lemma 2.7 it is in A. Let $h_{(p,\rho)}$ be the collapsing map. The partial order between indices

is defined as in the proof of Theorem 1.1, $(p, \rho) < (\bar{p}, \bar{\rho})$ if $p \subseteq \bar{p}$ and $\rho \subseteq \bar{\rho}$. The embedding between m_{i_1} and m_{i_2} (i_1, i_2 indices) is defined as in Theorem 1.1 ($i_1 \leq i_2$): $f_{i_1 i_2} = h_{i_2} \circ h_{i_1}^{-1}$. As in §1 one can verify that $f_{i_1 i_2} \in A$ and m is the directed limit of $\mathrm{DS}(m, k-1, \alpha)$. We can define now the transformed directed system $\mathrm{TDS}(m, k-1, \alpha)$ where the indices are the same as $\mathrm{DS}(m, k-1, \alpha)$ but the structure associated with the index i is $j(m_i)$. If $i \leq j$ then the map from $j(m_i)$ into $j(m_j)$ is $j(f_{ij})$.

Lemma 2.9. *The directed system* $\mathrm{TDS}(m, k-1, \alpha)$ *has a limit which is a well-founded* $k-1$ *iterable structure.*

Proof. We first need

Lemma 2.10. *Let* m *and* k *be as above and let* $D \subseteq m$ *be countable. Then there exists* $D \subseteq D' \subseteq m$, D' *countable such that for* $\rho < \alpha$ *the transitive collapse of* $H_{k-1}^m(D' \cup C_m \cup \rho)$ *is a* $k-1$ *mouse.*

Proof. No matter which $D \subseteq D'$ we take, if $H_{k-1}^m(D' \cup C_m \cup \rho)$ collapses to t, then t is a $k-1$ iterable structure. The only problem is finding a finite $p \subseteq t$ such that $t = H_k^t(p \cup \bar{\mu})$ where t is based at $\bar{\mu}$. By Lemma 2.6 we know that there exist some finite $p \subseteq m$, $m = H_k^m(p \cup \mu)$. Let $D' = H_\omega^m(C_m \cup p \cup D)$. We claim that D' is the required set. Let $\rho < \alpha$ and consider $B = H_{k-1}^m(D' \cup \rho)$. We claim that D' is a Σ_k elementary substructure of B. Let $\Phi(x, y)$ be a Π_{k-1} formula. If D' satisfies $\exists x \Phi(x, a)$, say $D' \models \Phi(b, a)$, we have $m \models \Phi(b, a)$, but $a, b \in B$, and B is a Σ_{k-1} elementary substructure of m. Hence $B \models \Phi(b, a)$, therefore $B \models \exists x \Phi(x, a)$. If $B \models \exists x \Phi(x, a)$ then by a similar argument $m \models \exists x \Phi(x, a)$ but then $D' \models \exists x \Phi(x, a)$.

By a previous claim if τ is a Σ_k Skolem term and if $a_1, \ldots, a_n \in D'$ then $\tau^{D'}(a_1, \ldots, a_n) = \tau^B(a_1, \ldots, a_n)$, but D' is an elementary substructure of m, hence $\tau^{D'}(a_1, \ldots, a_n) = \tau^m(a_1, \ldots, a_n) = \tau^B(a_1, \ldots, a_n)$. One can also easily get that $D' = H_k^{D'}(p \cup (D' \cap \mu))$. Letting $\alpha \in D'$, there is a Σ_k Skolem term τ and $\alpha_1, \ldots, \alpha_l \in \mu$ such that $\alpha = \tau^m(p, \alpha_1, \ldots, \alpha_l)$, but since $p \subseteq D'$, we can find $\alpha_1, \ldots, \alpha_n \in D'$. But applying D' Σ_k Skolem functions is the same as applying B Σ_k Skolem functions. Hence

$$D' \subseteq H_k^B(p \cup (D' \cap \mu)) \subseteq H_k^B(p \cup (B \cap \mu)).$$

But then

$$B \subseteq H_{k-1}^B(D' \cup \rho) \subseteq H_k^B(D' \cup (B \cap \mu)) \subseteq H_k^B(p \cup B \cap \mu).$$

If t is transitive collapse of B, then clearly $t = H_k^t(\bar{p} \cup \bar{\mu})$ where t is based at $\bar{\mu}$ and $\bar{p}$ is the transitive collapse of p. Lemma 2.10 is proved. □

Note. The proof of Lemma 2.10 also works for the case $k = \omega$, where we understand $k - 1 = \omega$.

Lemma 2.10 is one of the points where it is important that we required in the definition of a k mouse m that m will be the $k+1$ closure of some finite

set and μ, where m is based at μ, because if we would have used the more "natural" one requiring that m is the k closure then we would have to get that t is the $k-1$ closure of some finite set, and we could not have done it because m is the Σ_k closure of a finite set and μ and could not get down to $k-1$. (We were lucky to have Lemma 2.6 for our special m because without it m is just the $k+1$ closure and we are faced with the same difficulty.)

We resume the proof of Lemma 2.9. The claim that the directed limit of $\mathrm{TDS}(m, k-1, \alpha)$ is well founded is a special case of the claim that every iteration of this direct limit is well founded.

The following fact is easily verified:

Fact. If the ηth iteration of the directed limit of $\mathrm{TDS}(m, k-1, \alpha)$ is not well founded then there is a countable subsystem of $\mathrm{TDS}(m, k-1, \alpha)$ such that the ηth iteration of its directed limit is not well founded.

(Without loss of generality we can assume that η is countable.)

So assume that the indices of the directed subsystem whose existence is implies by the fact are $i_1, i_2, \ldots, i_n, \ldots$. Without loss of generality assume $i_1 \leq i_2 \leq \cdots \leq i_n \leq \cdots$. Say $i_k = (p_k, \rho_k)$. Define $D = \bigcup_{k<\omega} p_k$ and $\rho = \sup_{k<\omega} \rho_k$. Note that $\rho < \alpha$ since $\mathrm{cf}(\alpha) > \omega$. Denote m_{i_l} by m_l and $f_{i_l i_{l'}}$ by $f_{ll'}$. Let D' be the countable set $D \subseteq D' \subseteq m$ for which Lemma 2.10 holds and let t be the transitive collapse of $H^m_{k-1}(D' \cup \rho \cup C_m)$. Hence t is a mouse, and, by Lemma 2.7, $t \in A$.

Clearly one can embed m_{i_k} into t by an embedding which is Σ_{k-1}, and such that the indiscernibles of m_{ik} are moved to the indiscernibles of t. Let f_k be this embedding. Also $f_k \in A$ and the following diagram commutes for $l \leq l'$:

$$\begin{tikzcd} t & \\ m_l \arrow[u, "f_l"] \arrow[r, "f_{ll'}"] & m_{l'} \arrow[ul, "f_{l'}"'] \end{tikzcd}$$

Since this diagram is in A, we can apply j to it and get that the following diagram commutes:

$$\begin{tikzcd} j(t) & \\ j(m_l) \arrow[u, "j(f_l)"] \arrow[r, "j(f_{ll'})"] & j(m_{l'}) \arrow[ul, "j(f_{l'})"'] \end{tikzcd}$$

Hence the directed limit of $\langle j(m_l) | l < \omega \rangle$ can be embedded into $j(t)$. Also since the indiscernibles of $j(m_l)$ are mapped to the indiscernibles of $j(t)$, then ηth iterate of the directed limit of $\langle j(m_l) | l < \omega \rangle$ is mapped into the ηth iterate of $j(t)$. But t is a mouse, hence $j(t)$ is a mouse and its ηth iterate is well founded. Therefore the directed limit of $\langle j(m_l) | l < \omega \rangle$ is an iterable structure. Lemma 2.9 is proved. □

From now on the proof is almost a copy of the corresponding case in Theorem 1.1. Let $\mathscr{L}$ be the transitive isomorph of the directed limit of $\mathrm{TDS}(m, k-1, \alpha)$. As in §1 we can define an embedding $\tilde{j}: m \to \mathscr{L}$ such that $\tilde{j}$ extends $j \restriction \alpha$, and such that $\tilde{j}$ is a Σ_k embedding. Note that $\mathscr{L}$ is based at some $\kappa > j(\alpha) = \gamma$. Now recall that $\alpha \subseteq H_k^m(\rho \cup D)$ for some $\rho < \alpha$ and some countable $D \subseteq m$. Since $\tilde{j}$ is Σ_k we get

$$j''\alpha = H_k^{\mathscr{L}}(j''\rho \cup \tilde{j}''D) \cap j(\alpha).$$

But $X \cap \gamma = j''\alpha \quad X \cap j(\rho) = j''\rho$. (Note that $j(\rho) < \gamma$.) By induction on γ, $X \cap j(\rho)$ is a countable union of sets in K so let $X \cap j(\rho) = \bigcup_{k<\omega} B_k$, where without loss of generality $B_k \subseteq B_{k+1}$. Hence

$$X \cap \gamma = \bigcup_{l<\omega} \bigcup_{\substack{\tau\alpha\Sigma_k \\ \text{Skolem} \\ \text{term}}} \bigcup_{\substack{p \text{ a finite} \\ \text{subset of} \\ \tilde{j}''D}} \{\tau^{\mathscr{L}}(\rho_1, \dots, \rho_i, p) | \rho_1, \dots, \rho_i \in B_l\} \cap \gamma.$$

The proof of the present subcase will be finished if we show that

$$(*) \qquad \{\tau^{\mathscr{L}}(\rho_1, \dots, \rho_i, p) | \rho_1, \dots, \rho_i \in B_l\} \cap \gamma$$

is in K for all finite $p \subseteq \mathscr{L}$. Consider the function defined on $j(\rho)^i$:

$$F(\rho_1, \dots, \rho_i) = \begin{cases} \tau^{\mathscr{L}}(\rho_1, \dots, \rho_i, \vec{p}) & \text{if } \tau^{\mathscr{L}}(\rho_1, \dots, \rho_i, \vec{p}) \in \gamma, \\ 0 & \text{otherwise}, \end{cases}$$

where F is clearly definable over $\mathscr{L}$ by a generalized Σ_k formula (see §A). F can be coded as a subset of γ. $\mathscr{L}$ is an iterable structure based at $\kappa > \gamma$. Hence $F \in K$. Recall that $B_l \in K$, hence the set in $(*)$ is in K because it is $F''B_l$. Case IIa is complete.

Case IIb. $k = 1$. In this case the analogy with the similar case in §1 is not complete since we *do not* know that if $m = \langle L_\eta[U], \in, U\rangle$ then η is a limit ordinal. Here we will have to distinguish three subcases ($\mathrm{cf}(\eta) = \omega$, $\mathrm{cf}(\eta) > \omega$, η a successor ordinal).

Case IIb1. $\mathrm{cf}(\eta) = \omega$. Recall that $H_1^m(D \cup \rho) \supseteq \alpha$ for some countable $D \subseteq m$. Enumerate all finite subsets of D as $\langle p_i | i < \omega\rangle$ such that every finite subset of D appears infinitely often. Let $\langle \eta_i | i < \omega\rangle$ be a sequence cofinal in η, where we can assume without loss of generality that $p_i \subseteq L_{\eta_i}[U]$. Let $m_{l,i}$ for $i \le l$ be the transitive collapse of $H_1^{L_{\eta_l}[U]}(p_i \cup \rho)$. $m_{l,i}$ is clearly a 1-mouse and $m_{l,i} < m$. Note that, for any set $B \subseteq m$, $H_1^m(B) = \bigcup_{i<\omega} H_1^{L_{\eta_i}[U]}(B \cap L_{\eta_i}[U])$. Hence

$$(*) \qquad \alpha = \bigcup_{l<\omega} \bigcup_{i<l} H_1^{L_{\eta_l}[U]}(p_i \cup \rho) \cap \alpha = \left(\bigcup_{l<\omega} \bigcup_{i<l} H_1^{m_{li}}(p_i^l \cup \rho)\right) \cap \alpha$$

(the equality is because the collapse maps are identity on α), where p_i^l is the image of p_i under the collapse of m_{li}. For each m_{li} we can form the direct system $\mathrm{DS}(m_{li}, \alpha, 1)$ as we have done for Case IIa, we can define the

$\mathrm{TDS}(m_{li}, \alpha, 1)$, having a well-founded directed limit which is an iterable structure, M_{li}, and we can get a Σ_2 embedding $j_{li} : m_{li} \to M_{li}$ which extends j on α, hence:

$$H_1^{li}(j_{li}(p_i^l) \cup j''\rho) \cap \gamma = j''(H_1^{m_{li}}(p_i^l \cup \rho) \cap \alpha)$$

(where H_1^{li} is H_1 in the sense of M_{li}). As in Case IIa, $H_1^{li}(j_{li}(p_i^l) \cup j''\rho)$ is a countable union of sets in K. But

$$(*) \qquad X \cap \gamma = j''\alpha = \bigcup_{l<\omega} \bigcup_{i<l} j''(H_1^{m_{li}}(p_i \cup \rho) \cap \alpha)$$

is a countable union of sets in K. Case IIb1 is complete.

Case IIb2. $\mathrm{cf}(\eta) > \omega$. In this case we shall represent m as a directed limit of a directed system $\mathrm{DS}(m, 0, \alpha)$. The indices will be triples of the form $(p, \bar{\eta}, \delta)$, where $\bar{\eta} < \eta$, p is a finite subset of $L_{\bar{\eta}}$ and $\delta < \alpha$. The structure associated with the index $i = (p, \bar{\eta}, \delta)$ will be m_i which is the transitive collapse of $H_1^{L_\eta[U]}(p \cup C_m \cup \delta)$. Let h_i be the collapse map. The partial order on indices is defined by $j = (q, \chi, \beta) \geq i = (p, \bar{\eta}, \delta)$ if $q \supseteq p$, $\chi \geq \bar{\eta}$, $\beta \geq \delta$ and $\bar{\eta} \in q$. Clearly if $i \leq j$ as before

$$H_1^{L_\eta[U]}(p \cup \delta) \subseteq H_1^{L_\chi[U]}(q \cup \beta).$$

Hence we can define the map $f_{ij} = h_j \circ h_i^{-1}$ which is a Σ_0 embedding.

As in Case IIa we get $m_i \in A$ and $f_{ij} \in A$ for each i and j_0. Also m is clearly the directed limit of $\mathrm{DS}(m, 0, \alpha)$. Similarly to Case IIa we can define the transformed directed system $\mathrm{TDS}(m, 0, \alpha)$.

Lemma 2.11. *The directed limit of* $\mathrm{TDS}(m, 0, \alpha)$ *is a* 0 *iterable structure.*

Proof. The proof is very similar to the proof of Lemma 2.9. If some iterate of the directed limit of $\mathrm{TDS}(m, 0, \alpha)$ is not well founded, we can find an increasing sequence of indices $\langle i_k = (p_l, \eta_l, \delta_l) | l < \omega \rangle$ such that $i_k < i_{k+1}$ and such that an iterate of the directed limit of $\langle j(m_{i_l}) | l < \omega \rangle$ is not well founded. Let $D = \bigcup_{l<\omega} p_l$, $\bar{\eta} = \sup_{l<\omega} \eta_l$ and $\delta = \sup_{l<\omega} \delta_l$. (Note $\bar{\eta} < \eta$ since $\mathrm{cf}(\eta) > \omega$ and $\delta < \alpha$ since $\mathrm{cf}(\alpha) > \omega$.) We need a lemma corresponding to Lemma 2.10.

Lemma 2.12. *There exists a countable* $\overline{D} \subseteq m$, $D \subseteq \overline{D}$, $\bar{\eta} \in \overline{D}$, *and* $\bar{\eta} < \bar{\bar{\eta}} < \eta$ *such that the transitive collapse of* $H_1^{L_{\bar{\bar{\eta}}}[U]}(\overline{D} \cup \delta \cup C_m)$ *is a* 1*-mouse.*

Proof. Recall that, by Lemma 2.6, since $k = 1$ there is finite $p \subseteq m$ such that $m = H_1^m(p \cup \mu)$. Let $\overline{D}$ be $H_\omega^m(p \cup D \cup \{\bar{\eta}\})$. Since $\overline{D}$ is countable, there is $\bar{\eta} < \bar{\bar{\eta}} < \eta$ such that $\overline{D} \subseteq L_{\bar{\bar{\eta}}}[U]$. As in the proof of Lemma 2.10 we can show that $\overline{D}$ is a Σ_1 elementary substructure of $H_1^{L_{\bar{\bar{\eta}}}[U]}(\overline{D} \cup \delta \cup C_m)$. (Actually it is a Σ_2 substructure but Σ_1 is enough.) But since (as in Lemma 2.10)

$$\overline{D} = H_1^{\overline{D}}(p \cup (\overline{D} \cap \mu)) \subseteq H_1^{L_{\bar{\bar{\eta}}}[U]}(p \cup (\overline{D} \cap \mu)),$$

we get that if $t = H_1^{L_{\bar{\bar{\eta}}}[U]}(\overline{D} \cup \delta \cup C_m)$ then $t = H_1^t(p \cup (t \cap \mu))$. Hence the transitive collapse of t is a 1 mouse. Lemma 2.12 is proved. □

Let $\overline{D}$ and $\bar{\eta}$ be as in Lemma 2.12. The directed limit of $\langle m_{i_l} | l < \omega \rangle$ can be easily embedded into the transitive collapse of $H_1^{L_\eta[U]}(\overline{D} \cup \delta \cup C_m)$. This transitive collapse is a mouse smaller than m, hence as before it is in A. Denote it by t. $j(t)$ is a mouse and we can embed any iterate of the directed limit of $\langle j(m_{i_l}) | l < \omega \rangle$ into the corresponding iterate of $j(t)$, hence this previous iterate is well founded. This completes the proof of Lemma 2.11. □

From here on the proof is exactly like Case IIa. Let $\mathscr{L}$ be the 0 iterable structure which is the directed limit of $\mathrm{TDS}(m, 0, \alpha)$. $j \restriction \alpha$ can be extended to an embedding $\tilde{j} : m \to \mathscr{L}$ which can be shown to be Σ_1. We know that for some countable $D \subseteq m$ and $\rho < \alpha$

$$\alpha = \bigcup_{\substack{p \text{ a finite} \\ \text{subset of } D}} H_1^m(p \cup \rho) \cap \alpha .$$

Hence

$$\tilde{j}''\alpha = j''\alpha = X \cap \gamma = \bigcup_{\substack{p \text{ a finite} \\ \text{subset} \\ \text{of } D}} H_1^{\mathscr{L}}(\tilde{j}''p \cup j''\rho) \cap \gamma .$$

As in Case IIa we can show that $H_1^{\mathscr{L}}(\tilde{j}''p \cup j''\rho) \cap \gamma$ is a countable union of sets in K. Case IIb2 is complete.

Case IIb3. η is a successor ordinal, say $\eta = \bar{\eta} + 1$. In this case, as in previous cases, we represent m as the direct limit of a directed system where the embeddings are Σ_0. The indices of the directed system (denoted by $\mathrm{DS}(m, 0, \alpha)$) are (p, δ), where p is a finite subset of $L_{\bar{\eta}}[U]$ and $\delta < \alpha$. The structure associated with (p, δ) is obtained as follows: Consider the transitive collapse of $H_w^{L_\eta[U]}(p \cup C_m \cup \delta)$. It has the form $\langle L_{\bar{\eta}}[\overline{U}], \in, \overline{U} \rangle$. $m_{(p,\delta)}$ is $\langle L_{\bar{\eta}+1}[\overline{U}], \in, \overline{U} \rangle$. Note that since the C_m are included in $H_w^{L_\eta[U]}(p \cup C_m \cup \delta)$, $\overline{U}$ has a natural extension to $L_{\bar{\eta}+1}[\overline{U}]$. $m_{(p,\delta)}$ is Σ_0 embedded into m by embedding the elements of $L_{\bar{\eta}}[\overline{U}]$ into $L_\eta[U]$, using the inverse of the collapsing map, and then, for a definable subset of $L_{\bar{\eta}}[\overline{U}]$, map it to a subset of $L_\eta[U]$ defined by the same formula, using the corresponding parameters. (One can easily show by induction on Σ_0 formulas that this defines a Σ_0 embedding.) Let m_i be the structure associated with $i = (p, \delta)$ and f_i the Σ_0 embedding of m_i into m. $m_i \in A$ since if $m_i = \langle L_{\bar{\eta}+1}[\overline{U}], \in, \overline{U} \rangle$ then $m_i^- = \langle L_{\bar{\eta}}[\overline{U}], \in, \overline{U} \rangle$ is a ω mouse, smaller than m, based by minimality of m below α, hence it is in A. But if $m_i^- \in A$ clearly $m_i \in A$. If i, j are indices, $i \leq j$ if $i = (p, \delta)$ and $j = (q, \nu)$, where $p \subseteq q$ and $\delta \leq \nu$. If $i \leq j$, f_{ij} is $f_j^{-1} \circ f_i$. Clearly $f_{ij} \in A$. It is also easy to see that m is the directed limit of $\mathrm{DS}(m, 0, \alpha)$.

As in previous cases, we can define $\mathrm{TDS}(m, 0, \alpha)$ using j. Again we need

Lemma 2.13. $\mathrm{TDS}(m, 0, \alpha)$ *has a limit which is a well-founded iterable structure.*

Proof. The proof is almost a repetition of Lemmas 2.9 and 2.11, except that now we need

Lemma 2.14. *Let $D \subseteq L_{\bar{\eta}}[U]$, D countable, and let $\delta < \alpha$. Then there is $\overline{D}$, $\overline{D}$ countable, $D \subseteq \overline{D} \subseteq L_{\bar{\eta}}[U]$, such that the transitive collapse of $H_\omega^{L_{\bar{\eta}}[U]}(\overline{D} \cup C_m \cup \delta)$ is an ω mouse.*

Proof. Recall that since $k = 1$, there is (by Lemma 2.6) a finite $p \subseteq m$ such that $m = H_1^m(p \cup \mu)$. Let $\overline{\overline{D}} = H_\omega^m(D \cup p \cup C_m)$ and let $\overline{D} = \overline{\overline{D}} \cap L_{\bar{\eta}}[U]$. Let h^{-1} be the collapsing map on $\overline{D}$. The image of h^{-1} is a structure of the form $\langle L_{\bar{\eta}}[\overline{U}], \in, \overline{U}\rangle$ and as before h can be extended to an embedding of $\tilde{h}$ of $L_{\bar{\eta}+1}[\overline{U}]$ into m. $\tilde{h}$ is an elementary embedding becasue $L_{\bar{\eta}+1}[\overline{U}]$ is exactly the collapse of $\overline{\overline{D}}$. Let t be the transitive collapse of $H_\omega^{L_{\bar{\eta}}[U]}(\overline{D} \cup C_m \cup \delta)$. t has the form $\langle L_\nu[\overline{\overline{U}}], \in, \overline{\overline{U}}\rangle$, say based at $\bar{\mu}$, and the inverse of the collapse map, f, can be easily extended to an embedding, $\tilde{f}$, of $L_{\nu+1}[\overline{\overline{U}}]$ into m. One can easily show that $\tilde{f}^{-1}(\overline{\overline{D}})$ is a Σ_1 elementary substructure of $L_{\nu+1}[U]$. Also $\overline{\overline{D}} = H_1^{\overline{\overline{D}}}(p \cup (\overline{\overline{D}} \cap \mu))$. Hence

$$\tilde{f}^{-1}(\overline{\overline{D}}) \subseteq H_1^{L_{\nu+1}[\overline{\overline{U}}]}(\tilde{f}^{-1}(p) \cup \bar{\mu}).$$

Clearly $H_1^{L_{\nu+1}[\overline{\overline{U}}]}(f^{-1}(\overline{\overline{D}}) \cup \bar{\mu}) = L_{\nu+1}[\overline{\overline{U}}]$. Therefore

$$L_{\nu+1}[\overline{\overline{U}}] = H_1^{L_{\nu+1}[\overline{\overline{U}}]}(\tilde{f}^{-1}(p) \cup \bar{\mu})$$

which shows that $t = \langle L_\nu[\overline{\overline{U}}], \ldots\rangle$ is an ω mouse. This proves Lemma 2.14 and completes the proof of Lemma 2.13. □

The argument from now on is as in Case IIa by getting an embedding extending j from m into the directed limit of $\mathrm{TDS}(m, 0, \alpha)$ and arguing that this embedding is Σ_1. Case IIb3 is complete.

Case IIc. $k = \omega$. Let $D \subseteq m$, and let $\rho < \alpha$ witness the fact that α decomposes at m. Consider for each finite $p \subseteq D$ and $n < \omega$ the structure $m_{n,p}$ which is the transitive collapse of $H_n^m(p \cup \mu)$. Let $\bar{p}_n$ be the collapse of p in $m_{n,p}$. $m_{n,p}$ is clearly an n mouse and also

$$(*) \qquad \alpha = \bigcup_{n<\omega} \bigcup_{\substack{p \text{ a finite} \\ \text{subset of } D}} H_n^{m_{n,p}}(\bar{p}_n \cup \rho).$$

For each $m_{n,p}$ consider the directed system $\mathrm{DS}(m_{n,p}, n, \alpha)$. By minimality of k $(k = \omega)$ and by the fact that $m_{n,p} \leq m$, this directed system has all its structures and embeddings in A, hence we can define the $\mathrm{TDS}(m_{n,p}, n, \alpha)$ and prove that it has a well-founded iterable structure as its directed limit.

Denote the transitive isomorph of this directed limit by $\mathscr{L}_{p,n}$. We get a Σ_{n+1} embedding $j_{p,m}$ of $m_{n,p}$ into $\mathscr{L}_{p,n}$ which extends j. Hence (in view of $(*)$):

$$\begin{aligned} X \cap \gamma = j''\alpha &= \bigcup_{n<\omega} \bigcup_{\substack{p \text{ a finite} \\ \text{subset of } D}} j''(H_n^{m_{n,p}}(\bar{p}_n \cup \rho) \cap \alpha) \\ &= \bigcup_{n<\omega} \bigcup_{\substack{p \text{ a finite} \\ \text{subset of } D}} (j''_{n,p} H_n^{m_{n,p}}(\bar{p}_n \cup \rho)) \cap \gamma \\ &= \bigcup_{n<\omega} \bigcup_{\substack{p \text{ a finite} \\ \text{subset of } D}} H_n^{\mathscr{L}_{n,p}}(j_{n,p}(\bar{p}_n) \cup j''\rho) \cap \gamma . \end{aligned}$$

As in previous cases $H_n^{\mathscr{L}_{n,p}}(j_{n,p}(\bar{p}_n) \cup j''\rho) \cap \gamma$ is a countable union of sets in K, hence $X \cap \gamma$ is a countable union of sets in K. Case IIc is complete, which completes the proof of Theorem 2.1. □

3. Representing sets as the union of ω_1 sets in K

As noted in the introduction, the assumption made in Theoem 2.2 that there is no inner model with an Erdös cardinal cannot be weakened. What happens if we try to weaken the conclusion, namely "Every closed enough set is a union of ω_1 sets in K"? An immediate modification of the proof of Theorem 2.1 gives that assuming no inner model having a cardinal κ such that $\kappa \to (\omega_2)^{<\omega}$, then for each β we can define (in K) countably many functions such that every subset of β closed under these functions is a union of ω_1 sets in K and again this is the best possible result. However, now Baumgartner's argument for getting the counterexample down to small β like ω_n does not work, as indicated by the following theorem.

Theorem 3.1. *Assume there is no inner model with a measurable cardinal and let β be an ordinal $\beta < \omega_2$. Then in V one can define a countable set of functions on β such that every subset of β closed under those functions is the union of ω_1 sets in K.*

Proof. Consider the structure $\mathscr{L} = \langle H^K(\beta^{++}), \in, \beta\rangle$. Let $\langle g_i | i < \omega\rangle$ be a sequence of functions containing a sequence of Skolem functions for the structure $\mathscr{L}$, as well as the function $\alpha \to \mathrm{cf}(\alpha)$. (This is the *real* cofinality function, not the cofinality function in K.) Also among $\langle g_i | i < \omega\rangle$ we need a two-variable function $f(\alpha, \gamma)$ such that $f(\alpha, -) \restriction \mathrm{cf}(\alpha)$ maps $\mathrm{cf}(\alpha)$ cofinally into α for fixed α. We also require that among the g_i we have a binary function h such that $\langle \omega_2, h\rangle$ is a Jonsson algebra. (It is well known that one can find such a function.)

Main claim. Every subset of β closed under $\langle g_i | i < \omega\rangle$ is the union of ω_1 sets in K.

As in the proof of Theorem 2.1 there is an elementary substructure of $\langle H^K(\beta^{++}), \in, \beta\rangle$, whose intersection with β is X. Let $j : A \to \langle H(\beta^{++}), \in,$

$\beta\rangle$ be the inverse of the collapsing map of this elementary substructure, where for some ordinal δ, $j''\delta = X$. (δ is actually $j^{-1}(\beta)$.) The proof is now like the proof of Theorem 2.1, where now we prove by induction on $\gamma \in X \cup \{\beta\}$ that $X \cap \gamma$ is the union of ω_1 sets in K. If $\gamma < j(\omega_2)$ then $X \cap \gamma$ is of cardinality $\leq \omega_1$ and the claim is trivial. If $\gamma \geq j(\omega_2)$ we distinguish two cases:

Case I. $\gamma = \omega_2$. In this case $X \cap \omega_2$ is of cardinality ω_2, since X is closed under the function h, which makes ω_2 a Jonsson algebra. We get $X \cap \omega_2 = \omega_2$, hence our inductive step is trivial.

Case II. $\gamma > \omega_2$. Let $\alpha = j^{-1}(\gamma)$. As in Theorem 2.1 we can assume $\mathrm{cf}(\alpha) > \omega_1$. The proof is exactly like the proof of Theoem 2.1 except that now we have a weaker assumption, namely "No inner model with a measurable cardinal" rather than "No inner model with an Erdös cardinal". The only place in the proof of Theorem 2.1 we used the assumption was in the proof of Lemma 2.4, hence we need a substitute for that lemma.

Lemma 3.2. *Let m be a mouse based at some $\rho < \alpha$ such that $m \notin A$, and let $\langle m_\eta | \eta \leq \omega_2 \rangle$ be the first $\omega_2 + 1$ iterates of m where m_η is based and ρ_η. Then $\rho_\eta < \alpha$ for some $\eta < \omega_2$, but $\alpha \leq \rho_{\eta+1}$.*

Proof. It is enough to show that $\alpha \leq \rho_\eta$ for some $\eta < \omega_2$ because the minimal δ such that $\alpha \leq \rho_\delta$ must be successor. (Recall $\mathrm{cf}(\alpha) > \omega_1$ and $\rho_\delta = \sup_{\beta<\delta} \rho_\beta$ for limit δ.) So assume, hoping for a contradiction, that $\rho_\eta < \alpha$ for all $\eta < \omega_2$.

Lemma 3.3. *For each $\eta < \omega_2$, $A \models \rho_\eta$ is a regular cardinal.*

Proof. Assume otherwise. Then $A \models \exists$ a mouse $\bar{m}$ based at some $\bar{\rho} > \rho_\eta$ and $\bar{m} \models \rho_\eta$ is singular. Let λ be a regular cardinal larger than $\bar{\rho}$ and ρ_η, and iterate both m and $m_{\eta+1}$ to λ to get $\bar{m}_\lambda$ and m_λ. We clearly have $m_\lambda \in \bar{m}_\lambda$ or $m_\lambda = \bar{m}_\lambda$ or $\bar{m}_\lambda \in m_\lambda$. But $m_\lambda \models \rho_\eta$ is regular whereas $\bar{m}_\lambda \models \rho_\eta$ is singular, hence we cannot have $m_\lambda = \bar{m}_\lambda$ or $\bar{m}_\lambda \in m_\lambda$. Therefore $m_\lambda \in \bar{m}_\lambda$ and hence $m < \bar{m}$. $\bar{m} \in A$, m is based at an ordinal in A, hence $m \in A$ which is a contradiction. □

It follows that for each $\eta < \omega_2$

$$K \models j(\rho_\eta) \text{ is a regular cardinal.}$$

Note that $j(\rho_\eta) < \omega_{\omega_2}$ since $\beta < \omega_{\omega_2}$. Hence we must have for some $\mu < \omega_2$ that the set $\{\eta | \eta < \omega_2, \omega_\mu < j(\rho_\eta) < \omega_{\mu+1}\}$ is a final segment of ω_2. Let η_0 be such that $\omega_\mu < j(\rho_\eta) < \omega_{\mu+1}$ for $\eta_0 < \eta < \omega_2$.

Claim. $2 \leq \mu$. Assume otherwise, hence for all $\eta < \omega_2$, $j(\rho_\eta) < \omega_2$, but then $X \cap \omega_2$ is of cardinality ω_2, X is closed under the function h, hence $X \cap \omega_2 = \omega_2$. Therefore j is the identity on ω_2. In particular since $A \models \rho_\omega$ is regular we get $K \models j(\rho_\omega) = \rho_\omega$ is regular, but since $m \in K$, the sequence $\langle \rho_n | n < \omega \rangle$ is in K, a contradiction! Recall we are assuming that there is no

inner model with a measurable cardinal. Hence it follows from $(*)$ and the Jensen-Dodd covering theorem that $\mathrm{cf}(j(\rho_\eta)) = \omega_\mu$ for every $\eta \geq \eta_0$. (In particular ω_μ is regular.) Since X is closed under the cofinality function, $\omega_\mu \in X$. Let $\nu = j^{-1}(\omega_\mu)$. X is also closed under the function $f(\alpha, \gamma)$. Hence if $\bar{f}$ is the collapse of f, $\bar{f}$ is a monotone mapping of ν into ρ_η. Hence it follows that for every η, $\eta_0 < \eta < \omega_2$ $\mathrm{cf}(\rho_\eta) = \mathrm{cf}(\nu)$, but for limit η, $\mathrm{cf}(\rho_\eta) = \mathrm{cf}(\eta)$ (since $\rho_\eta = \sup\langle \rho_{\eta'} | \eta' < \eta \rangle$). Hence we immediately get a contradiction by picking $\eta_1, \eta_2 \leq \eta_0$ where $\mathrm{cf}(\eta_1) = \omega$, $\mathrm{cf}(\eta_2) = \omega_1$. This completes the proofs of Lemma 3.2 and Theorem 3.1. □

A. On the core model

In this section we present the basic definitions and terminology used concerning iterable structures, mice and the core model K. We supply almost no proofs because any reader familiar with the standard definitions [Do-Jen or Do] will have little difficulty translating our terminology into the standard one and supplying the proofs.

Definition A.1. An acceptable structure is a structure of the form $\langle L_\delta[U], \in, U \rangle$ where

$$L_\delta[U] \models U \text{ is a normal}, \kappa \text{ complete ultrafilter on some cardinal } \kappa.$$

For an acceptable structure the κ above is unique. We say that the acceptable structure is *based* at κ. Note that an acceptable structure has a Σ_1 well-ordering (which is uniformly definable over all acceptable structures). We expand our language by having a symbol for each canonical Skolem function on an acceptable structure. A Σ_n term is a term made up of Σ_n Skolem functions. A generalized Σ_n formula is a formula of the form $\Phi(\tau_1(x_1, \ldots, x_n), \ldots, \tau_k(x_1, \ldots, x_n))$ where Φ is a Boolean combination of Σ_n and τ_i are Σ_n Skolem terms. Note that by Lemma 0.1, Σ_n embeddings preserve also generalized Σ_n formulas.

Definition A.2. Given the acceptable structure $\mathscr{A} = \langle L_\delta[U], \in, U \rangle$, based at κ

(a) the language of $\mathscr{A}$, $L_{\mathscr{A}}$, is a first order language having symbols for $\in$, U and a constant for every member of $\mathscr{A}$. (c_a is the constant denoting $a \in \mathscr{A}$.) $\mathrm{Th}_{\mathscr{A}}$ is the theory of $\mathscr{A}$ in this language.

(b) The expanded language of $\mathscr{A}$ is $L^*_{\mathscr{A}}$, the language of $\mathscr{A}$ together with a sequence of ω many constants $\langle d_i | i < \omega \rangle$.

(c) A scheme for a Σ_n iteration of $\mathscr{A}$ $(1 \leq n \leq \omega)$ is a consistent complete set of generalized Σ_n formulas in $L^*_{\mathscr{A}}$, T, such that

(1) $\mathrm{Th}_{\mathscr{A}} \cap \{\text{generalized } \Sigma_n \text{ formulas}\} \subseteq T$,
(2) for $i < j$, "$d_i \in d_j$" $\in T$,
(3) for $i < \omega$, "$d_i < c_\kappa$" $\in T$,

(4) "The d_i form generalized Σ_n indiscernible satisfying Silver's remarkably condition." *Namely* : $\Phi(d_{i_1}, \dots, d_{i_k}) \in T$ (for $i_1 < \cdots < i_2$) iff $\Phi(d_1, \dots, d_k) \in T$, and if τ is a Σ_n Skolem term and "$\tau(d_1, \dots, d_k) < d_i$" $\in T$ then

$$\tau(d_1, \dots, d_{i-1}, d_i, \dots, d_k) = \tau(d_1, \dots, d_{i-1} d_{k+1}, \dots, d_{2k-i+1}) \in T.$$

(5) If τ is a Σ_n Skolem term then $U(\tau(d_{i_1}, \dots, d_{i_k})) \in T$ iff

$$\tau(d_{i_1}, \dots, d_{i_k}) \subseteq C_k \in T \text{ and } \forall i > i_k \text{ "}d_i \in \tau(d_{i_1}, \dots, d_{i_k})\text{"} \in T.$$

(d) Let T a scheme for a Σ_n iteration of $\mathscr{A}$. Let η be an ordinal. The ηth iterate of $\mathscr{A}$ by $T, \mathscr{A}_\eta, T$ (we usually drop the superscript T when it is understood from the context) is the structure which we obtain from a set of indiscernibles of order type η, when we apply Σ_n Skolem terms to them and when the underlying type of these indiscernibles is determined by the theory T.

Clearly, if $\mathscr{A}$ is acceptable and T is a scheme for a Σ_n iteration, $\mathscr{A}_0^T$ is isomorphic to $\mathscr{A}$, and also $\mathscr{A}$ is naturally embedded by a Σ_n embedding into $\mathscr{A}_\eta$. Moreover if $\eta \leq \eta'$ then $\mathscr{A}_\eta$ is Σ_n embedded into $A_{\eta'}$. If $\mathscr{A}_\eta^T$ is well founded we shall identify it with its transitive collapse which clearly has the form $\langle L_\delta[U_\eta], \in, U_\eta\rangle$. In this case we can naturally extend T to a Σ_n scheme for $\mathscr{A}_\eta^T$, T_η, such that for all η', $(\mathscr{A}_\eta^T)_{\eta'}^{T_\eta} = \mathscr{A}_{\eta+\eta'}$. Also if we denote κ_η as the image of c_κ in A_η then the sequence $\langle \kappa_\eta | \eta \in On\rangle$ forms a continuous increasing sequence.

Definition A.3. A Σ_n iterable structure $(1 \leq n \leq \omega)$ is an acceptable structure $\mathscr{A}$ together with some Σ_n scheme for iteration of $\mathscr{A}$, T, such for every ordinal η, $\mathscr{A}_\eta^T$ is well founded. (Many times we shall not mention T, and refer to $\mathscr{A}$ as the iterable structure where T is understood from the context.)

Definition A.4. x is a member of the core model K if for some iterable structure $\mathscr{A}$ based at κ we have $x \in \mathscr{A}$ and $\operatorname{rank}(x) < \kappa$.

Theorem A.1. *K is a transitive model of* ZFC + G.C.H.

Let $\langle \mathscr{A}, T\rangle$ be a Σ_ω iterable structure $\mathscr{A} = \langle L_\delta[U], \in, U\rangle$. Then there is $U \subseteq \overline{U}$ such that $\mathscr{A}_T^+ = \langle L_{\delta+1}[\overline{U}], \in, \overline{U}\rangle$ is an acceptable structure.

Recall that every member of $L_{\delta+1}[\overline{U}]$ is a definable subset of $L_\delta[U]$ so we have to define $\overline{U}$ on such subsets of ρ, where A is based at ρ. For this use T to define

$$\{x \subseteq \rho | \mathscr{A} \models \Phi(x)\} \in \overline{U} \quad \text{iff} \quad \Phi(d_1) \in T.$$

If $(\mathscr{A}, T)$ is an iterable structure and λ is a regular cardinal $\lambda > |\mathscr{A}|$, then $\mathscr{A}_\lambda$ has the form $\langle L_\alpha[\mathscr{F}_\lambda], \in, \mathscr{F}_\lambda \cap L_\alpha[\mathscr{F}_\lambda]\rangle$, where $\mathscr{F}_\lambda$ is the filter generated by the closed unbounded subsets of λ. This fact motivates the definition of the natural pre-well-ordering of iterable structures.

Definition A.5. Let $(\mathscr{A}, T)$, $(\mathscr{L}, S)$ be two iterable structures. We say that $(\mathscr{A}, T) \leq (\mathscr{L}, S)$ if for sufficiently large regular λ

$$\mathscr{A}_\lambda^T = \langle L_\alpha[\mathscr{F}_\lambda], \in, \mathscr{F}_\lambda \cap L_\alpha[\mathscr{F}_\lambda]\rangle, \qquad \mathscr{L}_\lambda^T = \langle L_\beta[\mathscr{F}_\lambda], \in, \mathscr{F}_\lambda \cap L_\beta[\mathscr{F}_\lambda]\rangle$$

and $\alpha \leq \beta$.

Definition A.6. Let $\langle \mathscr{A}, T\rangle$ be a Σ_n iterable structure. We say that $\langle \mathscr{A}, T\rangle$ is a realized iterable structure if A is based at ρ and for some infinite $C \subseteq \rho$

$$\Phi(d_1, \dots, d_k) \in T \leftrightarrow \text{For some cofinite subset of } C,\ C' \text{ and } \langle \alpha_1, \dots, \alpha_n\rangle, \text{ sequence of members of } C' A \vDash \Phi(\alpha_1, \dots, \alpha_k)$$

for every generalized Σ_n formula Φ in $L^*_{\mathscr{A}}$.

Note that one can easily prove that C as in Definition A.6 must have order type ω. Note also that for a realized iterable structure, $\langle \mathscr{A}, T\rangle$ can be constructed from $\langle A, C\rangle$, so many times we refer to $\langle \mathscr{A}, C\rangle$ as the iterable structure, or even to $\mathscr{A}$ itself when C is understood from the context. If $\langle \mathscr{A}, T\rangle$ is iterable then $A^T_{\eta+w}$ is clearly a realized iterable structure.

The main advantage of being a realized iterable structure is

Lemma A.2. (a) *Let $\langle \mathscr{A}, T\rangle$ be a Σ_n realized iterable structure* $(1 \leq n < w)$. *Then the canonical embedding of $\mathscr{A}$ into $\mathscr{A}_\eta^T$ is a Σ_{n+1} embedding.*

(b) *Let $\langle \mathscr{A}, T\rangle$ be a Σ_ω realized iterable structure and let j be the canonical embedding of $\mathscr{A}$ into $\mathscr{A}_\eta^T$. Then j can be naturally extended to a Σ_1 embedding of $\mathscr{A}_T^+$ into $(\mathscr{A}_\eta^T)^+_{T_\eta}$.*

Proof. (a) Let j be the embedding of $\mathscr{A}$ into $\mathscr{A}_\eta^T$, j is clearly Σ_n. Let $\Phi(x, \vec{y})$ be a Π_n formula and assume

$$\mathscr{A}_\eta^T \vDash \exists x \Phi(x, j(a_1), \dots, j(a_k)) \quad \text{for } a_1, \dots, a_k \in \mathscr{A}.$$

(The converse direction is trivial.) Every element of $\mathscr{A}_\eta^T$ is obtained as $\tau(c_{\alpha_1}, \dots, c_{\alpha_l})$ where τ is a Σ_n Skolem term and $\langle c_{\alpha_1}, \dots, c_{\alpha_l}\rangle$ is a finite sequence of the indiscernibles. We must have:

$$\Phi(\tau(c_1, \dots, c_l), c_{a_1}, \dots, c_{a_k}) \in T.$$

By $\mathscr{A}$ being a realized iterable structure we must have $b_1, \dots, b_l \in \mathscr{A}$ such that $\mathscr{A} \vDash \Phi(\tau(b_1, \dots, b_l), a_1, \dots, a_k)$, hence $\mathscr{A} \vDash \exists x \Phi(x, a_1, \dots, a_k)$.

(b) Define $\bar{j}$ from $\mathscr{A}_T^+$ into $(\mathscr{A}_\eta^T)^+_{T_\eta}$ by

$$\bar{j}(\{x | \mathscr{A} \vDash \Phi(x, \vec{a})\}) = \{x | \mathscr{A}_\eta^T \vDash \Phi(x, j(\vec{a}))\}.$$

We need a lemma which is similar to Lemma 1.10, however here it will be slightly more complicated because of the extra predicate U. As in §1, denote $A^{\psi, \vec{p}}_{\mathscr{A}} = \{x | \mathscr{A} \vDash \psi(x, \vec{p})\}$. For each ψ introduce a new predicate $T^\psi(\vec{p})$ whose meaning is $A^{\psi, \vec{p}}_{\mathscr{A}} \in \overline{U}$.

Lemma A.3. *Let $\psi(x_1, \dots, x_k)$ be a Σ_0 formula in the language of $\mathscr{A}_T^+$, and let $\psi_1, \dots, \psi_k$ be any formulas in the same language. Then there is a formula Φ which is a Boolean combination of formulas in the language of $\mathscr{A}$ and formulas of the form $T^{\psi_i}(\vec{p}_i)$ such that for all η (including $\eta = 0$)*

$$(\mathscr{A}_\eta^T)^+ \models \psi(a_{\mathscr{A}}^{\psi_1, \vec{p}_1}, \dots, U_{\mathscr{A}}^{\psi_k, \vec{p}_k}) \leftrightarrow \mathscr{A}_\eta^T \models \Phi(\vec{p}_1, \dots, \vec{p}_k).$$

Proof. The proof is exactly like Lemma 1.10. The only slightly different case is the quantifier case where ψ is $\exists y \in x_i \psi'(y, x_1, \dots, x_k)$. Then let Φ' be the formula satisfying the lemma for ψ' and $z \in y$, $\psi_1, \dots, \psi_k$. (We consider y to be $\vec{p}_0$.) Let Φ'' be the formula obtained from Φ' replacing $T^{z \in y}(y)$ by $U(y)$ and let Φ be the formula $\exists y(\psi_i(y, \vec{p}_i) \wedge \Phi''(y, \vec{p}_1, \dots, \vec{p}_n))$. It is of the right form because y does not appear in any predicate of the form $T^{\psi_i}(\vec{p}_i)$. □

It follows that $\bar{j}$ is a Σ_0 embedding and as in the proof of part (a) we can get that $\bar{j}$ is a Σ_1 embedding. Lemma A.2 is proved □

Our definition of "mouse" is slightly different from the ususal definition. (The reason for this difference is Lemmas 2.10 and 2.14.)

Definition A.7. (a) An n mouse m is an n-iterable $(1 < n < \omega)$ realized structure such that if m is based at ρ then $m = H_{n+1}^m(p \cup \rho)$ for some finite $p \subseteq m$.

(b) An ω mouse m is an ω-iterable realized structure such that if m is based at ρ then $m^+ = H_1^{m^+}(p \cup \rho)$ for some finite $p \subseteq m$.

Lemma A.4. (a) *If m is a mouse then $m \in K$.*

(b) *If $x \in K$ then there is a mouse based at $\rho > \operatorname{rank}(x)$ such that $x \in m$.*

The proof is very much like the standard proof. Note that we assumed that m is the Σ_{n+1} closure of ρ and some finite set (rather than the Σ_n closure as usual) but we get part (a) by using Lemma A.2.

Lemma A.5. *Let H be a transitive model of* ZF^-, *$m \in H$, m a mouse and t a mouse such that $t < m$ (in the pre-well-ordering of iterable structures) and t is based at ρ such that $\rho \in H$. Then $t \in H$.*

Proof. $t < m$ means that for regular λ large enough, for the λth iterate of t, t_λ, we have $t_\lambda \in m_\lambda$ where m_λ is the λth iterate of m. Assume m is an n mouse $(1 \leq n \leq \omega)$. Hence for some Σ_n Skolem term τ and $c_{\alpha_1}, \dots, c_{\alpha_l}$, $\alpha_i < \lambda$ (where $\langle c_\alpha | \alpha < \lambda \rangle$ are the indiscernibles generating m_λ over m), $t_\lambda = \tau(c_{\alpha_1}, \dots, c_{\alpha_l})$. Let j be the canonical embedding of t into t_λ, where j is a Σ_{k+1} embedding if t is a k mouse. (We shall assume $k < \omega$. The case $k = \omega$ is argued similarly.) $t = H_{k+1}^t(p \cup \rho)$. Hence $j''t = H_{k+1}^{t_\lambda}(j''p \cup \rho)$. (Note that j is the identity on ρ). Also $j''p \in m_\lambda$. Hence $j''\vec{p} = \vec{\mu}(c_{\beta_1}, \dots, c_{\beta_j})$ for some vector of Skolem terms $\vec{\mu}$ and $\beta_1, \dots, \beta_j < \lambda$. Without loss of generality we can assume $\langle c_{\beta_1}, \dots, c_{\beta_j} \rangle = \langle c_{\alpha_1}, \dots, c_{\alpha_l} \rangle = \vec{c}$. Let i be such that $c_{\alpha_1}, \dots, c_{\alpha_i} < \rho$ while $\rho \leq c_{\alpha_{i+1}}, \dots, c_{\alpha_l}$.

Let τ_1, τ_2 be two Σ_{k+1} Skolem terms. Then we have (for $\vec{\rho} = (\rho_1, \ldots, \rho_s)$ where $\rho_j < \rho$)

$$\tau_1^t(\vec{p}, \vec{\rho}) \in \tau_2^t(\vec{p}, \vec{\rho}) \quad \text{iff} \quad m_\lambda \models \tau_1^{\tau(\vec{c})}(\vec{\mu}(\vec{c}), \vec{\rho}) \in \tau_2^{\tau(\vec{c})}(\vec{\mu}(\vec{c}), \vec{\rho}).$$

Let $m_{\rho+\omega}$ be the $(\rho+\omega)$th iterate of m. Since $m \in H$, $\rho \in H$ and H is a model of ZF^-, $m_{\rho+\omega} \in H$. Note that $\langle c_\beta | \beta < \rho + \omega \rangle$ are the indiscernibles generating $m_{\rho+\omega}$. Let $\vec{d} = \langle c_{\alpha_1}, \ldots, c_{\alpha_i}, c_\rho c_{\rho+1}, \ldots, c_{\rho+l-i+1} \rangle$. By indiscerniblity we get

$$m_\lambda \models \tau_1^{\tau(\vec{c})}(\vec{\mu}(\vec{c}), \vec{\rho}) \in \tau_2^{\tau(\vec{c})}(\vec{\mu}(\vec{c}), \vec{\rho}) \quad \text{iff} \quad m_{\rho+\omega} \models \tau_1^{\tau(\vec{d})}(\vec{\mu}(\vec{d}), \vec{\rho}) \in \tau_2^{\tau(\vec{d})}(\vec{\mu}(\vec{d}), \vec{\rho}).$$

Therefore the set

$$B = \{(\tau_1, \tau_2, \vec{\rho}) | \tau_1^t(\vec{p}, \vec{\rho}) \in \tau_2^t(\vec{p}, \vec{\rho})\}$$

is definable in H from $m_{\rho+\omega}$ and $\vec{\alpha}$, but from B one can easily define a structure isomorphic to t. Since H is a model of ZF^- the structure of t is in H. Using completely similar arguments one can reconstruct in H the iteration scheme of t, hence $t \in H$. □

Note that Lemma A.5 does not claim that the sequence of indecernibles for t, C_t is in H. Only the structure t with its scheme for iteration. If we want to recover C_t as well, we need (in case t is an n mouse based at μ) that t is the Σ_{n+1} Skolem hull of $\rho \cup p \cup C_t$ for some p and ρ such that: $\rho < \mu$ and p is a finite subset of t. When we use Lemma A.5 in the proof of Theorem 2.1, we are only interested in having the mouse as an iterable structure as a member of A, so Lemma A.5 is sufficient.

Usually we assume that a mouse, m, is given by specifying the underlying structure and the set of indiscernibles witnessing the fact that m is realized, which we denote by C_m. Many times we shall consider Σ_k hulls $H_k^m(D)$, $D \subseteq m$. We shall always assume that if m is an n mouse then $k \leq n$ and that the set for which we take the Σ_k closure contains C_m. Hence when we consider the transitive collapse of $H_k^m(D)$ we have a natural scheme for k-iteration, using the collapses of the C_m. One can easily show that if $C_m \subseteq D$ then $H_k^m(D)$ is isomorphic to a realized k-iterable structure. (The fact that it is iterable can be proved by embedding its iterations into the iterations of m.) Note that the transitive collapse of $H_k^m(D)$ is not necessarily a mouse.

Also many times we shall consider directed systems, in which the elements are k mice and the embeddings are Σ_k. In all these cases we shall have that if $m \leq m'$ (in the directed system partial order) then the embedding form m to m' maps C_m onto $C_{m'}$. Hence if the directed limit is well founded we shall have a natural scheme for Σ_k iteration of it (by using the image of C_m).

References

[Ba] J. Baumgartner, Unpublished notes on the cardinality of clubs in $P_k(\lambda)$.

[Bo] G. L. Boolos, *On the semantics of the constructible levels*, Z. Math. Logik Grundlag. Math. **16** (1970), 139–148.

[De] K. J. Devlin, *Aspects of constructibility*, Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin and New York, 1973.

[De-Jen] K. Devlin and R. Jensen, *Marginalia to a theorem of Silver* (Proceedings of ISILC, Kiel 1975), Lecture Notes in Math., vol. 499, Springer-Verlag, Berlin and New York, 1975, pp. 115–142.

[Do] A. J. Dodd, *The core model*, Lecture Notes Series, No. 61, London Math. Soc., Cambridge Univ. Press, 1982.

[Do-Jen] A. J. Dodd and R. Jensen, *The core model*, Ann. Math. Logic **20** (1981), 43–75.

[Don-Jen-Ko] H. D. Donder, R. Jensen, and B. Koppelberg, *Some applications of the core model*, Set Theory and Model Theory (R. Jensen and A. Prestel, eds.), Lecture Notes in Math., vol. 872, Springer-Verlag, Berlin, Heidelberg and New York, 1981, pp. 55–97.

[Je 1] T. J. Jech, *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic **5** (1973), 165–198.

[Je 2] T. J. Jech, *Set theory*, Academic Press, New York, 1978.

[Jen] R. Jensen, *The fine structure of the constructible universe*, Ann. Math. Logic **4** (1972), 229–308.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE, HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, ISRAEL (Current address)